Note on Basic Topology

Zhen Yao

1. Euclidean Spaces

$\mathbb{R}^n$ is the n-field Cartesian product of $\mathbb{R}$, i.e., $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}\times\cdots\mathbb{R} = \{(x_1, x_2, \cdots, x_n)|x_i \in \mathbb{R}, i = 1, 2, \cdots, n\}$. Also, $\mathbb{R}^n$ is a linear space with respect to the addition and multiplication of points by scalars (i.e., real numbers) which are defined for $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n), c \in \mathbb{R}$ as follows:

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \\ cx &= (cx_1, cx_2, \cdots, cx_n) \end{aligned}$$

so that $x + y \in \mathbb{R}^n$ and $cx \in \mathbb{R}^n$. We also define the inner product (or scalar product) of x and y

$$(x, y) = x \cdot y = \sum_{i=1}^{n} x_i y_i$$

and the norm of x by

$$\|x\| = (x \cdot x)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$$

The structure now defined is called euclidean n-spaces.

Theorem 1.1. *Suppose $x, y, z \in \mathbb{R}^n$ and α is real. Then*

(a) $\|x\| \geq 0$;
(b) $\|x\| = 0$ *if and only if* $x = 0$;
(c) $\|\alpha x\| = |\alpha| \, \|x\|$;
(d) $\|x \cdot y\| \leq \|x\| \, \|y\|$, *this is called Cauchy-Schwarz inequality;*
(e) $\|x + y\| \leq \|x\| + \|y\|$, *this is called Triangle inequality;*
(f) $\|x - z\| \leq \|x - y\| + \|y - z\|$.

Proof. We only proof (d). If $x = (0, 0, \cdots, 0) = 0$, or $y = 0$, then it is obvious. If $x \neq 0$ and $y \neq 0$, then for $t \in \mathbb{R}$, we have

$$\begin{aligned} 0 \leq \|x + ty\| &= (x + ty, x + ty) \\ &= (x, x) + 2t(x, y) + t^2(y, y) \end{aligned}$$

And we know that $(x, x) + 2t(x, y) + t^2(y, y)$ is a quadratic function which is not negative. Hence, we have

$$\begin{aligned} \Delta = (2(x, y))^2 - 4(x, x)(y, y) \leq 0 \\ \Rightarrow \|(x, y)\| \leq \|x\| \, \|y\| \end{aligned}$$

□

2. Metric Spaces

Definition 2.1. *A set X, whose elements we call points, is said to be a metric space if with any two points x and y of X there is associated a real number $d(x,y):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ called the distance from x to y, which is defined as*

$$d(x,y)=\|x-z\|$$

which has the following properties

(a) $d(x,y)>0$ *if* $x\neq y$;
(b) $d(x,y)=0$ *if* $x=y$;
(c) $d(x,y)=d(y,x)$;
(d) $d(x,y)\leq d(x,z)+d(z,y)$.

Definition 2.2. *Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points in $\mathbb{R}^n$ and $y\in\mathbb{R}^n$. We say that $\{x_i\}_{i=1}^{\infty}$ converges to y if*

$$\lim_{i\to\infty}\|x_i-y\|=0$$

Then we write $\lim_{i\to\infty}x_i=y$. Equivalently, $\lim_{i\to\infty}x_i=y$ if

$$\forall\varepsilon>0,\exists N>0,\forall n>N,d(x_n,y)<\varepsilon.$$

Theorem 2.1. *Let $x_i=(x_{1i},x_{2i},\cdots,x_{ni})\in\mathbb{R}^n$, and $y=(y_1,y_2,\cdots,y_n)\in\mathbb{R}^n$. Then*

$$\lim_{i\to\infty}x_i=y$$

if and only if

$$\lim_{i\to\infty}x_{ki}=y_k,k=1,2,\cdots,n.$$

Proof. We have

$$\begin{aligned}\|x_i-y\|&=\sqrt{(x_{1i}-y_1)^2+\cdots+(x_{ni}-y_1)^2}\\&\geq\sqrt{(x_{ki}-y_k)^2}=|x_{ki}-y_k|\end{aligned}$$

Hence, $\|x_i-y\|\to0\Rightarrow|x_{ki}-y_k|\to0$.

On the other hand, if $|x_{ki}-y_k|$ as $i\to\infty$ for $k=1,2,\cdots,n$, then

$$\max_k|x_{ki}-y_k|\to0\quad\text{as}\quad i\to\infty$$

and hence

$$\begin{aligned}\|x_i-y\|&=\sqrt{(x_{1i}-y_1)^2+\cdots+(x_{ni}-y_1)^2}\\&\leq\sqrt{n\max_k(x_{ki}-y_k)^2}\\&\leq\sqrt{n}\max_k|x_{ki}-y_k|\to0.\end{aligned}$$

□

Definition 2.3. *Let (X,d) be a metric space and let $x_i\in X,i=1,2,\cdots,$ and $x\in X$. We say that the sequence $\{x_n\}_{i=1}^{\infty}$ converges to x, saying $\lim_{i\to\infty}x_i=x$ if $\lim_{i\to\infty}d(x_i,x)=0$.*

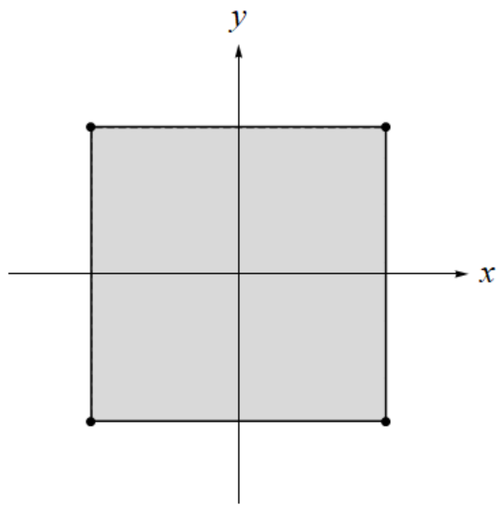

FIGURE 1. $B(0,1)$ in $(\mathbb{R}^2, \rho_1)$

Example 2.1. Examples of metric spaces:

(1)$(\mathbb{R}^n, \rho_1)$, where $\rho_1(x,y) = \max_i |x_i - y_i|$. And $B(0,1)$ in $(\mathbb{R}^2, \rho_1)$ is shown as above.

(2)$(\mathbb{R}^n, \rho_2)$, where $\rho_2(x,y) = \sum_{i=1}^n |x_i - y_i|$, this is called taxi metric or New York metric. And $B(0,1)$ in $(\mathbb{R}^2, \rho_2)$ is shown as bellow

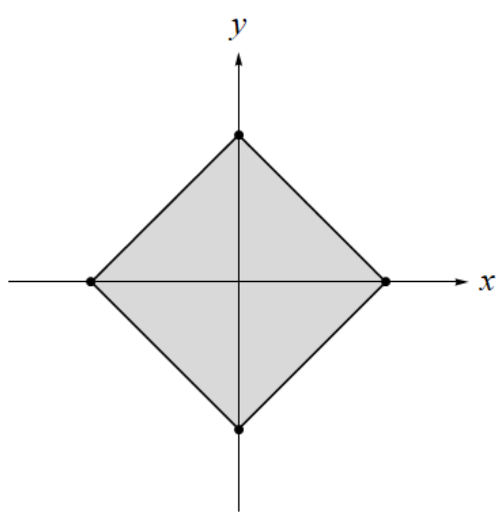

FIGURE 2. $B(0,1)$ in $(\mathbb{R}^2, \rho_2)$

(3)$(\mathbb{R}^n, \rho_3)$, where $\rho_3(x,y) = \|x - y\|$, this is called standard euclidean space.
(4)$(\mathbb{R}^n, \rho_4)$, where $\rho_4(x,y) = \|x - y\|^{1/2}$.
(5)(X, d), where X is arbitrary set and

$$d(x,y) = \begin{cases} 1, \text{if} & x \neq y \\ 0, \text{if} & x = y \end{cases}$$

This is called discrete metric space.

(6)One can prove that every continuous function on $[0,1]$ is bounded. This fact implies that (C,d), where $C([0,1]) = \{f : [0,1] \to \mathbb{R}, f \text{ is continuous}\}$ with $d(f,g) = \|f-g\|_\infty = \sup\{|f(x)-g(x)| : x, y \in [0,1]\}$ is a metric space.

(7)Let $l^1 = \{x = \{x_n\}_{n=1}^\infty : \sum_{n=1}^\infty |x_n| < \infty\}$, i.e., l^1 is the space of all absolutely convergent sequences. For $x = \{x_n\}_{n=1}^\infty, y = \{y_n\}_{n=1}^\infty \in l^1$, we define

$$d(x,y) = \sum_{n=1}^\infty |x_n - y_n|$$

We will prove that (l^1, d) is a metric space.

