MAT 120

Beta and Gamma Functions

1. Evaluate in terms of gamma function: (WEEK 2)

$$(i)\int_0^4 x^{3/2}(4-x)^{5/2}dx,\quad (ii)\int_0^b y^5\sqrt{b^2-y^2}\,dy,\quad (iii)\int_0^\infty e^{-t^2}dt,$$

$$(iv)\int_0^\infty x^5e^{-4x}dx,\quad (v)\int_0^\infty e^{-y^2}y^5dy,\quad (vi)\int_0^\infty e^{-x^2}dx,\quad (vii)\int_0^\infty x^6e^{-3x}dx,$$

$$(viii)\int_0^\infty e^{-x^2}x^9dx,\quad (ix)\int_0^\infty \sqrt{x}e^{-x^2}dx,\quad (x)\int_0^1 \frac{x^3}{\sqrt{1-x^3}}dx,$$

$$(xi)\int_0^1 \frac{1}{\sqrt{x\ln(1/x)}}dx,\quad (xii)\int_0^1\left(1-\frac{1}{x}\right)^{1/3}dx.$$

2. Evaluate in terms of beta function: (WEEK 3)

$$(i)\int_0^1 \frac{x^2}{\sqrt{1-x}}dx,\quad (ii)\int_0^1 x^7(1-x)^3dx,\quad (iii)\int_0^1\frac{1}{\sqrt{1-x^3}}dx,$$

$$(iv)\int_0^1 (1-x)^{1/2}x^3dx,\quad (v)\int_0^1 x^{5/2}(1-x)^{3/2}dx,$$

$$(vi)\int_0^a y^7\sqrt{a^4-y^4}\,dy,\quad (vii)\int_0^4 y^3\sqrt{64-y^3}\,dy,$$

$$(viii)\int_0^1 x^2(1-x^3)^{3/2}dx,\quad (ix)\int_0^\infty\frac{1}{1+x^4}dx.$$

3. Evaluate the following integrals: (WEEK 3)

$$(i)\int_0^\pi \sin^5\theta\cos^4\theta\,d\theta,\quad (ii)\int_0^\pi \sin^6\theta\cos^7\theta\,d\theta,$$

$$(iii)\int_0^{\pi/6}\sin^2 6x\cos^4 3x\,dx,\quad (iv)\int_0^{\pi/4}\sin^2 4\theta\cos^3 2\theta\,d\theta,$$

$$(v)\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta,\quad (vi)\int_0^{\pi/8}\sin^2 8x\cos^4 4x\,dx.$$

Formula

1. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad where \quad n > 0$

2. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad where \quad m > 0, n > 0.$

3. $\int_0^{\pi/2} \sin^p x \cos^q x dx = \dfrac{\Gamma(\frac{p+1}{2})\,\Gamma(\frac{q+1}{2})}{2\,\Gamma(\frac{p+q+2}{2})}$

4. $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

5. $\Gamma(n) = (n-1)!$

6. $\Gamma(n+1) = n\Gamma(n) = n!$

7. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

8. $\Gamma(1) = 1$

9. $\Gamma(\frac{p}{2}) = (\frac{p}{2}-1)(\frac{p}{2}-2)(\frac{p}{2}-3)\ldots\ldots\ldots\ldots\ldots\frac{1}{2}.\Gamma(\frac{1}{2})$

10. $\Gamma(n) = (n-1)(n-2)\ldots\ldots\ldots\ldots 3.2.1$

11. $\int_0^{\pi/2} 2sin^{2x-1}(t)cos^{2y-1}(t)dt = \beta(x,y)$

Gamma Beta Distribution Solution

1. (i) 8.4.32 $\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(6)}$

   (ii) $\frac{b^7}{2}\frac{\Gamma(3)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$

   (iii) ½ (1/2)

   (iv) $\frac{1}{4^6}$(6)

   (v) ½ (3)

   (vii) $\frac{1}{3^7}$ (7)

   (viii) ½ (5)

   (ix) ½ (3/4)

   (x) $\frac{1}{3}\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{4}{3}+\frac{1}{2}\right)}$

   (xi) $\sqrt{2}$ (1/2)

   (xii) $-\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma(2)}$

2. (i) β(3, ½)

   (ii) β(8, 4)

