**Homework 4 for Math 1540**
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**Problem 50.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Prove that if partial derivatives

$$\frac{\partial f}{\partial x_i}(x_0), \quad i = 1, 2, \ldots, n,$$

exist, then $f$ is differentiable at $x_0$.

*Proof.* Let $A = \left[\frac{\partial f}{\partial x_1}(x_0), \cdots, \frac{\partial f}{\partial x_n}(x_0)\right]$, and we need to prove that

$$\phi(h) = f(x_0 + h) - f(x_0) - Ah$$

satisfies $\frac{\phi(h)}{\|h\|} \to 0$ as $h \to 0$. $\phi(h)$ is convex and we denote by $h = (h_1, \cdots, h_n) = e_1 h_1 + \cdots + e_n h_n$, then we have

$$\begin{aligned}
\phi(h) &= \phi\left(\frac{1}{n}\sum_{i=1}^n h_i n e_i\right) \\
&\leq \frac{1}{n}\sum_{i=1}^n h_i n e_i = \sum_{i=1}^n h_i \frac{\phi(h_i n e_i)}{h_i n} \\
&\leq \|h\| \sum_{i=1}^n \left|\frac{\phi(h_i n e_i)}{h_i n}\right|,
\end{aligned}$$

similarly,

$$\phi(-h) \leq \|h\| \sum_{i=1}^n \left|\frac{\phi(-h_i n e_i)}{-h_i n}\right|.$$

Also, we have $0 = \phi\left(\frac{h+(-h)}{2}\right) \leq \frac{\phi(h)+\phi(-h)}{2}$, which implies $\phi(h) \leq -\phi(-h)$, then

$$0 \xleftarrow{h \to 0} -\sum_{i=1}^n \left|\frac{\phi(-h_i n e_i)}{-h_i n}\right| \leq \frac{\phi(h)}{\|h\|} \leq \sum_{i=1}^n \left|\frac{\phi(h_i n e_i)}{h_i n}\right| \xrightarrow{h \to 0} 0,$$

where in the last step we used the fact that partial derivatives $\frac{\partial f}{\partial x_i}(x_0), i = 1, 2, \ldots, n$ exist, then we have $\phi(te_i) = f(x_0 + te_i) - f(x_0) - \frac{\partial f}{\partial x_i}(x_0)t$ and

$$\lim_{t \to 0} \frac{\phi(te_i)}{t} = \lim_{t \to 0} \frac{f(x_0 + te_i) - f(x_0)}{t} - \frac{\partial f}{\partial x_i}(x_0) = 0.$$

□

**Problem 51.** Let $Q : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ be a linear mapping such that $Qf \geq 0$ whenever $f \in C^\infty(\mathbb{R}^n)$ satisfies $f(0) = 0$ and $f(x) \geq 0$ in a neighborhood of 0. Prove that there are real numbers $a_{ij}$, $b_i$ and $c$ such that

$$Qf = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) + \sum_{i=1}^n b_i \frac{\partial f}{\partial x_i}(0) + cf(0) \quad \text{for all } f \in C^\infty(\mathbb{R}^n).$$

*Proof.* We have

$$f(x) = f(0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(0) x_i + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) x_i x_j + \varphi(x),$$

and let $c = Q(0), b_i = Q(x_i), a_{ij} = \frac{1}{2}Q(x_ix_j)$, then we have

$$Qf = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(0) + \sum_{i=1}^{n} b_i \frac{\partial f}{\partial x_i}(0) + cf(0) + Q\varphi.$$

We need to prove that $Q\varphi = 0$. Since $\varphi(x) = o(|x|^2)$, then there exists $\varepsilon > 0$, such that $\varepsilon|x|^2 - \varphi(x) \geq 0$ near 0. Then, $Q\left(\varepsilon|x|^2 - \varphi(x)\right) \geq 0$, which implies $Q\varphi \leq \varepsilon Q\left(|x|^2\right)$, and then $Q\varphi \leq 0$ as $\varepsilon \to 0$. Similarly, $Q\left(\varepsilon|x|^2 + \varphi(x)\right) \geq 0$ and we have $-Q\varphi \leq \varepsilon Q\left(|x|^2\right)$. Then, $Q\varphi \geq 0$ and hence $Q\varphi = 0$. □

**Problem 52.** Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{x^2(y^4+2x)}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that $f$ is differentiable at $(0,0)$.

