**Homework 3 for Math 1530**
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**Problem 27.** Let $a_1, a_2, a_3, \ldots > 0$. Prove that if

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1\,,$$

then the series $a_1 + a_2 + a_3 + \ldots$ converges.

*Proof.* Since $\lim_{n\to\infty} n\Big(\frac{a_n}{a_{n+1}} - 1\Big) > 1$, then there exists a $r_1$ such that $\lim_{n\to\infty} n\Big(\frac{a_n}{a_{n+1}} - 1\Big) > r_1 > 1$. Then there exists an $N_1 > 0$, such that for $\forall n > N_1$, $\frac{a_n}{a_{n+1}} > 1 + \frac{r_1}{n}$.

We take $r_2$ such that $1 < r_2 < r_1$. And we consider function $f(x) = 1 + r_1 x - (1+x)^{r_2}$, which satisfies $f(0) = 0$. Also, $f'(x) = r_1 - r_2(1+x)^{r_2 - 1} > 0$ in a small neighborhood of $x = 0$. Then there exists an $N_2 > 0$ such that for $\forall n > N_2$, we have

$$\begin{aligned} &\frac{a_n}{a_{n+1}} > 1 + \frac{r_1}{n} > \left(1 + \frac{1}{n}\right)^{r_2} = \frac{(n+1)^{r_2}}{n^{r_2}} \\ \Rightarrow &(n+1)^{r_2} a_{n+1} < n^{r_2} a_n \end{aligned}$$

as $x$ substituted by $\frac{1}{n}$. Then for $n > N_2$, we have

$$a_n < \frac{N_2^{r_2} a_{N_2}}{n^{r_2}}$$

By comparison test, $\sum_{k=1}^{\infty} a_k$ converges since $r_2 > 1$ and $\sum_{n=1}^{\infty} 1/n^{r_2}$. ☐

**Problem 28.** Provide an example of a convergent series $a_1 + a_2 + a_3 + \ldots$, where $a_n > 0$, $n = 1, 2, 3, \ldots$ such that the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist.

*Proof.* We already know that series $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \cdots = 1$, which is convergent. Now we rearrange this series as

$$a_1 = \frac{1}{2^2}, a_2 = \frac{1}{2}, a_3 = \frac{1}{2^4}, a_4 = \frac{1}{2^3}, a_5 = \frac{1}{2^6}, \cdots$$

by substituting the positions between $2n$th and $(2n-1)$th. Then we have

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= 2, n \text{ is odd} \\ \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \frac{1}{8}, n \text{ is even} \end{aligned}$$

which means the limit does not exist. ☐

**Problem 29.** Prove that there is a sequence of positive integers $n_1 < n_2 < n_3 < \ldots$ such that the sequence $a_k = \sin n_k$ converges.

*Proof.* Based on Bolzano-Weierstrass Theorem, we can know that bounded sequence has a convergent subsequence. Also, $sinn$ is dense in $[-1, 1]$, then there exists a subsequence that converges to any value in $[-1, 1]$. Suppose we want a ubsequence that converges to $g \in [-1, 1]$. First, for $\forall \varepsilon > 0$, there exists $n_1$ such that $\|\sin n_1 - g\| < \varepsilon$. Then, starting from $n_1$, we could find a $n_2 > n_2$ such that $\|\sin n_2 - g\| < \varepsilon$ since $\sin n$ is dense in $[-1, 1]$. Repeating this process, and we can find $n_1 < n_2 < n_3 < \cdots$ such that $\lim_{n\to\infty} a_k = \sin n_k = g \in [-1, 1] \setminus \{0\}$. ☐

**Problem 30.** Prove that the series

$$\sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log\log n)^p}$$

diverges if $0 < p \le 1$ and converges if $p > 1$.

*Proof.* Based on Cauchy condensation test, the convergence of $\sum_{n=1}^{\infty} a_n$ is equivalent to the convergence of $\sum_{n=0}^{\infty} 2^n a_{2^n}$. Then we only need to consider $\sum_{n=2}^{\infty} 2^n a_{2^n}$ in this case, we have

$$\begin{aligned}\sum_{n=2}^{\infty} 2^n a_{2^n} &= \sum_{n=2}^{\infty} 2^n \frac{1}{2^n(\ln 2^n)(\ln\ln 2^n)^p} \\ &= \sum_{n=2}^{\infty} \frac{1}{(\ln 2^n)(\ln(n\ln 2))^p} \\ &= \frac{1}{\ln 2}\sum_{n=2}^{\infty} \frac{1}{n(\ln n + \ln(\ln 2))^p}\end{aligned}$$

we denote this sum by $A$. And we have

$$\frac{1}{\ln 2}\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \le A \le \frac{1}{\ln 2}\sum_{n=2}^{\infty} \frac{1}{n(\frac{1}{2}\ln n)^p} = \frac{2^p}{\ln 2}\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

since $\frac{1}{2}\ln n < \ln n + \ln(\ln 2) < \ln n$, for $n > 4$. And we already know that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if $p > 1$, and diverges if $0 < p \le 1$. So $A$ converges if $p > 1$, and diverges if $0 < p \le 1$. □

**Problem 31.** Prove that if the series $a_1 + a_2 + a_3 + \dots$ converges, where $a_n > 0$, $n = 1, 2, 3, \dots$, then the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} \quad \text{converges.}$$

*Proof.* Based on Cauchy-Schwarz inequality, we have

$$\left|\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}\right|^2 \le \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)$$

As we know, both series on the right side are convergent. Thus $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges. □

Definition. Let $a_1, a_2, a_3, \dots > 0$. We define the infinite product by

$$\prod_{n=1}^{\infty} a_n = \lim_{n\to\infty} a_1 a_2 \dots a_n\,.$$

We say that the infinite product *converges* if the limit is finite and *positive*. If the limit does not exist, equals 0 or $\infty$ then we say that the product *diverges*.

