

# Projective Integration of Nonlinear Collisional Kinetic Equation

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# Outline of the talk

## 1 Introduction

## 2 Toward a high order, explicit, uniformly stable time integrator

- Projective Integration (PI) on a nutshell
- Projective Forward Euler
- Toward high order (and beyond?)

## 3 Application to kinetic equations

- On collisional kinetic equations
- Examples of kinetic models
- PI for collisional kinetic equation

## 4 Numerical Methods

- Summary
- Fast spectral method for the Boltzmann operator

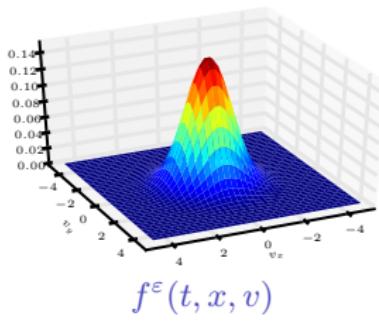
## 5 Numerical simulations

## 6 Conclusion

# The general setting

- **Mesoscopic:** Kinetic equations (Boltzmann, Vlasov, . . . )

$$\partial_t f + v \cdot \nabla_x f + K \cdot \nabla_v f = \mathcal{Q}(f)/\varepsilon$$



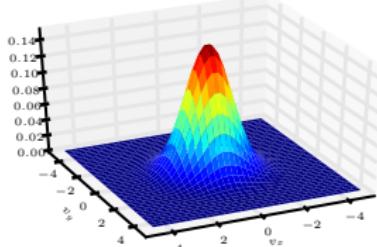
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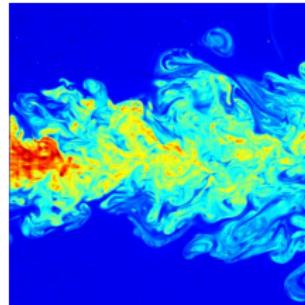
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$$\partial_t \mathbf{U} + \nabla_{\mathbf{x}} \cdot \mathbf{F}(\mathbf{U}) = \mathbf{G}(\mathbf{U})$$



$f^\varepsilon(t, x, v)$



$\rho(t, x), \mathbf{u}(t, x), E(t, x)$

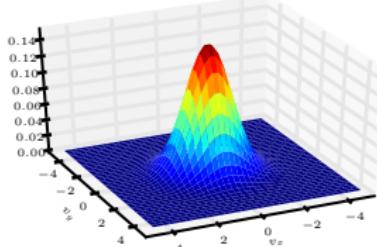
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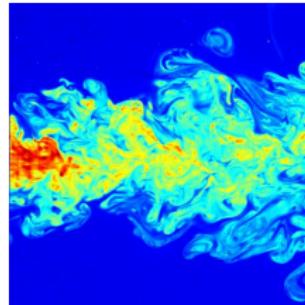
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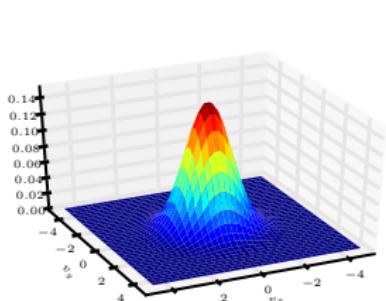
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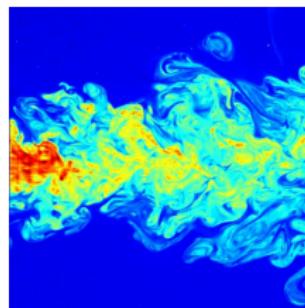
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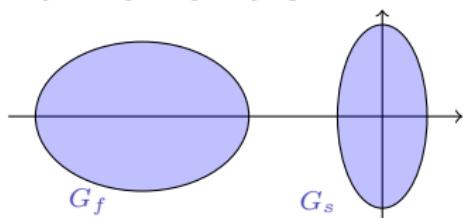
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**Numerical simulations:** E. Tadmor, B. Perthame, P. Degond, L. Pareschi, J. A. Carrillo, E. Sonnendrücker, A. Klar, S. Jin, F. Filbet, etc.

# Projective Integration “à la Gear and Kevrekidis”

We want to approximate the system of ODEs

$$\begin{cases} u'(t) = g(u(t)), & t > 0 \\ u(0) = u_0 \in \mathbb{R}^N, \end{cases}$$



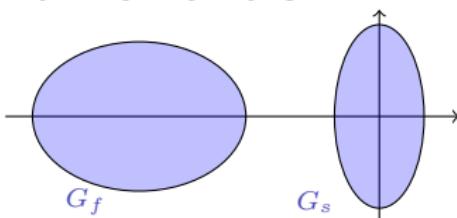
where  $\partial g / \partial u$  eigenvalues are clustered into two groups  $G_f, G_s \subset \mathbb{C}$ , separated by a gap ( $\sim$  stiffness):  $G_s$  is located in a neighborhood of the origin (slow components), and  $G_f$  far in the left-half plane (fast components).

**Consequence:**  $u$  is projected onto a low dimensional manifold in a very short time.

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**Formal idea:**

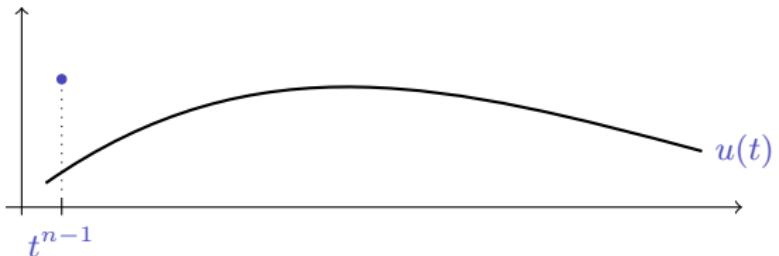
- Perform a number of small time steps of an inner integrator, corresponding to the fast rate of damping of  $u$  towards the equilibrium manifold.
- Extrapolate forward with a large time step, corresponding to the slow manifold.
- The inner integrator is explicit because its time steps  $\delta t$  are very small:

