Homework 8 for Math 1530
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Problem 84. Let (X, d) be a metric space. Prove that the set $A = \{x \in X : d(x, x_0) > 1\}$ is open, where $x_0 \in X$ is any fixed point.

Proof. For any $x \in A$, we can know that $d(x, x_0) - 1 > 0$, then there exists $r > 0$ such that $d(x, x_0) - 1 > r$. Then for any point $y \in B(x, r)$, we have $d(y, x_0) \geq d(x, x_0) - d(x, y) > d(x, x_0) - r > 1$, which implies that $y \in A$. Then, for any $x \in A$, there is an open ball $B(x, r)$ such that $B(x, r) \subset A$. Thus, A is open. $\square$

Problem 85. Show that the following sets are not compact, by exhibiting an open cover with no finite subcover.

(a) $\{x \in \mathbb{R}^n : |x| < 1\}$.

(b) $\mathbb{Z} \subset \mathbb{R}$.

Proof.

(a) Considering the collection of open covers $B = \left(0, 1 - \frac{1}{n}\right)$. Then this collection of open covers does not have a collection of finite subcovers. Thus, $\{x \in \mathbb{R}^n : |x| < 1\}$ is not compact.

(b) Considering the collection of open covers $B = \left(n, \frac{1}{2}\right)$. Then we can know this collection has no finite subcovers since $\mathbb{Z}$ is not bounded.

$\square$

Problem 86. Is it true that in a metric space the closed ball equals to the closure of the open ball, that is $\bar{B}(x, r) = \mathrm{cl}\,(B(x, r))$, where

$$B(x, r) = \{y : d(x, y) < r\} \quad \text{and} \quad \bar{B}(x, r) = \{y : d(x, y) \leq r\}?$$

Proof. It is not always true. Now consider the any set X, where $x, y \in X$ and a discrete metric space, where

$$d(x, y) = \begin{cases} 1, \text{if} & x \neq y \\ 0, \text{if} & x = y \end{cases}$$

Then the open unit ball of radius 1 around any point x: $B(x, 1)$ is the set $\{x\}$ and its closure $\mathrm{cl}\,(B(x, r))$ is also this set. But the closed ball $\bar{B}(x, y) = \{y : d(x, y) \leq 1\}$ is the whole set X. This is a counter example. $\square$

Problem 87. Let $(x_n)_{n=1}^{\infty}$ be a sequence of points in $\mathbb{R}^3$ such that $\|x_{n+1} - x_n\| \leq 1/(n^2 + n)$, $n \geq 1$. Show that (x_n) converges.

Proof. Prove by contradiction and suppose that $\{x_n\}_{n=1}^{\infty}$ dose not converges. Every convergent sequence in a metric space is a Cauchy sequence. Then since $\{x_n\}_{n=1}^{\infty}$ dose not converges, by definition we have $\exists \varepsilon > 0$, then for $\forall n > m$, we have $\|x_n - x_m\| \geq \varepsilon$.

Also, as n increases, for ε be given above, there exists n such that $1/(n^2+n) < \varepsilon$, denote the first n satisfying such property by N_1. Then, for $n > m \geq N_1$, we have $\|x_n - x_m\| \leq 1/(N_1^2+N_1) < \varepsilon$, which is a contradiction. $\square$

Problem 88. Prove that if K_1 and K_2 are nonempty compact and disjoint subsets of a metric space X, then the set $A = K_1 \cup K_2$ is disconnected.

Proof. We denote $U = \mathrm{cl}\,(K_1)$ and $V = \mathrm{cl}\,(K_2)$. Then we have $A \subset U \cup V$, $A \cap U \neq \varnothing$ and $A \cap V \neq \varnothing$. Since K_1 and K_2 are compact and disjoint subsets of a metric space X, then K_1 and K_2 are all closed and $K_1 \cap K_2 = \varnothing$. Then all limit points of K_1 and K_2 belong to K_1 and K_2 respectively, which means $\mathrm{cl}\,(K_1) \cap \mathrm{cl}\,(K_2) = \varnothing$. Then, $A \cap (U \cap V) = \varnothing$. By definition, A is disconnected. $\square$

Problem 89. Prove that $(\mathbb{R}^n, \varrho)$, where

$$\varrho(x, y) = \frac{\|x - y\|}{1 + \|x - y\|}$$

is a metric space.

Proof. We can verify as below:

(a) $\varrho(x, y) > 0$ if $x \neq y$ since $\|x - y\| > 0$.

(b) $\varrho(x, y) = 0$ if $x = y$ since $\|x - y\| = 0$.

(c) $\varrho(x, y) = \varrho(y, x)$.

(d) For $x, y, z \in \mathbb{R}^n$, we have

$$\begin{aligned}
\varrho(x, z) + \varrho(z, y) &= \frac{\|x - z\|}{1 + \|x - z\|} + \frac{\|y - z\|}{1 + \|y - z\|} \\
&\geq \frac{\|x - z\|}{1 + \|x - z\| + \|y - z\|} + \frac{\|y - z\|}{1 + \|x - z\| + \|y - z\|} \\
&= \frac{\|x - z\| + \|y - z\|}{1 + \|x - z\| + \|y - z\|} \\
&= 1 - \frac{1}{1 + \|x - z\| + \|y - z\|} \\
&\geq 1 - \frac{1}{1 + \|x - y\|} \\
&= \varrho(x, y)
\end{aligned}$$

Then $(\mathbb{R}^n, \varrho)$ is indeed a metric space. $\square$

Problem 90. Prove that every compact metric space is separable.