**Problem 32.** Prove that if $a_n > 0$, $n = 1, 2, \ldots$, then the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ converges. **Hint:** *You can use the inequality $e^x \geq 1 + x$ without proving it.*

*Proof.* Denote $\prod_{n=1}^{\infty}(1 + a_n)$ by $A$.
(1)If the series $\sum_{n=1}^{\infty} a_n$ converges, we have

$$\begin{aligned}
\ln A &= \ln(1 + a_1) + \cdots + \ln(1 + a_n) \\
&\leq \ln e^{a_1} + \cdots + \ln e^{a_n} \\
&= \sum_{n=1}^{\infty} a_n
\end{aligned}$$

Since $\sum_{n=1}^{\infty} a_n$ converges, then $\ln A$ converges. Thus, $A$ converges since log function is continuous.
(2)If $\prod_{n=1}^{\infty}(1 + a_n)$ converges, we can prove following inequality by induction

$$1 + \sum_{n=1}^{N} a_n \leq \prod_{n=1}^{N}(1 + a_n)$$

For $N = 1$, $1 + a_1 \leq 1 + a_1$, so it holds. Assume it also holds for $N = k$, then for $N = k + 1$, we have

$$\begin{aligned}
1 + \sum_{n=1}^{N+1} a_n &\leq \prod_{n=1}^{N}(1 + a_n) + a_n \\
&\leq \prod_{n=1}^{N}(1 + a_n) + \prod_{n=1}^{N}(1 + a_n)a_{n+1} \\
&= \prod_{n=1}^{N+1}(1 + a_n)
\end{aligned}$$

So We can know

$$\sum_{n=1}^{\infty} a_n \leq \prod_{n=1}^{\infty}(1 + a_n) - 1$$

which implies that $\sum_{n=1}^{\infty} a_n$ converges. $\square$

**Problem 33.** Prove that if $0 < a_n < 1, n = 1, 2, \ldots$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_n/(1 - a_n)$ converges.

*Proof.* (1)If $\sum_{n=1}^{\infty} a_n$ converges, then it implies that $\lim_{n\to\infty} a_n = 0$ then $\forall \varepsilon > 0$, $\exists N_1 > 0$ such that $\forall n > N_1$, $a_n < \varepsilon$. Since it is true for arbitrary $\varepsilon > 0$, then there exist an $N_2 > 0$, such that $a_n < \varepsilon < \frac{1}{2}$. Also, since $\sum_{n=1}^{\infty} a_n$ converges, then $\forall \varepsilon > 0$, $\exists N_3 > 0$, such that for $\forall n > N_3, \forall m > 0$, $|a_n + \cdots + a_{n+m}| < \varepsilon$. Now we set $N = \max\{N_1, N_2, N_3\}$, we have

$$\left| \frac{a_n}{1 - a_n} + \cdots + \frac{a_{n+m}}{1 - a_{n+m}} \right| \leq \frac{1}{2}(a_n + \cdots + a_{n+m}) \leq \frac{\varepsilon}{2}$$

since $a_n < \varepsilon < \frac{1}{2}$ for $n > N$. Then we proved that $\sum_{n=1}^{\infty} a_n/(1-a_n)$ converges.

(2)If $\sum_{n=1}^{\infty} a_n/(1-a_n)$ converges, then we have

$$\sum_{n=1}^{\infty} |a_n| < \sum_{n=1}^{\infty} a_n/(1-a_n)$$

since $0 < a_n < 1$ for $\forall n$. So $\sum_{n=1}^{\infty} a_n$ converges. □

**Problem 34.** Prove that if $0 < a_n < 1$, then the product $\prod_{n=1}^{\infty}(1-a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ converges.

*Proof.* (1)If $\sum_{n=1}^{\infty} a_n$ converges, then we have

$$\ln\left(\prod_{n=1}^{\infty}(1-a_n)\right) = \sum_{n=1}^{\infty} \ln(1-a_n) \leq \sum_{n=1}^{\infty} a_n$$

since $\ln(1-x) < x, 0 < x < 1$. Also, log function is continuous and we have that $\prod_{n=1}^{\infty}(1-a_n)$ converges.

(2)If $\prod_{n=1}^{\infty}(1-a_n)$ converges, we can know $\prod_{n=1}^{\infty} 1/(1-a_n)$ also converges, since $0 < a_n < 1$ which means $1 - a_n \neq 0$. Using inequality $e^{-x} > 1 - x$, we have $e^x < \frac{1}{1-x}$. Then we have

$$\sum_{n=1}^{\infty} a_n < \ln\left(\prod_{n=1}^{\infty} \frac{1}{1-a_n}\right)$$

Then $\sum_{n=1}^{\infty} a_n$ converges. The proof is complete. □