Note on Basic Topology
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1. Euclidean Spaces

$\mathbb{R}^n$ is the n-field Cartesian product of $\mathbb{R}$, i.e., $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}\times\cdots\mathbb{R} = \{(x_1, x_2, \cdots, x_n)|x_i \in \mathbb{R}, i = 1, 2, \cdots, n\}$. Also, $\mathbb{R}^n$ is a linear space with respect to the addition and multiplication of points by scalars (i.e., real numbers) which are defined for $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n), c \in \mathbb{R}$ as follows:

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \\ cx &= (cx_1, cx_2, \cdots, cx_n) \end{aligned}$$

so that $x + y \in \mathbb{R}^n$ and $cx \in \mathbb{R}^n$. We also define the inner product (or scalar product) of x and y

$$(x, y) = x \cdot y = \sum_{i=1}^{n} x_i y_i$$

and the norm of x by

$$\|x\| = (x \cdot x)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$$

The structure now defined is called euclidean n-spaces.

Theorem 1.1. *Suppose $x, y, z \in \mathbb{R}^n$ and α is real. Then*

(1) $\|x\| \geq 0$.
(2) $\|x\| = 0$ *if and only if* $x = 0$.
(3) $\|\alpha x\| = |\alpha| \, \|x\|$.
(4) $\|x \cdot y\| \leq \|x\| \, \|y\|$, *this is called Cauchy-Schwarz inequality.*
(5) $\|x + y\| \leq \|x\| + \|y\|$, *this is called Triangle inequality.*
(6) $\|x - z\| \leq \|x - y\| + \|y - z\|$.

Proof. We only proof (d). If $x = (0, 0, \cdots, 0) = 0$, or $y = 0$, then it is obvious. If $x \neq 0$ and $y \neq 0$, then for $t \in \mathbb{R}$, we have

$$\begin{aligned} 0 \leq \|x + ty\| &= (x + ty, x + ty) \\ &= (x, x) + 2t(x, y) + t^2(y, y) \end{aligned}$$

And we know that $(x, x) + 2t(x, y) + t^2(y, y)$ is a quadratic function which is not negative. Hence, we have

$$\begin{aligned} &\Delta = (2(x, y))^2 - 4(x, x)(y, y) \leq 0 \\ \Rightarrow &\|(x, y)\| \leq \|x\| \, \|y\| \end{aligned}$$

$\square$

2. Metric Spaces

Definition 2.1. *A set X, whose elements we call points, is said to be a metric space if with any two points x and y of X there is associated a real number $d(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ called the distance from x to y, which is defined as*

$$d(x,y) = \|x - z\|$$

which has the following properties

(1) $d(x,y) > 0$ *if* $x \neq y$.
(2) $d(x,y) = 0$ *if* $x = y$.
(3) $d(x,y) = d(y,x)$.
(4) $d(x,y) \leq d(x,z) + d(z,y)$.

Definition 2.2. *Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points in $\mathbb{R}^n$ and $y \in \mathbb{R}^n$. We say that $\{x_i\}_{i=1}^{\infty}$ converges to y if*

$$\lim_{i\to\infty} \|x_i - y\| = 0$$

Then we write $\lim_{i\to\infty} x_i = y$. Equivalently, $\lim_{i\to\infty} x_i = y$ if

$$\forall \varepsilon > 0, \exists N > 0, \forall n > N, d(x_n, y) < \varepsilon.$$

Theorem 2.1. *Let $x_i = (x_{1i}, x_{2i}, \cdots, x_{ni}) \in \mathbb{R}^n$, and $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$. Then*

$$\lim_{i\to\infty} x_i = y$$

if and only if

$$\lim_{i\to\infty} x_{ki} = y_k, k = 1, 2, \cdots, n.$$

Proof. We have

$$\begin{aligned} \|x_i - y\| &= \sqrt{(x_{1i} - y_1)^2 + \cdots + (x_{ni} - y_1)^2} \\ &\geq \sqrt{(x_{ki} - y_k)^2} = |x_{ki} - y_k| \end{aligned}$$

Hence, $\|x_i - y\| \to 0 \Rightarrow |x_{ki} - y_k| \to 0$.

On the other hand, if $|x_{ki} - y_k|$ as $i \to \infty$ for $k = 1, 2, \cdots, n$, then

$$\max_k |x_{ki} - y_k| \to 0 \quad \text{as} \quad i \to \infty$$

and hence

$$\begin{aligned} \|x_i - y\| &= \sqrt{(x_{1i} - y_1)^2 + \cdots + (x_{ni} - y_1)^2} \\ &\leq \sqrt{n \max_k (x_{ki} - y_k)^2} \\ &\leq \sqrt{n} \max_k |x_{ki} - y_k| \to 0. \end{aligned}$$

□

Definition 2.3. *Let (X, d) be a metric space and let $x_i \in X, i = 1, 2, \cdots,$ and $x \in X$. We say that the sequence $\{x_n\}_{i=1}^{\infty}$ converges to x, saying $\lim_{i\to\infty} x_i = x$ if $\lim_{i\to\infty} d(x_i, x) = 0$.*

Example 2.1. Examples of metric spaces:

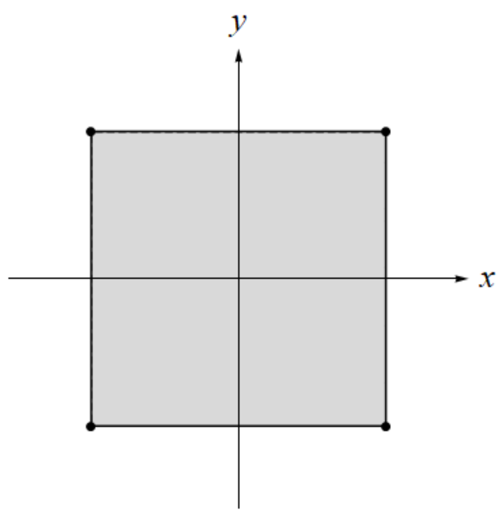

FIGURE 1. $B(0,1)$ in $(\mathbb{R}^2, \rho_1)$

(1) $(\mathbb{R}^n, \rho_1)$, where $\rho_1(x,y) = \max_i |x_i - y_i|$. And $B(0,1)$ in $(\mathbb{R}^2, \rho_1)$ is shown as above.

(2) $(\mathbb{R}^n, \rho_2)$, where $\rho_2(x,y) = \sum_{i=1}^n |x_i - y_i|$, this is called taxi metric or New York metric. And $B(0,1)$ in $(\mathbb{R}^2, \rho_2)$ is shown as bellow

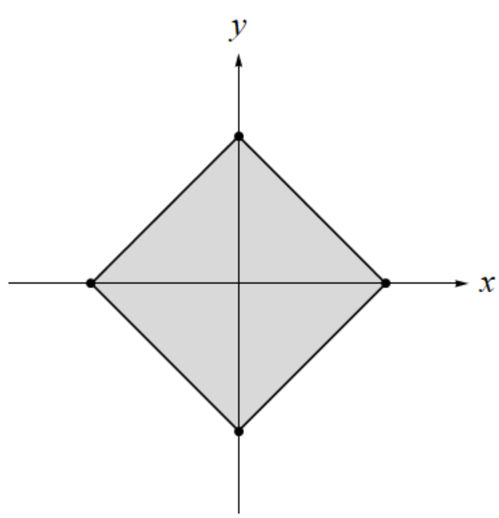

FIGURE 2. $B(0,1)$ in $(\mathbb{R}^2, \rho_2)$

(3) $(\mathbb{R}^n, \rho_3)$, where $\rho_3(x,y) = \|x - y\|$, this is called standard euclidean space.
(4) $(\mathbb{R}^n, \rho_4)$, where $\rho_4(x,y) = \|x - y\|^{1/2}$.
(5) (X, d), where X is arbitrary set and

$$d(x,y) = \begin{cases} 1, \text{if} & x \neq y \\ 0, \text{if} & x = y \end{cases}$$

This is called discrete metric space.