Proof. Suppose X is a compact metric space, and then immediately we have X is totally bounded. We need to prove that X contains a countable dense subset. Then for $\forall \varepsilon > 0$, there exists a finite covering of X by balls of radius ε.

Now we consider that X is covered by finite balls with radius 1, and we extract the center of each ball. And we denote the set without these centers of radius 1 by $B(X \setminus \{x\}, 1)$. Then consider finite balls with radius $\frac{1}{2}$ and there are finite such balls that cover X, and we extract the center of all such balls and denote the set by $B(X \setminus \{x\}, 1/2)$. We can continuous this process for ever $n, n \in \mathbb{N}$, and there are finite balls with radius $1/n$ covering X. And we can know that $\bigcup_{n=1}^{\infty} B(X \setminus \{x\}, 1/n)$ can cover X and this is countable union of dense subsets of X. ☐

Problem 91. Provide an example of a complete metric space that is not separable.

Proof. Take the metric space (X, d) where $X = \mathbb{R}$, and d is discrete metric. Then we can know that in discrete metric, every subset $S \subset X$ are closed and then $\operatorname{cl}(S) = S$. When $X = \mathbb{R}$, the only dense subset of $\mathbb{R}$ is itself, which is not countable. ☐

Problem 92. Let X be a complete metric space and let V_n, $n = 1, 2, 3, \ldots$ be open and dense sets. Prove that $\bigcap_{n=1}^{\infty} V_n$ is dense in X.

Proof. It suffices to show that for every open set $U \subset X$, we have $U \bigcap (\cap_{n=1}^{\infty} V_n) \neq \varnothing$.

Now we can define $U_n = (\cap_{1 \le i \le n} V_i) \bigcap U$. Then we have $\overline{U_n} \subset U_{n-1}$ and $\{U_n\}$ is decreasing sequence of open sets in the sense that $\operatorname{diam} U_n$ is decreasing. Now we choose $u_i \in U_i$ and then $\{u_i\}$ is a Cauchy sequence in X. Since X is a complete metric space, then every Cauchy sequence is convergent. Thus we have $\lim_{i \to \infty} u_i \to u^* \in X$. Then we can know that $U \bigcap (\cap_{n=1}^{\infty} V_n) \neq \varnothing$, then $\bigcap_{n=1}^{\infty} V_n$ is dense in X. ☐

Problem 93. Use previous problem to prove that the set of irrational numbers cannot be written as a union of countably many closed subsets of $\mathbb{R}$.

Proof. Prove by contradiction and suppose that $\mathbb{R} \setminus \mathbb{Q}$ can be written as a union of countably many closed subsets, we can assume $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n \in \mathbb{N}} F_n$, where F_n is closed in $\mathbb{R}$. Then

$$\mathbb{Q} = \mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} F_n = \bigcap_{n \in \mathbb{N}} (\mathbb{R} \setminus F_n) = \bigcap_{n \in \mathbb{N}} U_n$$

where $U_n = \mathbb{R} \setminus F_n$, which is open. Clearly, each of U_n is dense. Since $\mathbb{Q}$ is countable, we can write $\mathbb{Q} = \{q_n\}_{n \in \mathbb{N}}$ and set $V_n = U_n \setminus \{q_n\}$. Then V_n is also open and dense in $\mathbb{R}$, and we have

$$\bigcap_{n \in \mathbb{N}} V_n = \varnothing$$

which is contradicted with Problem 92. Then the proof is complete. ☐

Problem 94. Prove that ℓ^1 is a metric space, where

$$\ell^1 = \left\{ x = (x_1, x_2, \ldots) : \sum_{n=1}^{\infty} |x_i| < \infty \right\}, \quad d(x, y) = \|x - y\|_1 = \sum_{n=1}^{\infty} |x_n - y_n|.$$

Proof. We verify

(a) $d(x,y) > 0$ if $x \neq y$ since $x_i \neq y_i$ for some i and then $\sum_{n=1}^{\infty}|x_n - y_n|> 0$.

(b) $d(x,y) = 0$ if $x = y$ since $\|x_i - y_i\|= 0$ for all $i \in \mathbb{N}$.

(c) $d(x,y) = d(y,x)$.

(d) For $x, y, z \in \ell^1$, we have

$$\begin{aligned} d(x,z) + d(z,y) &= \sum_{n=1}^{\infty}|x_n - z_n|+\sum_{n=1}^{\infty}|z_n - y_n| \\ &= \sum_{n=1}^{\infty}|x_n - z_n|+|y_n - z_n| \\ &\geq \sum_{n=1}^{\infty}|x_n - y_n| \\ &= d(x,y) \end{aligned}$$

Thus, ℓ^1 is a metric space. □

Problem 95. Prove that ℓ^1 is complete.

Proof. We choose a Cauchy sequence $\left\{x_n = \left(x_1^{(n)}, x_2^{(n)}, \cdots\right)\right\}$ and then we have

$$\left|x_i^{(n)} - x_i^{(m)}\right| \leq \|x_n - x_m\|_1, i \in \mathbb{N}$$

then every $\{x_i\}$ is Cauchy sequence and then converges to a real number, denoted by z_i. Then we have $x_n \to z = (z_1, z_2, \cdots)$.