$$\delta t \simeq \mathcal{O}(\min |\lambda| : \lambda \in G_s)$$

# Projective Forward Euler (PFE) scheme

Gear, Kevrekidis, SINUM, 2003

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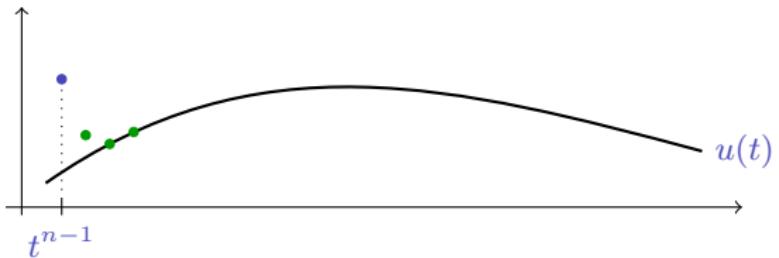
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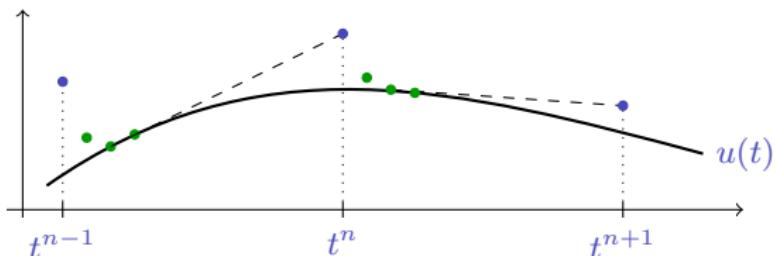
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**Outer integrator.** Let  $\Delta t$  be a regular time step, and  $u^n \simeq u(t^n)$ ,  $t^n := n\Delta t$

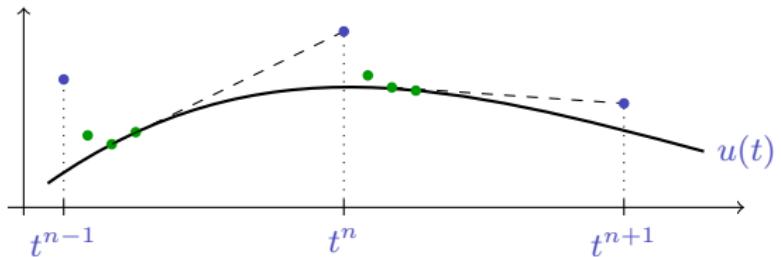
- Take  $K + 1$  **inner** steps of size  $\delta t$ , denoting by  $u^{n,k} \simeq u(t^n + k\delta t)$ .
- **Extrapolate** in time (**projective Forward Euler, PFE**) to compute  $u^{n+1} := u^{n+1,0}$

$$u^{n+1} = u^{n,K+1} + (\Delta t - (K + 1)\delta t) \frac{u^{n,K+1} - u^{n,K}}{\delta t}.$$

- **Iterate**

# Linear stability

$$\begin{cases} u'(t) = \lambda u(t), & t > 0 \\ u(0) = u_0 \in \mathbb{R}, \end{cases}$$



We have  $u^{n+1} = \sigma(\tau)u^n$  where  $\sigma(\tau) = ((M + 1)\tau - M)\tau^K$  and  $M = \Delta t/\delta t - (K + 1)$ .

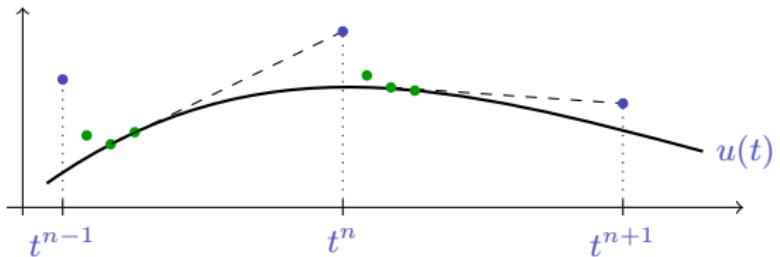
## Theorem (Gear, Kevrekidis, 2003, SINUM)

Let  $\mathcal{D}(\lambda, r) = \{z \in \mathbb{C} : |z - \lambda| \leq r\}$ . Then

$$|\sigma(\tau)| \leq 1 \Leftrightarrow \tau \in \mathcal{D}\left(1 - \frac{\delta t}{\Delta t}, \frac{\delta t}{\Delta t}\right) \cup \mathcal{D}\left(0, \left(\frac{\delta t}{\Delta t}\right)^{1/K}\right).$$

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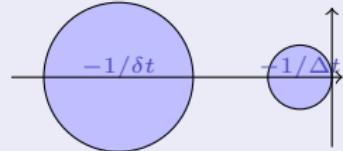
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**Corollary.** The PFE method is **linearly stable** if, and only if

$$\lambda \in \mathcal{D}\left(-\frac{1}{\Delta t}, \frac{1}{\Delta t}\right) \cup \mathcal{D}\left(-\frac{1}{\delta t}, \frac{1}{\delta t} \left(\frac{\delta t}{\Delta t}\right)^{1/K}\right)$$



# Projective Runge-Kutta method

Higher-order projective Runge-Kutta (**PRK**) methods can be constructed by replacing each time derivative evaluation  $k_s$  in a classical Runge-Kutta method by  $K+1$  steps of an inner integrator as follows:

$$\begin{aligned} s = 1 : \quad & \begin{cases} u^{n,k+1} &= u^{n,k} + \delta t g(u^{n,k}), \\ k_1 &= \frac{u^{n,K+1} - u^{n,K}}{\delta t} \end{cases} \quad 0 \leq k \leq K \\ 2 \leq s \leq S : \quad & \begin{cases} u_s^{n+c_s,0} &= u^{n,K+1} + (c_s \Delta t - (K+1)\delta t) \sum_{l=1}^{s-1} \frac{a_{s,l}}{c_s} k_l, \\ u_s^{n+c_s,k+1} &= u_s^{n+c_s,k} + \delta t g(u_s^{n+c_s,k}), \\ k_s &= \frac{u_s^{n+c_s,K+1} - u_s^{n+c_s,K}}{\delta t} \end{cases} \quad 0 \leq k \leq K \\ & u^{n+1} = u^{n,K+1} + (\Delta t - (K+1)\delta t) \sum_{s=1}^S b_s k_s. \end{aligned}$$

To ensure consistency, the Runge-Kutta matrix  $\mathbf{a} = (a_{s,i})_{s,i=1}^S$ , weights  $\mathbf{b} = (b_s)_{s=1}^S$ , and nodes  $\mathbf{c} = (c_s)_{s=1}^S$  satisfy the usual conditions  $0 \leq b_s \leq 1$  and  $0 \leq c_s \leq 1$ , as well as:

$$\sum_{s=1}^S b_s = 1, \quad \sum_{i=1}^{S-1} a_{s,i} = c_s, \quad 1 \leq s \leq S.$$

# A general Boltzmann-like equation

## Scaled form

Study of a **particle distribution function**  $f^\varepsilon(t, x, v)$ , depending on the time  $t > 0$ , space position  $x \in \Omega \subset \mathbb{R}^{d_x}$ ,  $d_x \in \{1, 2, 3\}$  and particle velocity  $v \in \mathbb{R}^{d_v}$ ,  $d_v \geq d_x$ , solution to

$$(1) \quad \begin{cases} \frac{\partial f^\varepsilon}{\partial t} + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} \mathcal{Q}(f^\varepsilon), \\ f^\varepsilon(0, x, v) = f_{in}(x, v), \end{cases}$$

where  $\mathcal{Q}$  is the **collision operator**, describing the microscopic collision dynamics between particles and  $\varepsilon$  is the **Knudsen number**, ratio between the mean free path between collisions and the typical length scale.