*Proof.* We have

$$f_x(0,0) = \lim_{s\to 0} \frac{f(s,0) - f(0,0)}{s} = \lim_{s \to 0} \frac{2s^3/s^2}{s} = 2$$
$$f_y(0,0) = \lim_{s\to 0} \frac{f(0,s) - f(0,0)}{s} = 0$$

and we need to show that the limit equals zero, which is

$$\lim_{(s,t)\to(0,0)} \frac{f(s,t) - f(0,0) - f_x(0,0)s - f_y(0,0)t}{\sqrt{s^2+t^2}}$$
$$= \lim_{(s,t)\to(0,0)} \frac{s^2t^4 - 2st^4}{(s^2+t^4)\sqrt{s^2+t^2}},$$

also,

$$\left|\frac{s^2t^4 - 2st^4}{(s^2+t^4)\sqrt{s^2+t^2}}\right| \leq \left|\frac{s^2t^4 - 2st^4}{2|s|t^2|t|}\right| = \left|\frac{|st|}{2} - t\right| \xrightarrow{(s,t)\to(0,0)} 0,$$

where we used the fact that $s^2 + t^4 \geq 2|s|t^2$ and $\sqrt{s^2+t^2} \geq |t|$. □

**Problem 53.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that the mixed partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist everywhere in $\mathbb{R}^2$, but

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

*Proof.* When $(x,y) \neq (0,0)$, with $f(x,y) = -f(y,x)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}$$
$$\frac{\partial f}{\partial y}(x,y) = -\frac{\partial f}{\partial x}(y,x) = -\frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2},$$

then $\frac{\partial^2 f}{\partial x\partial y}$, $\frac{\partial^2 f}{\partial y\partial x}$ exist at $(x,y)\neq(0,0)$. Also,

$$\frac{\partial^2 f}{\partial y\partial x}(0,0)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0)=\frac{\mathrm{d}}{\mathrm{d}y}\bigg|_{y=0}\left(\frac{\partial f}{\partial x}(0,y)\right)=\frac{\mathrm{d}}{\mathrm{d}y}\bigg|_{y=0}\left(-\frac{y^5}{y^4}\right)=-1,$$
$$\frac{\partial^2 f}{\partial x\partial y}(0,0)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0)=\frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=0}\left(\frac{\partial f}{\partial y}(x,0)\right)=\frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=0}\left(\frac{x^5}{x^4}\right)=1.$$

□

**Problem 54.** Prove that the function

$$f(x)=\begin{cases}\frac{x|y|}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

has all directional derivatives $D_v f(0,0)$ at the origin, but $f$ is not differentiable at $(0,0)$.

*Proof.* For $v=(a,b)\neq(0,0)$,

$$\begin{aligned}D_v f(0,0)&=\lim_{t\to0}\frac{f(ta,tb)-f(0,0)}{t}\\&=\lim_{t\to0}\frac{1}{t}\frac{ta|tb|}{|t|\sqrt{a^2+b^2}}\\&=\lim_{t\to0}\frac{a|b|}{\sqrt{a^2+b^2}},\end{aligned}$$

and $D_v f(0,0)$ is not linear with respect to $v$, thus, not differentiable. □

**Problem 55.** Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be a mapping of class $C^1$. Prove that if $\operatorname{rank} Df(x_0)=n$, then $f$ is injective in a neighborhood of $x_0$. Prove it directly, without using the inverse function theorem.

