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Advanced Calculus 2

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Final Exam

Due on May 1, 2020

Problem	Possible points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120	

You need 100 points so 2 problems is a bonus.

Problem 1. Suppose $f : \mathbb{R}_+^2 \to \mathbb{R}$ is a continuous function defined on

$$\mathbb{R}_+^2 = \{(x, y) : x \in \mathbb{R}, \ y > 0\}.$$

Assume also that the limits

$$g(u, v) = \lim_{t \to 0} \frac{f((u + t) \cos v, (u + t) \sin v) - f(u \cos v, u \sin v)}{t},$$

and

$$h(u, v) = \lim_{t \to 0} \frac{f(u \cos(v + t)), u \sin(v + t)) - f(u \cos v, u \sin v)}{t},$$

exist and define continuous functions g, h on the domain

$$D = \{(u, v) : u > 0, \ 0 < v < \pi\}.$$

Prove that the function f is differentiable on $\mathbb{R}_+^2$.

Proof. Let $\Phi : D \to \mathbb{R}_+^2$ defined as $\Phi(u, v) = (u \cos v, u \sin v)$ is diffeomorphism, since
$D\Phi(u, v) = \begin{bmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{bmatrix} = u > 0$. Also,

$$g(u, v) = \frac{\partial(f \circ \Phi)}{\partial u}, \quad h(u, v) = \frac{\partial(f \circ \Phi)}{\partial v}$$

and partial derivatives of $f \circ \Phi$ are continuous, and then $f \circ \Phi \in C^1$. Thus, $f = (f \circ \Phi) \circ \Phi^{-1} \in C^1$. $\qquad \square$

Problem 2. Let $f \in C^1(\mathbb{R})$ be a continuously differentiable function such that $|f'(x)| \leq 1/2$ for all $x \in \mathbb{R}$. Define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x, y) = (x + f(y), y + f(x)).$$

Prove that

  (1) g is a diffeomorphism,
  (2) $g(\mathbb{R}^2) = \mathbb{R}^2$,
  (3) the area $|g([0, 1]^2)|$ of the image of the unit square belongs to the interval $[3/4, 5/4]$.

Hint: Among other tools use the contraction principle.
Proof.

  (a) For part (a) and (b), it suffices to show that $\det Dg > 0$ on $\mathbb{R}^2$ and for every (x_0, y_0), there is a unique (x, y) such that $g(x, y) = (x_0, y_0)$. We have

$$\det Dg(x, y) = \begin{vmatrix} 1 & f'(y) \\ f'(x) & 1 \end{vmatrix} = 1 - f'(x)f'(y) \geq \frac{3}{4} > 0.$$

  To prove the existence of a unique solution to $g(x, y) = (x_0, y_0)$, use the contraction principle. Let

$$T(x, y) = -g(x, y) + (x, y) + (x_0, y_0)$$
$$= (x_0 - f(y), y_0 - f(x)).$$

Then,

$$T(x, y) - T(x', y') = (x_0 - f(y), y_0 - f(x)) - (x_0 - f(y'), y_0 - f(x'))$$
$$= (f(y') - f(y), f(x') - f(x))$$
$$= (f'(\xi)(y' - y), f'(\zeta)(x' - x)),$$

and then

$$|T(x, y) - T(x', y')| = \sqrt{|f(y') - f(y)|^2 + |f(x') - f(x)|^2}$$
$$\leq \frac{1}{2}\sqrt{|y' - y|^2 + |x' - x|^2}.$$

Thus, T is a contraction. Hence there is a unique fixed point (x, y) of T such that $(x_0 - f(y), y_0 - f(x)) = (x, y)$, and it follows that

$$g(x, y) = (x + f(y), y + f(x)) = (x_0, y_0).$$

Thus, g is a diffeomorphism.