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- Huge phase space (up to 7-D!)  $\Rightarrow$  Deterministic numerical simulations very costly!
- Stiff (possibly multi-scale), highly nonlinear problem  $\Rightarrow$  Impliciting almost impossible!

# Mathematical properties of the collision operator

- Conservation of mass, momentum and kinetic energy

$$\int_{\mathbb{R}^{d_v}} \mathcal{Q}(f)(v) dv = 0, \quad \int_{\mathbb{R}^{d_v}} \mathcal{Q}(f)(v) v dv = 0, \quad \int_{\mathbb{R}^{d_v}} \mathcal{Q}(f)(v) |v|^2 dv = 0;$$

- Dissipation of Boltzmann entropy

$$\int_{\mathbb{R}^{d_v}} \mathcal{Q}(f)(v) \log(f)(v) dv \leq 0;$$

- Explicit equilibria, known as Maxwellian distribution

$$\mathcal{Q}(f) = 0 \quad \Leftrightarrow \quad f = \mathcal{M}_{\rho, u, T} := \frac{\rho}{(2\pi T^{3/2})} \exp\left(-\frac{|v - u|^2}{2T}\right);$$

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- $0^{th}$  order fluid limit  $\varepsilon \rightarrow 0$  given by the compressible Euler system

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u} + \rho T \mathbf{I}) = \mathbf{0}_{\mathbb{R}^{d_v}}, \\ \partial_t E + \operatorname{div}_x(\mathbf{u}(E + \rho T)) = 0. \end{cases}$$

# The Boltzmann equation

It describes the non equilibrium behavior of a diluted gas of solid particles, interacting only via binary elastic collisions<sup>1</sup>



## Boltzmann collision operator

$$\mathcal{Q}_B(f)(v) = \int_{\mathbb{R}^{d_v} \times \mathbb{S}^{d_v-1}} [f'_* f' - f_* f] B(|v - v_*|, \cos \theta) d\sigma dv_*$$

where  $\cos \theta := (v - v_*) \cdot \sigma$ ,  $B$  is the collision kernel:  
 $B(r, s) = b(s)r^\alpha$  with  $\alpha \in [0, 1]$  and

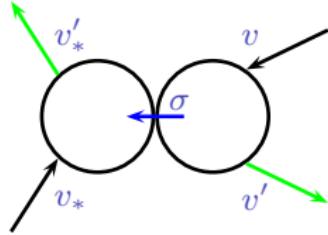
$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v^*}{2} - \frac{|v - v_*|}{2} \sigma.$$

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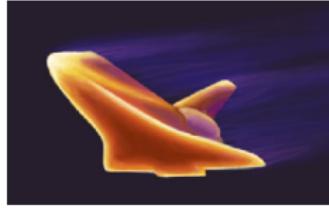
# The BGK equation

The **BGK**<sup>2</sup> equation replaces the quadratic Boltzmann operator by a **nonlinear relaxation** operator which mimics its main features.

## Applications

Microscale flow in MEMS, space shuttle atmospheric re-entry, . . . , but the simpler structure of the operator allows for easier computations (with a cost in accuracy)

### BGK operator



$$\mathcal{Q}_{BGK}(f)(v) = \nu(\rho) [\mathcal{M}_{\rho, u, T}(v) - f(v)],$$

where

$$(\rho, u, T) = \int_{\mathbb{R}^{dv}} f(t, x, v) \varphi(v) dv$$

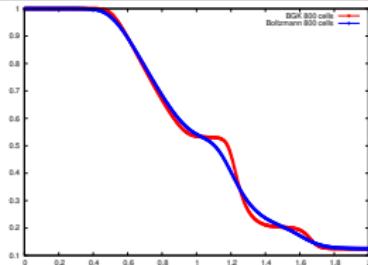
for  $\varphi(v) = (1, v, |v - u_f|^2)$  are the mass, velocity and local temperature of  $f$  and  $\mathcal{M}_{\rho, u, T}$  the associated Maxwellian distribution.

<sup>2</sup>Bhatnagar, Gross, Krook, *Phys. Rev.* (1954)

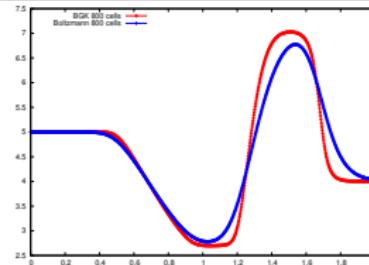
# Riemann problem (Sod's tube) $1D_x \times 2D_v$

BGK vs. Boltzmann

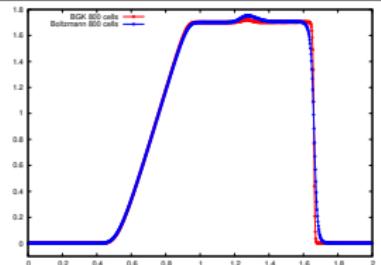
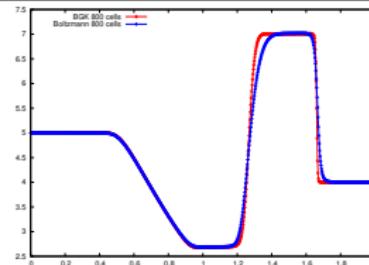
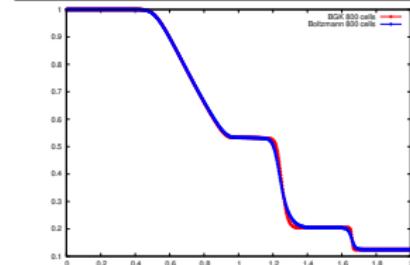
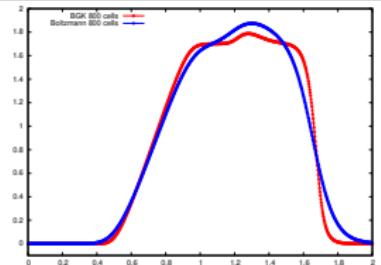
Density  $\rho$



Temperature  $T$



Velocity  $u_x$



BGK (red) and Boltzmann (blue) solutions for  $\varepsilon = 10^{-2}$  (top) and  $\varepsilon = 10^{-4}$ , at  $t = 0.15$  with 800 spatial cells and  $64^2$  velocity cells<sup>3</sup>