   (iii) $\frac{1}{3}$β(1/3, ½)

   (iv) β(4, 3/2)

   (v) β(7/2, 5/2)

   (vi) $\frac{a^{10}}{4}$β(2, 3/2)

   (vii) $4^4 . 8 . \frac{1}{3}$ β(4/3, 3/2)

   (viii) $\frac{1}{3}$β(1, 5/2)

   (ix) ½ $\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} = \frac{1}{4}\ \beta\left(\frac{1}{4},\frac{3}{4}\right)$

3. (i) $\frac{\Gamma(3)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}$

   (ii) $\frac{\Gamma\left(\frac{7}{2}\right)\Gamma(4)}{\Gamma\left(\frac{15}{2}\right)}$

   (iii) $\frac{4}{3}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{2}\right)}{2\Gamma(5)}$

   (iv) $\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(3)}{\Gamma\left(\frac{9}{2}\right)}$

   (v) $\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)}$

   (vi) $\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{2}\right)}{2\Gamma(5)}$

(1) $\int_0^4 x^{3/2}(4-x)^{5/2}\,dx$

$= \int_0^4 \left(\frac{x}{4}\right)^{\frac{3}{2}} \cdot 4^{\frac{3}{2}} \cdot \left(4\left(1-\frac{x}{4}\right)\right)^{\frac{5}{2}} dx$

$= 4^{\frac{3}{2}} \cdot \int_0^4 \left(\frac{x}{4}\right)^{\frac{3}{2}} \cdot 4^{5/2}\left(1-\frac{x}{4}\right)^{\frac{5}{2}} dx$

$= 4^{\frac{3}{2}} \cdot 4^{\frac{5}{2}} \cdot \int_0^4 \left(\frac{x}{4}\right)^{\frac{3}{2}} \cdot \left(1-\frac{x}{4}\right)^{5/2} dx$

$= 8 \cdot 32 \cdot \int_0^4 \left(\frac{x}{4}\right)^{\frac{3}{2}} \cdot \left(1-\frac{x}{4}\right)^{\frac{5}{2}} dx$

let,

$\frac{x}{4} = z$

$\Rightarrow \frac{dz}{dx} = \frac{1}{4}$

$\Rightarrow 4dz = dx$

limit

z	x
0	0
1	4

$= 4 \cdot 8 \cdot 32 \cdot \int_0^1 (z)^{3/2} \cdot (1-z)^{\frac{5}{2}} dz$

$= 4 \cdot 8 \cdot 32 \cdot \int_0^1 z^{\frac{5}{2}-1} \cdot (1-z)^{7/2-1} dz$

$= 4 \cdot 8 \cdot 32 \cdot \beta\left(\frac{5}{2}, 7/2\right)$

$= 4 \cdot 8 \cdot 32 \cdot \frac{\Gamma 5/2 \cdot \Gamma 7/2}{\Gamma 5/2 + 7/2}$

$= 4 \cdot 8 \cdot 32 \cdot \frac{\Gamma 5/2 \cdot \Gamma 7/2}{\Gamma 6}$

(III) $\int_0^\infty e^{-t^2}\, dt$

let, $t^2 = z \Rightarrow t = \sqrt{z}$

$\Rightarrow \frac{dz}{dt} = 2t$

$\Rightarrow dz = 2t\, dt$

$\Rightarrow dz = 2 . \sqrt{z} . dt$

$\Rightarrow \frac{1}{2\sqrt{z}} . dz = dt$

$= \frac{1}{2}\int_0^\infty e^{-z} . \frac{1}{\sqrt{z}} . dz$

$= \frac{1}{2}\int_0^\infty e^{-z} . z^{-\frac{1}{2}}\, dz$

$= \frac{1}{2}\int_0^\infty e^{-z} . z^{\frac{1}{2}-1}\, dz$

$= \frac{1}{2} . \Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2}$

(IV) ~~$\int_0^\infty e^{-t^2} dt$~~

$\int_0^\infty x^5 . e^{-4x}\, dx$

let, $4x = u \Rightarrow \frac{u}{4} = x$

$\Rightarrow \frac{du}{dx} = 4$

$\Rightarrow du = 4\, dx$

$\Rightarrow dx = \frac{1}{4} du$

$= \int_0^\infty e^{-4x} . x^5\, dx$

$= \int_0^\infty e^{-u} . \left(\frac{u}{4}\right)^5 . \frac{1}{4}\, du$

$= \left(\frac{1}{4}\right)^5 . \frac{1}{4} \int_0^\infty e^{-u} . u^5\, du$

$= \left(\frac{1}{4}\right)^6 . \int_0^\infty e^{-u} . u^{6-1}\, du$

$= \left(\frac{1}{4}\right)^6 . \Gamma 6 = \left(\frac{1}{4}\right)^6 \times 5 \times 4 \times 3 \times 2 \times 1 = \frac{15}{512}$ Ans