*Proof.* Since $\operatorname{rank} Df(x_0)=n$, then $M=f^{-1}(0)=\{x\in\mathbb{R}^n|f(x)=0\}$ is 0-dimensional submanifold. Thus, $M=\{0\}$, which implies $f$ is injective. □

**Problem 56.** Assume that $f:\mathbb{R}^n\to\mathbb{R}^m$ is differentiable and $\operatorname{rank} Df(x)=n$ for all $x\in\mathbb{R}^n$. Prove that if $S\subset\mathbb{R}^n$ is bounded, then for every $y\in\mathbb{R}^m$, the set

$$S\cap f^{-1}(y)=\{x\in S:\ f(x)=y\}$$

is finite.
**Remark.** Since we do not assume continuity of $Df$, we cannot use the inverse function theorem.

*Proof.* Since $f$ is differentiable and $\operatorname{rank} Df(x)=n$, then for any $x\in\mathbb{R}^m$, $f$ is injective in a neighborhood $U$ of $x$. Then for any $y\in\mathbb{R}^m$, there exists a unique $x\in\mathbb{R}^m$ in its neighborhood, and thus $S\cap f^{-1}(y)$ is finite. □

**Problem 57.** Let $\Omega=\{(x,y)\in\mathbb{R}^2:\ x>0, 0<y<2\pi\}$. Prove that the mapping $f:\Omega\to\mathbb{R}^2$, $f(x,y)=(x\cos y, x\sin y)$ is a diffeomorphism of $\Omega$ onto an open subset of $\mathbb{R}^2$. Find $f(\Omega)$. **Hint:** A picture will help. Have you seen a similar mapping in Calculus 3?

*Proof.* Since

$$Jf(x) = \det \begin{pmatrix} \cos y & -x\sin y \\ \sin y & -x\cos y \end{pmatrix} = x \neq 0,$$

the mapping $f$ is diffeomorphism and hence invertible in a neighborhood of any point $(x, y) \in \mathbb{R}^2$. Suppose $f(x_1, y_1) = f(x_2, y_2)$, then we have

$$x_1 \cos y_1 = x_2 \sin y_2$$
$$x_1 \sin y_1 = x_2 \sin y_2$$

which implies $x_1 = x_2$ and $y_1 = y_2$. Thus, $f$ is invertible and hence $f$ is diffeomorphism of $\Omega$ onto an open subset of $\mathbb{R}^2$. Also, $f(\Omega) = \mathbb{R}^2$. □

**Problem 58.** Find a diffeomorphism of $\mathbb{R}^2$ onto the open unit disc $x^2 + y^2 < 1$.

*Proof.* $f(x, y) = \left(\frac{x}{\sqrt{x^2+y^2+1}}, \frac{y}{\sqrt{x^2+y^2+1}}\right)$. □

**Problem 59.** Find a diffeomorphism of the upper half plane $y > 0$ onto the first quadrant $x > 0$, $y > 0$.

*Proof.* $f(x, y) = (e^x, y)$. □

**Problem 60.** Suppose that $f \in C^1(\mathbb{R})$ is such that $|f'(x)| < 1$ for all $x \in \mathbb{R}$. Prove that the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (x + f(y), y - f(x))$ is a diffeomorphism in a neighborhood of any point $(x, y) \in \mathbb{R}^2$.

*Proof.* Since

$$JF(x) = \det \begin{pmatrix} 1 & f'(y) \\ -f'(x) & 1 \end{pmatrix} = 1 + f'(x)f'(y) \neq 0,$$

the mapping $F$ is diffeomorphism and hence invertible in a neighborhood of any point $(x, y) \in \mathbb{R}^2$ □

**Problem 61.** Prove that a complex polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$ regarded as a function $P : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism in a neighborhood of $z_0 \in \mathbb{C}$ if and only if $P'(z_0) \neq 0$, where $P'(z) = na_0 z^{n-1} + (n-1)a_1 z^{n-2} + \ldots + a_1$.
*Proof.*