First we have $d(x,y) < \infty$ for $\forall x, y \in l^1$. And we have $|x_n - y_n| \leq |x_n| + |y_n|$, and hence

$$d(x,y) = \sum_{n=1}^\infty |x_n - y_n| \leq \sum_{n=1}^\infty |x_n| + \sum_{n=1}^\infty |y_n| < \infty$$

Now we have $(1) : d(x,y) \geq 0$ and $(2) : d(x,y) = d(y,x)$, which is obvious. And $(3) : d(x,y) = 0 \Leftrightarrow \forall x_n = y_n \Leftrightarrow x = y$. Finally, we have

$$|x_n - y_n| \leq |x_n - z_n| + |z_n - y_n|$$

and hence

$$\sum_{n=1}^\infty |x_n - y_n| \leq \sum_{n=1}^\infty |x_n - z_n| + \sum_{n=1}^\infty |z_n - y_n|$$
$$\Rightarrow d(x,y) \leq d(x,z) + d(z,y).$$

(8)Let $l^2 = \{x = \{x_n\}_{n=1}^\infty : \sum_{n=1}^\infty |x_n|^2 < \infty\}$. For $x = \{x_n\}_{n=1}^\infty, y = \{y_n\}_{n=1}^\infty \in l^1$, we define

$$d_2(x,y) = \sqrt{\sum_{n=1}^\infty (x_n - y_n)^2}$$

Thus, (l^2, d_x) is a metric space and this space is call Hilbert space.

Theorem 2.2. *If $x_n \to x$ and $y_n \to y$ in a metric space, then $d(x_n, y_n) \to d(x,y)$.*

Proof. The triangle inequality yields

$$d(x,y) \leq d(x,x_n) + d(x_n,y_n) + d(y_n,y)$$

Then, we have $d(x,y) - d(x_n,y_n) \leq d(x,x_n) + d(y_n,y)$. Also, we have

$$d(x_n,y_n) \leq d(x_n,x) + d(x,y) + d(y,y_n)$$

and then $d(x_n,y_n) - d(x,y) \leq d(x_n,x) + d(y,y_n)$. These two inequalities yield

$$|d(x,y) - d(x_n,y_n)| \leq d(x,x_n) + d(y_n,y) \to 0$$

and hence

$$|d(x,y) - d(x_n,y_n)| \to 0$$

which implies $d(x,y) \to d(x_n,y_n)$. □

3. Elements of Topology

Let (X, d) be a metric space. For $x \in X$ and $r > 0$, we define

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

and call it the ball of radius r centered at x. For example, if (X, d) is a discrete metric space, then $B(x, 1/2) = \{x\}$, $B(x, 1) = \{x\}$ and $B(x, 2) = X$.

Definition 3.1. *We say that a set $U \subset X$ is open if*

$$\forall x \in X, \exists r > 0, B(x, r) \subset U.$$

Definition 3.2 (Definition in Rudin's Principle of Mathematical Analysis)**.** *Let X be a metric space.*

(a)A neighborhood or a ball of x is a set $N_r(x)$ consisting of all y such that $d(x, y) < r$, for some $r > 0$. The number r is called the radius of $N_r(x)$.
(b)A point x is a limit point of set E if every neighborhood of x contains a point $y \neq x$ such that $y \in E$.
(c)If $x \in E$ and x is not a limit point of E, then x is called an isolated point of E.
(d)E is closed if every limit point of E is a point of E.
(e)A point x is an interior point of E if there is a neighborhood(or ball) $N_r(x)$ of x such that $N_r(x) \in E$.
(f)E is open if every point of E is an interior point of E.
(g)The complement of E (denoted by E^c) is the set of all points $x \in X$ such that $x \notin E$.
(h)E is perfect if E is closed and if every point of E is a limit point of E.
(i)E is bounded if there is a real number M and a point $x \in X$ such that $d(x, y) < M$ for all $y \in E$.
(j)E is dense in X if every point of X is a limit point of E, or a point of E(or both).

Theorem 3.1. *Every ball $B(x, r)$ is open.*

Proof. Let $B(x_0, r_0)$ be a ball. We will prove it is open. If $x \in B(x_0, y_0)$, then $d(x, x_0) < r_0$, then there exists a $r > 0$, such that

$$d(x, x_0) + r < r_0$$

We will now prove that $B(x, r) \subset B(x_0, r_0)$. Indeed, if $y \in B(x, r)$, then we have

$$\begin{aligned} d(x_0, y) &\leq d(x_0, x) + d(x, y) \\ &\leq d(x_0, x) + r < r_0 \end{aligned}$$

Since every point of $B(x, r)$ belongs to $B(x_0, r_0)$, we can conclude that $B(x, r) \subset B(x_0, r_0)$. According to Definition 3.1, we proved that $B(x_0, r_0)$ is open. □

Theorem 3.2. *Let (X, d) be a metric space, then*
(a)$\varnothing, X$ are open;
(b)Intersection of a finite family $U_1, \cdots, U_n \subset X$ of open sets, $\bigcap_{i=1}^n U_i$ is open;
(c)Union of an arbitrary family $U_i, i \in I$ of open sets, $\bigcup_{i\in I}^n U_i$ is open.

Proof. (a)This is obvious.

(b)Suppose $U_1, \cdots, U_n \subset X$ are open. Let $x \in \bigcap_{i=1}^n U_i$, we need to show $B(x, r) \subset$

$\bigcap_{i=1}^n U_i$ for some $r > 0$. We have

$$
\begin{aligned}
x \in U_1 &\Rightarrow B(x, r_1) \subset U_1, \text{for some}\, r_1 > 0\\
x \in U_2 &\Rightarrow B(x, r_2) \subset U_2, \text{for some}\, r_2 > 0\\
&\vdots\\
x \in U_n &\Rightarrow B(x, r_n) \subset U_n, \text{for some}\, r_n > 0
\end{aligned}
$$

Hence, we can pick $r = \min\{r_1, r_2 \cdots, r_n\}$, and it follows that $B(x, r) \subset \bigcap_{i=1}^n U_i$.

(c)Let $\{U_i\}_{i\in I}$ be an arbitrary family of open sets and let $x \in \bigcup_{i\in I} U_i$. Then there exists a $i_0 \in I$ such that $x \in U_{i_0}$, and hence $B(x, r) \subset U_{i_0} \subset \bigcup_{i\in I} U_i$ for some $r > 0$. The proof is complete. □

Definition 3.3. *Given $A \subset X$, where X is a metric space. The interior of the set A is defines as*

$$int A = \{x \in A : \exists r > 0, B(x, r) \subset A\}$$

Theorem 3.3. *$intA$ is always open. It is the largest open set contained in A in the sense that if $U \subset A$ is open, then $U \subset intA$.*

Proof.

(a)If $U \subset A$, then for $\forall x \in U$, there exists $r > 0$ such that $B(x, r) \subset U \subset \text{int}A$. This implies that $x \in \text{int}A$. Thus, $U \subset \text{int}A$.

(b)If $x \in \text{int}A$, then there exists $r > 0$ such that $B(x, r) \subset A$. Since $B(x, r)$ is open and $B(x, r) \subset A$, we have $\text{int}A$ is open. □

Definition 3.4. *Let (X, d) be a metric space. We say that $A \subset X$ is closed if $X \setminus A$ is open.*

Theorem 3.4 (Theorem 2.23 in Rudin's book)**.** *A set A is open if and only if it complement is closed.*

Proof. First, suppose A^c is closed. For $x \in A$, then $x \notin A^c$, and x is not a limit point of E^c. Then there exists $r > 0$ such that $B(x, r) \cap A^c = \varnothing$. Then, we have $B(x, r) \subset A$. Thus x is an interior point of A and it follows that A is open.

Next, suppose A is open. Let x be a limit point of A^c. Then every neighborhood of x contains a point of A^c, so x is not a interior point of A. Since A is open, then $x \notin A$, which means $x \in A^c$. Since x is a limit point of A^c, then A^c is closed. The proof is complete. □

Theorem 3.5. *Let (X, d) be a metric space, then*

(a)$\varnothing, X$ are closed;

(b)Intersection of an arbitrary family $U_i, i \in I$ of closed sets, $\bigcap_{i\in I}^n U_i$ is closed;

(c)Union of a finite family $U_1, \cdots, U_n \subset X$ of closed sets, $\bigcup_{i=1}^n U_i$ is open.

Proof. (a)$\varnothing$ is closed, since $X \setminus \varnothing = X$ is open. Also, X is closed, since $X \setminus X = \varnothing$ is open.

(b)Suppose $\{U_i\}_{i\in I}$ is an arbitrary family of closed sets. Then the set $X \setminus U_i$ are open, and we have

$$\bigcup_{i\in I}(X \setminus U_i) = X \setminus \bigcap_{i\in I} U_i$$

is open and hence $\bigcap_{i\in I} U_i$ is closed.

