Convergence Proofs for Participatory Budgeting Mechanisms

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Shared Definitions

- V = set of n agents (voters), assumed to vote independently (i.i.d.).
- S = set of m projects, indexed $1, \ldots, m$.
- For each project $j \in S$ there is a "true" quality level ℓ_j.
- For each project j, let
$$p_j = \Pr(\text{a random voter approves project } j),$$
where $p_j = f(\ell_j)$ for some strictly increasing function f. (So better-quality projects have higher approval probabilities.)
- Each project j receives
$$A_j = \text{number of approvals project } j \text{ gets from the } n \text{ voters},$$
where $A_j \sim \text{Binomial}(n, p_j)$ and $\mathbb{E}[A_j] = np_j$.
- $S^\star$ = true top-k set by quality:
$$S^\star \in \arg \max_{T \subseteq S, |T|=k} Q(T),$$
for some underlying quality function $Q(T)$.

1. Approval Voting (AV)

Definitions

- AV = Approval Voting rule (selects the k projects with the highest approval counts).

Proposition.

$$\Pr\left(Q(AV(V)) \in \arg\max_{T\subseteq S, |T|=k} Q(T)\right) \;\geq\; 1 - m^2 \exp\Big(-\tfrac{1}{2}nD^2\Big).$$

Proof. A mistake occurs only if $\exists j \in S^\star, \ell \notin S^\star$ such that $A_\ell \geq A_j$. Fix such a pair (j, ℓ) and define $D' := A_j - A_\ell$. Then $\mathbb{E}[D'] = n(p_j - p_\ell) \geq nD$.

For each voter i, define

$$X_i := \mathbf{1}\{\text{voter } i \text{ approves } j\} - \mathbf{1}\{\text{voter } i \text{ approves } \ell\}.$$

Then $X_i \in [-1, 1]$ and $D' = \sum_{i=1}^n X_i$. Applying Hoeffding's inequality:

$$\Pr(D' \leq 0) \leq \exp\Big(-\tfrac{1}{2}nD^2\Big).$$

By the union bound over all pairs (j, ℓ),

$$\Pr(S_{AV} \neq S^\star) \leq m^2 \exp\Big(-\tfrac{1}{2}nD^2\Big).$$

Hence

$$\Pr(S_{AV} = S^\star) \geq 1 - m^2 \exp\Big(-\tfrac{1}{2}nD^2\Big),$$

which tends to 1 exponentially fast as $n \to \infty$. $\square$

2. Proportional Approval Voting (PAV)

Definitions.

- PAV = Proportional Approval Voting, selecting $T \subseteq S$ with $|T| = k$ maximizing its PAV score.
- For a subset $T \subseteq S$, define per-voter contribution

$$\phi_T(A_i) = \sum_{t=1}^{|T \cap A_i|} \frac{1}{t},$$

  and total score

$$\mathrm{Score}_n(T) = \sum_{i=1}^{n} \phi_T(A_i).$$

- Expected per-voter score: $s(T) = \mathbb{E}[\phi_T(A_i)]$.
- Let $\mathcal{T}_k = \{T \subseteq S : |T| = k\}$ and define

$$D_s = \min_{T, T' \in \mathcal{T}_k,\, T' \neq T} |s(T) - s(T')|.$$

- Define the minimum approval-probability gap between any true winner and any non-winner as

$$D := \min_{j \in S^\star,\, \ell \notin S^\star} (p_j - p_\ell),$$

  and assume $D > 0$ (i.e., no exact ties).

Proposition. Assume $D_s > 0$. Then

$$\Pr\big(PAV(V) \in \arg\max_{T \in \mathcal{T}_k} s(T)\big) \;\geq\; 1 - \binom{m}{k} \exp\Big(-\frac{2nD_s^2}{(H_k)^2}\Big),$$

where $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

Proof. Each $\phi_T(A_i) \in [0, H_k]$. By Hoeffding's inequality, for each fixed T and $\varepsilon > 0$:

$$\Pr\big(|\mathrm{Score}_n(T) - ns(T)| \geq n\varepsilon\big) \leq 2\exp\Big(-\frac{2n\varepsilon^2}{(H_k)^2}\Big).$$