(c) Since $\det Dg = 1 - f'(x)f'(y) \in \left[\frac{3}{4}, \frac{5}{4}\right]$, then

$$\left|g([0, 1]^2)\right| = \int_{[0,1]^2} |\det Dg| \in \left[\frac{3}{4}, \frac{5}{4}\right].$$

$\square$

Problem 3. Prove that the tangent planes to the surface S defined by $x^2 + y^2 - z^2 = 1$ at the points $(x, y, 0) \in S$ are parallel to the z-axis.

Proof. The gradient vector of S at point $(x, y, 0)$ is $(2x, 2y, -2z)|_{z=0} = (2x, 2y, 0)$. And this vector is certainly orthogonal to the tangent planes to S at $(x, y, 0)$, and also it is orthogonal to z-axis. Thus, these tangent planes are parallel to the z-axis. $\square$

Problem 4. Prove that if $f : [0, 1] \to [0, 1]$ is a continuous function then its graph as a subset of $\mathbb{R}^2$ has measure zero.

Proof. Since f is continuous on a compact set, then f is uniformly continuous, i.e., for any $\varepsilon > 0$, there exists $\delta > 0$, such that if $|x - y| < \delta$, then $|f(x) - f(y)| \leq \varepsilon/2$. Now let $\delta = 1/n$ for some n such that above condition holds. Denote the set $\left[f\left(\frac{k}{n}\right), f\left(\frac{k+1}{n}\right)\right] \times \left[f\left(\frac{k}{n}\right) - \frac{\varepsilon}{2}, f\left(\frac{k}{n}\right) + \frac{\varepsilon}{2}\right]$ by $P_k, k = 0, 1, \cdots, n-1$ then the volume $|P_k| = \frac{\varepsilon}{n}$. Then, the graph G_f of f satisfies $G_f \subset \bigcup_{k=1}^n P_k$ and then

$$|G_f| = \sum_{i=1}^n |P_k| = n\frac{\varepsilon}{n} = \varepsilon \to 0.$$

Thus, the graph of f has measure zero. $\square$

Problem 5. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a continuous function. Let $K \subset \mathbb{R}^3$ be a compact set such that $|f(x) - f(y)| \leq 2012|x - y|^2$ for all $x, y \in K$. Prove that the set $f(K) \subset \mathbb{R}^2$ has measure zero as a subset of $\mathbb{R}^2$.

Proof. Since K is compact, then there exists $M > 0$ such that $K \subset [-M, M]^3$ and K can be covered by n^3 closed cubes Q_i of side-length $2M/n$, whose diameter is $2\sqrt{3}M/n$. Each such cube can be covered by a closed ball of radius $2\sqrt{3}M/n$ centered at any point of the cube. Then, K can be covered by no more than n^3 balls $B\left(x_i, 2\sqrt{3}M/n\right), x_i \in K$, such that $Q_i \cap K \neq \varnothing$ and $Q_i \cap K \subset B\left(x_i, 2\sqrt{3}M/n\right)$. Now,

$$f\left(B\left(x_i, 2\sqrt{3}M/n\right)\right) \subset B\left(f(x_i), 2012\left(2\sqrt{3}M/n\right)^2\right),$$

also, $K \subset \bigcup_i^k B\left(x_i, 2\sqrt{3}M/n\right), k \leq n^3$, then,

$$f(K) \subset \bigcup_i^k B\left(f(x_i), 2012\left(2\sqrt{3}M/n\right)^2\right).$$

Denote $2012\left(2\sqrt{3}M/n\right)^2$ by r_i, then

$$\sum_{i=1}^k r_i^2 \leq cn^{-4}n^3 = cn^{-1} \xrightarrow{n\to\infty} 0.$$

Thus, $f(K)$ has measure zero. $\qquad\square$

Problem 6. Assume that $f : [0,1] \to [0,1]$ is a continuous function such that the set $\{x \in [0,1] : f(x) = 1\}$ has measure zero. Prove directly (without using any results like monotone or dominated convergence theorem) that

$$\lim_{n\to\infty} \int_0^1 f(x)^n \, dx = 0.$$

Proof. Let $E = \{x \in [0,1] | f(x) = 1\}$. For fixed $\varepsilon > 0$, since E has zero measure, then $E \subset \bigcup_{i=1}^k I_i$, where I_i are open and $\sum_{i=1}^k |I_i| < \varepsilon/2$, i.e., $\bigcup_{i=1}^k I_i$ are a covering of E by open intervals of total lengths less than $\varepsilon/2$.