(c)Suppose the set $U_1, \cdots, U_n \subset X$ are closed. Then the sets $X\setminus U_i$ are open and hence

$$\bigcap_{i=1}^{n}(X \setminus U_i) = X \setminus \bigcup_{i=1}^{n} U_i$$

is open and it follows that $\bigcup_{i=1}^{n} U_i$ is closed. □

Definition 3.5 (Definition 2.26 in Rudin's book)**.** *If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points(or accumulation points) of E in X, the closure of E is the set $\bar{E} = E \bigcup E'$.*

Theorem 3.6. *In any metric space, the set $\bar{B}(x, r) = \{y \in X : d(x, y) \leq r\}$ is closed.*

Proof. We can prove this theorem by proving that $X \setminus \bar{B}(x_0, r_0) = \{y \in X : d(x_0, y) > r\}$ is open.If $x \in X \setminus \bar{B}(x_0, r_0)$, then $d(x_0, x) > r$ and hence there exists $r > 0$ such that

$$d(x_0, x) > r_0 + r$$

And with triangle inequality, for $\forall y \in B(x, r)$, we have $d(x_0, y) \geq d(x, x_0) - d(x, y) > r_0$, since $d(x, y) < r$. Thus, we have $B(x, r) \subset X \setminus \bar{B}(x_0, r_0)$, which implies that $X \setminus \bar{B}(x_0, r_0)$ is open. □

Theorem 3.7. *A set $A \subset U$ is closed if and only if the following implication is true: if a sequence $\{x_n\}_{n=1}^{\infty} \in A$ such that $x_n \to x$, then $x \in A$, i.e., if for every convergent sequence of A, its limit belongs to A.*

Proof. (a)Suppose A is closed. We need to prove that if $x_n \in A$, then $x \in A$. Suppose by contradiction that $x_n \in A \to x$, but $x \notin A$. Then, $x \in X \setminus A$. Since $X \setminus A$ is open, there exists a $\varepsilon > 0$ such that $B(x, \varepsilon) \subset X \setminus A$. Then $d(x_n, x) > \varepsilon$ for all n and thus x_n does not converges to x, which is a contradiction.

(b)Suppose that a set $A \subset X$ has the property that if $X_n \in A$ which converges to x, then $x \in A$. We need to prove that A is closed. And we only need to prove that $X \setminus A$ is open. Then, we need to prove that, for $x \in X \setminus A$

$$\exists \varepsilon > 0, B(x, \varepsilon) \subset X \setminus A$$

Suppose by contradiction that the above statement is not true, i.e.,

$$\forall \varepsilon > 0, B(x, \varepsilon) \not\subset X \setminus A$$

which means $B(x, \varepsilon) \bigcap A \neq \varnothing$. Then, taking $\varepsilon = 1/n$, $B(x, 1/n) \bigcap A \neq \varnothing$. Then we take $x_n \in B(x, 1/n) \bigcap A$. Then $x_n \in A$ and $d(x, x_n) < 1/n$, so $x_n \to x$. Since we assume $x \notin A$, and we get a contradiction. The proof is complete. □

Definition 3.6. *Let (X, d) be a metric space. We say that $x \in X$ is an accumulation point(or cluster point, or limit point) of a set $A \subset X$ if there is a sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_n \neq x$ and $x_n \to x$.*

Theorem 3.8. *$x \in X$ is an accumulation point if and only if every open set containing x contains an element of A different than x.*

Proof. (a)First, let $x \in U$ and U is open. Then $B(x, \varepsilon) \subset U$ for some $\varepsilon > 0$. Let $x_n \in A$, $x_n \neq x$ and $x_n \to x$. Then there exists n such that $x_n \in B(x, \varepsilon) \subset U$, where $x_n \neq x$.

(b)Second, for each ball $B(x, 1/n)$, there is a $x_n \in B(x, 1/n) \bigcap A$ and $x_n \neq x$. Then it follows that $x_n \to x$. The proof is complete. $\square$

Theorem 3.9. *A is closed if and only if all accumulation points of A belong to A.*

Proof. (a)First, suppose A is closed, then $X \setminus A$ is open. So if $x \notin A$, then $B(x, \varepsilon) \subset X \setminus A$ for some $\varepsilon > 0$. Then $B(x, \varepsilon)$ contains no point of A and hence x cannot be an accumulation point. Therefore, every accumulation point must belong to A.

(b)Suppose all accumulation points of A belong to A. We need to show that A is closed, it suffices to prove that $X \setminus A$ is open. Let $x \in X \setminus A$, then x is not an accumulation point of A, then there exists a open set U such that $x \in U$ and U contains no point of A. Hence, $U \subset X \setminus A$, and it follows $B(x, \varepsilon) \subset U \subset X \setminus A$ for some $\varepsilon > 0$. Thus, $X \setminus A$ is open. The proof is complete. $\square$

Theorem 3.10 (Theorem 2.27 in Rudin's book)**.** *The closure of A: cl(A) is intersection of all closed sets that contain A. Therefore, cl(A) is closed. Moreover, cl(A) is the smallest closed set that contains A in the sense that if E is another closed set such that $A \subset E$, then cl$(A) \subset E$.*

Proof. First, if $x \in X$ and $x \notin \mathrm{cl}(A)$, then x is neither a point of A nor a accumulation point of A. Hence, for x there exists a $B(x, \varepsilon) \bigcap A = \varnothing$, for some $\varepsilon > 0$. Then we have that $X \setminus \mathrm{cl}(A)$ is open, which implies $\mathrm{cl}(A)$ is closed.

Second, If E is closed and $A \subset E$, since $\mathrm{cl}(A)$ is the intersection of all closed sets that contain A, and E is in the family whose intersection we take, and hence $\mathrm{cl}(A) \subset E$. $\square$

Theorem 3.11. *The closure of $A \subset X$ is cl$(A) = \{x \in X | \exists x_n \in A, n = 1, 2, \cdots, x_n \to x\}$.*

Remark 3.1. *We do not assume $x_n \neq x$ here.*

Proof. If $x \in A$, then $x_n = x$ satisfies that $x_n \to x$. If x is an accumulation point of A, then there is a sequence $x_n \in A$ such that $x_n \to x$. Therefore, we have

$$\mathrm{cl}(A) \subset \{x \in X | \exists x_n \in A, x_n \to x\}$$

On the other hand, if $x_n \in A$ and $x_n \to x$, then either all $x_n \neq x$ and x is an accumulation point of A or $x_n = x$ for some n and $x \in A$, which implies

$$\{x \in X | \exists x_n \in A, x_n \to x\} \subset \mathrm{cl}(A)$$

The proof is complete. $\square$

3.1. **Boundary of a set.**

Definition 3.7. *Let (X, d) be a metric space, and $A \subset X$. Boundary of A is defined as $bd(A) = cl(A) \bigcap cl(X \setminus A)$.*

Theorem 3.12. *$x \in bd(A)$ if and only if there exists a sequence in A and a sequence of $X \setminus A$ such that they both converge to x.*

Proof. It is an obvious result following the definition above. □

Another theorem follows this theorem immediately.

Theorem 3.13. *$x \in bd(A)$ if and only if $\forall \varepsilon > 0$, $B(x, \varepsilon) \bigcap A \neq \varnothing$ and $B(x, \varepsilon) \bigcap (X \setminus A) \neq \varnothing$.*

3.2. **Complete metric space.**

Definition 3.8. *Let (X, d) be a metric space. We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of X is a Cauchy sequence if $\forall \varepsilon > 0$, there $\exists N > 0$, such that $\forall n, m \geq N$, $d(x_n, x_m) < \varepsilon$.*

Definition 3.9. *A sequence $\{x_n\}_{n=1}^{\infty} \in X$ is called bounded if $x_n \in B(x_0, R)$ for some ball $B(x_0, R)$ and $\forall n = 1, 2, 3, \cdots$.*

Theorem 3.14.

(a)Every convergent sequence in a metric space is a Cauchy sequence.

(b)Every Cauchy sequence in a metric space is bounded.

(c)If a subsequence of a Cauchy sequence in a metric space is convergent, then the whole sequence is convergent to the same limit.

Proof. (a) and (b) are obvious.

(c)Suppose $\{x_n\}$ is a Cauchy sequence and $x_{n_k} \to x$. We need to prove that $x_n \to x$. For $\{x_{n_k}\}$, we have $\forall \varepsilon > 0$, there exists a N_1 such that for $\forall k \geq N_1$, $d(x_{n_k}, x) < \varepsilon/2$. Also, since $\{x_n\}$ is a Cauchy sequence, then for $\forall \varepsilon > 0$, there exists a N_2 such that for $\forall n, m \geq N_1$, $d(x_n, x_m) < \varepsilon/2$. Take $N = \max\{N_1, N_2\}$, since $n_N \geq N$, we have for $n \geq N$,

$$d(x, x_n) \leq d(x_n, x_{n_N}) + d(x_{n_N}, x) < \varepsilon$$

which means $x_n \to x$. The proof is complete. □

Definition 3.10. *We say that a metric space is complete if every Cauchy sequence is convergent. For example, $\mathbb{R}$ is complete, but $\mathbb{Q}$ is not.*

Theorem 3.15. *$\mathbb{R}^n$ is complete.*

Proof. We have

$$\begin{aligned}
&\{x_k\}_k = ((x_{k1}, x_{k2}, \cdots, x_{kn})) \text{ is Cauchy sequence} \\
\Leftrightarrow &\{x_{ki}\}_k, i = 1, 2, \cdots, n \text{ is Cauchy sequence} \\
\Leftrightarrow &\{x_{ki}\}_k, i = 1, 2, \cdots, n \text{ is convergent} \\
\Leftrightarrow &\{x_k\}_k \text{ is convergent}
\end{aligned}$$

The proof is complete. □

Definition 3.11. *Let $\{x_n\}$ be a sequence in a metric space. We say that $x \in X$ is a cluster point of $\{x_n\}$ if x is the limit of a sbusequence of $\{x_n\}$.*

Remark 3.2.