<sup>3</sup>from Dimarco, Loubère, Narski, TR, JCP 2018

# Kinetic approximation of conservation laws

Let  $f^\varepsilon \in L^1_{x,v} ((1 + v) dv)$  solution to the kinetic equation

$$(2) \quad \partial_t f^\varepsilon + v \partial_x f^\varepsilon = \frac{1}{\varepsilon} (R[u^\varepsilon] - f^\varepsilon),$$

where  $u^\varepsilon = \int_{\mathbb{R}} f^\varepsilon(v) dv$  and

$$\int_{\mathbb{R}} R[u^\varepsilon](v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} u^\varepsilon \\ g(u^\varepsilon) \end{pmatrix}.$$

Then, when  $\varepsilon \rightarrow 0$ ,  $u^\varepsilon$  converges<sup>4</sup> toward  $u$ , solution to the scalar conservation law

$$(3) \quad \partial_t u + \partial_x g(u) = 0$$

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<sup>4</sup>Brenier 81-84; Lions, Perthame, Tadmor 91-94; Bouchut 99

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Discretizing (2) in  $v$  on a uniform grid and in  $x$  with upwind fluxes, one can prove

**Theorem (Lafitte, Leijon, Melis, Samaey, 2012-2014)**

Choosing the parameters of the **PFE scheme** as  $K = 2$ ,  $\delta t = \varepsilon$  and  $\Delta t$  as the hyperbolic CFL coming from (3) provides a  $\varepsilon$ -uniformly stable time integrator for (2), whose limit is a stable approximation to (3). It is also consistent in the **linear** case.

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**Theorem (Lafitte, Leijon, Melis, Samaey, 2012-2014)**

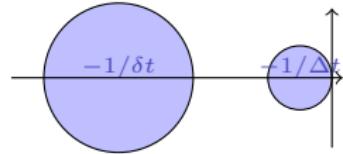
Choosing the parameters of the PFE scheme as  $K = 2$ ,  $\delta t = \varepsilon$  and  $\Delta t$  as the hyperbolic CFL coming from (3) provides a  $\varepsilon$ -uniformly stable time integrator for (2), whose limit is a stable approximation to (3).

**Core of the proof.** Compute the **slow** and **fast** eigenvalue branches:

$$\lambda^s = -\lambda_1^s \varepsilon + i \mu^s \varepsilon^2 + \mathcal{O}(\varepsilon^3), \quad \lambda^f = -\frac{1}{\varepsilon} - \lambda_1^f \varepsilon - i \mu^f \varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

then use the stability criterion of the PFE method: it is **linearly stable** if, and only if

$$\lambda \in \mathcal{D} \left( -\frac{1}{\Delta t}, \frac{1}{\Delta t} \right) \cup \mathcal{D} \left( -\frac{1}{\delta t}, \frac{1}{\delta t} \left( \frac{\delta t}{\Delta t} \right)^{1/K} \right)$$



# Spectrum of the linearized BGK and Boltzmann operators

And now, for something (slightly) different

Let  $\mathcal{M} = \mathcal{M}_{1,0,1}$  and define the **linearized** Boltzmann operator as

$$\mathcal{L}_{\mathcal{M}} g := \mathcal{M}^{-1} (\mathcal{Q}(\mathcal{M}, g) + \mathcal{Q}(g, \mathcal{M})) = K_{\mathcal{M}} g - \nu(v) g$$

where  $K_{\mathcal{M}}$  is a compact operator on  $L_v^2(\mathcal{M}^{-1} dv)$  and  $\nu$  is bounded by below.

Going to Fourier in space, one has the **linearized Boltzmann equation** (similar with BGK):

$$(4) \quad \partial_t g = \frac{1}{\varepsilon} K_{\mathcal{M}} g - (\nu(v)/\varepsilon + i \varepsilon \gamma \cdot v) g.$$

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**Theorem** (Grad '56; McLennan '65; Nicolaenko '71; **Ellis-Pinsky '75**; Degond-Lemou '97)

The spectrum of the RHS of (4) is composed of

- **fast modes:** Eigenvalues located at a distance at least  $1/\varepsilon$  on the left of the imaginary axis;
- **slow modes:** if  $|\varepsilon| \ll 1$ , exactly  $d_v + 2$  eigenvalues branches given by

$$\lambda^{(j)}(|\gamma|) := i \lambda_1^{(j)} \varepsilon |\gamma| - \lambda_2^{(j)} \varepsilon^2 |\gamma|^2 + \mathcal{O}(\varepsilon^3 |\gamma|^3),$$

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In the Boltzmann case, an **essential spectrum** also exists...

# THE FIRST AND SECOND FLUID APPROXIMATIONS TO THE LINEARIZED BOLTZMANN EQUATION (\*)

By Richard S. ELLIS and Mark A. PINSKY

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## 1. A. Introduction

Let  $p_\epsilon(t, x, \xi)$  be the solution of the Boltzmann equation :

$$(1.1) \quad \frac{\partial p}{\partial t} + \xi \cdot \text{grad } p = \frac{1}{\epsilon} Q p,$$

$$\lim_{t \downarrow 0} p(t, x, \xi) = f(x, \xi),$$

in a Euclidean domain  $D$ , where boundary conditions are prescribed on  $\partial D$  if  $D$  is finite. When  $\epsilon \rightarrow 0$ , a great simplification occurs in the solution of (1.1), known as a "contraction of the description". This is formally treated by the Chapman-Enskog expansion at the physical level of rigor [17].

To make this precise, Grad [8] first considered (1.1) in a cube  $D \subset \mathbb{R}^3$  with periodic boundary conditions, where  $Q$  is the linearized collision operator corresponding to a spherically symmetric potential function with a hard core. Using a priori estimates for (1.1), he proved that for  $f$  suitably smooth

$$(1.2) \quad T_\epsilon(t)f - E(t)f = 0(\epsilon) \quad (\epsilon \downarrow 0),$$

$$(1.3) \quad T_\epsilon\left(\frac{t}{\epsilon}\right)f - N_\epsilon\left(\frac{t}{\epsilon}\right)f = 0(\epsilon) \quad (\epsilon \downarrow 0),$$

where  $T_\epsilon(t)f = p_\epsilon$  is the solution of (1.1) and  $E(t), N_\epsilon(t)$  denote, respectively, the solution operators for the linear Euler and Navier-Stokes equations with viscosity and heat conduction coefficients proportional to  $\epsilon$ . These systems of partial differential equations are derived by means of the classical Chapman-Enskog-Hilbert expansion as applied to the linearized Boltzmann equation (1.1).

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(\*) Research supported in part by National Science Foundation Grant GP 28576.