By compactness of $[0,1]$, $[0,1] \setminus \bigcup_{i=1}^k I_i$ is closed and hence compact. Then f attains maximum M on this set, which is strictly less than 1, and hence $|f(x)| \leq M < 1$ for all $x \in [0,1] \setminus \bigcup_{i=1}^k I_i$. Then,

$$f(x) \leq \begin{cases} 1, & x \in \bigcup_{i=1}^k I_i, \\ M^n, & x \notin \bigcup_{i=1}^k I_i. \end{cases}$$

and then we have

$$\int_0^1 f^n \, dx \leq 1 \cdot \left|\sum_{i=1}^k I_i\right| + M^n \cdot 1,$$

$$\leq \frac{\varepsilon}{2} + M^n.$$

Since $M < 1$, then there exists $N > 0$ such that for all $n \geq N$, $M^n < \varepsilon/2$. Thus, for all $n \geq N$, $\int_0^1 f^n \, dx \leq \varepsilon \to 0$. $\qquad\square$

Problem 7. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ and $\mathbf{v}_i : \mathbb{R} \to \mathbb{R}^n$, $i = 1, 2, \ldots, n-1$, be C^∞ smooth functions such that for any $t \in \mathbb{R}$ the vectors

$$\gamma'(t), \mathbf{v}_1(t), \ldots, \mathbf{v}_{n-1}(t)$$

form an orthonormal basis of $\mathbb{R}^n$ (here we differentiate γ but **do not** differentiate $\mathbf{v}_i$, $i = 1, 2, \ldots, n-1$).

Consider the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi(x_1, \ldots, x_n) = \gamma(x_n) + \sum_{i=1}^{n-1} x_i \mathbf{v}_i(x_n).$$

(a) Find the derivative $D\Phi(x_1, \ldots, x_n)$;
(b) Prove that Φ is a diffeomorphism in a neighborhood of and point of the form $(0, \ldots, 0, x_n)$;
(c) Find the limit

$$\lim_{r \to 0} \frac{|\Phi(B^n(0, r))|}{|B^n(0, r)|},$$

where $B^n(0, r)$ denotes the ball of radius r centered at the origin and $|A|$ stands for the volume of the set A.

Proof.

(a) $D\Phi(x_1, \ldots, x_n) = \left(\mathbf{v}_1(x_n), \ldots, \mathbf{v}_{n-1}(x_n), \gamma'(x_n) + \sum_{i=1}^{n-1} x_i \mathbf{v}'_{n-1}(x_n) \right)$.
(b) $D\Phi(0, \ldots, 0, x_n) = (\mathbf{v}_1(x_n), \ldots, \mathbf{v}_{n-1}(x_n), \gamma'(x_n))$ is an orthogonal matrix and then $|D\Phi|(0, \ldots, 0, x_n) = \pm 1 \neq 0$. Thus, Φ is a diffeomorphism in a neighborhood of a point of the form $(0, \ldots, 0, x_n)$.

(c)

$$\begin{aligned}
\lim_{r \to 0} \frac{|\Phi(B^n(0, r))|}{|B^n(0, r)|} &= \lim_{r \to 0} \frac{\left| \int_{B^n} \gamma(x_n) \, dA + \sum_{i=1}^{n-1} \int_{B^n} x_i \mathbf{v}_i(x_n) \, dA \right|}{\int_{B^n} 1 \, dA} \\
&= \frac{\left| \int_{B^n} \gamma(0) \, dA + \sum_{i=1}^{n-1} \int_{B^n} 0 \mathbf{v}_i(0) \, dA \right|}{\left| \int_{B^n} 1 \, dA \right|} \\
&= |\gamma(0)|.
\end{aligned}$$