(1) *$x \in X$ is an accumulation point of a set $A \subset X$ if there is a sequence $\{x_n\}_{n=0}^{\infty} \in A$ such that $x_n \neq x$ and $x_n \to x$.*

(2) *$x \in X$ is a limit point of set $A \subset X$ if every neighborhood of x contains a point $y \neq x$ such that $y \in A$.*

Theorem 3.16. *The set of cluster points is closed.*

Proof. Suppose a_k is a cluster point of $\{x_n\}$ and $a_k \to a$. We need to prove that a is a cluster point of x_n. Each a_k is a cluster point of a subsequence x_{n_k}, then in any neighborhood of a there are infinitely many elements of x_n, and hence we can select a subsequence x_{n_k} that converges to a. ☐

Theorem 3.17. *Let $A \subset X$ be a closed subspace of a complete metric space (X, d), then (A, d) is a complete metric space as well.*

Proof. If $\{x_n\}$ is a Cauchy sequence in A, then it is a Cauchy sequence in X, so it converges to some point in X. Since A is closed, then $x \in A$, which proves that (A, d) is complete. ☐

Theorem 3.18. *In a metric space, $x_n \to x$ if and only if every subsequence of x_n has a further subsequence that converges to x.*

Proof. ($\Rightarrow$) This is obvious result of convergence.

($\Leftarrow$) Suppose that $\{x_n\}$ has the property above, but x_n does not converge, that is $\exists \varepsilon > 0$, for $\forall N > 0$, there exists $n \geq N$ such that $d(x_n, x) \geq \varepsilon$. Thus we can pick a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, x) \geq \varepsilon$$

Clearly, we can know $\{x_{n_k}\}$ has no sequence converging to x, which is a contradiction. ☐

3.3. **Compact spaces.**

Definition 3.12. *We say that a subset $A \subset X$ of a metric space is compact if every sequence in A has a subsequence converging to a point in A.*

Definition 3.13 (Definition 2.31 & 2.32 in Rudin's book)**.**

- *By an open cover of a set A in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $A \subset \bigcup_\alpha G_\alpha$.*
- *A subset A of a metric space X is said to be compact if every open vector of A contains a finite subcover.*

Proposition 3.1. *If A is compact, then A is bounded and closed.*

Proof. Let $A \subset X$ be compact. To prove that A is closed we need to prove that

$$A \ni x_n \to x \Rightarrow x \in A$$

Since $x_n \in A$ and A is compact, then it has a subsequence $\{x_{n_k}\}$ convergenting to a point x, then clearly $x \in A$.

Now we prove that A is bounded. Suppose A is not bounded, we fixed $x_0 \in X$ and find a sequence $\{x_n\} \in A$ such that $d(x_n, x_0) \geq n$. Then no subsequence of $\{x_n\}$ converges, which is a contradiction. $\square$

Theorem 3.19 (Heine-Borel Theorem)**.** *$A \subset \mathbb{R}^n$ is compact if and only if A is bounded and closed.*

Proof. ($\Rightarrow$) This is a result of proposition above.

($\Leftarrow$) We prove it for $n = 3$. Let $x_k \in A, k = 1, 2, 3, \cdots$. Since A is bounded, we can know that all three elements of $x_k = (x_{1k}, x_{2k}, x_{3k})$ are bounded in $\mathbb{R}$. Then the sequence $\{x_{1k}\}_k$ is bounded, so Bolzano-Weierstrass theorem, it has a convergent subsequence. Then $x_{1k_n} \to x_1$ and $x_1 \in A$ since A is closed. Similiarly, $\{x_{2k_n}\}$ also has a convergent subsequence $\{x_{2k_{n_m}}\}$ converging to $x_2 \in A$, and $\{x_{3k_{n_m}}\}$ also has a convergent subsequence $\{x_{3k_{n_{m_l}}}\}$ converging to $x_3 \in A$. Thus, we have

$$x_{k_{n_{m_l}}} = \left(x_{1k_{n_{m_l}}}, x_{2k_{n_{m_l}}}, x_{3k_{n_{m_l}}}\right) \to (x_1, x_2, x_3) \in A$$

Then A is compact. $\square$

Definition 3.14. *Let (X, d) be a metric space and $A \subset U$ a subset. We say that a family of open sets $\{U_i\}_{i \in I}$ forms a open covering of A if $A \subset \bigcup_{i \in I} U_i$. Now, $\{U_{i_k}\}_{i_k \in I}, k = 1, 2, \cdots, N$ forms a finite subcovering of A if $A \subset \bigcup_{i_k=1}^{N} U_{i_k}$.*

Theorem 3.20 (Bolzano-Weierstrass theorem)**.** *Let (X, d) be a metric space and $A \subset X$ a subset. Then (X, d) is compact if and only if every open covering of A has a finite subcovering.*

Remark 3.3. *We denote balls in metric space (X, d) and (A, d) by $B^X(x, r)$ and $B^A(x, r)$ respectively. Clearly, $B^A(x, r) = B^X(x, r) \bigcap A$.*

Before we prove the Bolzano-Weierstrass theorem, we need to mention some other theorem and lemma.

Theorem 3.21 (Theorem 2.30 in Rudin's book)**.** *$U \subset A$ is open in (A, d) if and only if there is $W \subset X$ open in (X, d) such that $U = W \bigcap A$.*

Proof. ($\Leftarrow$) Suppose $W \subset X$ open in X and $U = W \bigcap A$. Then for $\forall x \in U$, there exists a $r > 0$ such that $B^X(x, r) \subset W$ and hence $B(x, r)^X \bigcap A \subset U \bigcap A = U$, where $B(x, r)^X \bigcap A$ is a ball in A. Then U is open in (A, d).

($\Rightarrow$) Suppose $U \subset A$ is open in (A, d). Then for $\forall x \in U$, there exists a $r > 0$ such that $B^X(x, r) \bigcap A \subset U$, where $B^X(x, r) \bigcap A$ is a ball in A. Clearly, $U = \bigcup_{x \in X} B^X(x, r) \bigcap A$.

Now we set $W = \bigcup_{x \in X} B^X(x, r)$ and W is open in (X, d) and

$$W \bigcup A = \bigcup_{x \in X} B^X(x, r) \bigcap A = U$$

The proof is complete. $\square$

Corollary 3.21.1. *$E \subset A$ is closed in (A, d) is and only if there is a set $F \subset X$ closed in (X, d) such that $E = F \bigcap A$.*

Theorem 3.22. *A metric space X is compact if and only if every open covering of X has a finite subcovering.*

Proof. ($\Leftarrow$) Suppose that every open covering of X has a finite subcovering. We need to prove that every sequence $\{x_n\}$ in X has a convergent subsequence.

By contradiction, we suppose that $\{x_n\} \in X$ does not has convergent subseqnece(Note that $\{x_n\}$ has infinitely many different values, otherwise we would have a constant, and thus convergent subsequence). Therefore we can select a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \neq x_{n_l}$ for $k \neq l$ and $\{x_{n_k}\}$ does not converge. Observe that the set $\{x_{n_1}, x_{n_2}, x_{n_3}, \cdots\}$ is closed and the set has no accumulation point.

In particular, every x_{n_k} is not a accumulation point of the sequence $\{x_{n_k}\}$, and hence there is a $\varepsilon_k > 0$ such that the ball $B(x_{n_k}, \varepsilon_k)$ contains no points of this sequence other than x_{n_k}, i.e.,

$$x_{n_l} \notin B(x_{n_k}, \varepsilon_k), \text{if}\, l \neq k$$

Clearly, $X \setminus \{x_{n_1}, x_{n_2}, x_{n_3}, \cdots\}$ is open and hence

$$X = \bigcup_{k=1}^{\infty} B(x_{n_k}, \varepsilon_k) \cup (X \setminus \{x_{n_1}, x_{n_2}, x_{n_3}, \cdots\})$$

is an open covering of X. Thus, this covering has no finite subcovering.

($\Rightarrow$) We need a lemma.

