PAV Proof

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Definitions

- PAV = Proportional Approval Voting. Among all subsets $T \subseteq S$ with $|T| = k$, PAV returns a subset that maximizes the PAV score.
- V = set of n agents (voters).
- S = set of m projects, indexed $1, \dots, m$.
- Each voter i casts an approval ballot $A_i \subseteq S$, i.i.d. from other voters.
- For each project $j \in S$ there is a "true" quality level (ℓ_j).
- For each project $j \in S$, random voter $i \in V$ let

$$p_j \; = \; \Pr\big(j \in A_i\big)$$

  Assume $p_j = f(\ell_j)$ for some strictly increasing f – ie, better-quality projects have larger p_j.
- For a subset $T \subseteq S$ define the per-voter PAV contribution:

$$\phi_T(A_i) \; = \; \sum_{t=1}^{|T \cap A_i|} \frac{1}{t}.$$

  (Harmonic-vector weight: a voter who approves r chosen projects contributes $1 + 1/2 + \cdots + 1/r$ to the PAV score.)
- The empirical (total) PAV score of a set T on the n ballots is

$$\text{Score}_n(T) \; = \; \sum_{i=1}^{n} \phi_T(A_i).$$

- The expected per-voter PAV score of T is

$$s(T) \; = \; \mathbb{E}\big[\phi_T(A_i)\big],$$

  and the population (expected total) score is $ns(T)$.

- Let $\mathcal{T}_k := \{T \subseteq S : |T| = k\}$ denote all size-k subsets. Define the minimal difference between any chosen and any non-chosen projects' PAV scores:

$$D_s \ := \min_{T \in \mathcal{T}_k,\, T' \in \mathcal{T}_k,\, T' \neq T} \big(s(T) - s(T')\big)$$

Proposition. Assume $D_s > 0$. Then

$$\Pr\big(PAV(V) \in \arg\max_{T \in \mathcal{T}_k} \mathrm{Score}_n(T) \text{ equals a maximizer of } s(\cdot)\big) \ \geq\ 1 - \binom{m}{k} \exp\Big(-\frac{2nD_s^2}{(H_k)^2}\Big),$$

where $H_k := 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

Proof.

Per-voter contribution. For any set T we have $|T \cap A_i| \leq k$, hence the per-voter contribution

$$0 \leq \phi_T(A_i) \leq H_k := \sum_{t=1}^{k} \frac{1}{t}.$$

So each random variable $\phi_T(A_i)$ is bounded in the interval $[0, H_k]$.

Concentration for a fixed set T. For fixed $T \in \mathcal{T}_k$, the n terms $\phi_T(A_1), \ldots, \phi_T(A_n)$ are i.i.d. with mean $s(T)$ and range at most H_k. By Hoeffding's inequality, for any $\varepsilon > 0$,

$$\Pr\Big(|\mathrm{Score}_n(T) - ns(T)| \geq n\varepsilon\Big) = \Pr\Big(\Big|\sum_{i=1}^{n} \big(\phi_T(A_i) - s(T)\big)\Big| \geq n\varepsilon\Big) \leq 2\exp\Big(-\frac{2n^2\varepsilon^2}{n(H_k)^2}\Big) = 2\exp\Big(-\frac{2n\varepsilon^2}{(H_k)^2}\Big).$$

Concentration over all size-k sets. There are $\binom{m}{k}$ sets in $\mathcal{T}_k$. Using Union Bound, with probability at least

$$1 - 2\binom{m}{k} \exp\Big(-\frac{2n\varepsilon^2}{(H_k)^2}\Big)$$

for every $T \in \mathcal{T}_k$:

$$|\mathrm{Score}_n(T) - ns(T)| < n\varepsilon.$$

Choose ε as half the population gap. By definition, $D_s > 0$ is the minimal positive gap. Then,

$$\varepsilon \ =\ \frac{D_s}{2}.$$

Then for any maximizer $T^\star$ of $s(\cdot)$ and any other set $T \neq T^\star$,

$$\mathrm{Score}_n(T^\star) \ \geq\ ns(T^\star) - n\varepsilon \ =\ n\big(s(T^\star) - \varepsilon\big) \ =\ n\big(s(T) - \varepsilon + (s(T^\star) - s(T))\big) \ \geq\ n\big(s(T) + D_s - \varepsilon - \varepsilon\big) \ =\ n\big(s(T)\big).$$

Since $s(T^\star) - s(T) \geq D_s$ and both empirical scores are within $n\varepsilon$ of their means:

$\mathrm{Score}_n(T^\star) - \mathrm{Score}_n(T) \geq n\big((s(T^\star) - \varepsilon) - (s(T) + \varepsilon)\big) = n\big(s(T^\star) - s(T) - 2\varepsilon\big) \geq 0.$

Since $\varepsilon = D_s/2$, the last difference is ≥ 0. Thus, under the uniform concentration event, the empirical maximizers coincide with the population maximizers.

Probability bound. Using the union bound estimate with $\varepsilon = D_s/2$, we get

$$\Pr\big(\text{empirical maximizer equals population maximizer}\big) \;\geq\; 1 - 2\binom{m}{k}\exp\Big(-\frac{2n(D_s/2)^2}{(H_k)^2}\Big).$$

Simplifying:

$$\frac{2n(D_s/2)^2}{(H_k)^2} \;=\; \frac{nD_s^2}{2(H_k)^2}.$$

$$\Pr\big(PAV(V) \text{ is a maximizer of } s(\cdot)\big) \;\geq\; 1 - \binom{m}{k}\exp\Big(-\frac{2nD_s^2}{(H_k)^2}\Big)$$

Hence the probability tends to 1 exponentially fast in n whenever $D_s > 0$. $\square$