AV Proof

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Definitions

- AV = Approval Voting rule (selects the k projects with the highest approval counts).
- V = set of n agents (voters), who make decisions i.i.d. of each other.
- S = set of m projects, indexed $1, \dots, m$.
- For each project $j \in S$ there is a "true" quality level (ℓ_j).
- For each project j let

  $$p_j = \Pr(\text{a random voter approves project } j),$$

  Assume $p_j = f(\ell_j)$ for some strictly increasing f – ie, better-quality projects have larger p_j.
- For each project j let

  $$A_j = \text{number of approvals project } j \text{ receives from the } n \text{ voters},$$

  so $A_j \sim \text{Binomial}(n, p_j)$ and $\mathbb{E}[A_j] = np_j$.
- $AV(V)$ is the set of k projects chosen by approval voting.
- Let $S^\star$ = true top-k set by quality (any element of $\arg\max_{T \subseteq S, |T|=k} Q(T)$). Multiple such sets can exist, any one will suffice.
- Define the difference between any chosen and any non-chosen project:

  $$D \ := \ \min_{j \in S^\star,\ \ell \notin S^\star} (p_j - p_\ell).$$

  Assume $D > 0$ (if $D = 0$, there are exact ties in approval probabilities).

Proposition.

$$\Pr\left(Q(AV(V)) \in \arg\max_{T \subseteq S, |T|=k} Q(T)\right) \ \geq \ 1 - m^2 \exp\Big(-\tfrac{1}{2} n D^2\Big).$$

Proof.

Mistake Case AV makes a mistake only if there exists some ordered pair (j, ℓ) with $j \in S^\star,\ \ell \notin S^\star$ such that $A_\ell \geq A_j$.

Hence, the lower bound probability of such a mistake is defined as the probability that there exists $j \in S^\star$ and $\ell \notin S^\star$ with $A_\ell \geq A_j$.

Fix one pair (j, ℓ). Fix $j \in S^\star$ and $\ell \notin S^\star$. Consider the difference

$$D := A_j - A_\ell.$$

We have $\mathbb{E}[D] = n(p_j - p_\ell) \geq nD$ by definition of D.

D as a sum of independent bounded variables. For voter $i = 1, \dots, n$ define

$$X_i := \mathbf{1}\{\text{voter } i \text{ approves } j\} - \mathbf{1}\{\text{voter } i \text{ approves } \ell\}.$$

Then $X_i \in \{-1, 0, 1\}$, the X_i are independent across i, and

$$D = \sum_{i=1}^{n} X_i, \qquad \mathbb{E}[X_i] = p_j - p_\ell.$$

Hoeffding's Inequality. Each X_i is bounded in $[-1, 1]$. Hoeffding's inequality yields, for any $t > 0$,

$$\Pr\Big(\sum_{i=1}^{n} X_i - \mathbb{E}\big[\sum_{i=1}^{n} X_i\big] \leq -t\Big) \leq \exp\Big(-\frac{t^2}{2n}\Big).$$

Take $t = \mathbb{E}[D] \geq nD$. Then

$$\Pr(D \leq 0) = \Pr\Big(\sum_{i=1}^{n} X_i - \mathbb{E}[D] \leq -\mathbb{E}[D]\Big)$$

$$\leq \exp\Big(-\frac{(\mathbb{E}[D])^2}{2n}\Big) \leq \exp\Big(-\tfrac{1}{2}nD^2\Big).$$

Thus for this fixed ordered pair (j, ℓ) the probability that the outside project ℓ ties or beats the inside project j is at most $\exp(-\frac{1}{2}nD^2)$.

Union bound. AV makes a mistake only if there exists some ordered pair (j, ℓ) with $j \in S^\star,\ \ell \notin S^\star$ such that $A_\ell \geq A_j$. The number of such ordered pairs is $|S^\star| \cdot (m - |S^\star|) \leq m^2$. By the union bound,

$$\Pr\big(S_{\text{AV}} \neq S^\star\big) \leq \sum_{j \in S^\star} \sum_{\ell \notin S^\star} \Pr(A_\ell \geq A_j) \leq m^2 \exp\Big(-\tfrac{1}{2}nD^2\Big).$$

Therefore

$$\Pr\big(S_{\text{AV}} = S^\star\big) \geq 1 - m^2 \exp\Big(-\tfrac{1}{2}nD^2\Big),$$

and the RHS tends to 1 exponentially fast as $n \to \infty$. $\square$