Complex Variables

MATH463

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Contents

1 Review of Complex Numbers

Definition.

A **complex number** $z = x + iy \in \mathbb{C}$ consists of a **real part** $\operatorname{Re} z = x \in \mathbb{R}$ and an **imaginary part** $\operatorname{Im} z = y \in \mathbb{R}$.

$\mathbb{C}$ (the set of complex numbers) is equipped with **addition** and **multiplication**.

Let $z_1 = x_1 + iy_1$, $z_2 = x_z + iy_2 \in \mathbb{C}$. Then

$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$ and

$z_1 \cdot z_2 = (x_1 \cdot x_2 - y_1 \cdot y_2) + i\,(x_1 \cdot y_2 + y_1 \cdot x_2)$.

Example.

Try adding and multiplying two complex numbers. Addition has a simple geometric interpretation.

Definition.

A **field** is a set F equipped with and closed under two binary operators. The operators must be **associative**, **commutative**, and **distributive**. Each operator must also have an **inverse** and an **identity**.

$(\mathbb{C}, +, \cdot)$ forms a **field**. For any $z_1, z_2, z_3 \in \mathbb{C}$,

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3) \rightarrow$ **Associativity**.

$z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1 \rightarrow$ **Commutativity**.

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \rightarrow$ **Distributivity**.

$0 = 0 + 0i \rightarrow$ **Additive Identity**.

$1 = 1 + 0i \rightarrow$ **Multiplicative Identity**.

$x + iy \mapsto -x - iy \rightarrow$ **Additive Inverse**.

$0 \neq x + iy \mapsto \dfrac{x}{x^2 + y^2} - i\dfrac{y}{x^2 + y^2} \rightarrow$ **Multiplicative Inverse**.

Definition.

The **absolute value** (or **modulus**) of a complex number $z = x + iy$ is denoted by $|z| = \sqrt{x^2 + y^2}$. It is a nonnegative real number. It is gemetrically similar to the magnitude.

Definition.

The **complex conjugate** of $z = x + iy$ is $\overline{z} = x - iy$.

Geometric Interpretation: $|z|$ is the distance from z to 0. $\overline{z}$ is obtained by reflecting z over the real axis.

Basic Properties: For $z_1, z_2 \in \mathbb{C}$, $a \in \mathbb{R} > 0$,
$\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$,
$|z_1 z_2| = |z_1| \, |z_2|$ and $|a z_1| = a \, |z_1|$,
$|z_1 + z_2| \leq |z_1| + |z_2| \rightarrow$ **Triangle Inequality**. Equality occurs when 0, z_1, and z_2 are collinear.
$z_1 \overline{z_1} = |z_1|^2$.
By the last property, if $z_1 \neq 0$, $z_1^{-1} = \dfrac{1}{z_1} = \dfrac{\overline{z_1}}{z_1 \overline{z_1}} = \dfrac{\overline{z_1}}{|z_1|^2}$.

Example.
$i^{-1} = \dfrac{1}{i} = \dfrac{-i}{1} = -i$. $(4+3i)^{-1} = \dfrac{(4-3i)}{25}$.

Polar Form of Complex Numbers:

The **polar form** of a complex number $z = x + iy \coloneqq |z| (\cos\theta + i \sin\theta)$. This is basically the magnitude/modulus of the complex number multiplied by the direction it faces. Thus we can let $r = |z|$ and $e^{i\theta} = \cos\theta + i\sin\theta$, and we get $z = re^{i\theta}$. Notice if $z \neq 0$, then $\dfrac{z}{|z|}$ lies on the unit circle.

Definition.

θ is called the **argument** of complex number $z \neq 0$. If θ is an argument of z, then so is $\theta + 2n\pi$ for any $n \in \mathbb{Z}$.

The **principal argument** of $z \neq 0$ denoted by $\operatorname{Arg} z$ is the unique $-\pi < \theta \leq \pi$ that satisfies $z = |z| \, e^{i\theta}$.

