Name: ______________________________

Advanced Calculus 2

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Final Exam

Due on May 1, 2020

Problem	Possible points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120	

You need 100 points so 2 problems is a bonus.

Problem 1. Suppose $f : \mathbb{R}^2_+ \to \mathbb{R}$ is a continuous function defined on

$$\mathbb{R}^2_+ = \{(x,y) :\ x \in \mathbb{R},\ y > 0\}.$$

Assume also that the limits

$$g(u,v) = \lim_{t\to 0} \frac{f((u+t)\cos v, (u+t)\sin v) - f(u\cos v, u\sin v)}{t},$$

and

$$h(u,v) = \lim_{t\to 0} \frac{f(u\cos(v+t)), u\sin(v+t)) - f(u\cos v, u\sin v)}{t},$$

exist and define continuous functions g, h on the domain

$$D = \{(u,v) :\ u > 0,\ 0 < v < \pi\}.$$

Prove that the function f is differentiable on $\mathbb{R}^2_+$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 2. Let $f \in C^1(\mathbb{R})$ be a continuously differentiable function such that $|f'(x)| \leq 1/2$ for all $x \in \mathbb{R}$. Define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x,y) = (x + f(y), y + f(x)).$$

Prove that

(1) g is a diffeomorphism,
(2) $g(\mathbb{R}^2) = \mathbb{R}^2$,
(3) the area $|g([0,1]^2)|$ of the image of the unit square belongs to the interval $[3/4, 5/4]$.

Hint: Among other tools use the contraction principle.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 3. Prove that the tangent planes to the surface S defined by $x^2 + y^2 - z^2 = 1$ at the points $(x, y, 0) \in S$ are parallel to the z-axis.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 4. Prove that if $f : [0,1] \to [0,1]$ is a continuous function then its graph as a subset of $\mathbb{R}^2$ has measure zero.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 5. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a continuous function. Let $K \subset \mathbb{R}^3$ be a compact set such that $|f(x) - f(y)| \leq 2012|x-y|^2$ for all $x, y \in K$. Prove that the set $f(K) \subset \mathbb{R}^2$ has measure zero as a subset of $\mathbb{R}^2$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 6. Assume that $f : [0,1] \to [0,1]$ is a continuous function such that the set $\{x \in [0,1] :\ f(x) = 1\}$ has measure zero. Prove directly (without using any results like monotone or dominated convergence theorem) that

$$\lim_{n\to\infty} \int_0^1 f(x)^n \, dx = 0.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 7. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ and $\mathbf{v}_i : \mathbb{R} \to \mathbb{R}^n$, $i = 1, 2, \ldots, n-1$, be C^∞ smooth functions such that for any $t \in \mathbb{R}$ the vectors

$$\gamma'(t), \mathbf{v}_1(t), \ldots, \mathbf{v}_{n-1}(t)$$

form an orthonormal basis of $\mathbb{R}^n$ (here we differentiate γ but **do not** differentiate $\mathbf{v}_i$, $i = 1, 2, \ldots, n-1$).

Consider the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi(x_1, \ldots, x_n) = \gamma(x_n) + \sum_{i=1}^{n-1} x_i \mathbf{v}_i(x_n).$$

(a) Find the derivative $D\Phi(x_1, \ldots, x_n)$;

(b) Prove that Φ is a diffeomorphism in a neighborhood of and point of the form $(0, \ldots, 0, x_n)$;

(c) Find the limit

$$\lim_{r \to 0} \frac{|\Phi(B^n(0,r))|}{|B^n(0,r)|},$$

where $B^n(0,r)$ denotes the ball of radius r centered at the origin and $|A|$ stands for the volume of the set A.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 8. Prove that if $K \in C^1(\mathbb{R}^2 \setminus \{(0,0)\})$ satisfies the estimate

$$|\nabla K(x)| \leq \frac{1}{|x|^3} \quad \text{for all } x \neq (0,0)$$

then there is a constant $C > 0$ such that

$$\iint_{\{x \in \mathbb{R}^2 \colon |x| > 2|y|\}} |K(x-y) - K(x)| \, dx \leq C$$

for all $y \in \mathbb{R}^2$.

Hint: *Use the mean value theorem to estimate* $|K(x-y) - K(x)|$ *and then integrate in polar coordinates.*

Proof. WRITE YOUR SOLUTION HERE. □

Problem 9. Use Green's theorem to prove the following result: If the vertices of a polygon, in counterclockwise order, are (x_1, y_1), (x_2, y_2), …,(x_n, y_n), then the area of the polygon is

$$A = \frac{1}{2} \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i),$$

where we use notation $x_{n+1} = x_1$, $y_{n+1} = y_1$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 10. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^1 boundary and let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi(x,y) = (u(x,y), v(x,y))$, be a C^2 diffeomorphism. Prove that

$$\int_{\partial\Omega} u v_x \, dx + u v_y \, dy = \pm |\Phi(\Omega)|,$$

where $|\Phi(\Omega)|$ denotes the area of $\Phi(\Omega)$ and $\partial\Omega$ has positive orientation. Show on examples that both cases $+|\Phi(\Omega)|$ and $-|\Phi(\Omega)|$ are possible.

Proof. WRITE YOUR SOLUTION HERE. $\square$

Problem 11. Let f be a polynomial of total degree at most three in $(x, y, z) \in \mathbb{R}^3$. Prove that:

$$\int_{x^2+y^2+z^2\leq 1} f(x,y,z)\,dx\,dy\,dz = \frac{4\pi f((0,0,0))}{3} + \frac{2\pi\,(\Delta f)\,((0,0,0))}{15}.$$

Here $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is the Laplacian operator on $\mathbb{R}^3$.

Proof. WRITE YOUR SOLUTION HERE. $\square$

Problem 12. Suppose that $K \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ and

$$K(x) = \frac{K(x/\|x\|)}{\|x\|} \quad \text{for all } x \in \mathbb{R}^2 \setminus \{0\}.$$

(a) Prove that $\nabla K(tx) = t^{-2}\nabla K(x)$ for $x \neq 0$ and $t > 0$.

(b) Use the divergence theorem to prove that (on both sides we integrate vector valued functions)

$$\int_{\{1\leq\|x\|\leq 2019\}} \nabla K(x)\,dx = \int_{\partial\{1\leq\|x\|\leq 2019\}} K(x)\vec{\mathbf{n}}\,d\sigma(x).$$

(c) Prove that

$$\int_{\{1\leq\|x\|\leq 2019\}} \nabla K(x)\,dx = 0.$$

Hint. *Show first that $K(tx) = t^{-1}K(x)$ for $x \neq 0$, $t > 0$. In (a) differentiate $K(tx)$. Part (a) is not needed for parts (b) and (c).*

Proof. WRITE YOUR SOLUTION HERE. $\square$