(6) One can prove that every continuous function on $[0,1]$ is bounded. This fact implies that (C,d), where $C([0,1]) = \{f : [0,1] \to \mathbb{R}, f \text{ is continuous}\}$ with $d(f,g) = \|f-g\|_\infty = \sup\{|f(x)-g(x)| : x,y \in [0,1]\}$ is a metric space.

(7) Let $l^1 = \{x = \{x_n\}_{n=1}^\infty : \sum_{n=1}^\infty |x_n| < \infty\}$, i.e., l^1 is the space of all absolutely convergent sequences. For $x = \{x_n\}_{n=1}^\infty, y = \{y_n\}_{n=1}^\infty \in l^1$, we define

$$d(x,y) = \sum_{n=1}^\infty |x_n - y_n|$$

We will prove that (l^1, d) is a metric space.
First we have $d(x,y) < \infty$ for $\forall x,y \in l^1$. And we have $|x_n - y_n| \le |x_n| + |y_n|$, and hence

$$d(x,y) = \sum_{n=1}^\infty |x_n - y_n| \le \sum_{n=1}^\infty |x_n| + \sum_{n=1}^\infty |y_n| < \infty$$

Now we have $(1) : d(x,y) \ge 0$ and $(2) : d(x,y) = d(y,x)$, which is obvious. And $(3) : d(x,y) = 0 \Leftrightarrow \forall x_n = y_n \Leftrightarrow x = y$. Finally, we have

$$|x_n - y_n| \le |x_n - z_n| + |z_n - y_n|$$

and hence

$$\sum_{n=1}^\infty |x_n - y_n| \le \sum_{n=1}^\infty |x_n - z_n| + \sum_{n=1}^\infty |z_n - y_n|$$
$$\Rightarrow d(x,y) \le d(x,z) + d(z,y).$$

(8) Let $l^2 = \{x = \{x_n\}_{n=1}^\infty : \sum_{n=1}^\infty |x_n|^2 < \infty\}$. For $x = \{x_n\}_{n=1}^\infty, y = \{y_n\}_{n=1}^\infty \in l^1$, we define

$$d_2(x,y) = \sqrt{\sum_{n=1}^\infty (x_n - y_n)^2}$$

Thus, (l^2, d_x) is a metric space and this space is call Hilbert space.

Theorem 2.2. *If $x_n \to x$ and $y_n \to y$ in a metric space, then $d(x_n, y_n) \to d(x,y)$.*

Proof. The triangle inequality yields

$$d(x,y) \le d(x,x_n) + d(x_n,y_n) + d(y_n,y)$$

Then, we have $d(x,y) - d(x_n,y_n) \le d(x,x_n) + d(y_n,y)$. Also, we have

$$d(x_n,y_n) \le d(x_n,x) + d(x,y) + d(y,y_n)$$

and then $d(x_n,y_n) - d(x,y) \le d(x_n,x) + d(y,y_n)$. These two inequalities yield

$$|d(x,y) - d(x_n,y_n)| \le d(x,x_n) + d(y_n,y) \to 0$$

and hence

$$|d(x,y) - d(x_n,y_n)| \to 0$$

which implies $d(x,y) \to d(x_n,y_n)$. $\square$

3. Elements of Topology

Let (X,d) be a metric space. For $x \in X$ and $r > 0$, we define

$$B(x,r) = \{y \in X : d(x,y) < r\}$$

and call it the ball of radius r centered at x. For example, if (X,d) is a discrete metric space, then $B(x,1/2) = \{x\}$, $B(x,1) = \{x\}$ and $B(x,2) = X$.

Definition 3.1. *We say that a set $U \subset X$ is open if*

$$\forall x \in X, \exists r > 0, B(x,r) \subset U.$$

Definition 3.2 (Definition in Rudin's Principle of Mathematical Analysis)**.** *Let X be a metric space.*

(1) *A neighborhood or a ball of x is a set $N_r(x)$ consisting of all y such that $d(x,y) < r$, for some $r > 0$. The number r is called the radius of $N_r(x)$.*
(2) *A point x is a limit point of set E if every neighborhood of x contains a point $y \neq x$ such that $y \in E$.*
(3) *If $x \in E$ and x is not a limit point of E, then x is called an isolated point of E.*
(4) *E is closed if every limit point of E is a point of E.*
(5) *A point x is an interior point of E if there is a neighborhood(or ball) $N_r(x)$ of x such that $N_r(x) \in E$.*
(6) *E is open if every point of E is an interior point of E.*
(7) *The complement of E (denoted by E^c) is the set of all points $x \in X$ such that $x \notin E$.*
(8) *E is perfect if E is closed and if every point of E is a limit point of E.*
(9) *E is bounded if there is a real number M and a point $x \in X$ such that $d(x,y) < M$ for all $y \in E$.*
(10) *E is dense in X if every point of X is a limit point of E, or a point of E(or both).*

Theorem 3.1. *Every ball $B(x,r)$ is open.*

Proof. Let $B(x_0,r_0)$ be a ball. We will prove it is open. If $x \in B(x_0,y_0)$, then $d(x,x_0) < r_0$, then there exists a $r > 0$, such that

$$d(x,x_0) + r < r_0$$

We will now prove that $B(x,r) \subset B(x_0,r_0)$. Indeed, if $y \in B(x,r)$, then we have

$$\begin{aligned} d(x_0,y) &\leq d(x_0,x) + d(x,y) \\ &\leq d(x_0,x) + r < r_0 \end{aligned}$$

Since every point of $B(x,r)$ belongs to $B(x_0,r_0)$, we can conclude that $B(x,r) \subset B(x_0,r_0)$. According to Definition 3.1, we proved that $B(x_0,r_0)$ is open. □

Theorem 3.2. *Let (X,d) be a metric space, then*

(1) *$\varnothing, X$ are open.*
(2) *Intersection of a finite family $U_1, \cdots, U_n \subset X$ of open sets, $\bigcap_{i=1}^n U_i$ is open.*
(3) *Union of an arbitrary family $U_i, i \in I$ of open sets, $\bigcup_{i \in I}^n U_i$ is open.*

Proof.

(1) This is obvious.

(2) Suppose $U_1, \cdots, U_n \subset X$ are open. Let $x \in \bigcap_{i=1}^n U_i$, we need to show $B(x,r) \subset \bigcap_{i=1}^n U_i$ for some $r > 0$. We have

$$
\begin{aligned}
x \in U_1 &\Rightarrow B(x, r_1) \subset U_1, \text{for some}\, r_1 > 0 \\
x \in U_2 &\Rightarrow B(x, r_2) \subset U_2, \text{for some}\, r_2 > 0 \\
&\vdots \\
x \in U_n &\Rightarrow B(x, r_n) \subset U_n, \text{for some}\, r_n > 0
\end{aligned}
$$

Hence, we can pick $r = \min\{r_1, r_2 \cdots, r_n\}$, and it follows that $B(x,r) \subset \bigcap_{i=1}^n U_i$.

(3) Let $\{U_i\}_{i \in I}$ be an arbitrary family of open sets and let $x \in \bigcup_{i \in I} U_i$. Then there exists a $i_0 \in I$ such that $x \in U_{i_0}$, and hence $B(x,r) \subset U_{i_0} \subset \bigcup_{i \in I} U_i$ for some $r > 0$. The proof is complete.