Some Polar Properties:

If $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, then $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$. Geometric interpretation: To multiply two complex numbers, multiply their moduli and add their arguments.

If $z = re^{i\theta}$, then for any $n \in \mathbb{Z}$, $z^n = r^n e^{in\theta}$. Note that for negative n, $z^{-1} = \dfrac{1}{r} e^{-i\theta}$.

If $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, then $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$.

Example.

$(1+i)^{18} \mapsto \sqrt{2}^{18} e^{i18\pi/4}$ in polar. Then we want the argument to be the principal argument. $= 2^9 e^{i\pi/2}$. Now we can see $= 2^9 i$.

Roots of Complex Numbers: Any nonzero complex number has exactly n n^{th} roots.

If $z = re^{i\theta}$, then the distinct n^{th} roots of z are $r^{\frac{1}{n}} e^{i(\frac{\theta}{n} + \frac{2k\pi}{n})}$ for $k = 0, 1, \dots n-1$.

Geometric Interpretation: The n^{th} roots of z are located on the circle with center 0 and radius $|z|^{\frac{1}{n}}$ and they divide the circle into n equal parts.

Solutions to Quadratics: Given a quadratic polynomial $az^2 + bz + c$, where $a, b, c \in \mathbb{C}$, define $\Delta = b^2 - 4ac$. If $\Delta \neq 0$, it always has two square roots $\sqrt{\Delta}$ and $-\sqrt{\Delta}$. So the roots of the quadratic are $\dfrac{-b \pm \sqrt{\Delta}}{2a}$.

2 Complex Functions

We mainly deal with functions $f : S \subset \mathbb{C} \to \mathbb{C}$. These can be written as ...

$f(z) = w$, or $f(x + iy) = u(x, y) + iv(x, y)$, or $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$,

where $u, v : S \subset \mathbb{R}^2 \to \mathbb{R}$ are real functions defined on some domain in $\mathbb{R}^2$.

Definition.

Polynomial Functions: $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots a_0$, $a_i \in \mathbb{C}$, $a_n \neq 0$.

Rational Functions: $f(z) = \dfrac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomial functions. Domain of $f(z) = \mathbb{C} \setminus \{$ roots of $Q(z)\}$.

Exponential Functions: exp: $\mathbb{C} \to \mathbb{C}$ is defined by sending $z = x + iy$ to $e^x e^{iy} = e^x (\cos y \sin y)$, so $u(x, y) = e^x \cos y$, $v(x, y) = e^x \sin y$.

Absolute Value Function: $f(z) = |z| : \mathbb{C} \to \mathbb{C}$, $f(x + iy) = \sqrt{x^2 + y^2}$, so $u(x, y) = \sqrt{x^2 + y^2}$, $v(x, y) = 0$.

Principal Argument Function: $f(z) = \operatorname{Arg}(z) : \mathbb{C} \to \mathbb{C}$. $f(x + iy) = ?$, where ? depends on x and y. Case $x = y = 0$, $? = 0$. Case $x = 0$, $y \neq 0$, $? = \dfrac{\pi y}{2|y|}$. Case $x > 0$, $? = \tan \dfrac{y}{x}$. Case $x < 0$, $? = \pi + \tan^{-1} \dfrac{y}{x}$. Remember range of Arg is $(-\pi, \pi]$.

Conjugation function: $f(z) = \overline{z} : \mathbb{C} \to \mathbb{C}$, $f(x + iy) = x - iy$, so $u(x, y) = x$, $v(x, y) = -y$.

Note that graphing complex functions is impossible as it requires 4 dimensions! We usually graph with respect to z, and then to w.

Common Functions:

Translations. Let $c = a + bi \in \mathbb{C}$. Then $f(z) = z + c$.

Rotations. Let $\alpha \in \mathbb{R}$. Then $f(z) = ze^{i\alpha}$. When $\alpha = 0$, it is the identity function. This rotates CCW by α. Also $f(re^{i\theta}) = re^{i(\theta+\alpha)}$.