$\square$

Problem 8. Prove that if $K \in C^1(\mathbb{R}^2 \setminus \{(0, 0)\})$ satisfies the estimate

$$|\nabla K(x)| \leq \frac{1}{|x|^3} \quad \text{for all } x \neq (0, 0)$$

then there is a constant $C > 0$ such that

$$\iint_{\{x \in \mathbb{R}^2 : |x| > 2|y|\}} |K(x - y) - K(x)| \, dx \leq C$$

for all $y \in \mathbb{R}^2$.
Hint: *Use the mean value theorem to estimate $|K(x - y) - K(x)|$ and then integrate in polar coordinates.*

Proof. Denote the left hand side by I. If $y = 0$, then $I = 0$. We can assume $|y| > 0$, and the point of interval connecting x to $x - y$ are of form $x - ty, t \in [0, 1]$. Then, with $|x| > 2|y|$,

$$|x - ty| > |x| - |y| > \frac{|x|}{2}.$$

With mean value theorem,

$$|K(x-y) - K(x)| \le |\nabla K(x - ty)| \cdot |(x - y) - x|$$

$$\le \frac{|y|}{|x - ty|^3}$$

$$\le \frac{8|y|}{|x|^3}.$$

Thus,

$$\iint_{|x|>2|y|} |K(x-y) - K(x)|\, dx \le \iint_{|x|>2|y|} \frac{8|y|}{|x|^3}\, dx$$

$$= 8|y| \int_0^{2\pi} \int_{2|y|}^\infty \frac{1}{r^3} r\, dr d\theta$$

$$= 8|y| 2\pi \left(-r^{-1} \right) \Big|_{2|y|}^\infty = 8\pi.$$

$\square$

Problem 9. Use Green's theorem to prove the following result: If the vertices of a polygon, in counterclockwise order, are (x_1, y_1), (x_2, y_2), ...,(x_n, y_n), then the area of the polygon is

$$A = \frac{1}{2} \sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i),$$

where we use notation $x_{n+1} = x_1$, $y_{n+1} = y_1$.

Proof. By Green's theorem, the area is

$$A = \frac{1}{2} \int_C x\, dy - y\, dx = \frac{1}{2} \sum_{i=1}^n \int_{(x_i,y_i)}^{(x_{i+1},y_{i+1})} x\, dy - y\, dx.$$

Let $\alpha(t) = (x_i + t(x_{i+1} - x_i), y_i + t(y_{i+1} - y_i))$, $t \in [0, 1]$ is a parameterization of the segment connecting (x_i, y_i) to (x_{i+1}, y_{i+1}). Let $\gamma(t) = (x(t), y(t)) \to \mathbb{R}^2, t \in [a, b]$, then

$$\int_\gamma P\, dx + Q\, dy = \int_a^b (P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t))\, dt$$

Then,

$$\int_{(x_i,y_i)}^{(x_{i+1},y_{i+1})} x\, dy - y\, dx = \int_0^1 (x_i + t(x_{i+1} - x_i))(y_{i+1} - y_i) - (y_i + t(y_{i+1} - y_i))(x_{i+1} - x_i)\, dt$$

$$= \left(tx_i + \frac{t^2}{2}(x_{i+1} - x_i) \right)(y_{i+1} - y_i) \Big|_0^1$$

$$- \left(ty_i + \frac{t^2}{2}(y_{i+1} - y_i) \right)(x_{i+1} - x_i) \Big|_0^1$$

$$= x_i y_{i+1} - x_{i+1} y_i,$$

and hence,

$$A = \frac{1}{2} \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i).$$

$\square$

Problem 10. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^1 boundary and let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi(x, y) = (u(x, y), v(x, y))$, be a C^2 diffeomorphism. Prove that

$$\int_{\partial\Omega} uv_x \, dx + uv_y \, dy = \pm|\Phi(\Omega)| \,,$$

where $|\Phi(\Omega)|$ denotes the area of $\Phi(\Omega)$ and $\partial\Omega$ has positive orientation. Show on examples that both cases $+|\Phi(\Omega)|$ and $-|\Phi(\Omega)|$ are possible.