□

Definition 3.3. *Given $A \subset X$, where X is a metric space. The interior of the set A is defines as*

$$intA = \{x \in A : \exists r > 0, B(x,r) \subset A\}$$

Theorem 3.3. *$intA$ is always open. It is the largest open set contained in A in the sense that if $U \subset A$ is open, then $U \subset intA$.*

Proof.

(1) If $U \subset A$, then for $\forall x \in U$, there exists $r > 0$ such that $B(x,r) \subset U \subset \text{int}A$. This implies that $x \in \text{int}A$. Thus, $U \subset \text{int}A$.

(2) If $x \in \text{int}A$, then there exists $r > 0$ such that $B(x,r) \subset A$. Since $B(x,r)$ is open and $B(x,r) \subset A$, we have $\text{int}A$ is open.

□

Definition 3.4. *Let (X,d) be a metric space. We say that $A \subset X$ is closed if $X \setminus A$ is open.*

Theorem 3.4 (Theorem 2.23 in Rudin's book)**.** *A set A is open if and only if it complement is closed.*

Proof. First, suppose A^c is closed. For $x \in A$, then $x \notin A^c$, and x is not a limit point of E^c. Then there exists $r > 0$ such that $B(x,r) \cap A^c = \varnothing$. Then, we have $B(x,r) \subset A$. Thus x is an interior point of A and it follows that A is open.

Next, suppose A is open. Let x be a limit point of A^c. Then every neighborhood of x contains a point of A^c, so x is not a interior point of A. Since A is open, then $x \notin A$, which means $x \in A^c$. Since x is a limit point of A^c, then A^c is closed. The proof is complete. □

Theorem 3.5. *Let (X,d) be a metric space, then*

(1) *$\varnothing, X$ are closed.*

(2) *Intersection of an arbitrary family $U_i, i \in I$ of closed sets, $\bigcap_{i \in I}^n U_i$ is closed.*

(3) *Union of a finite family $U_1, \cdots, U_n \subset X$ of closed sets, $\bigcup_{i=1}^n U_i$ is open.*

Proof.

(1) $\varnothing$ is closed, since $X \setminus \varnothing = X$ is open. Also, X is closed, since $X \setminus X = \varnothing$ is open.
(2) Suppose $\{U_i\}_{i\in I}$ is an arbitrary family of closed sets. Then the set $X \setminus U_i$ are open, and we have
$$\bigcup_{i\in I}(X \setminus U_i) = X \setminus \bigcap_{i\in I} U_i$$
is open and hence $\bigcap_{i\in I} U_i$ is closed.
(3) Suppose the set $U_1, \cdots, U_n \subset X$ are closed. Then the sets $X \setminus U_i$ are open and hence
$$\bigcap_{i=1}^{n}(X \setminus U_i) = X \setminus \bigcup_{i=1}^{n} U_i$$
is open and it follows that $\bigcup_{i=1}^{n} U_i$ is closed.

□

Definition 3.5 (Definition 2.26 in Rudin's book)**.** *If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points(or accumulation points) of E in X, the closure of E is the set $\bar{E} = E \bigcup E'$.*

Theorem 3.6. *In any metric space, the set $\bar{B}(x,r) = \{y \in X : d(x,y) \leq r\}$ is closed.*

Proof. We can prove this theorem by proving that $X \setminus \bar{B}(x_0, r_0) = \{y \in X : d(x_0, y) > r\}$ is open.If $x \in X \setminus \bar{B}(x_0, r_0)$, then $d(x_0, x) > r$ and hence there exists $r > 0$ such that
$$d(x_0, x) > r_0 + r$$
And with triangle inequality, for $\forall y \in B(x,r)$, we have $d(x_0, y) \geq d(x, x_0) - d(x, y) > r_0$, since $d(x,y) < r$. Thus, we have $B(x,r) \subset X \setminus \bar{B}(x_0, r_0)$, which implies that $X \setminus \bar{B}(x_0, r_0)$ is open. □

Theorem 3.7. *A set $A \subset U$ is closed if and only if the following implication is true: if a sequence $\{x_n\}_{n=1}^{\infty} \in A$ such that $x_n \to x$, then $x \in A$, i.e., if for every convergent sequence of A, its limit belongs to A.*
Proof.
(1) Suppose A is closed. We need to prove that if $x_n \in A$, then $x \in A$. Suppose by contradiction that $x_n \in A \to x$, but $x \notin A$. Then, $x \in X \setminus A$. Since $X \setminus A$ is open, there exists a $\varepsilon > 0$ such that $B(x, \varepsilon) \subset X \setminus A$. Then $d(x_n, x) > \varepsilon$ for all n and thus x_n does not converges to x, which is a contradiction.
(2) Suppose that a set $A \subset X$ has the property that if $X_n \in A$ which converges to x, then $x \in A$. We need to prove that A is closed. And we only need to prove that $X \setminus A$ is open. Then, we need to prove that, for $x \in X \setminus A$
$$\exists \varepsilon > 0, B(x, \varepsilon) \subset X \setminus A$$
Suppose by contradiction that the above statement is not true, i.e.,
$$\forall \varepsilon > 0, B(x, \varepsilon) \not\subset X \setminus A$$
which means $B(x,\varepsilon) \bigcap A \neq \varnothing$. Then, taking $\varepsilon = 1/n$, $B(x, 1/n) \bigcap A \neq \varnothing$. Then we take $x_n \in B(x, 1/n) \bigcap A$. Then $x_n \in A$ and $d(x, x_n) < 1/n$, so $x_n \to x$. Since we assume $x \notin A$, and we get a contradiction. The proof is complete.

$\square$

Definition 3.6. *Let (X, d) be a metric space. We say that $x \in X$ is an accumulation point(or cluster point, or limit point) of a set $A \subset X$ if there is a sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_n \neq x$ and $x_n \to x$.*

Theorem 3.8. *$x \in X$ is an accumulation point if and only if every open set containing x contains an element of A different than x.*
Proof.

(1) First, let $x \in U$ and U is open. Then $B(x, \varepsilon) \subset U$ for some $\varepsilon > 0$. Let $x_n \in A$, $x_n \neq x$ and $x_n \to x$. Then there exists n such that $x_n \in B(x, \varepsilon) \subset U$, where $x_n \neq x$.
(2) Second, for each ball $B(x, 1/n)$, there is a $x_n \in B(x, 1/n) \bigcap A$ and $x_n \neq x$. Then it follows that $x_n \to x$. The proof is complete.

$\square$

Theorem 3.9. *A is closed if and only if all accumulation points of A belong to A.*
Proof.

(1) First, suppose A is closed, then $X \setminus A$ is open. So if $x \notin A$, then $B(x, \varepsilon) \subset X \setminus A$ for some $\varepsilon > 0$. Then $B(x, \varepsilon)$ contains no point of A and hence x cannot be an accumulation point. Therefore, every accumulation point must belong to A.
(2) Suppose all accumulation points of A belong to A. We need to show that A is closed, it suffices to prove that $X \setminus A$ is open. Let $x \in X \setminus A$, then x is not an accumulation point of A, then there exists a open set U such that $x \in U$ and U contains no point of A. Hence, $U \subset X \setminus A$, and it follows $B(x, \varepsilon) \subset U \subset X \setminus A$ for some $\varepsilon > 0$. Thus, $X \setminus A$ is open. The proof is complete.

$\square$

Theorem 3.10 (Theorem 2.27 in Rudin's book)**.** *The closure of A: cl(A) is intersection of all closed sets that contain A. Therefore, cl(A) is closed. Moreover, cl(A) is the smallest closed set that contains A in the sense that if E is another closed set such that $A \subset E$, then cl$(A) \subset E$.*
Proof.