(v) $\int_0^\infty e^{-y^2} \cdot y^5 \, dy$

$= \int_0^\infty e^{-z} \cdot (\sqrt{z})^5 \, dy$

$= \int_0^\infty e^{-z} \cdot z^{5/2} \cdot \frac{1}{2\sqrt{z}} \, dz$

$= \frac{1}{2} \int_0^\infty e^{-z} \cdot z^{5/2 - \frac{1}{2}} \, dz$

$= \frac{1}{2} \int_0^\infty e^{-z} \cdot z^{2} \, dz$

$= \frac{1}{2} \int_0^\infty e^{-z} \cdot z^{3-1} \, dz$

$= \frac{1}{2} \cdot \Gamma 3$ (Ans)

let,

$y^2 = z \Rightarrow y = \sqrt{z}$

$\Rightarrow \frac{dz}{dy} = 2y$

$\Rightarrow dz = 2y \cdot dy$

$\Rightarrow dy = \frac{1}{2y} \cdot dz$

$= \frac{1}{2\sqrt{z}} dz$

limit

y	z
0	0
∞	∞

(vii) $\int_0^\infty x^6 e^{-3x} \, dx$

$= \int_0^\infty e^{-3x} \cdot x^6 \, dx$

$= \frac{1}{3} \int_0^\infty e^{-p} \cdot (p/3)^6 \, dp$

$= \left(\frac{1}{3}\right)^6 \cdot \frac{1}{3} \int_0^\infty e^{-p} \cdot p^6 \, dp$

$= \left(\frac{1}{3}\right)^7 \cdot \Gamma 7$ (Ans)

let,

$+3x = p \Rightarrow \frac{p}{3} = x$

$\Rightarrow \frac{dp}{dx} = 3$

$\Rightarrow dx = \frac{1}{3} dp$

(VIII) $\int_0^\infty e^{-x^2} \cdot x^9 \, dx$

let, $x^2 = z \Rightarrow z = x^2 \Rightarrow x = \sqrt{z}$

$\Rightarrow \frac{dz}{dx} = 2x$

$\Rightarrow dx = \frac{1}{2x} dz$

$= \frac{1}{2\sqrt{z}} dz$

limit

x	z	
0	0	
∞	∞	

$= \frac{1}{2}\int_0^\infty e^{-z} \cdot (\sqrt{z})^9 \cdot \frac{1}{\sqrt{z}} \cdot dz$

$= \frac{1}{2}\int_0^\infty e^{-z} \cdot z^{9/2 - \frac{1}{2}} \, dz$

$= \frac{1}{2}\int_0^\infty e^{-z} \cdot z^4 \, dz$

$= \frac{1}{2}\Gamma 5$ (Ans)

(IX) $\int_0^\infty \sqrt{x} \cdot e^{-x^2} dx$

let, $x^2 = p \Rightarrow x = \sqrt{p}$

$\Rightarrow \frac{dp}{dx} = 2x$

$\Rightarrow \frac{1}{2\sqrt{p}} dp = dx$

$= \frac{1}{2}\int_0^\infty e^{-p} \cdot \sqrt{(\sqrt{p})} \cdot \frac{1}{\sqrt{p}} \, dp$

$= \frac{1}{2}\int_0^\infty e^{-p} \cdot p^{\frac{1}{4}} \cdot p^{-\frac{1}{2}} dp$

$= \frac{1}{2}\int_0^\infty e^{-p} \cdot p^{-\frac{1}{4}} dp$

$= \frac{1}{2}\int_0^\infty e^{-p} \cdot p^{-1/4 - 1 + 1} dp$

$= \frac{1}{2}\Gamma 3/4$ (Ans)