Proof. By Green's theorem, $\int_{\partial\Omega} P \, dx + Q \, dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, then

$$\int_{\partial\Omega} uv_x \, dx + uv_y \, dy = \iint_\Omega \frac{\partial}{\partial x}(uv_y) - \frac{\partial}{\partial y}(uv_x)$$

$$= \iint_\Omega u_x v_y + uv_{yx} - u_y v_x - uv_{xy}$$

$$= \iint_\Omega u_x v_y - u_y v_x,$$

where in the last step we used the fact that $\Phi \in C^2$. Moreover,

$$\iint_\Omega u_x v_y - u_y v_x = \iint_\Omega \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \iint_\Omega \det D\Phi.$$

In particular, with change of variable, we have

$$\int_{\Phi(\Omega)} f = \int_\Omega (f \circ \Phi)|\det D\Phi|,$$

setting $f = 1$ gives

$$\int_\Omega |\det D\Phi| = |\Phi(\Omega)|.$$

Thus,

$$\int_{\partial\Omega} uv_x \, dx + uv_y \, dy = \begin{cases} |\Phi(\Omega)|, & \det D\Phi > 0, \\ -|\Phi(\Omega)|, & \det D\Phi < 0. \end{cases}$$

Taking $\Phi(x, y) = (x, y)$, then the integral equals to $|\Phi(\Omega)|$, if we take $\Phi(x, y) = (-x, y)$, then the integral equals to $-|\Phi(\Omega)|$. $\square$

Problem 11. Let f be a polynomial of total degree at most three in $(x, y, z) \in \mathbb{R}^3$. Prove that:

$$\int_{x^2+y^2+z^2\leq 1} f(x, y, z) \, dx \, dy \, dz = \frac{4\pi f((0,0,0))}{3} + \frac{2\pi \, (\Delta f) \, ((0,0,0))}{15}.$$

Here $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is the Laplacian operator on $\mathbb{R}^3$.

Proof. With Taylor's formula,

$$f(x) = f(0) + \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(0)x_i + \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_i}(0)x_i x_i + \frac{1}{6}\sum_{i,j,k=1}^{3} \frac{\partial^3 f}{\partial x_i \partial x_i \partial x_k}(0)x_i x_i x_k + R.$$

and since f is a polynomial of degree at most 3, then the remainder $R = 0$. Also, with the property of odd function,

$$\int_B x_i \, dx = 0, \quad \int_B x_i x_i x_k \, dx = 0,$$

and

$$\int_B x_i x_i \, dx = 0, \ i \neq j.$$

Then,

$$\int_B f(x, y, z) \, dx \, dy \, dz = f(0) \underbrace{\int_B dx}_{4\pi/3} + \frac{1}{2}\sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}(0) \int_B x_i^2 \, dx,$$

also, with

$$\int_B x_1^2 \, dx = \int_B x_2^2 \, dx = \int_B x_3^2 \, dx = \frac{1}{3}\int_B |x|^2 \, dx$$

$$= \int_0^1 \int_{S(r)} r^2 \, d\sigma \, dr$$

$$= \int_0^1 r^2 \cdot 4\pi r^2 \, dr = \frac{4\pi}{5},$$

the proof is completed. $\qquad\square$

Problem 12. Suppose that $K \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ and

$$K(x) = \frac{K(x/\|x\|)}{\|x\|} \quad \text{for all } x \in \mathbb{R}^2 \setminus \{0\}.$$

(a) Prove that $\nabla K(tx) = t^{-2}\nabla K(x)$ for $x \neq 0$ and $t > 0$.

(b) Use the divergence theorem to prove that (on both sides we integrate vector valued functions)

$$\int_{\{1 \leq \|x\| \leq 2019\}} \nabla K(x) \, dx = \int_{\partial\{1 \leq \|x\| \leq 2019\}} K(x)\vec{n} \, d\sigma(x).$$

(c) Prove that

$$\int_{\{1 \leq \|x\| \leq 2019\}} \nabla K(x) \, dx = 0.$$

Hint. *Show first that $K(tx) = t^{-1}K(x)$ for $x \neq 0$, $t > 0$. In (a) differentiate $K(tx)$. Part (a) is not needed for parts (b) and (c).*

Proof.