(1) First, if $x \in X$ and $x \notin \mathrm{cl}(A)$, then x is neither a point of A nor a accumulation point of A. Hence, for x there exists a $B(x, \varepsilon) \bigcap A = \varnothing$, for some $\varepsilon > 0$. Then we have that $X \setminus \mathrm{cl}(A)$ is open, which implies $\mathrm{cl}(A)$ is closed.
(2) Second, If E is closed and $A \subset E$, since $\mathrm{cl}(A)$ is the intersection of all closed sets that contain A, and E is in the family whose intersection we take, and hence $\mathrm{cl}(A) \subset E$.

$\square$

Theorem 3.11. *The closure of $A \subset X$ is cl$(A) = \{x \in X | \exists x_n \in A, n = 1, 2, \cdots, x_n \to x\}$.*

Remark 3.1. *We do not assume $x_n \neq x$ here.*

Proof. If $x \in A$, then $x_n = x$ satisfies that $x_n \to x$. If x is an accumulation point of A, then there is a sequence $x_n \in A$ such that $x_n \to x$. Therefore, we have

$$\mathrm{cl}(A) \subset \{x \in X | \exists x_n \in A, x_n \to x\}$$

On the other hand, if $x_n \in A$ and $x_n \to x$, then either all $x_n \neq x$ and x is an accumulation point of A or $x_n = x$ for some n and $x \in A$, which implies

$$\{x \in X | \exists x_n \in A, x_n \to x\} \subset \mathrm{cl}(A)$$

The proof is complete. □

3.1. **Boundary of a set.**

Definition 3.7. *Let (X, d) be a metric space, and $A \subset X$. Boundary of A is defined as $bd(A) = cl(A) \bigcap cl(X \setminus A)$.*

Theorem 3.12. *$x \in bd(A)$ if and only if there exists a sequence in A and a sequence of $X \setminus A$ such that they both converge to x.*

Proof. It is an obvious result following the definition above. □

Another theorem follows this theorem immediately.

Theorem 3.13. *$x \in bd(A)$ if and only if $\forall \varepsilon > 0$, $B(x, \varepsilon) \bigcap A \neq \varnothing$ and $B(x, \varepsilon) \bigcap (X \setminus A) \neq \varnothing$.*

3.2. **Complete metric space.**

Definition 3.8. *Let (X, d) be a metric space. We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of X is a Cauchy sequence if $\forall \varepsilon > 0$, there $\exists N > 0$, such that $\forall n, m \geq N$, $d(x_n, x_m) < \varepsilon$.*

Definition 3.9. *A sequence $\{x_n\}_{n=1}^{\infty} \in X$ is called bounded if $x_n \in B(x_0, R)$ for some ball $B(x_0, R)$ and $\forall n = 1, 2, 3, \cdots$.*

Theorem 3.14. *Properties of Cauchy sequence.*

1. *Every convergent sequence in a metric space is a Cauchy sequence.*
2. *Every Cauchy sequence in a metric space is bounded.*
3. *If a subsequence of a Cauchy sequence in a metric space is convergent, then the whole sequence is convergent to the same limit.*

Proof.

1. (a) and (b) are obvious.
2. For (c), suppose $\{x_n\}$ is a Cauchy sequence and $x_{n_k} \to x$. We need to prove that $x_n \to x$. For $\{x_{n_k}\}$, we have $\forall \varepsilon > 0$, there exists a N_1 such that for $\forall k \geq N_1$, $d(x_{n_k}, x) < \varepsilon/2$. Also, since $\{x_n\}$ is a Cauchy sequence, then for $\forall \varepsilon > 0$, there exists a N_2 such that for $\forall n, m \geq N_1$, $d(x_n, x_m) < \varepsilon/2$. Take $N = \max\{N_1, N_2\}$, since $n_N \geq N$, we have for $n \geq N$,
$$d(x, x_n) \leq d(x_n, x_{n_N}) + d(x_{n_N}, x) < \varepsilon$$
which means $x_n \to x$. The proof is complete.

□

Definition 3.10. *We say that a metric space is complete if every Cauchy sequence is convergent. For example, $\mathbb{R}$ is complete, but $\mathbb{Q}$ is not.*

Theorem 3.15. $\mathbb{R}^n$ *is complete.*

Proof. We have

$$\begin{aligned} &\{x_k\}_k = ((x_{k1}, x_{k2}, \cdots, x_{kn})) \text{ is Cauchy sequence} \\ \Leftrightarrow &\{x_{ki}\}_k, i = 1, 2, \cdots, n \text{ is Cauchy sequence} \\ \Leftrightarrow &\{x_{ki}\}_k, i = 1, 2, \cdots, n \text{ is convergent} \\ \Leftrightarrow &\{x_k\}_k \text{ is convergent} \end{aligned}$$

The proof is complete. □

Definition 3.11. *Let $\{x_n\}$ be a sequence in a metric space. We say that $x \in X$ is a cluster point of $\{x_n\}$ if x is the limit of a sbusequence of $\{x_n\}$.*

Remark 3.2.

(1) *$x \in X$ is an accumulation point of a set $A \subset X$ if there is a sequence $\{x_n\}_{n=0}^{\infty} \in A$ such that $x_n \neq x$ and $x_n \to x$.*
(2) *$x \in X$ is a limit point of set $A \subset X$ if every neighborhood of x contains a point $y \neq x$ such that $y \in A$.*

Theorem 3.16. *The set of cluster points is closed.*

Proof. Suppose a_k is a cluster point of $\{x_n\}$ and $a_k \to a$. We need to prove that a is a cluster point of x_n. Each a_k is a cluster point of a subsequence x_{n_k}, then in any neighborhood of a there are infinitely many elements of x_n, and hence we can select a subsequence x_{n_k} that converges to a. □

Theorem 3.17. *Let $A \subset X$ be a closed subspace of a complete metric space (X, d), then (A, d) is a complete metric space as well.*

Proof. If $\{x_n\}$ is a Cauchy sequence in A, then it is a Cauchy sequence in X, so it converges to some point in X. Since A is closed, then $x \in A$, which proves that (A, d) is complete. □

Theorem 3.18. *In a metric space, $x_n \to x$ if and only if every subsequence of x_n has a further subsequence that converges to x.*

Proof. ($\Rightarrow$) This is obvious result of convergence.

($\Leftarrow$) Suppose that $\{x_n\}$ has the property above, but x_n does not converge, that is $\exists \varepsilon > 0$, for $\forall N > 0$, there exists $n \geq N$ such that $d(x_n, x) \geq \varepsilon$. Thus we can pick a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, x) \geq \varepsilon$$

Clearly, we can know $\{x_{n_k}\}$ has no sequence converging to x, which is a contradiction. □

3.3. **Compact spaces.**

Definition 3.12. *We say that a subset $A \subset X$ of a metric space is compact if every sequence in A has a subsequence converging to a point in A.*

Definition 3.13 (Definition 2.31 & 2.32 in Rudin's book)**.**

1. *By an open cover of a set A in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $A \subset \bigcup_\alpha G_\alpha$.*
2. *A subset A of a metric space X is said to be compact if every open vector of A contains a finite subcover.*

Proposition 3.1. *If A is compact, then A is bounded and closed.*

Proof. Let $A \subset X$ be compact. To prove that A is closed we need to prove that

$$A \ni x_n \to x \Rightarrow x \in A$$

Since $x_n \in A$ and A is compact, then it has a subsequence $\{x_{n_k}\}$ convergenting to a point x, then clearly $x \in A$.