Applying the union bound over $\binom{m}{k}$ possible T:

$$\Pr\big(\forall T \in \mathcal{T}_k :\; |\mathrm{Score}_n(T) - ns(T)| < n\varepsilon\big) \geq 1 - 2\binom{m}{k}\exp\Big(-\frac{2n\varepsilon^2}{(H_k)^2}\Big).$$

Choosing $\varepsilon = D_s/2$, all empirical maximizers coincide with population maximizers, since

$$\mathrm{Score}_n(T^\star) - \mathrm{Score}_n(T) \geq n(s(T^\star) - s(T) - 2\varepsilon) \geq 0.$$

Hence

$$\Pr(PAV(V) = T^\star) \geq 1 - \binom{m}{k}\exp\Big(-\frac{2nD_s^2}{(H_k)^2}\Big),$$

which tends to 1 exponentially fast as $n \to \infty$. □

3. Greedy Voting

Definitions.

$$V_k := \text{set of agents remaining at step } k, \quad V_1 = V,$$
$$V_{i+1} = V_i \cap C_i^c, \qquad S_{i+1} = S_i \cap (\text{expired choices})^c,$$

where C_i is the set of agents whose preferences were approved at step i. We aim to show

$$\lim_{n\to\infty} \Pr\big(\text{Greedy}(V) \in \operatorname{argmax} Q(S)\big) = 1.$$

Base Case ($k = 1$). At $k = 1$, Greedy(V) = AV(V), so the result holds by the AV proposition.

Inductive Step. Assume the proposition holds for some step k. We consider two cases:

Case 1: $V_{k+1} \neq \varnothing$. Then

$$\text{Greedy}(V_{k+1}) = \text{AV}(V_{k+1}, S_{k+1}),$$

which, by the inductive hypothesis, satisfies the proposition.

Case 2: $V_{k+1} = \varnothing$. Then

$$\text{Greedy}(S_{k+1}) = \text{Uniform}(S_{k+1}),$$

Lemma 1

$$\Pr(\text{Case 2 happens}) \to 0 \quad \text{as } n \to \infty.$$

Proof. Let p_i denote the probability that agent i's preference is approved at some step, with $p_i \in (0, 1)$. Define

$$p^* := \max_{i\in\{1,\dots,|S|\}} p_i.$$

The probability that all agents' approvals vanish (i.e. $V_m = \varnothing$) is given by the union of the disjoint events that each agent fails to approve at all steps:

$$\Pr(V_m = \varnothing) = \Pr\left(\bigcup_{i=1}^{|S|} \text{agent } i \text{ unapproved}\right) \le \sum_{i=1}^{|S|} (1 - p_i)^n.$$

Since $(1 - p_i)^n \le (1 - p^*)^n$ for all i,

$$\Pr(V_m = \varnothing) \le |S|(1 - p^*)^n.$$

Taking the limit,

$$\lim_{n\to\infty} \Pr(V_m = \varnothing) \le \lim_{n\to\infty} |S|(1 - p^*)^n = 0.$$

Thus, as $n \to \infty$, Case 2 occurs with probability 0. □

Lemma 2

For $k = 1$,

$$\text{GreedyHelper}(S, V, k = 1) = \text{ApprovalVoting}(S, V, k = 1).$$

Proof. By definition, at $k = 1$, GreedyHelper selects the element

$$s^* = \arg\max_{s \in S} \sum_{v \in V} \mathbf{1}\{s \in \mathrm{pref}(v)\},$$

that is, the candidate $s \in S$ receiving the largest number of approvals from agents in V.

The Approval Voting mechanism likewise chooses

$$s_{\mathrm{AV}} = \arg\max_{s \in S} \sum_{v \in V} \mathbf{1}\{s \in \mathrm{pref}(v)\}.$$

Therefore, for $k = 1$, the two mechanisms are equivalent:

$$\mathrm{GreedyHelper}(S, V, 1) = \mathrm{ApprovalVoting}(S, V, 1).$$

$\square$

Conclusion. By Lemmas 1–2 and induction,

$$\lim_{n \to \infty} \Pr\big(\mathrm{Greedy}(V) \in \mathrm{argmax}\, Q(S)\big) = 1.$$

$\square$