Homework 2 for Math 1530
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Problem 15. Find $\sup A$ and $\inf A$, where

$$A = \left\{ \frac{n^2+2n-3}{n+1} : \ n = 1, 2, 3, \ldots \right\}.$$

Proof. We have

$$\frac{n^2+2n-3}{n+1} = \frac{(n+1)^2-4}{n+1} = (n+1) - \frac{4}{n+1}$$

Substituting $n+1$ as $t > 1$, we can define a function $f(t) = t - 4/t$. Then, we have

$$f'(t) = 1 + \frac{4}{t^2} > 0$$

which means that $f(t)$ is increasing as $t \to \infty$. Thus, we have $\sup A = +\infty$.

On the other hand, since $f(t)$ is increasing, we can have $\inf A = f(2) = 0$. □

Problem 16. Prove that is $A, B \subset \mathbb{R}$ are bounded and non-empty, then

$$\sup(A+B) = \sup A + \sup B, \quad \text{where} \quad A+B = \{x+y : \ x \in A, \ y \in B\}.$$

Proof. (1) First, we set $M = \sup A$ and $N = \sup B$. Then for $\forall x \in A$ and $\forall y \in B$, we have $x + y \leq M = N$. Thus, $M + N$ is an upper bound of $A + B$.

(2)Since $M = \sup A$ and $N = \sup B$, then for $\forall \varepsilon > 0$, $\exists x \in A$ such that $x > M - \frac{\varepsilon}{2}$ and $\exists y \in B$ such that $y > N - \frac{\varepsilon}{2}$. Thus, we have

$$x + y > M + N - \varepsilon$$

which means $M + N$ is the least upper bound of $A + B$. The proof is complete. □

Problem 17. Prove that is $A, B \subset (0, \infty)$ are bounded and non-empty, then

$$\sup(A \cdot B) = \sup A \cdot \sup B, \quad \text{where} \quad A \cdot B = \{xy : \ x \in A, \ y \in B\}.$$

Proof. (1)First, we set $M = \sup A$ and $N = \sup B$. Then for $\forall x \in A$ and $\forall y \in B$, we have $x \cdot y \leq M \cdot N$ since $A, B \subset (0, \infty)$. Thus, MN is an upper bound of $A \cdot B$.

(2)Second, for $\forall \varepsilon > 0$, $\exists x \in A$ such that $x > M - \frac{\varepsilon}{2N}$, and $\exists y \in B$ such that $y > N - \frac{\varepsilon}{2M}$. Thus, we have

$$\begin{aligned} x \cdot y &> \left(M - \frac{\varepsilon}{2N}\right)\left(N - \frac{\varepsilon}{2M}\right) \\ &> MN - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{MN} \\ &> MN - \varepsilon \end{aligned}$$

which means that MN is a least upper bound of $a \cdot B$. The proof is complete. □

Problem 18. Prove that if $\lim_{n\to\infty} a_n = \infty$, then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = \infty.$$

Proof. Since $\lim_{n\to\infty} = \infty$, then $\forall M \in \mathbb{R}$, there exist an n_0 such $\forall n > 2n_0, a_n > 2M$, then we have

$$\begin{aligned}\frac{a_1 + a_2 + \cdots + a_{n_0}}{n} &> \frac{a_1 + \cdots + a_{n0} + 2M(n - n_0)}{n} \\ &= 2M - \frac{2n_0 M}{n} + \frac{a_1 + \cdots + a_{n_0}}{n} \\ &> 2M - M = M\end{aligned}$$

Then the proof is complete. □

Problem 19. Prove that if $\lim_{n\to\infty} a_n = g \in \mathbb{R}$ and $a_n > 0$ for all n, then

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = g.$$

Proof. (1)Since $\lim_{n\to\infty} a_n = g$, we can know that there exists $M > 0$ such that for all n

$$|a_n - g| \le M \Rightarrow g - M \le a_n \le g + M$$

Also, based on the definition of limit, we can know that for $\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n > n_0$, $|a_n - g| < \varepsilon$. Using arithmetic-geometric mean inequality, we have

$$\begin{aligned}\sqrt[n]{a_1 \cdots a_n} &\le \frac{a_1 + \cdots + a_n}{n} \\ \sqrt[n]{a_1 \cdots a_n} - g &\le \frac{a_1 + \cdots + a_n}{n} - g\end{aligned}$$