Now we prove that A is bounded. Suppose A is not bounded, we fixed $x_0 \in X$ and find a sequence $\{x_n\} \in A$ such that $d(x_n, x_0) \geq n$. Then no subsequence of $\{x_n\}$ converges, which is a contradiction. $\square$

Theorem 3.19 (Heine-Borel Theorem)**.** *$A \subset \mathbb{R}^n$ is compact if and only if A is bounded and closed.*
Proof.

1. ($\Rightarrow$) This is a result of proposition above.
2. ($\Leftarrow$) We prove it for $n = 3$. Let $x_k \in A, k = 1, 2, 3, \cdots$. Since A is bounded, we can know that all three elements of $x_k = (x_{1k}, x_{2k}, x_{3k})$ are bounded in $\mathbb{R}$. Then the sequence $\{x_{1k}\}_k$ is bounded, so Bolzano-Weierstrass theorem, it has a convergent subsequence. Then $x_{1k_n} \to x_1$ and $x_1 \in A$ since A is closed. Similiarly, $\{x_{2k_n}\}$ also has a convergent subsequence $\{x_{2k_{n_m}}\}$ converging to $x_2 \in A$, and $\{x_{3k_{n_m}}\}$ also has a convergent subsequence $\{x_{3k_{n_{m_l}}}\}$ converging to $x_3 \in A$. Thus, we have

$$x_{k_{n_{m_l}}} = \left(x_{1k_{n_{m_l}}}, x_{2k_{n_{m_l}}}, x_{3k_{n_{m_l}}}\right) \to (x_1, x_2, x_3) \in A$$

Then A is compact.

$\square$

Definition 3.14. *Let (X, d) be a metric space and $A \subset U$ a subset. We say that a family of open sets $\{U_i\}_{i \in I}$ forms a open covering of A if $A \subset \bigcup_{i \in I} U_i$. Now, $\{U_{i_k}\}_{i_k \in I}, k = 1, 2, \cdots, N$ forms a finite subcovering of A if $A \subset \bigcup_{i_k=1}^{N} U_{i_k}$.*

Theorem 3.20 (Bolzano-Weierstrass theorem)**.** *Let (X, d) be a metric space and $A \subset X$ a subset. Then (X, d) is compact if and only if every open covering of A has a finite subcovering.*

Remark 3.3. *We denote balls in metric space (X, d) and (A, d) by $B^X(x, r)$ and $B^A(x, r)$ respectively. Clearly, $B^A(x, r) = B^X(x, r) \bigcap A$.*

Before we prove the Bolzano-Weierstrass theorem, we need to mention some other theorem and lemma.

Theorem 3.21 (Theorem 2.30 in Rudin's book)**.** *$U \subset A$ is open in (A, d) if and only if there is $W \subset X$ open in (X, d) such that $U = W \bigcap A$.*
Proof.

(1) ($\Leftarrow$) Suppose $W \subset X$ open in X and $U = W \bigcap A$. Then for $\forall x \in U$, there exists a $r > 0$ such that $B^X(x, r) \subset W$ and hence $B(x, r)^X \bigcap A \subset U \bigcap A = U$, where $B(x, r)^X \bigcap A$ is a ball in A. Then U is open in (A, d).
(2) ($\Rightarrow$) Suppose $U \subset A$ is open in (A, d). Then for $\forall x \in U$, there exists a $r > 0$ such that $B^X(x, r) \bigcap A \subset U$, where $B^X(x, r) \bigcap A$ is a ball in A. Clearly, $U = \bigcup_{x \in X} B^X(x, r) \bigcap A$.
Now we set $W = \bigcup_{x \in X} B^X(x, r)$ and W is open in (X, d) and

$$W \bigcup A = \bigcup_{x \in X} B^X(x, r) \bigcap A = U$$

The proof is complete.

□

Corollary 3.21.1. *$E \subset A$ is closed in (A, d) is and only if there is a set $F \subset X$ closed in (X, d) such that $E = F \bigcap A$.*

Theorem 3.22. *A metric space X is compact if and only if every open covering of X has a finite subcovering.*
Proof.

(1) ($\Leftarrow$) Suppose that every open covering of X has a finite subcovering. We need to prove that every sequence $\{x_n\}$ in X has a convergent subsequence.

By contradiction, we suppose that $\{x_n\} \in X$ does not has convergent subseqnece(Note that $\{x_n\}$ has infinitely many different values, otherwise we would have a constant, and thus convergent subsequence). Therefore we can select a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \neq x_{n_l}$ for $k \neq l$ and $\{x_{n_k}\}$ does not converge. Observe that the set $\{x_{n_1}, x_{n_2}, x_{n_3}, \cdots\}$ is closed and the set has no accumulation point.

In particular, every x_{n_k} is not a accumulation point of the sequence $\{x_{n_k}\}$, and hence there is a $\varepsilon_k > 0$ such that the ball $B(x_{n_k}, \varepsilon_k)$ contains no points of this sequence other than x_{n_k}, i.e.,

$$x_{n_l} \notin B(x_{n_k}, \varepsilon_k), \text{if}\, l \neq k$$

Clearly, $X \setminus \{x_{n_1}, x_{n_2}, x_{n_3}, \cdots\}$ is open and hence

$$X = \bigcup_{k=1}^{\infty} B(x_{n_k}, \varepsilon_k) \cup (X \setminus \{x_{n_1}, x_{n_2}, x_{n_3}, \cdots\})$$

is an open covering of X. Thus, this covering has no finite subcovering.
(2) ($\Rightarrow$) We need a lemma.

Lemma 3.23. *Let $\bigcup_{i \in I} U_i$ be an open covering of a compact metric space of X such that $X = \bigcup_{i \in I} U_i$. Then there is $r > 0$(called Lebesgue number of the covering) such that $\forall x \in X$, $\exists i \in I$ such that $B(x, r) \subset U_i$.*

Proof. Prove by contradiction. Then we can find $x_n \in X$ such that $B(x_n, 1/n)$ is not contained in any of the open set U_i. Since X is compact, $\{x_n\}$ has a cinvergent subsequence $x_{n_k} \to x_0 \in U_{i_0}$ for some $i_0 \in I$. Then we have $B(x_0, \varepsilon) \subset U_{i_0}$ for some $\varepsilon > 0$. Since $x_n \to x_0$, it is clear that $B(x_n, 1/n) \subset B(x_0, \varepsilon)$, which is a contradiction. □

We need one definition and one more lemma.

Definition 3.15. *A metric space X is said totally bounded if $\forall \varepsilon > 0$, there exists a finite covering of X by balls of radius ε.*

Lemma 3.24. *If a metric space X is compact, then x is totally bounded.*

Proof. Prove by contradiction. Then there is a $\varepsilon > 0$ such that no finite family of balls with radius ε covers X. Let $x_1 \in X$, then $B(x_1, \varepsilon) \neq X$. Then there exists $x_2 \notin B(x_1, \varepsilon)$, and $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \neq X$. Then there exists $x_3 \notin B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$, and $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \neq X$. We can continue this process and construct a sequence $\{x_1, x_2, x_3, \cdots\}$ in X such that $d(x_k, x_l) \geq \varepsilon$ for $k \neq l$. Clearly, $\{x_n\}$ has no convergent subsequence, which is contradicted with compactness of X. □

Now we continue the proof of the implication ($\Rightarrow$) of theorem 3.22.
($\Rightarrow$) Suppose that X is compact, i.e., every sequence in x has a convergent subsequence. We need to prove that every open covering of X has a finite subcovering. Let $X = \bigcup_{i \in I} U_i$ and let $r > 0$ be a Lebesgue number of the covering. Since X is totally bounded, X has finite coverings by balls of radius r, i.e.,

$$X = \bigcup_{i=1}^{N} B(x_i, r)$$

By the definition of Lebesgue number, we have $B(x_i, r) \subset U_{k_i}$. Then,

$$X = \bigcup_{i=1}^{N} B(x_i, r) \subset \bigcup_{i=1}^{N} U_{k_i}$$

which gives a finite open subcovering of X. The proof of theorem 3.22 is complete. This also completes the proof is Bolzano-Weierstrass theorem.