Lemma 3.23. *Let $\bigcup_{i\in I} U_i$ be an open covering of a compact metric space of X such that $X = \bigcup_{i\in I} U_i$. Then there is $r > 0$(called Lebesgue number of the covering) such that $\forall x \in X$, $\exists i \in I$ such that $B(x, r) \subset U_i$.*

Proof. Prove by contradiction. Then we can find $x_n \in X$ such that $B(x_n, 1/n)$ is not contained in any of the open set U_i. Since X is compact, $\{x_n\}$ has a cinvergent subsequence $x_{n_k} \to x_0 \in U_{i_0}$ for some $i_0 \in I$. Then we have $B(x_0, \varepsilon) \subset U_{i_0}$ for some $\varepsilon > 0$. Since $x_n \to x_0$, it is clear that $B(x_n, 1/n) \subset B(x_0, \varepsilon)$, which is a contradiction. ☐

We need one definition and one more lemma.

Definition 3.15. *A metric space X is said totally bounded if $\forall \varepsilon > 0$, there exists a finite covering of X by balls of radius ε.*

Lemma 3.24. *If a metric space X is compact, then x is totally bounded.*

Proof. Prove by contradiction. Then there is a $\varepsilon > 0$ such that no finite family of balls with radius ε covers X. Let $x_1 \in X$, then $B(x_1, \varepsilon) \neq X$. Then there exists $x_2 \notin B(x_1, \varepsilon)$, and $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \neq X$. Then there exists $x_3 \notin B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$, and $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \neq X$. We can continue this process and construct a sequence $\{x_1, x_2, x_3, \cdots\}$ in X such that $d(x_k, x_l) \geq \varepsilon$ for $k \neq l$. Clearly, $\{x_n\}$ has no convergent subsequence, which is contradicted with compactness of X. ☐

Now we continue the proof of the implication ($\Rightarrow$) of theorem 3.22.

($\Rightarrow$) Suppose that X is compact, i.e., every sequence in x has a convergent subsequence. We need to prove that every open covering of X has a finite subcovering.

Let $X = \bigcup_{i\in I} U_i$ and let $r > 0$ be a Lebesgue number of the covering. Since X is totally bounded, X has finite coverings by balls of radius r, i.e.,

$$X = \bigcup_{i=1}^{N} B(x_i, r)$$

By the definition of Lebesgue number, we have $B(x_i, r) \subset U_{k_i}$. Then,

$$X = \bigcup_{i=1}^{N} B(x_i, r) \subset \bigcup_{i=1}^{N} U_{k_i}$$

which gives a finite open subcovering of X. The proof of theorem 3.22 is complete. This also completes the proof is Bolzano-Weierstrass theorem. $\square$

Theorem 3.25. *A metric space X is compact if and only if it is complete and totally bounded.*

Proof. ($\Rightarrow$) Suppose X is compact. Then X is totally bounded. To prove it is complete, let $\{x_n\}$ be a Cauchy sequence in X. The by compactness, we can know some subsequence of $\{x_{n_k}\}$ is convergent, such that $x_{n_k} \to x_0$. Then $\{x_n\}$ also converges to x_0. Thus, X is complete.

($\Leftarrow$) Suppose that X is complete and totally bounded. We need to prove that every sequence $\{x_n\}$ in X has a convergent subsequence.

Since X is totally bounded , then it has a finite open covering of balls with radius 1, i.e.,

$$X = \bigcup_{i=1}^{N_1} B(x_i^{(1)}, 1)$$

Hence, infinitely many elements of sequence $\{x_n\}$ belong to at least one of the balls, saying $B(x_{i_1}^{(1)}, 1)$. This ball has a finite open covering by balls with radius of $\frac{1}{2}$, i.e.,

$$B(x_{i_1}^{(1)}, 1) = \bigcup_{i=1}^{N_2} B\left(x_i^{(2)}, \frac{1}{2}\right) \cap B(x_{i_1}^{(1)}, 1)$$

Still, infinitely many elements of the sequence belong to at least one of the set on the right hand side, saying $B\left(x_{i_2}^{(2)}, \frac{1}{2}\right) \cap B(x_{i_1}^{(1)}, 1)$. And this set has a finite open covering by balls of radius of $\frac{1}{3}$, i.e.,

$$B\left(x_{i_2}^{(2)}, \frac{1}{2}\right) \cap B(x_{i_1}^{(1)}, 1) = \bigcup_{i=1}^{N_3} B\left(x_i^{(3)}, \frac{1}{3}\right) \cap B\left(x_{i_2}^{(2)}, \frac{1}{2}\right) \cap B(x_{i_1}^{(1)}, 1)$$

Still, infinitely many elements of the sequence belong to at least one of the set on the right hand side, and we continue this process and pick a subsequence from the given sequence $\{x_n\}$ such that

$$\begin{aligned}
x_{n_1} &\in B(x_{i_1}^{(1)}, 1) \\
x_{n_2} &\in B\left(x_{i_2}^{(2)}, \frac{1}{2}\right) \cap B(x_{i_1}^{(1)}, 1) \\
x_{n_3} &\in B\left(x_{i_3}^{(3)}, \frac{1}{3}\right) \cap B\left(x_{i_2}^{(2)}, \frac{1}{2}\right) \cap B(x_{i_1}^{(1)}, 1) \\
&\vdots
\end{aligned}$$

Therefore, for $k, ; l \geq N$, we have $x_{n_k}, x_{n_l} \in B\left(x_{i_N}^{(2)}, \frac{1}{N}\right)$ and hence

$$d(x_{n_k}, x_{n_l}) < \frac{2}{N}$$

Thus the sequence is Cauchy sequence and therefore convergent, it follows that X is complete. ☐

Theorem 3.26. *If $F_k \subset X$ are nonempty compact sets such that $F_1 \supset F_2 \supset F_3 \supset \cdots$, then $\bigcap_{k=1}^{\infty} F_k \neq \varnothing$.*

Proof. Let $x_k \in F_k$ for $k = 1, 2, 3, \cdots$. Then $X_n \in F_k$ for all $n \geq k$. In particular, $x_n \in F_1$ for all n. Hence x_n has a convergent subsequence in F_1(F_1 is compact), saying $x_{n_l} \to x_0 \in F_1$. Now, we have $x_{n_l} \in F_k$ for all $n_l \geq k$, and hence the limit x_0 of $\{x_{n_l}\}$ must belong to every F_k, i.e., $x_0 \in \bigcup_{k=1}^{\infty} F_k$ which proves the intersection is not empty. ☐

Remark 3.4. *The claim is not true if F_k are open or closed but unbounded, for example*

$$\bigcap_{k=1}^{\infty} \left(0, \frac{1}{k}\right) = \varnothing$$

$$\bigcap_{k=1}^{\infty} [k, \infty) = \varnothing$$

3.4. **Cantor set.**

Definition 3.16. *By the segment (a, b) we mean the set of all real numbers x such that $a < x < b$. By the interval $[a, b]$ we mean the set of all real numbers x such that $a \leq x \leq b$.*

The set which we are now constructing shows that there exists perfect set in $\mathbb{R}^1$ which contain no segment.

(1)Let E_0 be the interval $[0, 1]$.

(2)Remove the segment $\left(\frac{1}{3}, \frac{2}{3}\right)$, and let E_1 be the union of the interval of

$$\left[0, \frac{1}{3}\right], \left[\frac{1}{3}, 1\right].$$

(3)Remove the middle thirds of these two intervals and let E_2 be the union of

$$\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{1}{3}\right], \left[\frac{6}{9}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right].$$

(4)Continue this way and we can get a sequence of compact sets E_n, such that $E_1 \supset E_2 \supset E_3 \supset \cdots$ and E_n is the union of 2^n intervals with length 3^{-n}.

We show the Cantor set in seven iterations as below:

FIGURE 3. Cantor set in seven iterations

The set

$$P = \bigcap_{n=1}^{\infty} E_1$$

is called the Cantor set. P is clearly compact and P is not empty. And one can prove that P is uncountable.

3.5. **Connected sets.**

Definition 3.17.

(1) $\varphi : [a,b] \to X$ *is called continuous if* $x_n \to t$*, then* $\varphi(x_n) \to \varphi(x)$ *for every sequence* $\{x_n\}$ *in* $[a,b]$*.*
(2) *If* $A \subset X$ *is a subset of a metric space, a continuous path connecting* $x, y \in A$ *inside* A *is any continuous function* $\varphi : [a,b] \to A$ *such that* $\varphi(a) = x, \varphi(b) = y$*.*
(3) *A set* A *is called path connected if every two points in* A *can be connected by a continuous path inside* A*.*

Definition 3.18. *A set* $A \subset X$ *is called disconnected if there exists open sets* U, V *in* X *such that*
(1) $A \subset U \cup V$*;*
(2) $A \cap U \neq \varnothing$ *and* $A \cap V \neq \varnothing$*;*
(3) $A \cap (U \cap V) = \varnothing$

Moreover, A *is called connected if it is not disconnected.*

Disconnected set is shown as below, A is the grey area, which is contained in two open sets U and V, which are plotted by dotted line:

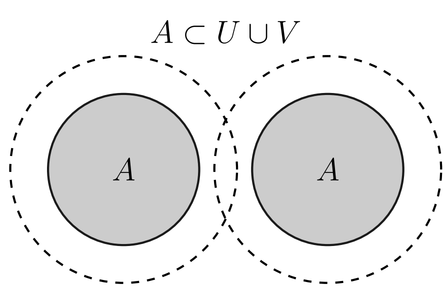

Figure 4. Disconnected set

Exercise 3.1. *Prove that the space* X *is connected if and only if the only subsets of* X *that are open and closed at the same time are* $\varnothing$ *and* X*.*

Proof. ($\Rightarrow$) Suppose X is connected. We want to show that if $E \subset X$ is open and closed at the same time, then $E = X$ or $E = \varnothing$.