(1) If $P$ is a diffeomorphism in a neighborhood of $z_0 = x_0 + iy_0 \in \mathbb{C}$, we can write $P$ as $P(z_0) = u(x_0, y_0) + iv(x_0, y_0)$, where $u, v$ are real functions. Then, with the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$, we have

$$\begin{aligned} JP(z_0) &= \det \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix} \\ &= u_x(z_0)v_y(z_0) - u_y(z_0)v_x(z_0) \\ &= u_x^2(z_0) + u_y^2(z_0) = v_x^2(z_0) + v_y^2(z_0) \neq 0, \end{aligned}$$

which implies $u_x, u_y, v_x, v_y \neq 0$. Also, with Wirtinger derivatives, we have

$$\begin{aligned}\frac{\partial P}{\partial z} &= \frac{1}{2}\left(\frac{\partial P}{\partial x} - i\frac{\partial P}{\partial y}\right)\\ &= \frac{1}{2}\left(u_x + iv_x - i(u_y + iv_y)\right)\\ &= \frac{1}{2}\left((u_x + v_y) + i(v_x - u_y)\right)\\ &= u_x + iv_x,\end{aligned}$$

and then $P'(z_0) \neq 0$.

(2) If $P'(z_0) \neq 0$, then $u_x(z_0), v_x(z_0)$. Thus, $JP(z_0) \neq 0$, which implies $P$ is a diffeomorphism in a neighborhood of $z_0 \in \mathbb{C}$.

□

**Problem 62.** Let $f : \mathbb{R} \to \mathbb{R}$ be $C^1$ and let

$$\begin{aligned} u &= f(x)\\ v &= -y + xf(x).\end{aligned}$$

If $f'(x_0) \neq 0$, show that this transformation is locally invertible near $(x_0, y_0)$ and the inverse has the form

$$\begin{aligned} x &= g(u)\\ y &= -v + ug(u).\end{aligned}$$

*Proof.* Since

$$\det\begin{pmatrix} u_x & u_y \\ v_x & v_y\end{pmatrix} = -f'(x_0) \neq 0,$$

then $F : (x, y) \to (u, v)$ is invertible in an open neighborhood $U$ of $(x_0, y_0)$ and for an open neighborhood $W$ of $(u(x_0), v(y_0))$, $F^{-1} : W \to U$ is of class $C^1$. Then there exists $g$ such that $f^{-1}(u) = x = g(u)$ and then $y = -v + ug(u)$. □

**Problem 63.** Prove that the system of equations

$$\begin{cases} xyz + x^2 + y = 0\\ z + x^2y^2z^2 = 0\end{cases}$$

has a solution of the form $y = y(x), z = z(x)$ in a neighborhood of $(0, 0, 0)$.

*Proof.* Denote

$$\begin{aligned} F_1(x, y, z) &= xyz + x^2 + y\\ F_2(x, y, z) &= z + x^2y^2z^2\end{aligned}$$

and we have

$$\Delta = \begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z}\\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z}\end{vmatrix} = \begin{vmatrix} xz + 1 & xy\\ 2x^2z^2y & 1 + 2x^2y^2z\end{vmatrix} = 1 + xz + 2x^2y^2z$$

and at point $(0, 0, 0)$, we have $\Delta(0, 0, 0) = 1 \neq 0$. Then with implicit function theorem, the system has a solution of the form $y = y(x), z = z(x)$ in a neighborhood of $(0, 0, 0)$. □

**Problem 64.** Let $F(x, y) = x^3y^2 + 3x^2y^3 - xy + 2x - y^2 + 1$, $(x, y) \in \mathbb{R}^2$. Prove that there exist functions $g, h \in C^\infty$ defined on an open neighborhood $U \subset \mathbb{R}$ of 0, such that $F(x, g(x)) = 0 = F(x, h(x))$ and $g(x) < h(x)$ for every $x \in U$. Find $g'(0)$, $h'(0)$.