Now we need to show that z is in ℓ^1. We have

$$\begin{aligned} \|z\|= \lim_{N\to\infty}\sum_{i=1}^{N}|z_i| &= \lim_{N\to\infty}\left(\lim_{n\to\infty}\sum_{i=1}^{N}\left|x_i^{(n)}\right|\right) \\ &= \lim_{n\to\infty}\left(\lim_{N\to\infty}\sum_{i=1}^{N}\left|x_i^{(n)}\right|\right) \end{aligned}$$

where we interchange the order of limit since it is the sum of finite numbers. Since $\{x_n\}$ is Cauchy sequence, then it is bounded. Then for some $M > 0$, we have $\|x_n\|< M$ for all n. Thus, for any N, we have

$$\sum_{i=1}^{N}\left|x_i^{(n)}\right| \leq \sum_{i=1}^{\infty}\left|x_i^{(n)}\right| = \|x_n\|< M$$

Then we take $n \to \infty$, we have

$$\sum_{i=1}^{N}|z_i| \leq \|x_n\|< M$$

Since this holds for arbitrary N, we can know that $\|z\|< M$. Thus, $z \in \ell^1$, which implies ℓ^1 is complete. □

Problem 96. Prove that ℓ^1 is separable.

Proof. For $x = (x_1, x_2, \cdots) \in \ell^1$, we have $\sum_{i=1}^{\infty} |x_i| < \infty$. Then, we can know that there exists a $N > 0$, such that for $i > N$, we have $\sum_{i=N+1}^{\infty} |x_i| < \varepsilon/2$. Now take a sequence $\{z_1, z_2, \cdots, z_N, 0, 0, \cdots\}$, $z_1, \cdots, z_N \in \mathbb{Q}$ satisfying $\sum_{i=1}^{N} |z_i - x_i| < \varepsilon/2$. Denote $z = (z_1, z_2, \cdots, z_N, 0, 0, \cdots)$ and we have

$$\|x - z\|_1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore, $x \in \ell^1$ can be approximated by elements of a countable subset $\{z_1, \cdots, z_N, 0, \cdots\}$, which consisting of rational numbers and 0. Now we set $Z_j = \{z_1, \cdots, z_j, 0, \cdots\}, z_1, \cdots, z_j \in \mathbb{Q}$ and then clearly, $\bigcup_{j=1}^{n} Z_j$ is a countable union of countable sets. Thus, ℓ^1 is separable. □

Problem 97. Prove that if $x \in \ell^1$ and $r > 0$, then the closed ball in ℓ^1

$$\bar{B}(x, 1) = \{z \in \ell^1 : \|x - z\|_1 \leq 1\}$$

is not compact.[1]

Proof. Consider the element $e_i = \left(0, \cdots, 0, \underbrace{1/2}_{i\,\mathrm{th}}, 0, \cdots\right)$, $i \in \mathbb{N}$. Then the sequence $\{e_n\}_{n=0}^{\infty}$ does not have convergent subsequence in ℓ^1, since $\|e_n - e_m\|_1 = 1$ for all $n, m \in \mathbb{N}$. □

Problem 98. Let

$$\ell^\infty = \left\{x = (x_1, x_2, \ldots) : \sup_n |x_n| < \infty\right\} \qquad d(x, y) = \|x - y\|_\infty = \sup_n |x_n - y_n|.$$

Prove that the metric space ℓ^∞ is not separable.

Proof. Consider the element $x_I = (x_1^I, x_2^I, \cdots) \in \ell^1$ and for any subset I of positive integers $\mathbb{N}$, x_i^I is defined by

$$x_i^I = \begin{cases} 1, \text{if} & i \in I \\ 0, \text{if} & i \notin I \end{cases}$$

Then we have $d(x_I, x_J) = 1$ for different subset I and J. Then we consider the collection of balls with radius 1/2:

$$\mathbb{M} = \left\{B\left(x_I, \frac{1}{2}\right), I \subset \mathbb{N}\right\}$$

and this is an uncountable collection of disjoint open balls. Now set S be a dense subset in ℓ^∞, then each ball in $\mathbb{M}$ must contain at least one point of S, and these points are all disjoint, which means S is uncountable infinite. Thus, ℓ^∞ is not separable. □

[1] This provides an example of a complete metric space where bounded and closed sets are not necessarily compact.

Problem 99. Prove that for every separable metric space (X, d) there is an isometric embedding $\kappa : X \to \ell^\infty$. *Hint: Let $x_0 \in X$ and let $\{x_i\}_{i=1}^\infty$ be a countable and a dense subset. For each $x \in X$ consider a sequence $(d(x, x_i) - d(x_i, x_0))_{i=1}^\infty$.*

Proof. Consider the map $\kappa : X \to (d(x, x_i) - d(x_i, x_0))_{i=1}^\infty \in \ell^\infty$, then we have

$$\begin{aligned} d_{\ell^\infty}(x, y) &= \sup_i |d(x, x_i) - d(x_i, x_0) - d(y, x_i) + d(x_i, x_0)| \\ &= \sup_i |d(x, x_i) - d(y, x_i)| \\ &\leq d(x, y) \end{aligned}$$

Then there exists a constant $c > 0$ such that $d_{\ell^\infty}(x, y) < cd(x, y)$, which means κ is an isometric embedding. □

Problem 100. Let $X \subset \mathbb{R}^n$ be a compact set. Prove that the set

$$Y = \{y \in \mathbb{R}^n : |x - y| = 2019 \text{ for some } x \in X\}$$

is compact.

Proof. For every $y \in Y$, we have $|x - y| = 2019$ for some $x \in X$. Then we can know that y lies on the ball centered at x with radius 2019. Then Y is bounded, since if not, there exists $y \in Y$ such that $|x - y| > 2019$, which is a contradiction.