□

Theorem 3.25. *A metric space X is compact if and only if it is complete and totally bounded.*
Proof.

(1) ($\Rightarrow$) Suppose X is compact. Then X is totally bounded. To prove it is complete, let $\{x_n\}$ be a Cauchy sequence in X. The by compactness, we can know some subsequence of $\{x_{n_k}\}$ is convergent, such that $x_{n_k} \to x_0$. Then $\{x_n\}$ also converges to x_0. Thus, X is complete.
(2) ($\Leftarrow$) Suppose that X is complete and totally bounded. We need to prove that every sequence $\{x_n\}$ in X has a convergent subsequence.

Since X is totally bounded , then it has a finite open covering of balls with radius 1, i.e.,

$$X = \bigcup_{i=1}^{N_1} B(x_i^{(1)}, 1)$$

Hence, infinitely many elements of sequence $\{x_n\}$ belong to at least one of the balls, saying $B(x_{i_1}^{(1)}, 1)$. This ball has a finite open covering by balls with radius of $\frac{1}{2}$, i.e.,

$$B(x_{i_1}^{(1)}, 1) = \bigcup_{i=1}^{N_2} B\left(x_i^{(2)}, \frac{1}{2}\right) \cap B(x_{i_1}^{(1)}, 1)$$

Still, infinitely many elements of the sequence belong to at least one of the set on the right hand side, saying $B\left(x_{i_2}^{(2)}, \frac{1}{2}\right) \cap B(x_{i_1}^{(1)}, 1)$. And this set has a finite open covering by balls of radius of $\frac{1}{3}$, i.e.,

$$B\left(x_{i_2}^{(2)}, \frac{1}{2}\right) \cap B(x_{i_1}^{(1)}, 1) = \bigcup_{i=1}^{N_3} B\left(x_i^{(3)}, \frac{1}{3}\right) \cap B\left(x_{i_2}^{(2)}, \frac{1}{2}\right) \cap B(x_{i_1}^{(1)}, 1)$$

Still, infinitely many elements of the sequence belong to at least one of the set on the right hand side, and we continue this process and pick a subsequence from the given sequence $\{x_n\}$ such that

$$\begin{aligned}
x_{n_1} &\in B(x_{i_1}^{(1)}, 1) \\
x_{n_2} &\in B\left(x_{i_2}^{(2)}, \frac{1}{2}\right) \cap B(x_{i_1}^{(1)}, 1) \\
x_{n_3} &\in B\left(x_{i_3}^{(3)}, \frac{1}{3}\right) \cap B\left(x_{i_2}^{(2)}, \frac{1}{2}\right) \cap B(x_{i_1}^{(1)}, 1) \\
&\vdots
\end{aligned}$$

Therefore, for $k, ; l \geq N$, we have $x_{n_k}, x_{n_l} \in B\left(x_{i_N}^{(2)}, \frac{1}{N}\right)$ and hence

$$d(x_{n_k}, x_{n_l}) < \frac{2}{N}$$

Thus the sequence is Cauchy sequence and therefore convergent, it follows that X is complete.

□

Theorem 3.26. *If $F_k \subset X$ are nonempty compact sets such that $F_1 \supset F_2 \supset F_3 \supset \cdots$, then $\bigcap_{k=1}^{\infty} F_k \neq \varnothing$.*

Proof. Let $x_k \in F_k$ for $k = 1, 2, 3, \cdots$. Then $X_n \in F_k$ for all $n \geq k$. In particular, $x_n \in F_1$ for all n. Hence x_n has a convergent subsequence in F_1(F_1 is compact), saying $x_{n_l} \to x_0 \in F_1$. Now, we have $x_{n_l} \in F_k$ for all $n_l \geq k$, and hence the limit x_0 of $\{x_{n_l}\}$ must belong to every F_k, i.e., $x_0 \in \bigcup_{k=1}^{\infty} F_k$ which proves the intersection is not empty. □

Remark 3.4. *The claim is not true if F_k are open or closed but unbounded, for example*

$$\bigcap_{k=1}^{\infty}\left(0,\frac{1}{k}\right)=\varnothing$$
$$\bigcap_{k=1}^{\infty}[k,\infty)=\varnothing$$

3.4. **Cantor set.**

Definition 3.16. *By the segment (a,b) we mean the set of all real numbers x such that $a<x<b$. By the interval $[a,b]$ we mean the set of all real numbers x such that $a\le x\le b$.*

The set which we are now constructing shows that there exists perfect set in $\mathbb{R}^1$ which contain no segment.

(1) Let E_0 be the interval $[0,1]$.
(2) Remove the segment $\left(\frac{1}{3},\frac{2}{3}\right)$, and let E_1 be the union of the interval of

$$\left[0,\frac{1}{3}\right],\left[\frac{1}{3},1\right].$$

(3) Remove the middle thirds of these two intervals and let E_2 be the union of

$$\left[0,\frac{1}{9}\right],\left[\frac{2}{9},\frac{1}{3}\right],\left[\frac{6}{9},\frac{7}{9}\right],\left[\frac{8}{9},1\right].$$

(4) Continue this way and we can get a sequence of compact sets E_n, such that $E_1\supset E_2\supset E_3\supset\cdots$ and E_n is the union of 2^n intervals with length 3^{-n}.

We show the Cantor set in seven iterations as below:

Figure 3. Cantor set in seven iterations

The set

$$P=\bigcap_{n=1}^{\infty}E_1$$

is called the Cantor set. P is clearly compact and P is not empty. And one can prove that P is uncountable.

3.5. **Connected sets.**

Definition 3.17.

(1) *$\varphi : [a, b] \to X$ is called continuous if $x_n \to t$, then $\varphi(x_n) \to \varphi(x)$ for every sequence $\{x_n\}$ in $[a, b]$.*
(2) *If $A \subset X$ is a subset of a metric space, a continuous path connecting $x, y \in A$ inside A is any continuous function $\varphi : [a, b] \to A$ such that $\varphi(a) = x, \varphi(b) = y$.*
(3) *A set A is called path connected if every two points in A can be connected by a continuous path inside A.*

Definition 3.18. *A set $A \subset X$ is called disconnected if there exists open sets U, V in X such that*

(1) *$A \subset U \cup V$;*
(2) *$A \cap U \neq \varnothing$ and $A \cap V \neq \varnothing$;*
(3) *$A \cap (U \cap V) = \varnothing$*

Moreover, A is called connected if it is not disconnected.

Disconnected set is shown as below, A is the grey area, which is contained in two open sets U and V, which are plotted by dotted line:

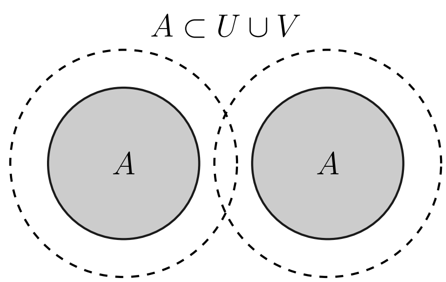

FIGURE 4. Disconnected set

Exercise 3.1. *Prove that the space X is connected if and only if the only subsets of X that are open and closed at the same time are $\varnothing$ and X.*

Proof.

(1) ($\Rightarrow$) Suppose X is connected. We want to show that if $E \subset X$ is open and closed at the same time, then $E = X$ or $E = \varnothing$.