(X) $\int_0^1 \frac{x^3}{\sqrt{1-x^3}}\,dx$

$= \int_0^1 x^3 . (1-x^3)^{-\frac{1}{2}}\,dx$

$= \int_0^1 z . (1-z)^{-\frac{1}{2}} . \frac{1}{3} . \frac{1}{z^{2/3}}\,dz$

$= \frac{1}{3}\int_0^1 z^{1-\frac{2}{3}} (1-z)^{-\frac{1}{2}}\,dz$

$= \frac{1}{3}\int_0^1 z^{4/3-1} (1-z)^{\frac{1}{2}-1}\,dz$

$= \frac{1}{3} . \beta\left(\frac{4}{3}, \frac{1}{2}\right)$

$= \frac{1}{3} . \frac{\Gamma 4/3 . \Gamma 1/2}{\Gamma 4/3 + 1/2}$ Ans

let,

$x^3 = z \Rightarrow x = \sqrt[3]{z}$

$\Rightarrow \frac{dz}{dx} = 3x^2$

$\Rightarrow \frac{dz}{dx} = 3x^2$

$\Rightarrow dz = 3x^2\,dx$

limit

x	z
0	0
1	1

$\Rightarrow \frac{1}{3x^2} . dz = dx$

$\Rightarrow \frac{1}{3.(\sqrt[3]{z})^2}\,dz = dx$

$\Rightarrow \frac{1}{3} . \frac{1}{z^{2/3}}\,dz = dx$

(Xii) $\int_0^1 \left(1-\frac{1}{x}\right)^{1/3} dx$

$= \int_0^1 \left(\frac{x-1}{x}\right)^{\frac{1}{3}} dx$

$= \int_0^1 \left(\frac{1}{x}\right)^{\frac{1}{3}} (x-1)^{\frac{1}{3}}\,dx$

$= -\int_0^1 \left(\frac{1}{x}\right)^{\frac{1}{3}} (1-x)^{\frac{1}{3}}\,dx$

$= -\int_0^1 (x^{-1})^{\frac{1}{3}} (1-x)^{\frac{1}{3}}\,dx$

$= -\int_0^1 x^{-\frac{1}{3}} (1-x)^{\frac{1}{3}}\,dx$

$\frac{1}{3} = \cancel{m}-1$

$\Rightarrow m = 1+\frac{1}{3}$

$= \frac{4}{3}$

$\frac{1}{2} - 1 + 1$

$= \frac{1}{2} \cancel{+}1$

$x - 1 = m$

$\Rightarrow x = \frac{1}{m} + 1$

$= \frac{2}{m}$

$$= -\int_0^1 x^{2/3-1} \cdot (1-x)^{\frac{4}{3}-1} \, dx$$

$$= -\frac{\Gamma 2/3 \; \Gamma 4/3}{\Gamma 4/3 + 2/3}$$

$$= -\frac{\Gamma 2/3 \; \Gamma 4/3}{\Gamma 2}$$ (Ans)

(2)

(i) $\int_0^1 \frac{x^2}{\sqrt{1-x}} \, dx$

$$= \int_0^1 x^2 \cdot (1-x)^{-\frac{1}{2}} \, dx$$

$$= \int_0^1 x^{3-1} (1-x)^{\frac{1}{2}-1} \, dx$$

$$= \beta\left(3, \frac{1}{2}\right)$$

$$= \frac{\Gamma 3 \; \Gamma \frac{1}{2}}{\Gamma 3 + \frac{1}{2}}$$ A

(11) $\int_0^1 x^7 (1-x)^3 \, dx$

$= \int_0^1 x^{8-1} \cdot (1-x)^{4-1} \, dx = \beta(8,4)$

$= \frac{\Gamma 8 \cdot \Gamma 4}{\Gamma 12}$

(iii) $\int_0^1 \frac{1}{\sqrt{1-x^3}}\,dx$

$= \int_0^1 (1-x^3)^{-1/2}\,dx$

$= \int_0^1 (1-p)^{-\frac{1}{2}} \cdot \frac{1}{3} \cdot \frac{1}{p^{2/3}} \cdot dp$