Since we already know that if $\lim_{n\to\infty} a_n = g$, then $\lim_{n\to\infty} \frac{a_1+\cdots+a_n}{n} = g$, thus we have

$$\sqrt[n]{a_1 \cdots a_n} - g < \varepsilon$$

(2)Now we want to prove $-\varepsilon < \sqrt[n]{a_1 \cdots a_n} - g$. And we already know $\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n > n_0$, $|a_n - g| < \varepsilon \Rightarrow g - \varepsilon < a_n < g + \varepsilon$, thus we have

$$\begin{aligned}(a_1 \cdots a_{n_0-1})(g-\varepsilon)^{n-n_0+1} \le a_1 \cdots a_n \le (a_1 \cdots a_{n_0-1})(g+\varepsilon)^{n-n_0+1} \\ (g-M)^{n_0-1}(g-\varepsilon)^{n-n_0+1} \le a_1 \cdots a_n \le (g+M)^{n_0-1}(g+\varepsilon)^{n-n_0+1}\end{aligned}$$

which yields

$$\begin{aligned}(g^{n_0-1} - (n_0-1)g^{n_0-2}M + \cdots)(g^{n-n_0+1} - (n-n_0+1)g^{n-n_0}\varepsilon + \cdots) &\le a_1 \cdots a_n \\ g^n - (n-n_0+1)g^{n-1}\varepsilon &\le a_1 \cdots a_n\end{aligned}$$

Now we just have to change the condition that $\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n > n_0$, $|a_n - g| < \varepsilon / \left(g^{n-1}(n - n_0 + 1)\right)$, then we have $g^n - \varepsilon < a_1 \cdots a_n$. Then we have $\lim_{n\to\infty} a_1 \cdots a_n = g^n$, which gives us $\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = g$. □

Problem 20. Prove that the sequence defined by

$$a_1 = 0, \qquad a_{n+1} = \sqrt{6 + a_n} \text{ for } n \ge 1$$

is convergent and find its limit.
Remark. It is natural to denote the limit of this sequence by

$$\sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots}}}$$

Proof. We assume the sequence is convergent and its limit is g. Then we have

$$\begin{aligned} &a_{n+1} = \sqrt{6 + a_n}, a_0 = 0 \\ &\Rightarrow g = \sqrt{6+g} \\ &\Rightarrow g = 3 \end{aligned}$$

Now we have to prove 3 is the limit of this sequence.

(1)Since $a_{n+1} = \sqrt{6+a_n} \Rightarrow a_{n+1} > a_n$, this sequence is increasing.

(2)Now we need to prove this sequence is bounder above. Assume that $a_n \leq 3$, then we have $a_{n+1} \leq \sqrt{6+3} = 3$, which means this sequence is bounded above and the limit is 3. □

Problem 21. Prove that

$$2\cos\left(\frac{\pi}{2^{n+1}}\right) = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n \text{ square roots}}.$$

Proof. Based on the previous problem, we can know that $\lim \sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}} = 2$. And as $n \to \infty$, $\lim_{n\to\infty} 2\cos\left(\frac{\pi}{2^{n+1}}\right) = 2\cos 0 = 2$. The proof is complete. □

Problem 22. Find the limit

$$\lim_{n\to\infty} 2^n \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}}_{n \text{ square roots}}.$$

That is not a typo. We have one "−" and the rest are "+" signs.

Proof. Substituting $\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}$ with $2\cos\left(\frac{\pi}{2^n}\right)$ and set the limit of above sequence as A, we have

$$\begin{aligned} A &= \lim_{n\to\infty} 2^n \sqrt{2 - 2\cos\left(\frac{\pi}{2^n}\right)} \\ &= \lim_{n\to\infty} 2^n \sqrt{4\sin^2 \frac{\pi}{2^{n+1}}} \\ &= \lim_{n\to\infty} 2^{n+1} \sin\frac{\pi}{2^{n+1}} \\ &= \pi \end{aligned}$$

□

Problem 23. Find the limit $\lim_{n\to\infty} \dfrac{n}{e^{1+\frac{1}{2}+\cdots+\frac{1}{n}}}$.