Suppose by contraction that there exists a $E \subset X$ such that $E \neq X$, $E \neq \varnothing$ and E is open and closed at the same time. Then $U = E$, $V = X \setminus E$ are both open. And we have $X \subset U \cup V$, $X \cap U \neq \varnothing$, $X \cap V \neq \varnothing$, $X \cap U = V \neq \varnothing$ and $X \cap (U \cap V) = X \cap E \cap (X \setminus E) = \varnothing$. Thus, X is disconnected, which is a contradiction.

(2) ($\Leftarrow$) Suppose that the only subsets of X that are open and closed at the same time are $\varnothing$ and X. We need to show that X is connected.

By contradiction that X is not connected. Then, there exists two open sets U and V such that $X = U \cup V$, $X \cap U \neq \varnothing$, $X \cap V \neq \varnothing$, $X \cap U = V \neq \varnothing$. Then $U \neq \varnothing$ and it is also closed since $X \setminus U = V$ is open, then this is a contradiction with the fact that X and $\varnothing$ are only sets that are open and closed at the same time.

$\square$

Theorem 3.27. *If $A \subset X$ is path connected, then A is connected.*

Proof. Prove by contradiction and suppose A is path connected but not connected. Then $A \subset U \cup V$, $A \cap U \neq \varnothing$, $A \cap V \neq \varnothing$, and $A \cap (U \cap V) = \varnothing$.

Let $x \in A \cap U$, $y \in A \cap V$, and let $\varphi : [a, b] \to A$ be a continuous path such that $\varphi(a) = x, \varphi(b) = y$. Define $c = \sup\{t \in [a, b] | \, \varphi([a, t]) \subset A \cap U\}$. If $\varphi(c) \in A \cap U$, then $c \neq b$ (otherwise it would implies $\varphi(c) = \varphi(b) = y \in A \cap U$, which is impossible).

Since $\varphi(c) \in A \cap U$ and $c < b$, it follows from the continuity of φ that there exists a $\varepsilon > 0$ such that $\varphi([c, c + \varepsilon]) \subset U$, which contradicts the definition of c. Therefore, $\varphi(c) \in A \cup V$. Then by the same argument as above, that there exists a $\varepsilon > 0$ such that $\varphi([c - \varepsilon, c]) \in V$ implying $\varphi(c - \varepsilon) \in V$, which contradicts the definition of c. $\square$

Theorem 3.28. *If A is connected, then the closure $cl(A)$ of A is also connected.*

Proof. Prove by contradiction and suppose that A is connected while $\mathrm{cl}(A)$ is not closed. Then there exists open sets U, V such that $\mathrm{cl}(A) \subset U \cup V, \mathrm{cl}(A) \cap U \neq \varnothing, \mathrm{cl}(A) \cap V \neq \varnothing$, and $\mathrm{cl}(A) \cup (U \cup V) = \varnothing$. Then we have

(1) $A \subset U \cup V$.
(2) $A \cup U \neq \varnothing$.
(3) $A \cup V \neq \varnothing$.
(4) $A \cup (U \cup V) = \varnothing$.

which means that A is disconnected. Contradiction.

The properties (1) and (4) are obvious. Now we prove property (2). Since $\mathrm{cl}(A) \cap U \neq \varnothing$, then there is a $x \in \mathrm{cl}(A) \cap U$. Hence there is a sequence $\{x_n\} \in A$ such that

$$x_n \to x \in \mathrm{cl}(A) \cap U$$

Since U is open, then there exists sufficiently large $N > 0$ such that $\forall n > N$, $x_n \in A \cap U$, which proves that $A \cap U \neq \varnothing$. $\square$

Example 3.1. The graph of the function $y = \sin \frac{1}{x}, 0 \leq x \leq \pi$, i.e., the set

$$G = \left\{ (x, \sin \frac{1}{x}) | \, 0 < x \leq \pi \right\}$$

is path connected, hence G is connected.

Therefore, $\mathrm{cl}(G)$ is connected. However, $\mathrm{cl}(G)$ is the union of G and the segment $[-1, 1]$ on the $y-$axis. It is clear that this set is not path connected. $\mathrm{cl}(G)$ is an example of a set which is connected but not path connected. The plot of $y = \sin \frac{1}{x}, 0 \leq x \leq \pi$ is shown as below:

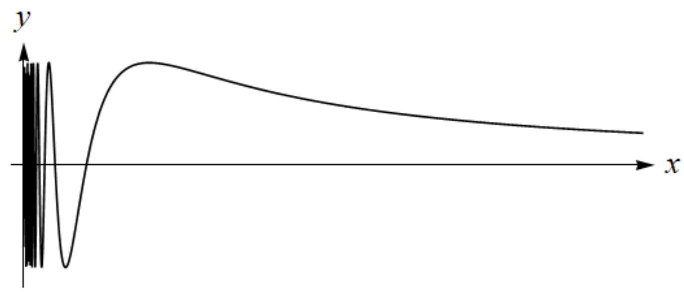

FIGURE 5. $y = \sin(1/x)$

Theorem 3.29 (Theorem 2.47 in Rudin's book)**.** *A subset E of the real line $\mathbb{R}^1$ is connected if and only if it has the following property: If $x \in E, y \in E$ and $x < z < y$, then $z \in E$.*
Proof.

(1) ($\Rightarrow$) Prove by contradiction and suppose that if there exists a $z \in (x, y)$ and $z \in E$, then $E = U \cup V$, where

$$U = E \cap (-\infty, z), V = E \cap (x, \infty)$$

Since $x \in U, y \in V$, U and V are not nonempty. Since $U \subset (-\infty, z)$ and $V \subset (z, \infty)$, then they are separated. Also, we can have $E \cap (U \cap V) = \varnothing$, which means E is not connected. This is a contradiction.

(2) ($\Leftarrow$) Supposed that E is not connected. Then there exist sets U and V separating E such that $E \subset U \cup V, E \cap U \neq \varnothing, E \cap V \neq \varnothing$ and $E \cap (U \cap V) = \varnothing$. Now we pick $x \in U$ and $y \in V$ and without losing generality, assume that $x < y$. Define

$$z = \sup\{(U \setminus V) \cap [x, y]\}$$

Then $z \in \mathrm{cl}(U)$ and $z \notin V$. In particular, $x \leq z < y$. If $z \notin U \setminus V$, then z certainly does not belong to E. If $z \in U \setminus V$, then $z \notin \mathrm{cl}(V)$, hence there exists a z_1 such that $z < z_1 < y$ and $z_1 \notin V$. Then $x < z_1 < y$ and $z \notin E$. This is a contradiction.

□

3.6. **Continuity.**

Definition 3.19. *Let $(X, d), (Y, \rho)$ be two metric spaces and $A \subset X$. Consider a mapping $f : A \to Y$. If x_0 is an accumulation point of A, then we say that $\lim_{x \to x_0} f(x) = b \in Y$, if*

$$\forall \varepsilon > 0, \exists \delta > 0, 0 < d(x, x_0) < \delta \Rightarrow \rho(f(x), b) < \varepsilon$$

Equivalently, $\lim_{x \to x_0} f(x) = b$, if

$$x_0 \neq x_n (x_n \in A) \to x_0 \Rightarrow f(x_n) \to b$$

Definition 3.20. *We say that a mapping $f : A \to Y$ is continuous at point $x_0 \in A$, if $A \ni x_n \to x_0$ then $f(x_n) \to f(x_0)$(no need of condition: $x_n \neq x_0$).*

If x_0 is not an accumulation point of A, i.e., x_0 is an isolated point, then f is always continuous at x_0.

Definition 3.21. *We say that $f : A \to Y$ is continuous if f is continuous at every point in A. Equivalently, $f : A \to Y$ is continuous if for every $x \in A$ and $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that*

$$d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon.$$

This is Definition 4.5 in Rudin's book.