*Proof.* Suppose $F(0, y_0) = 0$, then we have $y_0 = -1$ or $y_0 = 1$. Also, we have $F_y(0, y) = -2y$, then $F_y(0, 1) \neq 0$ and $F_y(0, -1) \neq 0$. With implicit function theorem, there exists a neighborhood $U$ of 0 and a neighborhood $V$ of $y_0$ such that for every $x \in U$, there is exactly one $y \in V$ satisfying $F(x, f(x)) = 0$.

For $y_0 = -1$, we have $F(x, g(x)) = 0$, where $g : U \to V_1$ and $V_1$ is a neighborhood of $-1$. Also, for $y_0 = 1$, we have $F(x, h(x)) = 0$, where $h : U \to V_2$ and $V_2$ is a neighborhood of 1. And $V_1 \cap V_2 = \varnothing$, since if it is not, it contradicts with there is only one $y$ such that $F(x, y) = 0$. Then we have $g(x) < h(x)$.

Also, consider $F(x, g(x)) = 0$, we have $F_x(0, g(0)) = -g(0) + 2 - 2g(0)g'(0) = 0$, with $g(0) = -1$ we have $g'(0) = -\frac{3}{2}$. Similarly, we have $h'(0) = \frac{1}{2}$. □

**Problem 65.** Let $F$ be as in Problem 64. Prove that there is a function $g \in C^\infty$ defined on an open neighborhood $U$ of 0 such that $F(g(y), y) = 0$ for every $y \in U$. Find $g'(0)$.

*Proof.* Suppose $F(x_0, 0) = 0$, then we have $x_0 = -\frac{1}{2}$. Also, we have $F_x(x_0, 0) = 3 \neq 0$. With implicit function theorem, there exists a neighborhood $V$ of 0 and a neighborhood $U$ of $x_0$ such that for every $y \in V$, there is exactly one $x \in U$ satisfying $F(g(y), y) = 0$.

Also, consider $F(g(y), y) = 0$, we have $F_y(g(0), 0) = -g(0) + 2g'(0) = 0$, with $g(0) = -\frac{1}{2}$, we have $g'(0) = -\frac{1}{4}$. □

**Problem 66.** Suppose that $F : \mathbb{R}^3 \to \mathbb{R}$ is of class $C^1$. $F(0, 0, 0) = 0$, $F_x(0, 0, 0) \neq 0$, $F_y(0, 0, 0) \neq 0$, $F_z(0, 0, 0) \neq 0$. The implicit function theorem implies that the equation $F(x, y, z) = 0$ can uniquely be solved in a neighborhood of the point $(0, 0, 0)$ as $x = x(y, z)$ or $y = y(x, z)$ or $z = z(x, y)$. Prove that at every point in some neighborhood of $(0, 0, 0)$ we have

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1\,.$$

*Proof.* Since $F(x, y, z(x, y)) = 0$, we have

$$F_x(x, y, z(x, y))\frac{\partial x}{\partial x} + F_y(x, y, z(x, y))\frac{\partial y}{\partial x} + F_z(x, y, z(x, y))\frac{\partial z}{\partial x} = 0,$$

which implies $F_x + F_z\frac{\partial z}{\partial x} = 0$ and hence $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$. Similarly, $F(x, y(x, z), z) = 0$ yields $\frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$ and $F(x(y, z), y, z) = 0$ yields $\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}$. Thus,

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = (-1)^3\frac{F_x}{F_z}\frac{F_z}{F_y}\frac{F_y}{F_x} = -1.$$

□

**Problem 67.** Let $F = F(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be of class $C^1$ such that $\partial F/\partial y \neq 0$ on $\mathbb{R}^2$. Prove that if the set $S = \{(x, y) \,|\, F(x, y) = 0\}$ is nonempty, then it is of the form $S = \{(x, g(x)) \,|\, x \in U\}$, where $g : U \to \mathbb{R}$ is a $C^1$ function defined on an open set $U \subset \mathbb{R}$.