$= \frac{1}{3}\int_0^1 \frac{1}{p^{2/3}} \cdot (1-p)^{-1/2}\,dp$

$= \frac{1}{3} \cdot \int_0^1 p^{-2/3} \cdot (1-p)^{-1/2}\,dp$

$= \frac{1}{3}\int_0^1 p^{\frac{1}{3}-1} (1-p)^{\frac{1}{2}-1}\,dp$

$= \frac{1}{3} \cdot \beta\left(\frac{1}{3}, \frac{1}{2}\right)$ ✓

let,

$x^3 = p \Rightarrow x = \sqrt[3]{p}$

$\Rightarrow \frac{dp}{dx} = 3x^2$

$\Rightarrow \frac{dp}{3x^2} = dx$

$\Rightarrow dx = \frac{1}{3 \cdot (\sqrt[3]{p})^2}\,dp$

$= \frac{1}{3 \cdot p^{2/3}} \cdot dp$

Limit

x	p
0	0
1	1

$x - 1 = -2/3$

$\Rightarrow x = \frac{1}{3}$

(iv) $\int_0^1 x^3 (1-x)^{\frac{1}{2}}\,dx$

$= \int_0^1 x^{4-1} (1-x)^{\frac{3}{2}-1}\,dx$

$= \beta(4, 3/2)$

(v) $\int_0^1 x^{5/2}(1-x)^{3/2}\,dx$

$= \int_0^1 x^{7/2-1}(1-x)^{5/2-1}\,dx$

$= \beta(7/2, 5/2)$

limit	
y	u
a	1
0	0

(vi) $\int_0^a y^7\sqrt{a^4-y^4}\,dy$

$= \int_0^a y^7 \cdot \sqrt{a^4\left(1-\frac{y^4}{a^4}\right)}\,dy$

$= a^2\int_0^a y^7\sqrt{1-y^4/a^4}\,dy$

$= a^2\int_0^1 (a\sqrt[4]{u})^7 \cdot \sqrt{1-u} \cdot \frac{a}{4u^{3/4}}\,du$

$= a^2\int_0^1 a^7 \cdot u^{7/4} \cdot (1-u)^{\frac{1}{2}} \cdot \frac{a}{4} \cdot u^{-\frac{3}{4}}\,du$

$= \frac{a^{10}}{4}\int_0^1 u^{\cancel{1}} \cdot (1-u)^{\frac{1}{2}}\,du$

$= \frac{a^{10}}{4}\int_0^1 u^{2}(1-u)^{\frac{1}{2}}\,du$

$= \frac{a^{10}}{4}\int_0^1 u^{2-1}(1-u)^{\frac{3}{2}-1}\,du$

$= \frac{a^{10}}{4}\beta\left(2, \frac{3}{2}\right)$

let,

$\left(\frac{y}{a}\right)^4 = u$

$\Rightarrow \frac{y^4}{a^4} = u$

$\Rightarrow a^4u = y^4$

$\Rightarrow \sqrt[4]{a^4u} = y$

$\Rightarrow y = a\cdot\sqrt[4]{u}$

$\Rightarrow \frac{du}{dy} = \frac{d}{dy}\cdot\left(\frac{1}{a^4}\cdot y^4\right)$

$\Rightarrow \frac{du}{dy} = \frac{1}{a^4}\frac{d}{dy}y^4$

$\Rightarrow \frac{du}{dy} = \frac{1}{a^4}4y^3$

$\Rightarrow \frac{du}{dy} = 4y^3\cdot\frac{1}{a^4}$

$\Rightarrow \frac{a^4}{4y^3}\,du = dy$

$\Rightarrow \frac{a^4}{4(a\cdot\sqrt[4]{u})^3}\,du = dy$

$\Rightarrow \frac{a^3 \cdot a}{4a^3\cdot u^{3/4}}\,du = dy$

(VII) $\int_0^4 y^3\sqrt{64-y^3}\,dy$

$= \int_0^4 y^3\sqrt{64(1-y^3/64)}\ dy$

$= 8\int_0^4 y^3\sqrt{1-\frac{y^3}{64}}\ dy$

$= 8\int_0^1 (4\cdot\sqrt[3]{m})^3\cdot\sqrt{1-\frac{y^3}{64}}\ \cdot 64\cdot\frac{1}{3}(4\sqrt[3]{m})^{-2}\cdot dm$