Proof. Take ln on this sequence and we have

$$\begin{aligned} \lim_{n\to\infty} \ln\left(\frac{n}{e^{1+\frac{1}{2}+\cdots+\frac{1}{n}}}\right) &= \lim_{n\to\infty} \ln n - (\ln e + \ln e^{\frac{1}{2}} + \cdots + \ln e^{\frac{1}{n}}) \\ &= -\gamma \end{aligned}$$

which means $\lim_{n\to\infty} \frac{n}{e^{1+\frac{1}{2}+\cdots+\frac{1}{n}}} = e^{-\gamma}$. $\square$

Problem 24. Find the limit $\lim\limits_{n\to\infty} \sin\left(2\pi\sqrt{n^2+1}\right)$.

Proof.

$$\begin{aligned}
\lim_{n\to\infty} \sin\left(2\pi\sqrt{n^2+1}\right) &= \lim_{n\to\infty} \sin\left(2\pi n\sqrt{1+\frac{1}{n^2}}\right) \\
&= \lim_{n\to\infty} \sin\left(2\pi n\right) \\
&= 0
\end{aligned}$$

$\square$

Problem 25. Prove that the sequence $\sin n$ is divergent.

Proof. Suppose that $\sin n$ is convergent and the limit is g. Then we have $\lim \sin(2n) = g \Rightarrow 2\sin(n)\cos(n) = g$. Since $\sin^2(n)+\cos^2(n) = 1$, we have $\cos n = \sqrt{1-g^2}$. Using $\lim \sin(2n) = g \Rightarrow 2\sin(n)\cos(n) = g$, we have

$$2g\sqrt{1-g^2} = g \Rightarrow g = -1, 0, 1$$

Also we have $\cos(2n) = \cos^2(n) - \sin^2(n) = 1-2g^2$, and $\cos(2n) = \sqrt{1-\sin^2(2n)} = \sqrt{1-g^2}$, then

$$1-2g^2 = \sqrt{1-g^2} \Rightarrow g = 0$$

Now consider $\sin(n+1) = g$, which gives $\sin n\cos 1 + \cos n \sin 1 = g$, then we have

$$g\cos 1 + \sin 1 = g$$

where $g \neq 0$. This contradicts with above result. So $\sin n$ is divergent. $\square$

Problem 26. Prove that the sequence

$$\left(1+\frac{1}{n}\right)^{n+1}$$

is decreasing.

Proof. (1)Set $f(n) = \left(1+\frac{1}{n}\right)^{n+1} = e^{\ln\left(1+\frac{1}{n}\right)^{n+1}} = \exp\left((n+1)\ln\left(1+\frac{1}{n}\right)\right)$, then we set a function

$$f(x) = \exp\left((x+1)\ln\left(1+\frac{1}{x}\right)\right)$$

where $x \in [1,\infty)$, and

$$\begin{aligned}
f'(x) &= f(x)\left(\ln\left(1+\frac{1}{x}\right) - \frac{x+1}{x^2+x}\right) \\
&= f(x)\left(\ln\left(1+\frac{1}{x}\right) - \frac{1}{x}\right)
\end{aligned}$$

Now we need to determine the value of $\left(\ln\left(1+\frac{1}{x}\right) - \frac{1}{n}\right) = g(x)$, then we have

$$g'(x) = -\frac{1}{x^2+x} + \frac{1}{x^2} > 0, x \geq 1$$

which means $g(x)$ is increasing on $(1,\infty)$. Also, $\lim_{x\to\infty} g(x) = 0$, which implies that $f'(x) < 0$ for $x \in [1,\infty)$. Then we can know that $f(x)$ is decreasing on $[1,\infty)$, substituting x by n shows that the sequence is decreasing. The proof is complete.

(2)Another better approach is:

$$\begin{aligned}
\frac{f(n)}{f(n+1)} &= \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}} \frac{1}{1+\frac{1}{n+1}} \\
&= \left(1+\frac{1}{n^2+2n}\right)^{n+1} \frac{1}{1+\frac{1}{n+1}} \\
&\geq \left(1+\frac{n+1}{n^2+2n}\right) \frac{1}{1+\frac{1}{n+1}} \\
&\geq \left(1+\frac{n+1}{n^2+2n+1}\right) \frac{1}{1+\frac{1}{n+1}} \\
&= 1
\end{aligned}$$

The third step follows from $(1+x)^n \geq 1+nx$. Then we know $f(n) \geq f(n+1)$, which implies it is a decreasing sequence. $\square$