(a) Note that

$$K(tx) = \frac{K(tx/\|tx\|)}{\|tx\|} = t^{-1}\frac{K(x/\|x\|)}{\|x\|} = t^{-1}K(x).$$

Then,

$$t^{-1}\frac{\partial K}{\partial x_i}(x) = \frac{\partial}{\partial x_i}(t^{-1}K(x)) = \frac{\partial}{\partial x_i}(K(tx)) = \frac{\partial K}{\partial x_i}(tx) \cdot t,$$

which gives,

$$\frac{\partial K}{\partial x_i}(tx) = t^{-2}\frac{\partial K}{\partial x_i}(x).$$

Thus,

$$\nabla K(tx) = t^{-2}\nabla K(x).$$

(b) With divergence theorem, $\int_\Omega \boldsymbol{\nabla}\cdot\vec{F} = \int_{\partial\Omega} \vec{F}\cdot\vec{\mathbf{n}}$, if $F(x) = (K(x), 0)$, then $\boldsymbol{\nabla}\cdot F = \frac{\partial K}{\partial x_1}$ and then

$$\int_{\{1\le\|x\|\le 2019\}} \frac{\partial K}{\partial x_1} = \int_{\partial\{1\le\|x\|\le 2019\}} F \cdot \vec{\mathbf{n}}\, d\sigma(x) = \int_{\partial\{1\le\|x\|\le 2019\}} K(x)n_1\, d\sigma(x),$$

where in the last step we used $F \cdot \vec{\mathbf{n}} = (K(x), 0) \cdot (n_1, n_2) = K(x)n_1$.
   Similarly, if $F(x) = (0, K(x))$, then $\boldsymbol{\nabla} \cdot F = \frac{\partial K}{\partial x_2}$ and then

$$\int_{\{1\le\|x\|\le 2019\}} \frac{\partial K}{\partial x_2} = \int_{\partial\{1\le\|x\|\le 2019\}} K(x)n_2\, d\sigma(x).$$

Thus,

$$\int_{\{1\le\|x\|\le 2019\}} \nabla K(x)\, dx = \int_{\partial\{1\le\|x\|\le 2019\}} K(x)\vec{\mathbf{n}}\, d\sigma(x).$$

(c)

$$\int_{\{1\le\|x\|\le 2019\}} \nabla K(x)\, dx = \int_{\partial\{1\le\|x\|\le 2019\}} K(x)\vec{\mathbf{n}}\, d\sigma(x)$$

$$= -\int_{\|x\|=1} K(x)\frac{x}{\|x\|}\, d\sigma(x) + \int_{\|x\|=2019} K(x)\frac{x}{\|x\|}\, d\sigma(x).$$

Note that $K(2019x) = 2019^{-1}K(x)$. Use standard parameterization of the circle, we have

$$\int_\gamma f\, dxdy = \int_a^b f(x(t), y(t))\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt,$$

where $\gamma : [a, b] \to \mathbb{R}$ and $|\dot{\gamma}(t)| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$. Take $\gamma(t) = 2019(\cos t, \sin t)$, then

$$\int_{\|x\|=2019} K(x)\frac{x}{\|x\|}\, d\sigma(x) = \int_0^{2\pi} K(2019\cos t, 2019\sin t)\frac{(2019\cos t, 2019\sin t)}{2019}2019\, dt$$

$$= \int_0^{2\pi} K(\cos t, \sin t)\frac{(\cos t, \sin t)}{1}\, dt$$

$$= \int_{\|x\|=1} K(x)\frac{x}{\|x\|}\, d\sigma(x).$$

Thus,

$$\int_{\{1\leq\|x\|\leq 2019\}} \nabla K(x)\, dx = 0.$$

□