# PI for the BGK and Boltzmann equations

The spectrum of the linearized BGK operator is composed of

- Eigenvalues located at a distance at least  $1/\varepsilon$  on the left of the imaginary axis;
- If  $|\varepsilon| \ll 1$ ,  $d_v + 2$  eigenvalues branches given by

$$\lambda^{(j)}(|\gamma|) := i \lambda_1^{(j)} \varepsilon |\gamma| - \lambda_2^{(j)} \varepsilon^2 |\gamma|^2 + \mathcal{O}(\varepsilon^3 |\gamma|^3),$$

- **Fast** (exponential?) rate of damping of the solution to the full BGK equation toward Maxwellian distribution  $\Rightarrow$  **Linear regime**  $\Rightarrow$  Taking the same parameters for the **PFE** scheme as before  $K = 2$ ,  $\delta t = \varepsilon$  and  $\Delta t$  as the hyperbolic CFL coming from the compressible Euler dynamics will give an  **$\varepsilon$ -stable, uniformly accurate, explicit** time integrator for the BGK equation!

<sup>5</sup>Gear, Kevrekidis, SINUM 2004; Melis, Samaey, JSC 2018

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- In the Boltzmann case, an **essential spectrum** is also present: need to use **Telescopic Projective Integration**<sup>5</sup>, which brings a  $\log(1/\varepsilon)$  dependency on  $K$ .

<sup>5</sup>Gear, Kevrekidis, SINUM 2004; Melis, Samaey, JSC 2018

# Summary of the numerical solvers

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} \mathcal{Q}(f)$$

- Introduce a cartesian grid  $\mathcal{V}$  of  $\mathbb{R}^{d_v}$  by  $\mathcal{V} = \{v_k = k\Delta v + a, k \in \mathcal{K}\}$  and denote the discrete collision invariants on  $\mathcal{V}$  by  $m_k = (1, v_k, \frac{1}{2}|v_k|^2)$ .
- Set for any  $v_k$ :

$$f_k(x, t) \approx f(x, v_k, t).$$

The fluid quantities are then obtained from  $f_k$  :

$$U(x, t) = \sum_k m_k f_k(x, t) \Delta v.$$

- The discrete velocity model becomes a set of  $N$  equations for  $f_k$

$$\partial_t f_k + v_k \cdot \nabla_x f_k = \mathcal{Q}(f_k),$$

where the term  $\mathcal{Q}(f_k)$  couples all the equations.

- Free transport term  $\text{div}_x(v_k f_k)$  computed with WENO reconstruction.
- PRK time stepping.

# Fast spectral discretization of the Boltzmann operator

The main idea is to use the so-called Carleman representation:

$$\mathcal{Q}_B(f) = \int_{\mathbb{R}^{dv}} \int_{\mathbb{R}^{dv}} \tilde{B}(x, y) \delta(x \cdot y) [f(v + y) f(v + x) - f(v + x + y) f(v)] dx dy,$$

with

$$\tilde{B}(|x|, |y|) = 2^{dv-1} B \left( \sqrt{|x|^2 + |y|^2}, \frac{|x|}{\sqrt{|x|^2 + |y|^2}} \right) (|x|^2 + |y|^2)^{-\frac{dv-3}{2}}.$$

It yields the spectral quadrature formula

$$\hat{\mathcal{Q}}_k = \sum_{\substack{l, m = -N \\ l+m=k}}^N \hat{\beta}_F(l, m) \hat{f}_l \hat{f}_m, \quad k = -N, \dots, N$$

where  $\hat{\beta}_F(l, m) = \mathcal{B}_F(l, m) - \mathcal{B}_F(m, m)$  are given by

$$\mathcal{B}_F(l, m) = \int_{\mathcal{B}_0(R)} \int_{\mathcal{B}_0(R)} \tilde{B}(x, y) \delta(x \cdot y) e^{i(l \cdot x + m \cdot y)} dx dy.$$

# Fast spectral discretization II

- Now, we look for a convolution structure. The aim is to approximate each  $\hat{\beta}_F(l, m)$  by a sum

$$\hat{\beta}_F(l, m) \simeq \sum_{p=1}^A \alpha_p(l) \alpha'_p(m)$$

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An example, the two dimensional case:

- Make the decoupling assumption

$$\tilde{B}(x, y) = a(|x|) b(|y|);$$

satisfied if e.g.  $\tilde{B}$  is constant (2D Maxwellian molecules, 3D hard spheres).

- This gives

$$\mathcal{B}_F(l, m) = \int_0^\pi \phi_R^2(l \cdot e_\theta) \phi_R^2(m \cdot e_{\theta+\pi/2}) d\theta, \quad \phi_R^2(s) = 2R \operatorname{sinc}(Rs).$$

- A regular discretization of  $M$  equally spaced points gives

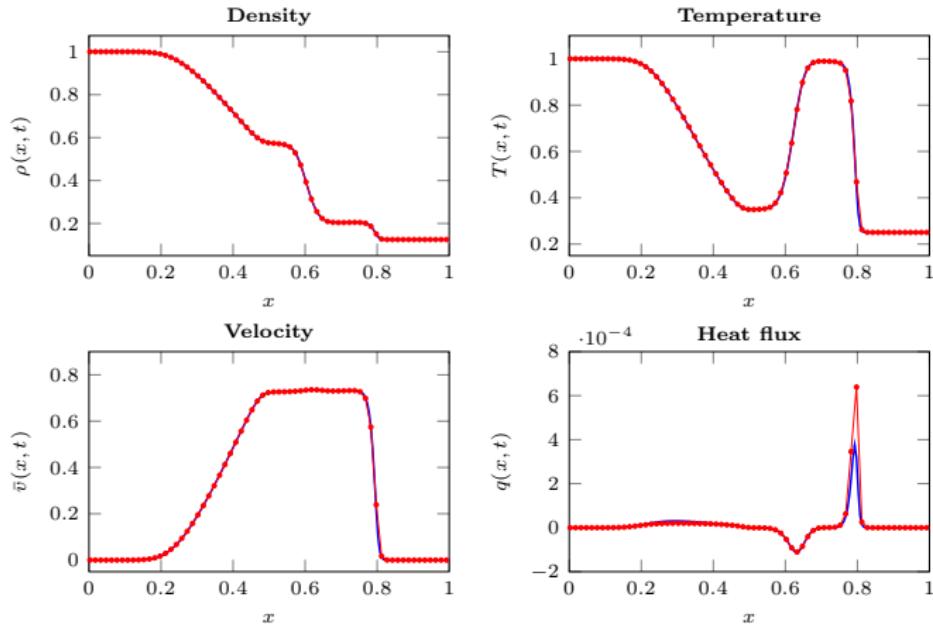
$$\mathcal{B}_F(l, m) = \frac{\pi}{M} \sum_{p=0}^{M-1} \alpha_p(l) \alpha'_p(m), \quad \alpha_p(l) = \phi_R^2(l \cdot e_{\theta_p}), \quad \alpha'_p(m) = \phi_R^2(m \cdot e_{\theta_p+\pi/2})$$

# $1D_x - 1D_v$ BGK

Sod shock tube problem, PRK4 vs. RK4, WENO 3 in  $x$

First moments of the solution to the BGK equation with  $\nu = 1$   
 RK4 time integrator (red dots) vs. the new PRK4 integrator (blue lines):

PRK4 is 130 times more efficient!