*Proof.* For $(x_0, y_0) \in S$, then $F(x_0, y_0) = 0$ and $F_y(x_0, y_0) \neq 0$. Then there exists a neighborhood $U$ of $x_0$ and a neighborhood $V$ of $y_0$ such that for any $x \in U$, there exists only one $y \in V$, $F(x, g(x)) = 0$. And $S$ in the neighborhood $U \times V$ of $(x_0, y_0)$ is a graph of a function $y = g(x)$. □

**Problem 68.** Prove that the equation $xe^z = y(z + x)$ defines $z$ as a function of $(x, y)$ in a neighborhood of the point $(x_0, y_0, z_0) = (2, 1, 0)$. Then find the Taylr polynomial of degree 2 of the function $z = z(x, y)$ centered at the point $(2, 1)$.

*Proof.* Let $F(x, y, z) = y(z + x) - xe^z$, then $F(2, 1, 0) = 0$ and $F_z(2, 1, 0) = -1 \neq 0$. Then there is a neighborhood of $U$ of $(2, 1)$ and a neighborhood $V$ of 0, such that for any $z \in V$, there exists only one $(x, y) \in U$, $F(x, y, z(x, y)) = 0$. Thus, the equation defines $z$ as a function of $(x, y)$ in a neighborhood of the point $(2, 1, 0)$.

Also, Taylor series of $z(x, y)$ is

$$z(x, y) = z(2, 1) + \frac{\partial z}{\partial x}(2, 1)(x - 2) + \frac{\partial z}{\partial y}(2, 1)(y - 1),$$

and we can compute for $\frac{\partial z}{\partial x}(2, 1)$ and $\frac{\partial z}{\partial y}(2, 1)$ by taking derivative of $F(x, y, z(x, y))$ with respect to $x$ and $y$, which yields $\frac{\partial z}{\partial x}(2, 1) = 0$, $\frac{\partial z}{\partial y}(2, 1) = 2$. Thus, we have

$$z(x, y) = 2(y - 1).$$

□

**Problem 69.** Show that there is a polynomial $P(x, y, z)$ of order 4 such that the set $P(x, y, z) = 0$ is a torus. Show that the gradient of $P$ is nonzero at every point of the torus and conclude that the torus is locally a graph of a smooth function of two variables.

*Proof.* Considering polynomial $P(x, y, z) = (x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2)$, $R > r$, then $P(x, y, z) = 0$ is a torus. And we have

$$\Delta P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 4(x^2 + y^2 + z^2 + R^2 - r^2)x - 8R^2 x \\ 4(x^2 + y^2 + z^2 + R^2 - r^2)y - 8R^2 y \\ 4(x^2 + y^2 + z^2 + R^2 - r^2)z \end{bmatrix} \neq 0, \forall x, y, z$$

since $\Delta P = 0$ only at $(0, 0, 0)$ but $(0, 0, 0)$ is not a point in this torus. Then at least one of $P_x, P_y$ and $P_z$ is not zero, thus the torus is locally a graph of a smooth function of two variables. □

**Problem 70.** Show that there is no polynomial $P(x, y, z)$ of order less than 4 such that the set $P(x, y, z) = 0$ is a torus.

*Proof.* □

**Problem 71.** The cylinder $(x - 1)^2 + y^2 = 1$ intersects with the sphere $x^2 + y^2 + z^2 = 4$ along a curve. This curve has a self-intersection (the curve looks like ”8”). Find the angle at which the curve intersects with itself.

*Proof.* The curve intersects itself at point $(2,0,0)$, and denote $x-1=\cos\theta$, then we have $y=\sin\theta$ and $z=\pm\sqrt{2-2\cos\theta}$. The the curve can be represented as

$$F(t)=\{(1+\cos\theta,\sin\theta,\pm\sqrt{2-2\cos\theta})\}$$

Then the tangent vectors at point $(2,0,0)$ are

$$\begin{aligned} v &= \left(-\sin\theta,\cos\theta,\pm\frac{\sin\theta}{\sqrt{2-2\cos\theta}}\right)\bigg|_{\theta=0} \\ &= (0,1,0)\,. \end{aligned}$$

Then the angle at which the curve intersects with itself is $\theta=0$. $\square$