Suppose by contraction that there exists a $E \subset X$ such that $E \neq X$, $E \neq \varnothing$ and E is open and closed at the same time. Then $U = E$, $V = X \setminus E$ are both open. And we have $X \subset U \cup V$, $X \cap U \neq \varnothing$, $X \cap V \neq \varnothing$, $X \cap U = V \neq \varnothing$ and $X \cap (U \cap V) = X \cap E \cap (X \setminus E) = \varnothing$. Thus, X is disconnected, which is a contradiction.

($\Leftarrow$) Suppose that the only subsets of X that are open and closed at the same time are $\varnothing$ and X. We need to show that X is connected.

By contradiction that X is not connected. Then, there exists two open sets U and V

such that $X = U \cup V$, $X \cap U \neq \varnothing$, $X \cap V \neq \varnothing$, $X \cap U = V \neq \varnothing$. Then $U \neq \varnothing$ and it is also closed since $X \setminus U = V$ is open, then this is a contradiction with the fact that X and $\varnothing$ are only sets that are open and closed at the same time. ☐

Theorem 3.27. *If $A \subset X$ is path connected, then A is connected.*

Proof. Prove by contradiction and suppose A is path connected but not connected. Then $A \subset U \cup V$, $A \cap U \neq \varnothing$, $A \cap V \neq \varnothing$, and $A \cap (U \cap V) = \varnothing$.

Let $x \in A \cap U$, $y \in A \cap V$, and let $\varphi : [a, b] \to A$ be a continuous path such that $\varphi(a) = x, \varphi(b) = y$. Define $c = \sup\{t \in [a, b] | \, \varphi\left([a, t]\right) \subset A \cap U\}$. If $\varphi(c) \in A \cap U$, then $c \neq b$ (otherwise it would implies $\varphi(c) = \varphi(b) = y \in A \cap U$, which is impossible).

Since $\varphi(c) \in A \cap U$ and $c < b$, it follows from the continuity of φ that there exists a $\varepsilon > 0$ such that $\varphi\left([c, c + \varepsilon]\right) \subset U$, which contradicts the definition of c. Therefore, $\varphi(c) \in A \cup V$. Then by the same argument as above, that there exists a $\varepsilon > 0$ such that $\varphi([c - \varepsilon, c]) \in V$ implying $\varphi(c - \varepsilon) \in V$, which contradicts the definition of c. ☐

Theorem 3.28. *If A is connected, then the closure cl(A) of A is also connected.*

Proof. Prove by contradiction and suppose that A is connected while $\mathrm{cl}(A)$ is not closed. Then there exists open sets U, V such that $\mathrm{cl}(A) \subset U \cup V, \mathrm{cl}(A) \cap U \neq \varnothing, \mathrm{cl}(A) \cap V \neq \varnothing$, and $\mathrm{cl}(A) \cup (U \cup V) = \varnothing$. Then we have

(1) $A \subset U \cup V$.
(2) $A \cup U \neq \varnothing$.
(3) $A \cup V \neq \varnothing$.
(4) $A \cup (U \cup V) = \varnothing$.

which means that A is disconnected. Contradiction.

The properties (1) and (4) are obvious. Now we prove property (2). Since $\mathrm{cl}(A) \cap U \neq \varnothing$, then there is a $x \in \mathrm{cl}(A) \cap U$. Hence there is a sequence $\{x_n\} \in A$ such that

$$x_n \to x \in \mathrm{cl}(A) \cap U$$

Since U is open, then there exists sufficiently large $N > 0$ such that $\forall n > N$, $x_n \in A \cap U$, which proves that $A \cap U \neq \varnothing$. ☐

Example 3.1. The graph of the function $y = \sin \frac{1}{x}, 0 \leq x \leq \pi$, i.e., the set

$$G = \left\{(x, \sin \frac{1}{x}) | \, 0 < x \leq \pi\right\}$$

is path connected, hence G is connected.

Therefore, $\mathrm{cl}(G)$ is connected. However, $\mathrm{cl}(G)$ is the union of G and the segment $[-1, 1]$ on the $y-$axis. It is clear that this set is not path connected.

$\mathrm{cl}(G)$ is an example of a set which is connected but not path connected. The plot of $y = \sin \frac{1}{x}, 0 \leq x \leq \pi$ is shown as below:

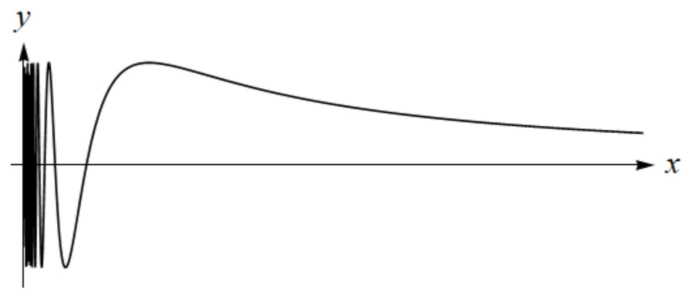

FIGURE 5. $y = \sin(1/x)$

Theorem 3.29 (Theorem 2.47 in Rudin's book)**.** *A subset E of the real line $\mathbb{R}^1$ is connected if and only if it has the following property: If $x \in E, y \in E$ and $x < z < y$, then $z \in E$.*

Proof. ($\Rightarrow$) Prove by contradiction and suppose that if there exists a $z \in (x, y)$ and $z \in E$, then $E = U \cup V$, where

$$U = E \cap (-\infty, z), V = E \cap (x, \infty)$$

Since $x \in U, y \in V$, U and V are not nonempty. Since $U \subset (-\infty, z)$ and $V \subset (z, \infty)$, then they are separated. Also, we can have $E \cap (U \cap V) = \varnothing$, which means E is not connected. This is a contradiction.

($\Leftarrow$) Supposed that E is not connected. Then there exist sets U and V separating E such that $E \subset U \cup V, E \cap U \neq \varnothing, E \cap V \neq \varnothing$ and $E \cap (U \cap V) = \varnothing$. Now we pick $x \in U$ and $y \in V$ and without losing generality, assume that $x < y$. Define

$$z = \sup\{(U \setminus V) \cap [x, y]\}$$

Then $z \in \mathrm{cl}(U)$ and $z \notin V$. In particular, $x \leq z < y$. If $z \notin U \setminus V$, then z certainly does not belong to E. If $z \in U \setminus V$, then $z \notin \mathrm{cl}(V)$, hence there exists a z_1 such that $z < z_1 < y$ and $z_1 \notin V$. Then $x < z_1 < y$ and $z \notin E$. This is a contradiction. □

3.6. **Continuity.**

Definition 3.19. *Let $(X, d), (Y, \rho)$ be two metric spaces and $A \subset X$. Consider a mapping $f : A \to Y$. If x_0 is an accumulation point of A, then we say that $\lim_{x \to x_0} f(x) = b \in Y$, if*

$$\forall \varepsilon > 0, \exists \delta > 0, 0 < d(x, x_0) < \delta \Rightarrow \rho(f(x), b) < \varepsilon$$

Equivalently, $\lim_{x \to x_0} f(x) = b$, *if*

$$x_0 \neq x_n (x_n \in A) \to x_0 \Rightarrow f(x_n) \to b$$

Definition 3.20. *We say that a mapping $f : A \to Y$ is continuous at point $x_0 \in A$, if $A \ni x_n \to x_0$ then $f(x_n) \to f(x_0)$(no need of condition: $x_n \neq x_0$).*

If x_0 is not an accumulation point of A, i.e., x_0 is an isolated point, then f is always continuous at x_0.

Definition 3.21. *We say that $f : A \to Y$ is continuous if f is continuous at every point in A. Equivalently, $f : A \to Y$ is continuous if for every $x \in A$ and $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that*

$$d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon.$$

This is Definition 4.5 in Rudin's book.

Example 3.2.

(1) If X is a discrete metric space, then every function $f : X \to Y$ is continuous.
(2) $f : \mathbb{Z} \to Y$ or $f : \mathbb{N} \to Y$ is always continuous.