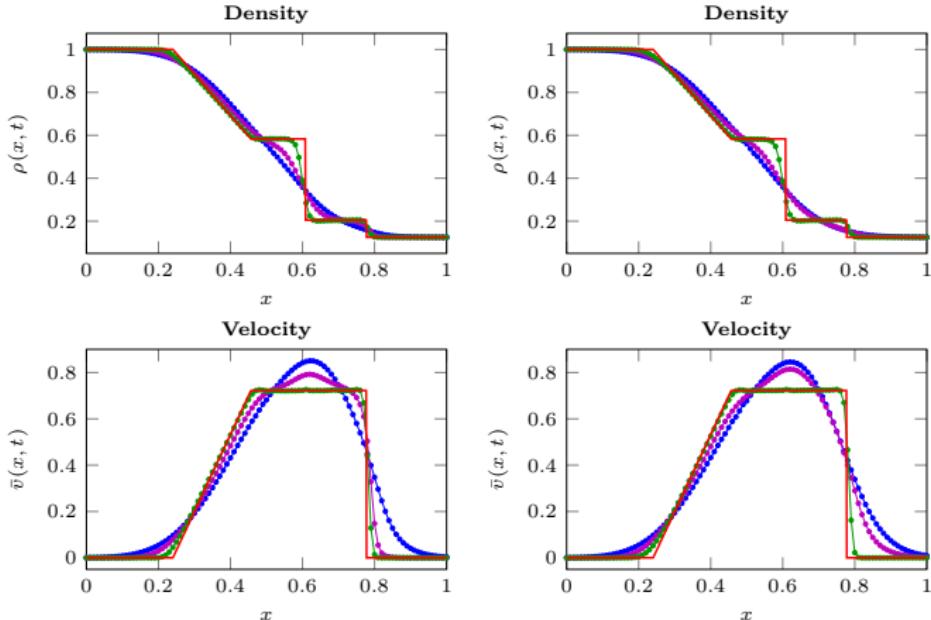


$$\Delta t = 0.4 \Delta x, \Delta x = 0.01, N_v = 80, K = 2 \text{ and } \delta t = \varepsilon, \text{ for } \varepsilon = 10^{-4}$$

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First moments of the solution to the BGK equation with  $\nu = 1$  (left) and  $\nu = \rho$  (right)

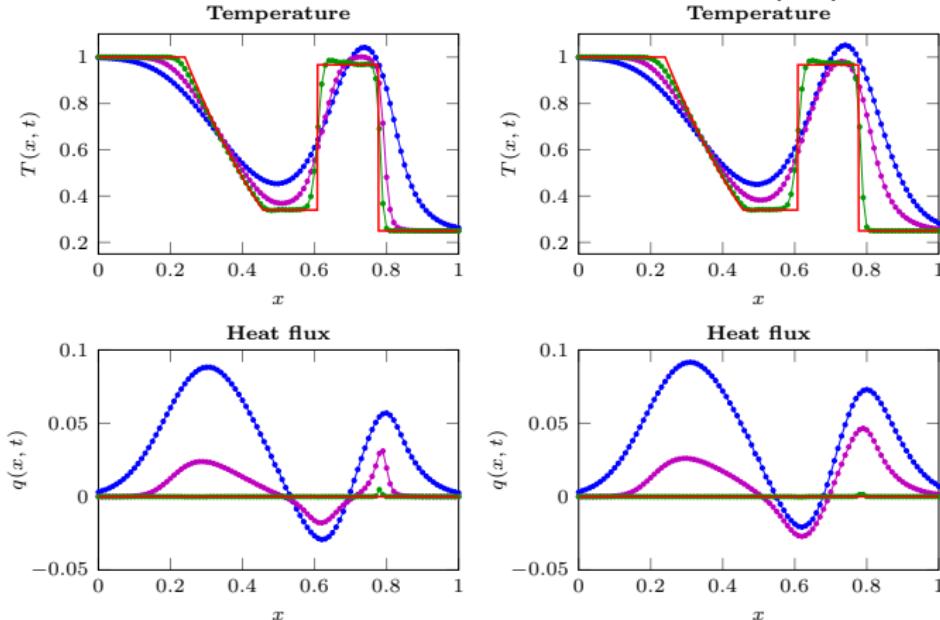


$\Delta t = 0.4\Delta x$ ,  $\Delta x = 0.01$ ,  $N_v = 80$ ,  $K = 2$  and  $\delta t = \epsilon$ , for  $\epsilon = 10^{-1}$  (blue dots),  $10^{-2}$  (purple dots),  $10^{-5}$  (green dots); red line: compressible Euler limit  $\epsilon = 0$ .

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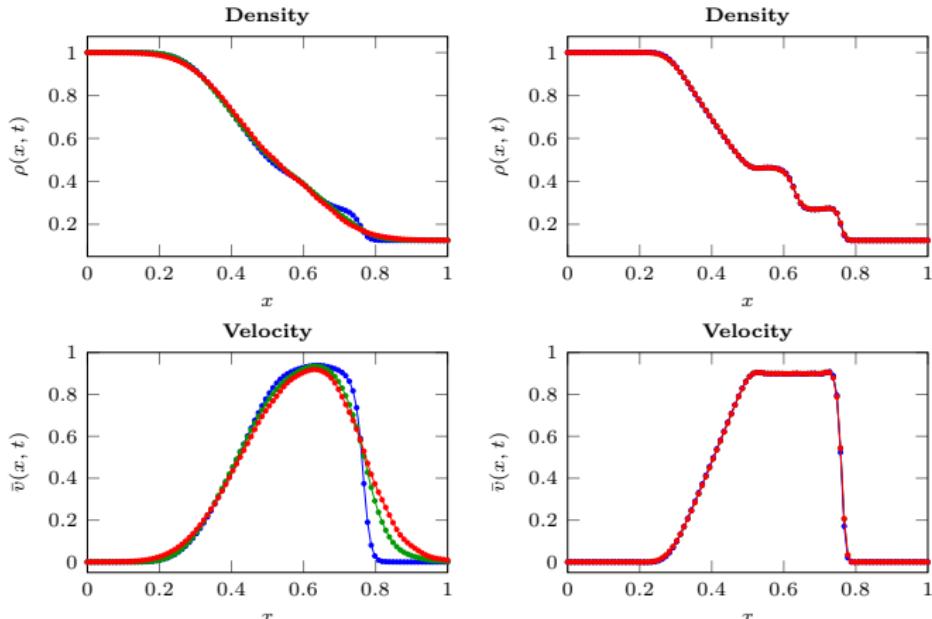


