1 Pen and Paper Tasks

1) Calculate the eigendecomposition of the following matrix:

$$A = \begin{bmatrix} 9 & 1 \\ 8 & 7 \end{bmatrix}$$

Solution:

Say $\mathbf{v}$ is an eigenvector of A, and λ its corresponding eigenvalue. We know that, by definition:

$$A\mathbf{v} = \lambda\mathbf{v} \implies A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \implies (A - \lambda I)\mathbf{v} = \mathbf{0}$$

$\mathbf{v}$ has a non-trivial solution only if $\det(A - \lambda I) = 0$

$$det(A - \lambda I) = det\left(\begin{bmatrix} 9-\lambda & 1 \\ 8 & 7-\lambda \end{bmatrix}\right) = 0 \implies (9-\lambda)(7-\lambda) - 8 = 0$$

Solving the above equation, we get the eigenvalues $\lambda_1 = 11$ and $\lambda_2 = 5$

Let $B = (A - \lambda I)$

For the first eigenvalue $\lambda_1 = 11$,

$$B = \begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix}$$

We know that $B\mathbf{v} = \mathbf{0}$

$$\begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve the above linear system by simple substitution method or Gaussian elimination to get a vector of the form: $\mathbf{u}_1 = \begin{bmatrix} v_1 \\ 2v_1 \end{bmatrix}$

Since there are infinitely many solutions, there are infinitely many eigenvectors. We can plug in any arbitrary value for v_1 to get a valid eigenvector.

Solve similarly for λ_2 to get its corresponding eigenvector, $\mathbf{u}_2 = \begin{bmatrix} -v_1 \\ 4v_1 \end{bmatrix}$

Now that we have the eigenvectors, the eigendecomposition is:

$$A = Q\Lambda Q^{-1}$$

Where Λ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, $\Lambda_{ii} = \lambda_i$, and Q is the is the square n x n matrix whose ith column is the eigenvector u_i of Λ (in this case, n = 2).

Arbitrarily plugging in 1 as the value for v_1 for both eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$, we get:

$$Q = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

Invert the 2x2 matrix Q to get:

$$Q^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

Hence the eigendecomposition of A is :

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 11 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

2) Use the eigendecomposition of A to show how you can efficiently compute A^{10} (you don't have to show the final value of the matrix).

Solution:

$$A^{10} = AAA \dots A = Q\Lambda Q^{-1} Q\Lambda Q^{-1} \dots Q\Lambda Q^{-1} = Q\Lambda^{10} Q^{-1}$$

$$A^{10} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 11^{10} & 0 \\ 0 & 5^{10} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

The solution above is enough to get full points. Final result:

$$A^{10} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 25937424601 & 0 \\ 0 & 9765625 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 17294871609 & 4321276496 \\ 34570211968 & 8652318617 \end{bmatrix}$$

3) You are given the following matrix:

$$A = \begin{bmatrix} 4 & 8 & 2 \\ 8 & 41 & 24 \\ 2 & 24 & 21 \end{bmatrix}$$

$\det(A) = 400$
One of the eigen values is 1. Find the other two. **Hint:** You don't have to calculate the eigenvalues from scratch. Use the properties of eigenvalues.

Solution:

Properties of eigenvalues we shall use are:

1. The trace of a square matrix A is equal to the sum of its eigenvalues.
2. The determinant of a square matrix A is equal to the product of its eigenvalues.

Applying the first property, we get: $\lambda_1 + \lambda_2 + \lambda_3 = 66$

Applying the second, we get: $\lambda_1 \lambda_2 \lambda_3 = 400$

We know that $\lambda_1 = 1$. Plug this value into the two equations above and solve to get

$\lambda_2 = \frac{65+5\sqrt{105}}{2}$, $\lambda_3 = \frac{65-5\sqrt{105}}{2}$ (Decimal : $\lambda_2 = 58.11737\dots$, $\lambda_3 = 6.88262\dots$)

4) You are given the following matrix:

$$A = \begin{bmatrix} 100 & 100 & 100 \\ 99 & 99 & 102 \\ 98 & 98 & 104 \end{bmatrix}$$

$\det(A) = 0$
Find the eigenvalues of A.

Solution:

If the sum of entries of each of the rows of a square matrix A are the same, then the common row sum is an eigenvalue of A. Therefore, 300 is an eigenvalue of A in this case.

Since the determinant is zero, one of the eigenvalues is also zero, because the determinant is equal to the product of eigenvalues of A.

Lastly, since the trace of A should be equal to the sum of its eigenvalues, we get:

$\lambda_1 + \lambda_2 + \lambda_3 = 303$

Which gives $\lambda_3 = 3$ when we plug in 300 and 0 for λ_1 and λ_2