Theorem 3.30 (Theorem 4.8 in Rudin's book)**.** *A mapping $f : X \to Y$ is continuous if and only if for every open set $U \subset Y$, $f^{-1}(U)$ is an open set in X.*

Proof. ($\Rightarrow$) Suppose f is continuous and $U \subset Y$ is a open set. We want to prove that $f^{-1}(U)$ is open.

If $x \in f^{-1}(U)$, then $f(x) \in U$ and hence $B(f(x), \varepsilon) \subset U$ for some $\varepsilon > 0$. It follows from the continuity of f there exists a $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$, if $d_X(x, y) < \delta$. Hence, $B(x, \delta) \subset f^{-1}(U)$ and then $f^{-1}(U)$ is open.

($\Leftarrow$) Suppose now that $f^{-1}(U)$ is open for every open set $U \subset Y$. We need to prove that f is continuous.

Let $\varepsilon > 0$ be given, then $B(f(x_0), \varepsilon)$ is open. Hence $x_0 \in f^{-1}(B(f(x_0), \varepsilon))$, which is open. Then there exists a $\delta > 0$ such that $B(x_0, \delta) \subset f^{-1}(B(f(x_0), \varepsilon))$. And hence if $d_X(x_0, x) < \delta$, then $d_Y(f(x_0), f(x)) < \varepsilon$, which proves the continuity of f. $\square$

Definition 3.22. *We say that a mapping $f : X \to Y$ is $L-$ Lipschitz if $\forall x, y \in X$, $d_Y(f(x), f(y)) \leq L d_X(x, y)$.*

We say that a mapping $f : X \to Y$ is Lipschitz if it is $L-$ Lipschitz for some $L > 0$.

Proposition 3.2. *Every Lipschitz mapping is continuous.*

Proof. Suppose $f : X \to Y$ is Lipschitz, then for $x_n \to x$, we have

$$d_Y(f(x_n), f(x)) \leq L d_X(x_n, x) \to 0$$

and hence $f(x_n) \to f(x)$. $\square$

Now we use different method to prove this theorem by showing that $f^{-1}(U)$ is open for every open set $U \subset Y$.

Proof. Let $U \subset Y$ be given, and let $x \in f^{-1}(U)$. We will prove that

$$B\left(x, \frac{\varepsilon}{L}\right) \subset f^{-1}(U)$$

where ε is taken such that $B(f(x), \varepsilon) \subset U$. Now we have

$$y \in B\left(x, \frac{\varepsilon}{L}\right) \Rightarrow d_X(x, y) < \frac{\varepsilon}{L}$$

with f being Lipschitz mapping, we have

$$d_Y(f(x), f(y)) \leq L d_X(x, y) \leq \varepsilon$$

then $f(y) \in B(f(x), \varepsilon) \subset U$. Then $B\left(x, \frac{\varepsilon}{L}\right) \in f^{-1}(U)$. $\square$

Example 3.3 (Riemann function)**.** f is defined as below

$$f(x) = \begin{cases} 0, \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \\ \dfrac{1}{q}, \text{if} \quad x \in \mathbb{Q}, x = \dfrac{p}{q}, q > 0 \end{cases}$$

and where the greatest common divisor of p and q is 1. Then f is continuous at all irrational points and discontinuous at all rational points.

Example 3.4. $f(x, y)$ is defined as below

$$f(x) = \begin{cases} 0, \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \\ 1, \text{if} \quad x \in \mathbb{Q} \end{cases}$$

and f is discontinuous everywhere.

Remark 3.5. *Practical way of proving continuity of a function of two(or more) variables at a given point is based on the following observation: If*

$$|f(x, y) - L| \leq g(x, y) \to 0$$

where $(x, y) \to (x_0, y_0), (x, y) \neq (x_0, y_0)$*, then* $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$*.*

Theorem 3.31. *If* $f : X \to Y$ *is continuous and* $A \subset X$ *is compact, then* $f(A) \subset Y$ *is compact.*

Proof. Let $y_n \in f(A), n = 1, 2, \cdots$ be a sequence. Then $y_n = f(x_n)$ for some $x_n \in A$. Since A is compact, then there is a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0 \in A$. Hence, $y_{n_k} = f(x_{n_k}) \to f(x_0) \in f(A)$. □

Theorem 3.32. *If* $f : X \to Y$ *is continuous and* $A \subset X$ *is connected,then* $f(A) \subset Y$ *is connected.*

Proof. Suppose $f(A)$ is not connected. Then there exist open sets U and V such that $f(A) \subset U \cup V, f(A) \cap U \neq \varnothing, f(A) \cap V \neq \varnothing$ and $f(A) \cap (U \cap V) = \varnothing$.

Then we have $A \subset f^{-1}(U), A \subset f^{-1}(U), A \cap f^{-1}(U) \neq \varnothing, A \cap f^{-1}(V) \neq \varnothing$ and $A \cap (f^{-1}(U) \cap f^{-1}(V)) = \varnothing$, which implies that A is disconnected. This is a contradiction. □

Theorem 3.33. *If* $f : X \to Y$ *is continuous and* $A \subset X$ *is path connected,then* $f(A) \subset Y$ *is also path connected.*

Proof. Let $y_1, y_2 \in f(A)$. Then $y_1 = f(x_1), y_2 = f(x_2)$ for some $x_1, x_2 \in A$. Since A is connected, then there exists a continuous mapping $\varphi : [a, b] \to A$, such that $\varphi(a) = x_1$ and $\varphi(b) = x_2$. Then we have $\psi = f \circ \varphi : [a, b] \to f(A)$ and $\psi(a) = y_1$, $\psi(b) = y_2$, this is continuous path connecting y_1 and y_2 in $f(A)$. Thus, $f(A)$ is path connected. □

Theorem 3.34. *If* $f : A \to \mathbb{R}$ *is continuous, where* $A \subset X$ *is compact, then* f *attains maximum and minimum in* A*, i.e.,*

$$\exists x_1 \in A, \forall x \in A, f(x_1) \geq f(x)$$
$$\exists x_2 \in A, \forall x \in A, f(x_2) \leq f(x)$$

Proof. There is a sequence $\{x_n\} \in A$ such that $f(x_n) \to \sup\{f(x)|x \in A\}$. Since A is compact, then there exists a subsequence $\{x_{n_k}\}$ of x_n such that $x_{n_k} \to x_1$. Then we have $f(x_1) = \sup\{f(x)|x \in A\}$.

Similar argument works with minimum. □

Corollary 3.34.1. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded.*

Example 3.5. Consider $f(x) = \frac{1}{x}, x \in (0, 1]$ is continuous but unbounded on its domain, since $(a, b]$ is not closed, so the corollary does not apply.

Now we classify all connected subsets.

Lemma 3.35. *If $A \subset X$ is connected, and $a < c < b, a, b \in A$, then $c \in A$.*

Proof. By contradiction and suppose $c \notin A$. Then we can know that $A \subset (-\infty, c) \cup (c, \infty), A \cap (-\infty, c) \neq \varnothing, A \cap (c, \infty) \neq \varnothing$ and $A \cap ((-\infty, c) \cap (c, \infty)) = \varnothing$. Thus, A is disconnected, which is a contradiction. □

Theorem 3.36. *$A \subset \mathbb{R}$ is connected if and only if A is an interval.*

Proof. ($\Leftarrow$) Any interval is connected since it is path connected.

($\Rightarrow$) Let $a = \inf\{x|x \in A\}$ and $b = \sup\{x|x \in A\}$. Then there exist sequence $\{a_n\} \in A$ and $\{b_n\} \in A$ such that $a_n \to a$ and $b_b \to b$. It follows from Lemma 3.35 that $[a_n, b_n] \subset A$. Hence $(a, b) = \bigcup_{n=1}^{\infty}[a_n, b_n] \subset A$. With the definition of a and b that no number less than a belongs to A and no number bigger than b belongs to A. Thus, A is an interval with endpoints a and b, □

Theorem 3.37 (Intermediate Value Theorem)**.** *If $f : A \to \mathbb{R}$ is continuous, $A \subset X$ is connected and $f(x) = a < c < b = f(y)$, then there exists a $z \in A$ such that $f(z) = c$.*

Proof. Since f is continuous, then $f(A)$ is an interval that contains a and b, so it must contains c. □

Theorem 3.38. *Let $f : A \to \mathbb{R}^n$, $f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$. Then f is continuous if and only if the functions f_i are continuous for $i = 1, 2, \cdots, n$.*

Proof. If $x_k \to x_0$, then $(f_1(x_k), f_2(x_k), \cdots, f_n(x_k)) \to (f_1(x_0), f_2(x_0), \cdots, f_n(x_0))$ if $f_i(x_k) \to f_i(x_0)$. □

Now we show a new method to prove the Arithmetic-Geometric mean inequality.

Theorem 3.39 (Arithmetic-Geometric mean inequality)**.** *Let $x_1, \cdots, x_n \geq 0$, then*

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}$$

and the equality holds if and only if $x_1 = \cdots = x_n$.