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# $1D_x - 2D_v$ BGK vs. Boltzmann

Sod shock tube problem, PRK4 time integrator, WENO 2 in  $x$ , fast spectral in  $v$

First moments of BGK equation with  $\nu = 1$  (blue),  $\nu = \rho$  (green) and Boltzmann (red)



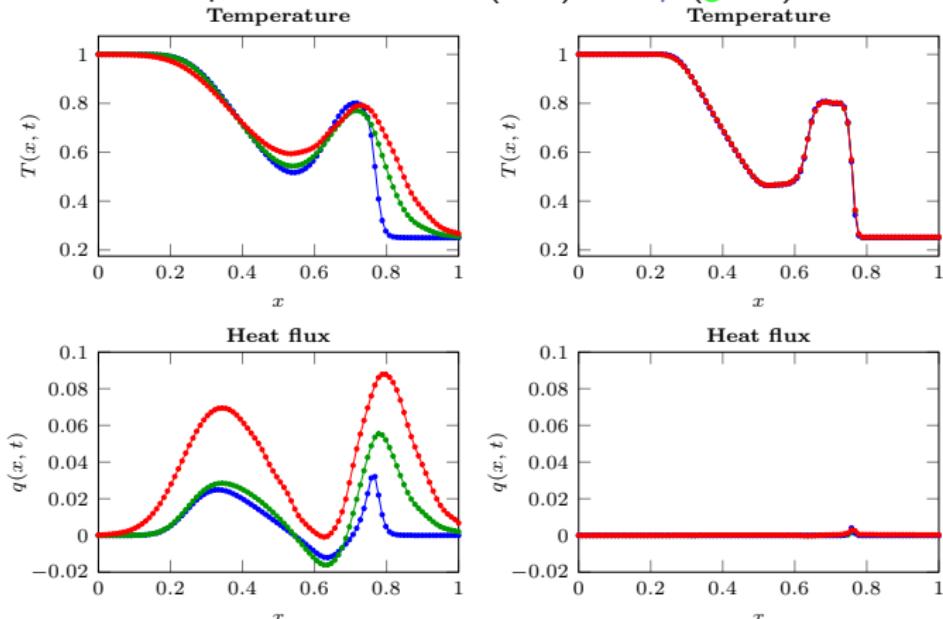
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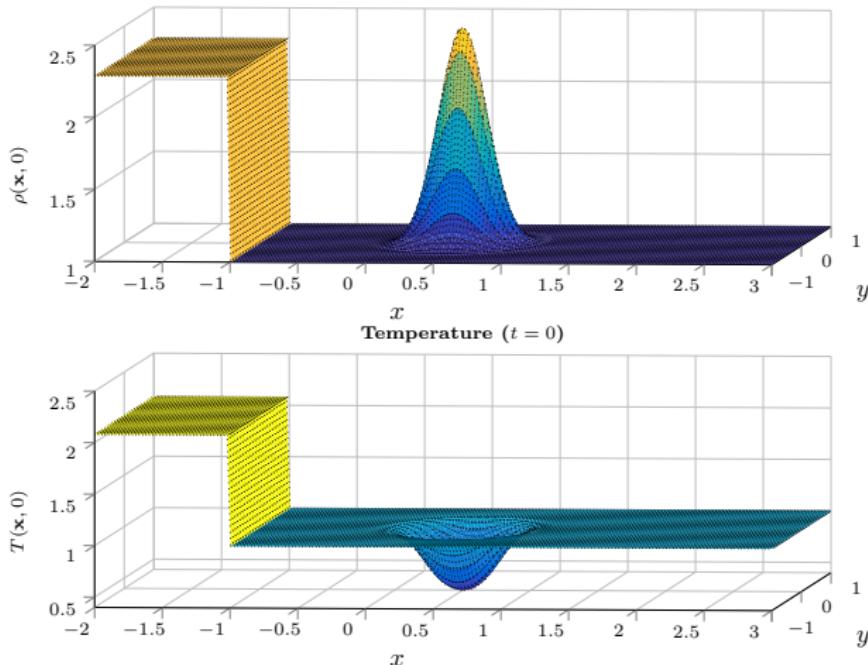


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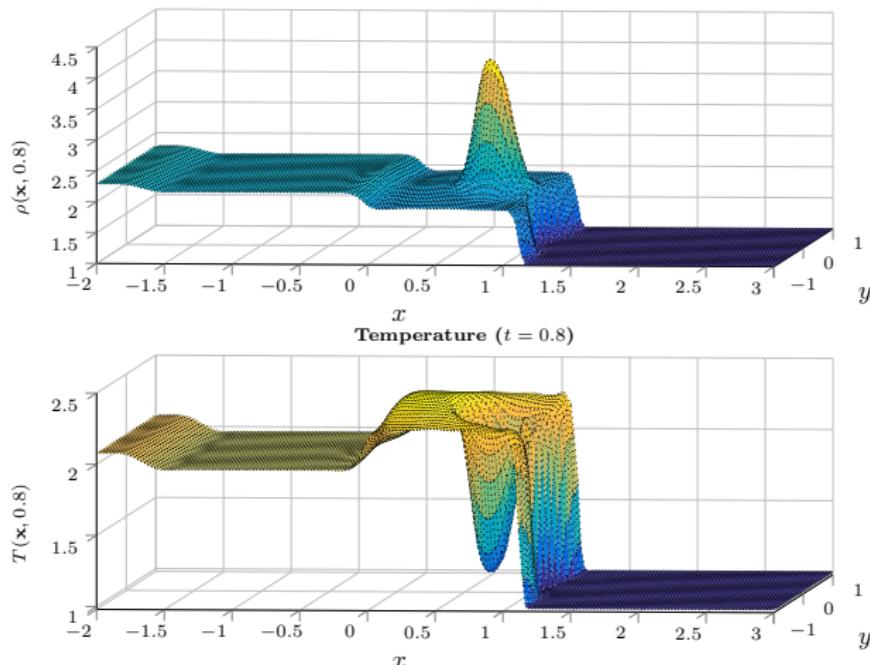
Shock-Bubble interaction, PRK4 time integrator, WENO 2 in  $x$ ,  $\varepsilon = 10^{-5}$ ,  $\nu = 1$



$\Delta t = 0.4\Delta x$ ,  $N_x = 200 \times 25$ ,  $N_v = 32^2$ ,  $K = 2$  and  $\delta t = \varepsilon$ .  
PRK4 is 133 times more efficient than RK4.

