Submission for Computer Vision Exercise 1

Mazen Amria (5974747)

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1 Pen and Paper task

1.1 Eigendecomposition of 2×2 matrix

First, let's find the eigenvalues of A.

$$\begin{aligned}
0 &= \det(A - \lambda I) \\
0 &= \det\left(\begin{bmatrix} 9 & 1 \\ 8 & 7 \end{bmatrix} - \lambda I\right) \\
0 &= \det\left(\begin{bmatrix} 9-\lambda & 1 \\ 8 & 7-\lambda \end{bmatrix}\right) \\
0 &= (9-\lambda)\cdot(7-\lambda) - 8 \cdot 1 \\
0 &= \lambda^2 - 16\lambda + 55 \\
0 &= (\lambda - 11)\cdot(\lambda - 5)
\end{aligned}$$

Hence, the eigenvalues are $\begin{bmatrix} 11 & 5 \end{bmatrix}$, now we find the corresponding eigenvectors:

$$\begin{aligned}
0 &= (A - \lambda_1 I)\cdot \mathbf{v_1} \\
0 &= \begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix} \cdot \mathbf{v_1}
\end{aligned}$$

We solve for $\mathbf{v_1}$ and get $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$. Similarly for $\mathbf{v_2}$ by substituting $\lambda = 5$, we get $\begin{bmatrix} \frac{1}{\sqrt{17}} & \frac{-4}{\sqrt{17}} \end{bmatrix}$.

$$\begin{aligned}
A = \begin{bmatrix} 9 & 1 \\ 8 & 7 \end{bmatrix} &= Q\Lambda Q^{-1} \\
&= \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{17}} \\ \frac{2}{\sqrt{5}} & \frac{-4}{\sqrt{17}} \end{bmatrix} \begin{bmatrix} 11 & 0 \\ 0 & 5 \end{bmatrix} \cdot Q^{-1} \\
&= \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{17}} \\ \frac{2}{\sqrt{5}} & \frac{-4}{\sqrt{17}} \end{bmatrix} \begin{bmatrix} 11 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{4\sqrt{5}}{6} & \frac{\sqrt{5}}{6} \\ \frac{2\sqrt{17}}{6} & \frac{-\sqrt{17}}{6} \end{bmatrix}
\end{aligned}$$

1.2 Computing A^{10} using eigendecomposition

$$\begin{aligned}
A^{10} &= \left(Q\Lambda Q^{-1}\right)^{10} \\
&= \left(Q\Lambda Q^{-1}\right)\left(Q\Lambda Q^{-1}\right)... \\
&= Q\Lambda^{10} Q^{-1} \\
&= \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{17}} \\ \frac{2}{\sqrt{5}} & \frac{-4}{\sqrt{17}} \end{bmatrix} \begin{bmatrix} 11^{10} & 0 \\ 0 & 5^{10} \end{bmatrix} \begin{bmatrix} \frac{4\sqrt{5}}{6} & \frac{\sqrt{5}}{6} \\ \frac{2\sqrt{17}}{6} & \frac{-\sqrt{17}}{6} \end{bmatrix}
\end{aligned}$$

You got it, no need to evaluate.

1.3 Properties of symmetric matrices and eigenvalues

The following matrix has $\det(A) = 400$, and one of the eigenvalues $\lambda_1 = 1$, find the other two.

$$A = \begin{bmatrix} 4 & 8 & 2 \\ 8 & 41 & 24 \\ 2 & 24 & 21 \end{bmatrix}$$

Let's obtain two equations:

$$\begin{aligned} \det(A) &= \prod_{\lambda \in \Lambda} \lambda \\ 400 &= \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \\ 400 &= \lambda_2 \cdot \lambda_3 \end{aligned} \tag{1}$$

and,

$$\begin{aligned} \mathrm{Tr}(A) &= \sum_{\lambda \in \Lambda} \lambda \\ 4 + 41 + 21 &= \lambda_1 + \lambda_2 + \lambda_3 \\ 65 &= \lambda_2 + \lambda_3 \end{aligned} \tag{2}$$

Solving the two equations, we get $\lambda_{2,3} = \frac{65 \pm 5\sqrt{105}}{2}$.

1.4 Properties of singular matrices

The following matrix has $\det(A) = 0$, find its eigenvalues.

$$A = \begin{bmatrix} 100 & 100 & 100 \\ 99 & 99 & 102 \\ 98 & 98 & 104 \end{bmatrix}$$

We start with $\det(A - \lambda I) = 0$:

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 100 & 100 & 100 \\ 99 & 99 & 102 \\ 98 & 98 & 104 \end{bmatrix} - \lambda I \right) \\ &= \det\left(\begin{bmatrix} 100 - \lambda & 100 & 100 \\ 99 & 99 - \lambda & 102 \\ 98 & 98 & 104 - \lambda \end{bmatrix} \right) \\ &= 0 \end{aligned}$$

We get the characterstic polynomial:

$$\begin{aligned} 0 &= -\lambda^3 + 303\lambda^2 - 900\lambda \\ 0 &= \lambda(-\lambda^2 + 303\lambda - 900) \end{aligned}$$

We solve for λ and get $\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda = 300$.