Proof. Let $x_1 + \cdots + x_n = a$. We assume that $a > 0$ or otherwise the inequality is obvious.

Consider the set $A = \{(z_1, \cdots, z_n) \in \mathbb{R}^n | z_1 + \cdots + z_n = a, z_1, \cdots, z_n \geq 0\}$. The set is bounded and closed, and therefore compact. Hence the function $f(z_1, \cdots, z_n) = \sqrt[n]{z_1, \cdots, z_n}$ has maximum in A at some point $(z_1^0, \cdots, z_n^0) \in A$. We will prove that $z_1^0 = \cdots = z_n^0$.

Observe first that $z_1^0, \cdots, z_n^0 \geq 0$ because 0 cannot be maximum of f. Suppose that $z_i^0 \neq z_j^0$ for some $i \neq j$, then

$$\left(\frac{z_i^0 + z_j^0}{2}\right)^2 > z_i^0 z_j^0$$

and therefore

$$\sqrt[n]{z_1^0 \cdots \left(\frac{z_i^0+z_j^0}{2}\right) \cdots \left(\frac{z_i^0+z_j^0}{2}\right) \cdots z_n^0} > \sqrt[n]{z_1^0 \cdots z_j^0 \cdots z_j^0 \cdots z_n^0}$$

and $\left(z_1^0, \cdots, \frac{z_i^0+z_j^0}{2}, \cdots, \frac{z_i^0+z_j^0}{2}, \cdots, z_n^0\right) \in A$, this is contradicted with the definition of $(z_1^0, \cdots, z_j^0, \cdots, z_j^0, \cdots, z_n^0)$. Thus $z_1^0 = \cdots = z_n^0 = \frac{a}{n}$, and we have

$$\sqrt[n]{x_1 \cdots x_n} \leq \sqrt[n]{z_1^0 \cdots z_n^0} = \frac{a}{n} = \frac{x_1 + \cdots + x_n}{n}$$

□

3.7. **Uniform continuity.**

Definition 3.23. *$f : A \to Y, A \subset X$ is called uniformly continuous, if for $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that for $\forall x, y \in A$, if $d_X(x,y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.*

Proposition 3.3. *If $f : (a,b) \to \mathbb{R}$ is uniformly continuous, where (a,b) is a bounded interval, then f is bounded.*

Proof. By the definition of uniformly continuity, we take $\varepsilon = 1$. Then there is a $\delta > 0$ such that if $|x - y| \leq \delta$, then $|f(x) - f(y)| < 1$. Choose N such that $N\delta > \frac{b-a}{2}$.

Let $x \geq \frac{a+b}{2}$, consider a sequence $x_k = \frac{a+b}{2} + \delta k, k = 0, 1, 2 \cdots$. Then there is $n < N$ such that $x_n \leq x$ but $x_{n+1} > x$. Thus we have

$$\begin{aligned}
\left| f(x) - f\left(\frac{a+b}{2}\right)\right| &= |(f(x_n) - f(x_{n_1})) + (f(x_{n_1}) - f(x_{n_2})) + \cdots + (f(x_1) - f(x_{n_0}))| \\
&\leq |f(x_n) - f(x_{n_1})| + |f(x_{n_1}) - f(x_{n_2})| + \cdots + |f(x_1) - f(x_{n_0})| \\
&\leq n + 1 \leq N
\end{aligned}$$

which proves f is bounded. □

Exercise 3.2. *Suppose $f : A \to \mathbb{R}$ is uniformly continuous, where $A \subset X$ is a bounded subset of a metric space. Dose it follow that f is bounded?*

No. And we will provide a counterexample.

Proof. If X is a discrete metric space, then any function $f : X \to \mathbb{R}$ is uniformly continuous. Indeed, let arbitrary $\varepsilon > 0$, then take $\delta = 1$, then if $d_X(x,y) < \delta = 1$, it follows that $x = y$ and $|f(x) - f(y)| = 0 < \varepsilon$.

Now take $X = \mathbb{Z}$ with the metric metric. Then X is bounded and and $f : X \to \mathbb{R}, f(n) = n$. Then f is uniformly continuous, but not bounded. □

Definition 3.24. *We say that a function $f : A \to Y$, where $A \subset X$ is $\alpha-$Hölder continuous, $\alpha > 0$, if $\exists c > 0$ such that*

$$d_Y(f(x), f(y)) \leq c d_X(x,y)^\alpha$$

for all $x, y \in A$.

Theorem 3.40. *If $f : A \to Y$ is continuous and $A \subset X$ is compact, then f is uniformly continuous.*

Proof. Prove by contradiction, and suppose $\exists \varepsilon > 0$, $\forall \delta > 0$, there exist $x, y \in A$ such that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

In particular, there exist sequences $\{x_n\}, \{y_n\} \in A$ such that $d_X(x_n, y_n) < \frac{1}{n}$ and $d_Y(f(x_n), f(y_n)) \geq \varepsilon$. Since A is compact, there is a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0 \in A$. Since $d_X(x_{n_k}, y_n) < \frac{1}{n_k}$, we also have $y_{n_k} \to x_0$. Hence, with f being continuous, $f(x_{n_k}) \to f(x_0)$ and $f(y_{n_k}) \to f(x_0)$. Therefore, $d_Y(f(x_{n_k}), f(Y_{n_k})) \to 0$, which contradicts the inequality $d_Y(f(x_{n_k}), f(Y_{n_k})) > \varepsilon$. $\square$

Recall that for $f : \mathbb{R}^n \to \mathbb{R}^m$, the graph is defined by

$$G_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | = \mathbb{R}^{n+m} | y = f(x)\}$$

Exercise 3.3. *Prove that if f is continuous, then G_f is a closed subset of $\mathbb{R}^{n+m}$. Is converse implication true?*

Proof. (1) Let f be continuous. We need to prove that if $G_f \ni (x_k, y_k) \to (x_0, y_0)$, then $(x_0, y_0) \in G_f$. We have $(x_k, y_k) \in G_f$, then $y_k = f(x_k)$. Now we have $x_k \to x_0 \in \mathbb{R}$, then $f(x_k) \to y_0$. Since f is continuous, then $f(x_k) \to f(x_0)$. Hence $y_0 = f(x_0)$ and therefore $(x_0, y_0) = (x_0, f(x_0)) \in G_f$.

(2) The converse implication is false. For example, consider function

$$f(x) = \begin{cases} \dfrac{1}{x}, \text{if} \quad x \neq 0 \\ 0, \text{if} \quad x = 0 \end{cases}$$

which is discontinuous. But its graph is a closed subset of $\mathbb{R}^3$, which is shown as below. $\square$

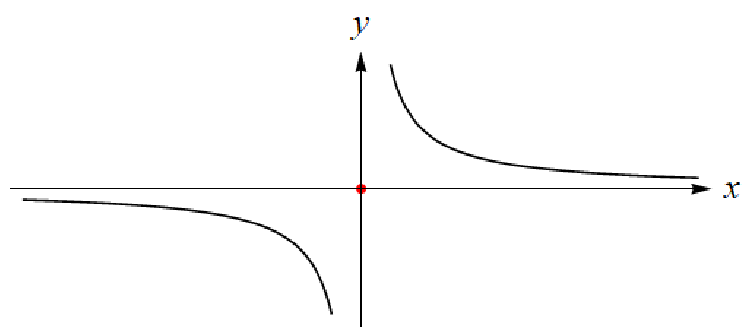

FIGURE 6. $y = \sin(1/x)$

Exercise 3.4. *Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is bounded, then f is continuous if and only if G_f is a closed subset of $\mathbb{R}^{n+m}$.*

Proof. ($\Rightarrow$) is proved in the previous exercise.

($\Leftarrow$) Suppose G_f is closed, then we want to prove that if $x_n \to x_0$, then $f(x_n) \to f(x_0)$. We prove it by contradiction and assume that there is a sequence $x_n \to x_0$ such that $\lim_{x_n \to x_0} f(x_n) \neq f(x_0)$.

Then there is a $\varepsilon > 0$, and for $\forall N > 0$ such that if $n > N$, i.e., $\forall n > 0$, we will have $|f(x_n) - f(x_0)| \geq \varepsilon$. Then we can find a convergent subsequence $\{x_{n_k}\}$ such that $\forall k$, $|f(x_{n_k}) - f(x_0)| \geq \varepsilon$. The sequence $f(x_{n_k})$ is bounded in $\mathbb{R}^m$, then it has a convergent subsequence $\{f(x_{n_{k_l}})\}$ converging to y_0. And now we have

$$G_f \ni \Big(x_{n_{k_l}}, f(x_{n_{k_l}})\Big) \to (x_0, y_0)$$

and since G_f is closed, then $(x_0, y_0) \in G_f$ and $y_0 = f(x_0)$. Therefore $f(x_{n_{k_l}}) \to f(x_0)$, which is a contradiction. $\square$