# $2D_x - 2D_v$ BGK

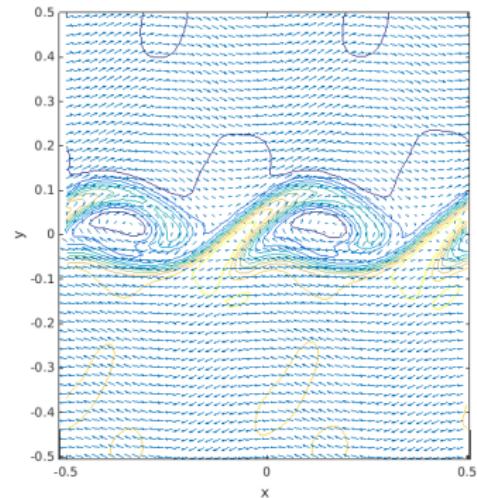
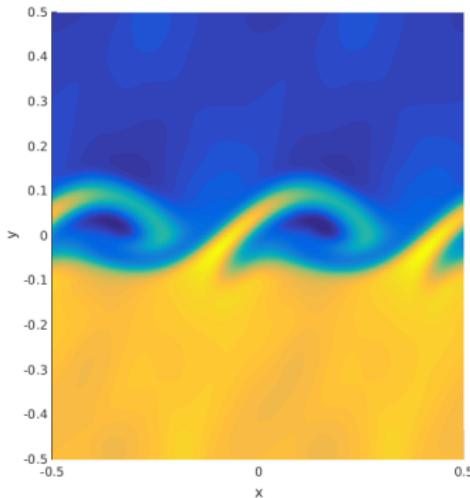
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# $2D_x - 2D_v$ BGK

Kelvin-Helmholtz instability, TPRK4 time integrator, WENO 3 in  $x$ ,  $\varepsilon = 5 \cdot 10^{-5}$ ,  $\nu = \rho$



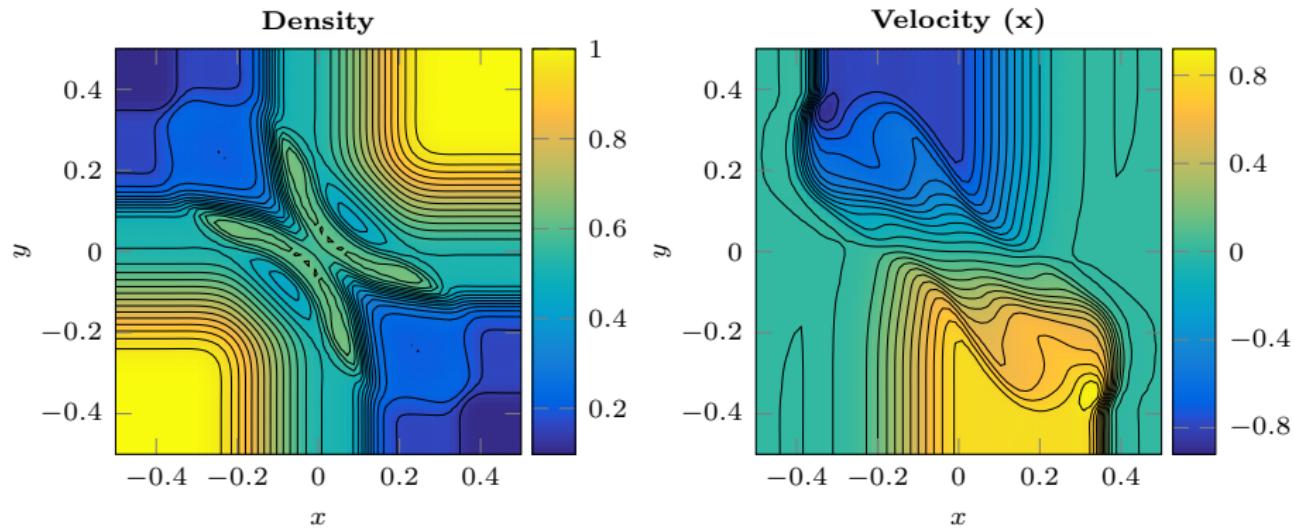
$\Delta t = 0.4\Delta x$ ,  $N_x = 100^2$ ,  $N_v = 32^2$ ,  $K = 3$  and  $\delta t = \varepsilon$ .

Density (left) and stream lines (right)

TPRK4 is 22 times more efficient than RK4.

# $2D_x - 2D_v$ Boltzmann

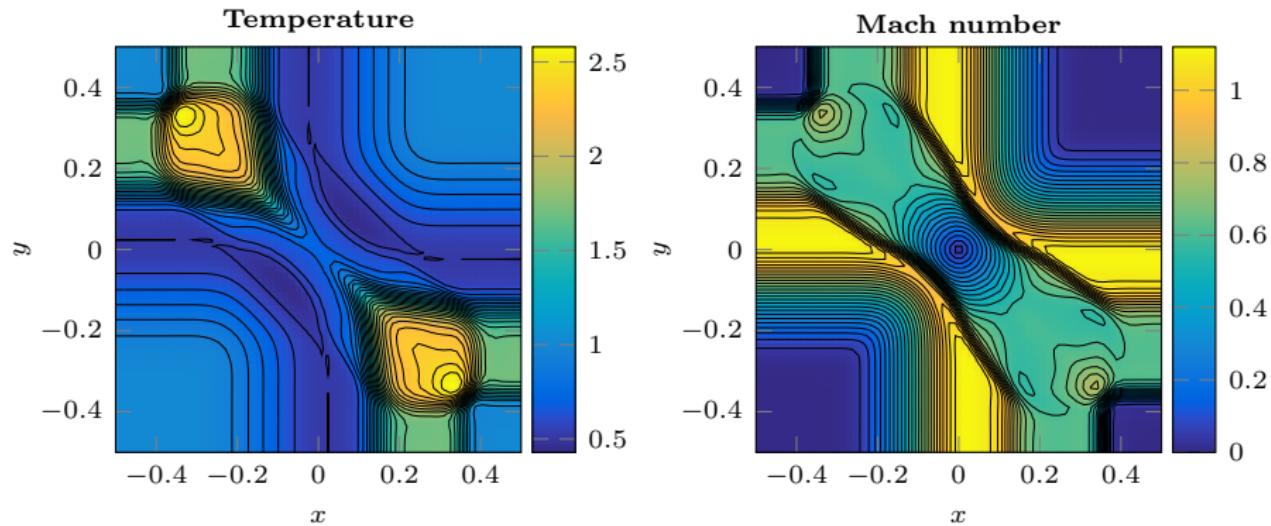
Double Sod shock, TPRK4 time integrator, WENO 2 in  $x$ ,  $\varepsilon = 5 \cdot 10^{-5}$



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 TPRK4 is 7 times more efficient than RK4.

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# Conclusion

- We have built and implemented deterministic, high order, explicit solvers for nonlinear kinetic equations;
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Thank you for your attention !