PROJECTIVE INTEGRATION FOR INELASTIC BOLTZMANN EQUATION

1. Boltzmann Equation with Inelastic collision

1.1. **The inelastic collision operator.** Assume two particles with velocities v and v_* are going to collide. During the collision, there is some loss of momentum in the impact direction $\omega \in S^{d-1}$ (S^{d-1} is the unit sphere in $\mathbb{R}^d$), resulting in the post-collisional velocities v' and v'_*. Let e stand for the restitution coefficient or inelasticity parameter, then

$$(v' - v'_*) \cdot \omega = -e[(v - v_*) \cdot \omega], \quad 0 \le e \le 1. \tag{1}$$

Using (1), v' and v'_* can be represented as

$$\begin{cases} v' = v - \dfrac{1+e}{2}[(v - v_*) \cdot \omega]\omega, \\ v'_* = v_* + \dfrac{1+e}{2}[(v - v_*) \cdot \omega]\omega. \end{cases} \tag{2}$$

which is ω-representation. One can easily verify the conservation of momentum and dissipation of energy:

$$v' + v'_* = v + v_*; \quad |v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -\frac{1-e^2}{2}[(v - v_*) \cdot \omega]^2 \le 0. \tag{3}$$

For numerical purpose, it is convenient to consider σ-representation:

$$\begin{cases} v' = \dfrac{v + v_*}{2} + \dfrac{1-e}{4}(v - v_*) + \dfrac{1+e}{4}|v - v_*|\sigma, \\ v'_* = \dfrac{v + v_*}{2} - \dfrac{1-e}{4}(v - v_*) - \dfrac{1+e}{4}|v - v_*|\sigma, \end{cases} \tag{4}$$

where σ is another unit vector on S^{d-1} and is related to ω as

$$(g \cdot \omega)\omega = \frac{1}{2}(g - |g|\sigma). \tag{5}$$

Then the collision has the following weak form:

$$\int_{\mathbb{R}^d} Q(f,f)(v)\,\varphi(v)\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{S^{d-1}} B_\sigma(|g|, \sigma \cdot \hat{g}) f f_* \left(\varphi' + \varphi'_* - \varphi - \varphi_*\right) \mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}v_*, \tag{6}$$

where $g = v - v_*$, and (v', v'_*) are given by (4). The collision kernel B_σ may take various forms depending on the types of interactions. The most commonly used form is the variable hard sphere (VHS) model:

$$B_\sigma(|g|, \sigma \cdot \hat{g}) = C_\lambda |g|^\lambda, \quad 0 \le \lambda \le 1, \tag{7}$$

where $C_\lambda > 0$ is some constant. Two special cases are: Maxwell molecules ($\lambda = 0$) and hard spheres ($\lambda = 1$).

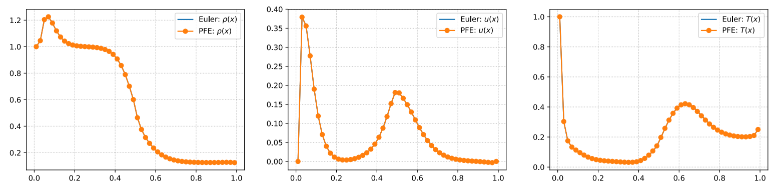

FIGURE 1. Solutions with $e = 0.2$. PFE vs. Euler

1.2. **The inelastic Boltzmann equation.** The inelastic Boltzmann equation reads:

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q_{\mathrm{in}}(f) \tag{8}$$

2. Projective Ingerator

Consider an ODE system:

$$\begin{cases} \dfrac{du}{dt} = g(u(t)), \quad t > 0 \\ u(0) = u_0. \end{cases} \tag{9}$$

where the dynamic has a scale separation, that is, the eigenvalues of $\frac{dg}{du}$ are clustered into two groups separated by a gap.