Suppose the sequence $\{y_n\}_{n=1}^\infty \in Y$, and $y_n \to y^*$. It suffices to show that $y^* \in Y$. Indeed, we have

$$\begin{aligned} |y^* - x| &\leq |y_n - x| + |y^* - y_n| \to 2019 \\ |y^* - x| &\geq |y_n - x| - |y^* - y_n| \to 2019 \end{aligned}$$

as $n \to \infty$. Then we can know that $y^* \in Y$. Now we proved that Y is bounded and closed, Y is compact follows naturally. □

Problem 101. Construct an example of a decreasing family of connected sets

$$C_1 \supset C_2 \supset C_3 \supset \ldots,$$

such that the intersection $\bigcap_{i=1}^\infty C_i$ is disconnected. (It is enough if you define C_i on a picture.)

Proof. We can define C_n as below

$$C_n = (\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \cup \{(x, y) | x \geq n, 0 \leq y \leq 1\}$$

Then C_n contains two horizontal lines and part of the regions between them, and it is clear C_n is connected. However, the intersection of C_n is just two parallel lines, which is not connected. □

Problem 102. Let $(f_n)_{n=1}^\infty$, $f_n : [0, 1] \to \mathbb{R}$ be sequence of continuous functions such that

(a) $f_n(x) \geq 0$ for all x and n,

(b) $f_{n+1} \leq f_n$ for all n,

(c) $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R}$.

Prove that $f_n \rightrightarrows 0$ converges uniformly to 0.

Proof. Given $\varepsilon > 0$, it suffices to prove that there exists $N > 0$, such that if $\forall n > N$ and $\forall x \in [0,1]$, then $0 \leq f_n(x) < \varepsilon$.

For any $x \in [0,1]$, let N_x be the least integer such that $f_{N_x}(x) < \varepsilon$. Then for $n > N_x$, $f_n(x) < \varepsilon$. Since f_{N_x} is continuous function, then there exists an open neighborhood $U_x \in [0,1]$ of x such that for every $z \in U_x$, $f_{N_x}(z) < \varepsilon$.

Since $[0,1]$ is compact, then there exists a finite open covering such that $[0,1] \subset U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_k}$. Now we pick $N = \max\{N_{x_1}, N_{x_2}, \cdots, N_{x_k}\}$, where N_{x_j} is the least integer such that $f_{N_{x_j}}(x_j) < \varepsilon$. Then if $n > N$ and for $x \in [0,1]$, then $x \in U_{x_i}$ for some $i \in \{1,2,\cdots,k\}$, then we have $0 \leq f_n(x) \leq f_N(x) \leq f_{N_{x_i}(x)} < \varepsilon$. Thus, f_n converges uniformly to 0. □

Problem 103. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a norm, that is for all $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

(a) $F(x) \geq 0$ and $F(x) = 0$ if and only if $x = 0$,

(b) $F(x+y) \leq F(x) + F(y)$,

(c) $F(tx) = |t|F(x)$.

Prove that there are constants $A, B > 0$ such that

$$A\|x\| \leq F(x) \leq B\|x\| \quad \text{for all } x \in \mathbb{R}^n.$$

Proof.

(a) We claim that F is bounded on unit sphere $\{\|x\| = 1\}$.
Let $\{e_1, e_2, \cdots, e_n\}$ be orthonormal basis for $\mathbb{R}^n$, then any $x \in \mathbb{R}^n$ can be written as

$$x = \sum_{i=1}^{n} c_i e_i$$

If $\|x\| = 1$, then we have $|c_i| \leq 1$. And we have

$$F(x) = F\left(\sum_{i=1}^{n} c_i e_i\right) \leq \sum_{i=1}^{n} |c_i| F(e_i) \leq \sum_{i=1}^{n} F(e_i) = B$$

Then there exists a $B > 0$.

(b) Now we claim F is continuous.
If $x \neq y$, then we have $y = x + \|y-x\| \cdot \frac{y-x}{\|y-x\|}$. Thus, we have

$$\begin{aligned} &F(y) \leq F(x) + \|y-x\| F\left(\frac{y-x}{\|y-x\|}\right) \\ \Rightarrow &F(y) - F(x) \leq B\|y-x\| \end{aligned}$$

Now we switch x and y, then we have $F(x)-F(y) \leq B\|y-x\|$. Thus we have $|F(x)-F(y)| \leq B\|y-x\|$, which implies F is continuous.

Now we complete the proof. Since F is continuous, so it obtains its minimum A on the compact unit sphere, i.e.,

$$\begin{aligned} A &= \inf_{\|x\|=1} F(x) = F(x_0) > 0 \\ \Rightarrow & A \leq F(x) \leq B, \|x\|= 1 \end{aligned}$$

Now if $\|x\|\neq 0$ is any point in $\mathbb{R}^n$, then

$$\begin{aligned} F(x) &= F\left(\|x\|\cdot\frac{x}{\|x\|}\right) = \|x\|\cdot F\left(\frac{x}{\|x\|}\right) \\ \Rightarrow & A\|x\| \leq F(x) \leq B\|x\| \end{aligned}$$

$\square$

Problem 104. Prove that if X is a metric space and $f : X \times [0,1] \to \mathbb{R}$ is continuous, then

$$g : X \to \mathbb{R}, \quad g(x) = \sup_{t\in[0,1]} f(x,t)$$

is continuous.

Proof. Prove by contradiction and suppose g is not continuous, i.e., there exists a $\varepsilon > 0$, for $\forall\delta > 0$, $\exists x_0 \in [0,1]$ such that if $d(x, x_0) > \delta$, then $|g(x) - g(x_0)| \geq \varepsilon$.

