Proof of Convergence for the Greedy Voting Mechanism

Greedy Proof

Definitions

Let

$$V_k := \text{set of agents left at step } k,$$

with $V_1 = V$. Define the update rules

$$V_{i+1} = V_i \cap C_i^c, \qquad S_{i+1} = S_i \cap (\text{expired choices})^c,$$

where C_i is the set of agents whose preferences were approved on step i.

We aim to prove

$$\lim_{n\to\infty} \Pr\left(\text{Greedy}(V) \in \operatorname{argmax} Q(S)\right) = 1.$$

Base Case: $k = 1$

Since $V_1 = V$, it follows that

$$\text{Greedy}(V) = \text{AV}(V),$$

by identical process at the first step. Hence, the proposition holds for $k = 1$.

Inductive Step

Assume the proposition holds for some step k. We consider two cases:

Case 1: $V_{k+1} \neq \varnothing$. Then

$$\text{Greedy}(V_{k+1}) = \text{AV}(V_{k+1}, S_{k+1}),$$

which, by the inductive hypothesis, satisfies the proposition.

Case 2: $V_{k+1} = \varnothing$. Then

$$\text{Greedy}(S_{k+1}) = \text{Uniform}(S_{k+1}),$$

which does not satisfy the property above. We will show that the probability of this case tends to zero as $n \to \infty$.

Lemma 1

$$\Pr(\text{Case 2 happens}) \to 0 \quad \text{as } n \to \infty.$$

Proof. Let p_i denote the probability that agent i's preference is approved at some step, with $p_i \in (0, 1)$. Define

$$p^* := \max_{i \in \{1,\ldots,|S|\}} p_i.$$

The probability that all agents' approvals vanish (i.e. $V_m = \varnothing$) is given by the union of the disjoint events that each agent fails to approve at all steps:

$$\Pr(V_m = \varnothing) = \Pr\left(\bigcup_{i=1}^{|S|} \text{agent } i \text{ unapproved}\right) \leq \sum_{i=1}^{|S|} (1 - p_i)^n.$$

Since $(1 - p_i)^n \leq (1 - p^*)^n$ for all i,

$$\Pr(V_m = \varnothing) \leq |S|(1 - p^*)^n.$$

Taking the limit,

$$\lim_{n\to\infty} \Pr(V_m = \varnothing) \leq \lim_{n\to\infty} |S|(1 - p^*)^n = 0.$$

Thus, as $n \to \infty$, Case 2 occurs with probability 0. □

Lemma 2

For $k = 1$,

$$\text{GreedyHelper}(S, V, k = 1) = \text{ApprovalVoting}(S, V, k = 1).$$

Proof. By definition, at $k = 1$, the `GreedyHelper` procedure selects the element

$$s^* = \arg\max_{s \in S} \sum_{v \in V} \mathbf{1}\{s \in \text{pref}(v)\},$$

that is, the candidate $s \in S$ receiving the largest number of approvals from agents in V.

The Approval Voting mechanism likewise chooses

$$s_{\text{AV}} = \arg\max_{s \in S} \sum_{v \in V} \mathbf{1}\{s \in \text{pref}(v)\}.$$

Therefore, for $k = 1$, the two mechanisms are equivalent:

$$\text{GreedyHelper}(S, V, 1) = \text{ApprovalVoting}(S, V, 1).$$

□

Conclusion

By Lemma 1, as $n \to \infty$, the inductive step proceeds via Case 1 with probability 1. By Lemma 2 and the base case, the Greedy mechanism coincides with Approval Voting at the first step.

Therefore, by the *Principle of Mathematical Induction*,

$$\lim_{n \to \infty} \Pr \big(\text{Greedy}(V) \in \operatorname{argmax} Q(S) \big) = 1.$$

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