Reflections. Horizontal axis, $f(z) = \overline{z}$. Vertical axis, $f(z) = -\overline{z}$.

$f(z) = z^2$. From the polar perspective, we have $f(re^{i\theta}) = r^2 e^{2i\theta}$, which means f squares the moduli and doubles the arguments. A circle centered at 0 with radius r traversed once gets mapped to a circle centered at 0 with radius r^2 traversed twice.

$f(z) = e^z = e^x e^{iy}$. The visual effect is difficult to explain. Refer to Needham's "Visual Complex Analysis" for further details.

3 Limits

For a complex function f, we say $\lim_{z\to z_0} f(z) = w_0$ if for any $\epsilon > 0$ there exists $\delta > 0$, such that for any $z \in U$ satisfying $0 < |z - z_0| < \delta$ we have $|f(z) - f(z_0)| < \epsilon$. In English, this means we can get arbitrarily close to w_0 via $f(z)$ as long as z is sufficiently close to z_0. If no such w_0 exists in the w-plane, we say the limit does not exist.

Theorem.

If $f(z)$ is a polynomial function and z_0 is in the domain of $f(z)$, then $\lim_{z\to z_0} f(z) = f(z_0)$. If it's not in the domain, the limit does not exist.

Theorem.

Suppose $f(z), g(z)$ are two complex functions and $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$. Then ...

- $\lim_{z\to z_0}(f(z) + g(z)) = w_0 + w_1$
- $\lim_{z\to z_0}(f(z)g(z)) = w_0 w_1$
- If $w_1 \neq 0$, then $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{w_0}{w_1}$

Theorem.

Suppose $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$, then $\lim_{z\to z_0} f(z) = w_0 \iff$

$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$, $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$

4 Notion of ∞ for Complex Numbers

Suppose we mapped the complex plane to a unit sphere via **stereographic projection**. It would be $\{x^2 + y^2 = 1\} \setminus \{0, 0, 1\}$. We say $z \to \infty$ if the corresponding points on the sphere approach the north pole. There is no notion of $-\infty$, a complex number z approaches ∞ no matter the direction in the complex plane.

Theorem.

Following useful facts are true:

- $z \to \infty \iff |z| \to \infty$
- $\lim_{z \to z_0} f(z) = \infty \iff \displaystyle\lim_{z \to z_0} \frac{1}{f(z)} = 0$
- $\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f(\frac{1}{z}) = w_0$
- $\displaystyle\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$

5 Continuous Functions

Suppose $f : U \subset \mathbb{C} \to \mathbb{C}$ is a complex function. We say f is continuous at $z_0 \in U$ if $\lim_{z \to z_0} f(z) = f(z_0)$.

Theorem.

Following useful facts are true:

- Sum, product, and quotient (if defined at z_0) of continous functions at z_0 is continous at z_0.
- If f is continuous at z_0 and g is continuous at $f(z_0)$, then $g \circ f$ is continuous at z_0.
- If $f(z) = u(x, y) + iv(x, y)$ then f is continuous at $z_0 = x_0 + iy_0 \iff u$ and v are continuous at (x_0, y_0).
- If f is continuous at z_0, then so is $\overline{f}$ and $|f|$.
- If $f : S \subset \mathbb{C} \to \mathbb{C}$ is continuous and S is closed and bounded, then f is bounded, i.e. there exists $M > 0$ s.t. for any $z \in S$, $|f(z)| \leq M$.

6 Derivatives

Suppose $f : U \subset \mathbb{C} \to \mathbb{C}$ is a complex function, and there exists a small open disc $D = \{z \in \mathbb{C}$ s.t. $|z - z_0| < \epsilon\} \subset U$. The derivative of f at z_0 is $f'(z_0) = \lim_{h \to 0} \dfrac{f(z_0 + h) - f(z_0)}{h}$ (if the limit exists).

Other notation: $\dfrac{df}{dz}(z_0)$ or $\dfrac{dw}{dz}(z_0)$ if $w = f(z)$.