$= 8\int_0^1 4^3\cdot m\cdot\sqrt{1-\frac{y^3}{64}}\ \cdot 4^3\cdot\frac{1}{3}\,4^{-2}(\sqrt[3]{m})^{-2}dm$

$= 8\int_0^1 4^4\cdot\frac{1}{3}\cdot m^{1-\left(\frac{2}{3}\right)}\cdot\sqrt{1-m}\ dm$

$= 4^4\cdot 8\cdot\frac{1}{3}\int_0^1 m^{\frac{1}{3}}\ (1-m)^{\frac{1}{2}}\,dm$

$= 4^4\cdot 8\cdot\frac{1}{3}\int_0^1 m^{4/3-1}\cdot(1-m)^{\frac{3}{2}-1}\ dm$

$= 4^4\cdot 8\cdot\frac{1}{3}\int_0^1 m^{4/3-1}\ (1-m)^{3/2-1}\ dm$

$= 4^4\cdot 8\cdot\frac{1}{3}\cdot\beta\left(4/3,\ 3/2\right)$ (A)

limit

y	m
0	0
4	1

let,

$\frac{y^3}{64} = m \Rightarrow 64m = y^3 \Rightarrow 4\sqrt[3]{m} = y$

$\Rightarrow \frac{dm}{dy} = \frac{1}{64}\cdot 3y^2$

$\Rightarrow dm = \frac{1}{64}\,3y^2\cdot dy$

$\Rightarrow \frac{dm}{\frac{1}{64}\cdot 3\cdot(4\sqrt[3]{m})^2} = dy$

(VIII) $\int_0^1 x^2 (1-x^3)^{3/2} dx$

$x^3 = p \Rightarrow \sqrt[3]{p} = x$

$\Rightarrow \frac{dp}{dx} = 3x^2$

$\Rightarrow \frac{1}{3 \cdot (\sqrt[3]{p})^2} dp = dx$

$\Rightarrow \frac{1}{3 \cdot p^{2/3}} dp = dx$

$= \int_0^1 x^2 (1-p)^{3/2} \cdot \frac{1}{3} \cdot p^{-2/3} \cdot dp$

$= \frac{1}{3} \int_0^1 p^{2/3} (1-p)^{3/2} \cdot p^{-2/3} dp$

$= \frac{1}{3} \int_0^1 p^{1-1} (1-p)^{\frac{3}{2}} dp$

$= \frac{1}{3} \int_0^1 p^{1-1} (1-p)^{\frac{5}{2}-1} dp$

$= \frac{1}{3} \beta(1, \frac{5}{2})$ Ans

limit

x	p
0	0
1	1

③

(i) $\int_0^{\pi} \sin^5\theta \cos^4\theta \, d\theta$

$= \int_0^{\pi/2} \sin^5 2x \cdot \cos^4 2x \cdot 2\,dx$

$= 2\int_0^{\pi/2} \sin^5 2x \cdot \cos^4 2x \cdot dx$

$= \cancel{2} \cdot \dfrac{\Gamma\frac{5+1}{2} \cdot \Gamma\frac{4+1}{2}}{\cancel{2}\,\Gamma\frac{5+4+2}{2}}$

$= \dfrac{\Gamma 6/2 \cdot \Gamma 5/2}{\Gamma 11/2}$

$= \dfrac{\Gamma 3 \cdot \Gamma 5/2}{\Gamma 11/2}$

Let

$\cancel{\theta = \frac{x}{2}}$

$\cancel{\Rightarrow \frac{d\theta}{dx} =}$

$\Rightarrow 2x = \theta$

$x = \frac{\theta}{2}$

$\Rightarrow \frac{dx}{d\theta} = \frac{1}{2}$

$\Rightarrow 2dx = d\theta$

limit

θ	x
0	0
π	$\pi/2$

(ii) $\int_0^{\pi} \sin^6\theta \cos^7\theta \, d\theta$

$= 2\int_0^{\pi/2} \sin^6 2x \cos^7 2x \, dx$

$= \cancel{2} \cdot \dfrac{\Gamma\frac{6+1}{2} \cdot \Gamma\frac{7+1}{2}}{\cancel{2}\,\Gamma\frac{6+7+2}{2}}$