Fix such ε and pick $\delta = 1/n$, then there exists x_n such that if $d(x_n, x_0) < 1/n$, then $|g(x_n) - g(x_0)| \geq \varepsilon$, which implies

$$\left|\sup_t f(x_n,t) - \sup_t f(x_0,t)\right| \geq \varepsilon$$

then there exist $t_n, t_0 \in [0,1]$ such that $f(x_n,t_n) = \sup_t f(x_n,t)$, $f(x_0,t_0) = \sup_t f(x_0,t)$. Then

$$|f(x_n,t_n) - f(x_0,t_0)| \geq \varepsilon$$

where $x_n \to x_0$. Since $\{t_n\}$ is a bounded sequence in $[0,1]$, then there exists a convergent subsequence $\{t_{n_k}\}$ such that $t_{n_k} \to s$, and then $f(x_{n_k}, t_{n_k}) \to f(x_n, s)$. Then we have

$$\begin{aligned} f\left(x_{n_k}, t_{n_k}\right) &= \sup_t f\left(x_{n_k}, t\right) \geq f\left(x_{n_k}, t_0\right) \\ f(x_n, t_0) &= \sup_t f\left(x_n, t\right) \geq f(x_n, s) \end{aligned}$$

Then we have

$$f(x_n,t_0) \leftarrow f\left(x_{n_k},t_0\right) \leq f\left(x_{n_k},t_{n_k}\right) \to f(x_n,s) \leq f(x_n,t_0)$$

which means $f\left(x_{n_k}, t_{n_k}\right) \to f(x_n, t_0)$, and this is a contradiction to the assumption above. $\square$

Problem 105. Prove that is $A \subset X$ is a dense subset of a metric pace X, and $f : A \to \mathbb{R}$ is continuous, then there is a unique function $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in A$. Prove then that F is uniformly continuous.

Proof.

(a) Since A is dense, then any $x \in X$ is a limit point of A, i.e., we can pick a sequence $\{a_k^x\} \in A$ such that $a_k^x \to x$. Since f is continuous on X, then for $\forall \varepsilon > 0$ and $x \in X$, there exists $\delta_x > 0$ such that if $d(x, y) < \delta_x, y \in A$, then $|f(x) - f(y)| < \varepsilon$. For such δ_x, we can find a $N > 0$, such that if $\forall l, k > N$, then $d\left(a_k^x, a_l^x\right) < \delta_x$, and hence

$$|f(a_k^x) - f(a_l^x)| < \varepsilon$$

then we know that $\{f(a_k^x)\}_{k=1}^{\infty}$ is a Cauchy sequence. Therefore, it is convergent.

Now we define

$$F(x) = \lim_{k \to \infty} f(a_k^x)$$

And we define $\delta = \min\{\delta_x | x \in X\}$. Then for any $x, y \in X$, if $d(x, y) < \delta$, then there exists $K > 0$ such that for $\forall k > K$, we have $d\left(a_k^x, a_k^y\right) < \delta$ and $a_k^x \to x, a_k^y \to y$. Then

$$|F(x) - F(y)| < \varepsilon,$$

and thus, $F(x)$ is uniformly continuous.

(b) It remains to show that F is unique. Uniqueness of F means that if $F_1, F_2 : X \to Y$ are continuous such that $F_1(x) = F_2(x) = f(x)$ for all $x \in A$, then $F_1(x) = F_2(x)$ for all $x \in X$. Indeed, if $x \in X$, by then density of A in X, there is a sequence $A \ni x_n \to x$ and the continuity of F_1 and F_2 yields

$$F_1(x) = \lim_{n \to \infty} F_1(x_n) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} F_2(x_n) = F_2(x).$$

□

Problem 106. Let $f : A \to X$ be a mapping between a dense subset $A \subset \mathbb{R}^n$ and a complete metric space (X, d). Assume that $d(f(x), f(y)) \leq |x - y|$ for all $x, y \in A$.

(a) Prove that there is a mapping $F : \mathbb{R}^n \to X$ such that $d(F(x), F(y)) \leq |x - y|$ for all $x, y \in \mathbb{R}^n$ and $F(x) = f(x)$ whenever $x \in A$.

(b) Provide an example showing that the claim in (a) is not true if we do not assume that the space (X, d) is complete.

Proof.

(a) Since $A \subset \mathbb{R}^n$ is dense, then any $x \in \mathbb{R}^n$ is a limit point of A. Then we can find a sequence $\{a_k^x\}_{k=1}^{\infty} \in A$ such that $a_k^x \to x$. Also, for $\forall \varepsilon > 0$ and $\forall x, y \in \mathbb{R}^n$, there exists a $\delta = \varepsilon$, such that if $|x - y| < \delta$, then $d(F(x), F(y)) \leq |x - y| < \varepsilon$. For such ε, we could find $N > 0$, such that if $\forall k, l > K$, then $|a_l^x - a_k^x| < \varepsilon$, and hence

$$|f(a_l^x) - f(a_k^x)| < \varepsilon$$

then we know that $\{f(a_k^x)\}_{k=1}^{\infty}$ is a Cauchy sequence. Since X is a complete metric space, then this Cauchy sequence converges.

