Complex Variables
MATH463

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Contents

1 Review of Complex Numbers

Definition.

A **complex number** $z = x + iy \in \mathbb{C}$ consists of a **real part** $\operatorname{Re} z = x \in \mathbb{R}$ and an **imaginary part** $\operatorname{Im} z = y \in \mathbb{R}$.

$\mathbb{C}$ (the set of complex numbers) is equipped with **addition** and **multiplication**.

Let $z_1 = x_1 + iy_1$, $z_2 = x_z + iy_2 \in \mathbb{C}$. Then

$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$ and

$z_1 \cdot z_2 = (x_1 \cdot x_2 - y_1 \cdot y_2) + i\,(x_1 \cdot y_2 + y_1 \cdot x_2)$.

Example.

Try adding and multiplying two complex numbers. Addition has a simple geometric interpretation.

Definition.

A **field** is a set F equipped with and closed under two binary operators. The operators must be **associative**, **commutative**, and **distributive**. Each operator must also have an **inverse** and an **identity**.

$(\mathbb{C}, +, \cdot)$ forms a **field**. For any $z_1, z_2, z_3 \in \mathbb{C}$,

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3) \to$ **Associativity**.

$z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1 \to$ **Commutativity**.

$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3 \to$ **Distributivity**.

$0 = 0 + 0i \to$ **Additive Identity**.

$1 = 1 + 0i \to$ **Multiplicative Identity**.

$x + iy \mapsto -x - iy \to$ **Additive Inverse**.

$0 \neq x + iy \mapsto \dfrac{x}{x^2 + y^2} - i \dfrac{y}{x^2 + y^2} \to$ **Multiplicative Inverse**.

Definition.

The **absolute value** (or **modulus**) of a complex number $z = x + iy$ is denoted by $|z| = \sqrt{x^2 + y^2}$. It is a nonnegative real number. It is gemetrically similar to the magnitude.

Definition.

The **complex conjugate** of $z = x + iy$ is $\overline{z} = x - iy$.

Geometric Interpretation: $|z|$ is the distance from z to 0. $\overline{z}$ is obtained by reflecting z over the real axis.

Basic Properties: For $z_1, z_2 \in \mathbb{C}$, $a \in \mathbb{R} > 0$,
$\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$,
$|z_1 z_2| = |z_2 z_1|$ and $|a z_1| = a\,|z_1|$,
$|z_1 + z_2| \leq |z_1| + |z_2| \to$ **Triangle Inequality**. Equality occurs when 0, z_1, and z_2 are collinear.
$z_1 \overline{z_1} = |z_1|^2$.
By the last property, if $z_1 \neq 0$, $z_1^{-1} = \dfrac{1}{z_1} = \dfrac{\overline{z_1}}{z_1 \overline{z_1}} = \dfrac{\overline{z_1}}{|z_1|^2}$.

Example.
$i^{-1} = \dfrac{1}{i} = \dfrac{-i}{1} = -i$. $(4+3i)^{-1} = \dfrac{(4-3i)}{25}$.

Polar Form of Complex Numbers:

The **polar form** of a complex number $z = x + iy := |z|\,(\cos\theta + i \sin\theta)$. This is basically the magnitude/modulus of the complex number multiplied by the direction it faces. Thus we can let $r = |z|$ and $e^{i\theta} = \cos\theta + i\sin\theta$, and we get $z = re^{i\theta}$. Notice if $z \neq 0$, then $\dfrac{z}{|z|}$ lies on the unit circle.

Definition.

θ is called the **argument** of complex number $z \neq 0$. If θ is an argument of z, then so is $\theta + 2n\pi$ for any $n \in \mathbb{Z}$.

The **principal argument** of $z \neq 0$ denoted by $\operatorname{Arg} z$ is the unique $-\pi < \theta \leq \pi$ that satisfies $z = |z|\, e^{i\theta}$.

Some Polar Properties:

If $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, then $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$. Geometric interpretation: To multiply two complex numbers, multiply their moduli and add their arguments.

If $z = re^{i\theta}$, then for any $n \in \mathbb{Z}$, $z^n = r^n e^{in\theta}$. Note that for negative n, $z^{-1} = \dfrac{1}{r} e^{-i\theta}$.

If $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, then $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$.

Example.

$(1+i)^{18} \mapsto \sqrt{2}^{18} e^{i18\pi/4}$ in polar. Then we want the argument to be the principal argument. $= 2^9 e^{i\pi/2}$. Now we can see $= 2^9 i$.

Roots of Complex Numbers: Any nonzero complex number has exactly n n^{th} roots.

IF $z = re^{i\theta}$, then the distinct n^{th} roots of z are $r^{\frac{1}{n}}e^{i(\frac{\theta}{n}+\frac{2k\pi}{n})}$ for $k = 0, 1, \ldots n-1$.

Geometric Interpretation: The n^{th} roots of z are located on the circle with center 0 and radius $|z|^{\frac{1}{n}}$ and they divide the circle into n equal parts.

Solutions to Quadratics: Given a quadratic polynomial $az^2 + bz + c$, where $a, b, c \in \mathbb{C}$, define $\Delta = b^2 - 4ac$. If $\Delta \neq 0$, it always has two square roots $\sqrt{\Delta}$ and $-\sqrt{\Delta}$. So the roots of the quadratic are $\dfrac{-b \pm \sqrt{\Delta}}{2a}$.