2.1. **The scheme.**

- Inner integrator: forward Euler method with small time step δt:
$$u^{k+1} = u^k + \delta t g(u^k).$$
- Extrapolate in time:
$$u^{n+1} = u^{n,K+1} + (\Delta t - (K+1)\delta t)\frac{u^{n,K+1} - u^{n,K}}{\delta t}.$$

3. Numerical

We consider the Riemann problem (Sod shock tube problem) as test.

3.1. **Example.** The initial data are given by the Maxwellian distributions computed from the macroscopic quantities

$$(\rho_l, u_l, T_l) = (1, 0, 1), \qquad \text{if } 0 \le x \le 0.5, \tag{10}$$
$$(\rho_r, u_r, T_r) = (0.125, 0, 0.25) \qquad \text{if } 0.5 < x \le 1. \tag{11}$$

We solve the $1D_x \times 2D_v$ inelastic Boltzmann equation with different εs and es.

3.1.1. $\varepsilon = 10^{-2}$. We test the PFE (projective forward euler) scheme compared with the explicite Euler scheme. $\Delta x = 0.02$, for PFE scheme $\Delta t = 0.004$, $\delta t = 10^{-3}$ and for Euler scheme $\Delta t = 0.002$. The macroscopic quantities and solution at $T = 0.1$ are shown in Figure 1.

3.1.2. $\varepsilon = 10^{-4}$. We perfom the same numerical experiment as above. However, since Δt of the Euler scheme is restricted by ε which is very small in this case. We can only run the PFE with a bigger time step $\Delta t = 0.002$.

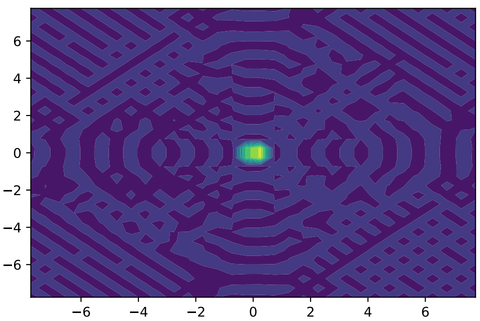

FIGURE 2. Solution f in v when $e = 0.2$.

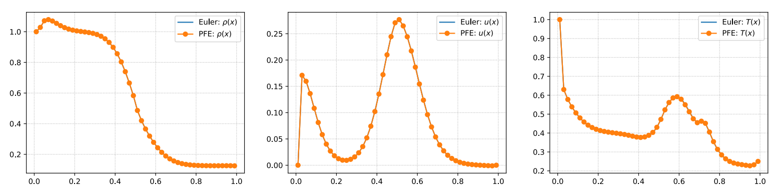

FIGURE 3. Solutions with $e = 0.8$. PFE vs. Euler

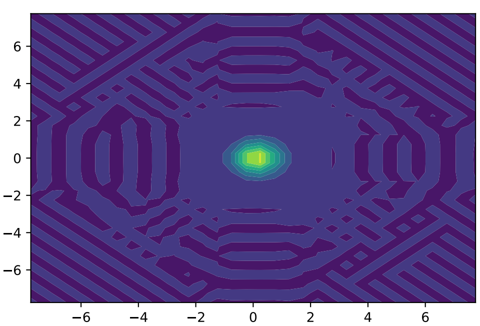

FIGURE 4. Solution f in v when $e = 0.8$.

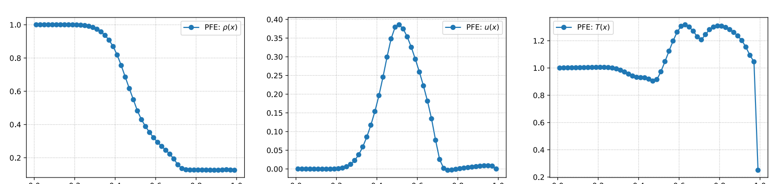

FIGURE 5. Solutions with $e = 1.0$. PFE

References