$= \dfrac{\Gamma 7/2 \cdot \Gamma 4}{\Gamma 15/2}$

Let

$2x = \theta$

$\Rightarrow x = \frac{\theta}{2}$

$\Rightarrow \frac{dx}{d\theta} = \frac{1}{2}$

$\Rightarrow d\theta = 2dx$

limit

x	θ
0	0
$\pi/2$	π

$$\int_0^{\pi/6} \sin^2 6x \cdot \cos^4 3x \, dx$$

$$= \int_0^{\pi/6} (\sin 2\cdot 3x)^2 \cdot \cos^4 3x \, dx$$

$$= \int_0^{\pi/6} (2\sin 3x \cdot \cos 3x)^2 \cdot \cos^4 3x \, dx$$

$$= \int_0^{\pi/6} 4 \cdot \sin^2 3x \cdot \cos^6 3x \, dx$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2 p \cdot \cos^6 p \, dx$$

$$= \frac{4}{3} \cdot \frac{\Gamma\frac{2+1}{2} \cdot \Gamma\frac{6+1}{2}}{2\,\Gamma\frac{2+6+2}{2}}$$

$$= \frac{4}{3} \cdot \frac{\Gamma 3/2 \cdot \Gamma 7/2}{2 \cdot \Gamma 10/2}$$

$$= \frac{4}{3} \cdot \frac{\Gamma 3/2 \cdot \Gamma 7/2}{2\Gamma 5}$$

let,

$3x = p$

$\Rightarrow \frac{dp}{dx} = 3$

$\Rightarrow dp = 3dx$

$\Rightarrow dx = \frac{1}{3} dp$

limit

x	p
0	0
$\pi/6$	$\pi/2$

(iv) $\int_0^{\pi/4} \sin^2 4\theta \cdot \cos^3 2\theta \, d\theta$

$= \int_0^{\pi/4} (2\sin 2\theta \cdot \cos 2\theta)^2 \cdot \cos^3 2\theta \, d\theta$

$= 4\int_0^{\pi/4} \sin^2 2\theta \cdot \cos^5 2\theta \, d\theta$

let,

$2\theta = x$

$\Rightarrow \frac{dx}{d\theta} = 2$

$\Rightarrow dx = 2d\theta$

$\Rightarrow d\theta = \frac{dx}{2}$

limit

x	θ
0	0
$\pi/2$	$\pi/4$

$= \frac{4}{2}\int_0^{\pi/2} \sin^2 x \cdot \cos^5 x \, dx$

$= \cancel{2} \cdot \frac{\Gamma\frac{2+1}{2} \cdot \Gamma\frac{5+1}{2}}{\cancel{2} \cdot \Gamma\frac{2+5+2}{2}}$

$= \frac{\Gamma 3/2 \cdot \Gamma 6/2}{\Gamma 9/2}$

$= \frac{\Gamma 3/2 \cdot \Gamma 3}{\Gamma 9/2}$ (Ans)

(v) $\int_0^{\pi/2} \sin^4\theta \cdot \cos^2\theta \, d\theta$

$= \dfrac{\Gamma\frac{4+1}{2} \cdot \Gamma\frac{2+1}{2}}{2\Gamma\frac{4+2+2}{2}}$

$= \dfrac{\Gamma 5/2 \cdot \Gamma 3/2}{2 \cdot \Gamma 4}$

(vi) $\int_0^{\pi/8} \sin^2 8x \cos^4 4x \, dx$

$= \int_0^{\pi/8} (2\sin 4x \cdot \cos 4x)^2 \cdot \cos^4 4x \, dx$

$= 4\int_0^{\pi/8} (\sin^2 4x) \cdot \cos^6 4x \, dx$

$= \frac{4}{4}\int_0^{\pi/2} \sin^2 m \cdot \cos^6 m \, dm$

$= \int_0^{\pi/2} \sin^2 m \cdot \cos^6 m \, dm$

$= \dfrac{\Gamma\frac{2+1}{2} \cdot \Gamma\frac{6+1}{2}}{2 \cdot \Gamma\frac{2+6+2}{2}} = \dfrac{\Gamma 3/2 \cdot \Gamma 7/2}{2 \cdot \Gamma 5}$

Let,

$4x = m$

$\Rightarrow \frac{dm}{dx} = 4$

$\Rightarrow dx = \frac{dm}{4}$

limit

x	m
0	0
$\pi/8$	$\pi/2$