Now we can define

$$F(x) = \lim_{k\to\infty} f(a_k^x)$$

and we can compute for

$$\begin{aligned} d(F(x), F(y)) &= d\left(\lim_{k\to\infty} f(a_k^x), \lim_{k\to\infty} f(a_k^y)\right) \\ &\le \left|\lim_{k\to\infty} a_k^x, \lim_{k\to\infty} a_k^y\right| \\ &\le |a_k^x, x| + |x, y| + |y, a_k^y| \to |x, y| \end{aligned}$$

Then we have $d(F(x), F(y)) \le |x, y|$ for $x, y \in \mathbb{R}^n$.

For $x \in A$, we have $F(x) = \lim_{k\to\infty} f(a_k^x) = f(x)$, since $\{f(a_k^x)\}$ is Cauchy sequence and $a_k^x \to x$. If not, then there exists $\varepsilon > 0$, and $\forall\delta > 0$, $\exists K$ such that if $\forall k > K$, $|a_k^x - x| < \delta$, then $|f(a_k^x) - f(x)| \ge \varepsilon$. We can take $\delta = \varepsilon$, then this is contradicted with $d(f(a_k^x), f(x)) \le |a_k^x - x| < \varepsilon$.

(b) Let $A = \mathbb{Q}^n, X = \mathbb{Q}$. Define for $x = (x_1, x_2, \cdots, x_n) \in A$, $f(x) = x_1$. Then we have

$$d(f(x), f(y)) = |x_1 - y_1| \le |x - y|.$$

For $x = (\sqrt{2}, 0, \cdots, 0) \in \mathbb{R}^n$, we have a sequence $\{x_n = (a_n, 0, \cdots, 0)\} \subset A$ such that $\lim_{n\to\infty} a_n = \sqrt{2}$. For $F : \mathbb{R}^n \to X = \mathbb{Q}$, then we have $F(x) = q$ for some $q \in \mathbb{Q}$. Then, we have

$$\lim_{n\to\infty} d(F(x), F(x_n)) = \lim_{n\to\infty} |q - a_n| = |q - \sqrt{2}| \ne 0,$$

however, $|x - x_n| \to 0$, then this is a contradiction and we find the example.

□

Problem 107. Show that the Hilbert cube

$$\mathcal{H} = \{x = (x_1, x_2, \ldots) : 0 \le x_n \le 2^{-n} \text{ for each } n \in \mathbb{N}\}$$

is compact when equipped with the ℓ^1 metric $d(x, y) = \sum\limits_{n=1}^{\infty} |x_n - y_n|$.

Proof. Let $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, \cdots\right)$, and with diagonal method, we can find a subsequence $\left\{x^{(n_k)}\right\}$ such that $\left\{x_i^{(n_k)}\right\}$ converges for $\forall i \in \mathbb{N}$, in the sense $x_i^{(n_k)} \to x_i$, where $0 \le x_i^{(n_k)} \le 2^{-i}$. Thus we have $0 \le x_i \le 2^{-i}$, which implies that $x = (x_1, x_2, \cdots) \in \mathcal{H}$.

It remains to prove that $x^{(n_k)} \xrightarrow{l^1} x$. Given $\varepsilon > 0$, and we can find a $N_1 > 0$, such that

$$\sum_{i=N_1+1}^{\infty} 2^{-i} < \varepsilon$$

since the series $\sum_{i=n}^{\infty} 2^{-i}$ is a decreasing sequence as n increases, which convegeing to 0. Then we can have

$$\sum_{i=N_1+1}^{\infty} \left| x_i^{(n_k)} - x_i \right| < \sum_{i=N_1+1}^{\infty} 2^{-i} < \varepsilon$$

Since $x_i^{(n_k)} \to x_i$ for $\forall i \in \mathbb{N}$, then there exists $N_2 > 0$ such that for all $k > N_2$, $\left| x_i^{(n_k)} - x_i \right| < \varepsilon / N_1, i \leq N_1$. Thus, now we take $N = N_1 + N_2$, then for all $k > N$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \left| x_i^{(n_k)} - x_i \right| &= \sum_{i=1}^{N_1} \left| x_i^{(n_k)} - x_i \right| + \sum_{i=N_1+1}^{\infty} \left| x_i^{(n_k)} - x_i \right| \\
&< N_1 \frac{\varepsilon}{N} + \varepsilon \\
&< N \frac{\varepsilon}{N} + \varepsilon \\
&< 2\varepsilon \\
&\Rightarrow x^{(n_k)} \xrightarrow{l^1} x
\end{aligned}$$

The proof is complete. □

Problem 108. Let $f_n : \mathbb{R}^k \to \mathbb{R}^m$ be continuous maps $(n = 1, 2, \ldots)$ Let $K \subset \mathbb{R}^k$ be compact. Prove that if $f_n \rightrightarrows f$ uniformly on K, then the set

$$S = f(K) \cup \bigcup_{n=1}^{\infty} f_n(K) \quad \text{is compact.}$$

Proof. It suffices to prove that S is bounded and closed.

(a) First, we prove that S is bounded. Since f is continuous and K is compact, then we have $f(K)$ is also compact, thus bounded. Since f_n uniformly converges to f, then for $\forall \varepsilon > 0$, there exists $N > 0$ and $\delta > 0$ such that for $\forall n \geq N$ and $\forall x \in K$, $\|f_n(x) - f(x)\| \leq \varepsilon$. Then this also holds for $\varepsilon = 1$ for $n \geq N$. Then $\bigcup_{n=N}^{\infty} f_n(K)$ is also bounded since it is the set of all points that within distance 1 to a compact set $f(K)$. Also, $\bigcup_{n=0}^{N-1} f_n(K)$ is also bounded since it is finite sum of compact sets.

(b) Second, we prove that S is closed. For every sequence $\{y_i\}_{i=1}^{\infty} \in S$ such that $y_i \to y$, we need to prove that $y \in S$. If infinitely many y_i's belong to $f(K)$ or $f_n(K)$ for some $n \in \mathbb{N}$, then y_i converges to a point in $f(K)$ or $f_n(K)$ since both are compact sets, which implies $y \in S$.

Otherwise, if every $f_n(K)$ only contians finite components of $\{y_i\}$, then there is a subsequence $\{y_{i_j}\}_{j=1}^{\infty}$ such that $y_{i_j} \in f_{n_{i_j}}(K)$, and $y_{i_j} = f_{n_{i_j}}(x_{i_j}), x_{i_j} \in K$. Since K is compact, then x_{i_j} has a convergent subsequence $\{x_{i_{j_l}}\}$ such that $x_{i_{j_l}} \to x \in K$. And since f_n uniformly converges to f, then we have

$$y \leftarrow y_{i_{j_l}} = f_{n_{i_{j_l}}}\left(x_{i_{j_l}}\right) \to f(x) \in f(K) \subset S$$

Thus, $y = f(x) \in S$.

The proof is complete. □

Problem 109. Let $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$ be a sequence of continuous functions on a metric space X such that the series $\sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in X$ and

$$\sup_{x\in X}\left(\sum_{n=1}^{\infty} f_n(x)^2\right)^{1/2} < \infty.$$

Prove that if a series of real numbers c_n, $n = 1, 2, \ldots$ satisfies $\sum_{n=1}^{\infty} c_n^2 < \infty$, then the series

$$\sum_{n=1}^{\infty} c_n f_n(x)$$

converges uniformly to a continuous function.

Proof. Define $f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$, and we can prove that $f(x)$ also converges for $x \in X$. Indeed, with Cauchy–Schwarz inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty} c_n f_n(x) &\leq \left(\sum_{n=1}^{\infty} c_n^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} f_n(x)^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=1}^{\infty} c_n^2\right)^{\frac{1}{2}} \sup_{x\in X}\left(\sum_{n=1}^{\infty} f_n(x)^2\right)^{\frac{1}{2}} < \infty\end{aligned}$$

It remains to prove that $f(x) = \lim_{n\to\infty} \sum_{n=1}^{\infty} c_n f_n(x)$ is a continuous function. Since $\sum_{n=1}^{\infty} c_n f_n(x) < \infty$, then $\lim_{n\to\infty} c_n f_n = 0$. Thus, for every $\varepsilon > 0$, there exists $N > 0$, such that for $n > N$, $\sum_{n=N+1}^{\infty} c_n f_n(x) < \infty$. Also, for the same ε, we can choose $\delta > 0$ such that if $|x - y| < \delta$, then

$$|f_n(x) - f_n(y)| < \frac{\varepsilon^2}{N\left(\sum_{n=1}^{\infty} c_n^2\right)}$$

for all $n = 1, 2, \cdots$. Indeed, we could find such δ since f_n's are continuous functions. Thus, if $|x - y| < \delta$, we have

$$\begin{aligned}|f(x) - f(y)| &\leq \sum_{n=1}^{\infty} c_n \left|f_n(x) - f_n(y)\right| \\ &= \sum_{n=1}^{N} c_n \left|f_n(x) - f_n(y)\right| + \sum_{n=N+1}^{\infty} c_n \left|f_n(x) - f_n(y)\right| \\ &\leq \left(\sum_{n=1}^{N} c_n^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} \left|f_n(x) - f_n(y)\right|^2\right)^{\frac{1}{2}} + \varepsilon \\ &\leq 2\varepsilon\end{aligned}$$

Thus, f is a continuous function as defined above. The proof is complete. □

Second Proof of Exercise 109. We can find a A such that

$$\sup_{x\in X}\left(\sum_{n=1}^{\infty} f_n(x)^2\right)^{1/2} \leq A < \infty$$

Also for $\forall\varepsilon > 0$, there exists $N_0 > 0$ such that for $M > N > N_0$, we have

$$\sum_{n=N}^{M} c_n^2 < \frac{\varepsilon^2}{A^2}$$

Then we have

$$\begin{aligned}\left|\sum_{n=N}^{M} c_n f_n(x)\right| &\leq \left(\sum_{n=N}^{M} c_n^2\right)^{\frac{1}{2}}\left(\sum_{n=N}^{M} f_n(x)^2\right)^{\frac{1}{2}} \\ &< \left(\frac{\varepsilon^2}{A^2}\right)^{\frac{1}{2}} = \varepsilon\end{aligned}$$

For such $x \in X$, $f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$ converges. We fix N and let $M \to \infty$, then we have

$$\left|f(x) - \sum_{n=1}^{N-1} c_n f_n(x)\right| = \left|\sum_{n=N}^{\infty} c_n f_n(x)\right| \leq \varepsilon$$

Thus, for $\forall\varepsilon > 0$, there exits $N_0 > 0$ such that for $\forall N > N_0$ and $\forall x \in X$, we have

$$\left|f(x) - \sum_{n=1}^{N-1} c_n f_n(x)\right| \leq \varepsilon$$

which implies $\sum_{n=1}^{\infty} c_n f_n(x) \rightrightarrows f(x)$. □

Problem 110. A graph of a mapping $f : X \to Y$ is defined as

$$G_f = \{(x, y) \in X \times Y : \, y = f(x)\}.$$

Prove that if X is a metric space and Y is a compact metric space, then the map $f : X \to Y$ is continuous if and only if G_f is a closed subset of $X \times Y$.

Proof.

(a) ($\Rightarrow$) We can pick a sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that $y_n = f(x_n)$. Since Y is compact, then there is a subsequence $\{y_{n_k}\}$ converging to a point in Y, denoted by y. Then we have $y_{n_k} \to y$, and if Y is compact, then it is closed, which implies that $y \in Y$. Also, we can find a $x \in X$ such that $x = f(y)$. With f being continuous, we can claim that $x_{n_k} \to x$. Thus, $(x_{n_k}, y_{n_k}) \to (x, y) \in G_f$, which implies that G_f is closed.

(b) ($\Leftarrow$) Suppose G_f is a closed subset of $X \times Y$, then convergent sequence $\{(x_n, y_n)\} \in G_f$ converges to a point in G_f, denoted by (x, y), where $y_n = f(x_n)$. Then we have $(x_n, f(x_n)) \to (x, f(x)) \in G_f$. Since every convergent sequence in metric space is Cauchy sequence, then for every $\varepsilon > 0$, we can find $\delta > 0$, such that for $\forall x, x_n \in X$, if $d_X(x, x_n) < \delta$, then $d_Y(f(x), f(x_n)) < \varepsilon$. Thus, f is continuous.

$\square$

Problem 111. Let (X, d) be a compact metric space and $z \in Z$. Let $T : X \to X$ be a mapping that satisfies $d(x, y) \leq d(T(x), T(y))$ for all $x, y \in X$, that is the distances are non-decreasing under the mapping T. Define $\{x_n\}$ by

$$x_1 = T(z) \quad \text{and} \quad x_{n+1} = T(x_n) \text{ for } n \geq 1.$$

Prove that there is a subsequence of $\{x_n\}$ which converges to z.

Proof. Prove by contradiction and suppose that there is no subsequence of $\{x_n\}$ converging to z. Then we have $d(x_n, z) \geq \varepsilon, \forall n \in \mathbb{N}$. Let $n > M$, then

$$\begin{aligned} d\left(T^n(z), T^m(z)\right) &\geq d\left(T^{n-1}(z), T^{m-1}(z)\right) \\ &\geq \cdots \\ &\geq d\left(T^{n-m}(z), z\right) \\ &\geq \varepsilon \end{aligned}$$

but X is compact, then $\{x_n\}$ should have convergent subsequence, which is a contradiction. The proof is complete. $\square$

Problem 112. Let (X, d) be a compact metric space and $f : X \to \mathbb{R}$ be a continuous function. Prove that for any $\varepsilon > 0$, there is $C > 0$ such that

$$|f(x) - f(y)| \leq Cd(x, y) + \varepsilon \quad \text{for all } x, y \in X.$$

Proof. Since f is continuous function, then for $\forall \varepsilon > 0$, there exists a $\delta > 0$, such that if $d(x, y) < \delta$, then $|f(x) - f(y)| < \varepsilon$. Then we can find an $r > 0$ such that $|f(x) - f(y)| \leq \varepsilon - r$. Thus we have

$$\begin{aligned} |f(x) - f(y)| &\leq \varepsilon - \frac{r}{d(x, y)} d(x, y) \\ &\leq \varepsilon - \frac{r}{\delta} d(x, y) \end{aligned}$$

we can define $C = -\frac{r}{\delta}$. Thus, we actually find the C for the ε above. $\square$

Problem 113. Let (X, d) be a metric space and $f : X \to X$ be a contraction mapping. Prove that if a non-empty and compact set $K \subset X$ satisfies $f(K) = K$, then K contains exactly one point.

Proof. Prove by contradiction and suppose K has more than one point. Then K must has at least two points x_1 and x_2. Without losing generality, we can assume $K = \{x_1, x_2\}$. Since f is a contraction mapping, then we have $d(f(x_1), f(x_2)) < d(x_1, x_2)$. Also, $f(K) = K$, then there are only two choices: one is that $f(x_1) = x_1, f(x_2) = x_2$ and another one is $f(x_1) = x_2, f(x_2) = x_1$. In both case we have $d(f(x_1), f(x_2)) = d(x_1, x_2)$, which is a contradiction. $\square$

Problem 114. Let (X, d) be a compact metric space. Prove that if $f : X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$, $x \neq y$, then, there is a unique point $x \in X$ such that $f(x) = x$.

Proof. First, it is easy to see that there is at most one fixed point. Indeed, if $x_1 \neq x_2$ are two distinct fixed points, then we have $d(x_1, x_2) = d(f(x_1), f(x_2)) < d(x_1, x_2)$, which is a contradiction.

It remains to show that there is $x_0 \in X$ such that $f(x_0) = x_0$. Let $\alpha = \inf_{x \in X} d(x, f(x))$. Since X is compact and $x \mapsto d(x, f(x))$ is continuous, then infimum is attained, that is $\alpha = d(x_0, f(x_0))$ for some $x_0 \in X$. Suppose the contrary that $f(x_0) \neq x_0$, then

$$\alpha \leq d(f(x_0), f(f(x_0))) < d(x_0, f(x_0)) = \alpha,$$

which is a contradiction. $\square$

Problem 115. Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - f(y)| < |x - y| \quad \text{for all } x, y \in \mathbb{R},\ x \neq y.$$

and f has no fixed point. You can find an explicit formula for f, but you do not have to. It is enough if you find a convincing argument that such a function exists. You do not have to be very precise, but your argument has to be convincing.

Proof. Take $f(x) = \ln(1 + e^x)$. $\square$