Example 3.2.

(1) If X is a discrete metric space, then every function $f : X \to Y$ is continuous.
(2) $f : \mathbb{Z} \to Y$ or $f : \mathbb{N} \to Y$ is always continuous.

Theorem 3.30 (Theorem 4.8 in Rudin's book)**.** *A mapping $f : X \to Y$ is continuous if and only if for every open set $U \subset Y$, $f^{-1}(U)$ is an open set in X.*

Proof.

(1) ($\Rightarrow$) Suppose f is continuous and $U \subset Y$ is a open set. We want to prove that $f^{-1}(U)$ is open.

If $x \in f^{-1}(U)$, then $f(x) \in U$ and hence $B(f(x), \varepsilon) \subset U$ for some $\varepsilon > 0$. It follows from the continuity of f there exists a $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$, if $d_X(x, y) < \delta$. Hence, $B(x, \delta) \subset f^{-1}(U)$ and then $f^{-1}(U)$ is open.

(2) ($\Leftarrow$) Suppose now that $f^{-1}(U)$ is open for every open set $U \subset Y$. We need to prove that f is continuous.

Let $\varepsilon > 0$ be given, then $B(f(x_0), \varepsilon)$ is open. Hence $x_0 \in f^{-1}(B(f(x_0), \varepsilon))$, which is open. Then there exists a $\delta > 0$ such that $B(x_0, \delta) \subset f^{-1}(B(f(x_0), \varepsilon))$. And hence if $d_X(x_0, x) < \delta$, then $d_Y(f(x_0), f(x)) < \varepsilon$, which proves the continuity of f.

□

Definition 3.22. *We say that a mapping $f : X \to Y$ is $L-$ Lipschitz if $\forall x, y \in X$, $d_Y(f(x), f(y)) \leq L d_X(x, y)$.*

We say that a mapping $f : X \to Y$ is Lipschitz if it is $L-$ Lipschitz for some $L > 0$.

Proposition 3.2. *Every Lipschitz mapping is continuous.*

Proof. Suppose $f : X \to Y$ is Lipschitz, then for $x_n \to x$, we have

$$d_Y(f(x_n), f(x)) \leq L d_X(x_n, x) \to 0$$

and hence $f(x_n) \to f(x)$. □

Now we use different method to prove this theorem by showing that $f^{-1}(U)$ is open for every open set $U \subset Y$.

Proof. Let $U \subset Y$ be given, and let $x \in f^{-1}(U)$. We will prove that

$$B\left(x, \frac{\varepsilon}{L}\right) \subset f^{-1}(U)$$

where ε is taken such that $B(f(x), \varepsilon) \subset U$. Now we have

$$y \in B\left(x, \frac{\varepsilon}{L}\right) \Rightarrow d_X(x, y) < \frac{\varepsilon}{L}$$

with f being Lipschitz mapping, we have

$$d_Y(f(x), f(y)) \le L d_X(x, y) \le \varepsilon$$

then $f(y) \in B(f(x), \varepsilon) \subset U$. Then $B\left(x, \frac{\varepsilon}{L}\right) \in f^{-1}(U)$. $\square$

Example 3.3 (Riemann function)**.** f is defined as below

$$f(x) = \begin{cases} 0, \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \\ \dfrac{1}{q}, \text{if} \quad x \in \mathbb{Q}, x = \dfrac{p}{q}, q > 0 \end{cases}$$

and where the greatest common divisor of p and q is 1. Then f is continuous at all irrational points and discontinuous at all rational points.

Example 3.4. $f(x, y)$ is defined as below

$$f(x) = \begin{cases} 0, \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \\ 1, \text{if} \quad x \in \mathbb{Q} \end{cases}$$

and f is discontinuous everywhere.

Remark 3.5. *Practical way of proving continuity of a function of two(or more) variables at a given point is based on the following observation: If*

$$|f(x, y) - L| \le g(x, y) \to 0$$

where $(x, y) \to (x_0, y_0), (x, y) \neq (x_0, y_0)$*, then* $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$.

Theorem 3.31. *If* $f : X \to Y$ *is continuous and* $A \subset X$ *is compact, then* $f(A) \subset Y$ *is compact.*

Proof. Let $y_n \in f(A), n = 1, 2, \cdots$ be a sequence. Then $y_n = f(x_n)$ for some $x_n \in A$. Since A is compact, then there is a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0 \in A$. Hence, $y_{n_k} = f(x_{n_k}) \to f(x_0) \in f(A)$. $\square$

Theorem 3.32. *If* $f : X \to Y$ *is continuous and* $A \subset X$ *is connected,then* $f(A) \subset Y$ *is connected.*

Proof. Suppose $f(A)$ is not connected. Then there exist open sets U and V such that $f(A) \subset U \cup V, f(A) \cap U \neq \varnothing, f(A) \cap V \neq \varnothing$ and $f(A) \cap (U \cap V) = \varnothing$.

Then we have $A \subset f^{-1}(U), A \subset f^{-1}(U), A \cap f^{-1}(U) \neq \varnothing, A \cap f^{-1}(V) \neq \varnothing$ and $A \cap (f^{-1}(U) \cap f^{-1}(V)) = \varnothing$, which implies that A is disconnected. This is a contradiction. $\square$

Theorem 3.33. *If* $f : X \to Y$ *is continuous and* $A \subset X$ *is path connected,then* $f(A) \subset Y$ *is also path connected.*

Proof. Let $y_1, y_2 \in f(A)$. Then $y_1 = f(x_1), y_2 = f(x_2)$ for some $x_1, x_2 \in A$. Since A is connected, then there exists a continuous mapping $\varphi : [a, b] \to A$, such that $\varphi(a) = x_1$ and $\varphi(b) = x_2$. Then we have $\psi = f \circ \varphi : [a, b] \to f(A)$ and $\psi(a) = y_1$, $\psi(b) = y_2$, this is continuous path connecting y_1 and y_2 in $f(A)$. Thus, $f(A)$ is path connected. $\square$

Theorem 3.34. *If $f : A \to \mathbb{R}$ is continuous, where $A \subset X$ is compact, then f attains maximum and minimum in A, i.e.,*

$$\exists x_1 \in A, \forall x \in A, f(x_1) \geq f(x)$$
$$\exists x_2 \in A, \forall x \in A, f(x_2) \leq f(x)$$

Proof. There is a sequence $\{x_n\} \in A$ such that $f(x_n) \to \sup\{f(x)|x \in A\}$. Since A is compact, then there exists a subsequence $\{x_{n_k}\}$ of x_n such that $x_{n_k} \to x_1$. Then we have $f(x_1) = \sup\{f(x)|x \in A\}$.

Similar argument works with minimum. □

Corollary 3.34.1. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded.*

Example 3.5. Consider $f(x) = \frac{1}{x}, x \in (0, 1]$ is continuous but unbounded on its domain, since $(a, b]$ is not closed, so the corollary does not apply.

Now we classify all connected subsets.

Lemma 3.35. *If $A \subset X$ is connected, and $a < c < b, a, b \in A$, then $c \in A$.*

Proof. By contradiction and suppose $c \notin A$. Then we can know that $A \subset (-\infty, c) \cup (c, \infty), A \cap (-\infty, c) \neq \varnothing, A \cap (c, \infty) \neq \varnothing$ and $A \cap ((-\infty, c) \cap (c, \infty)) = \varnothing$. Thus, A is disconnected, which is a contradiction. □

Theorem 3.36. *$A \subset \mathbb{R}$ is connected if and only if A is an interval.*
Proof.

(1) ($\Leftarrow$) Any interval is connected since it is path connected.
(2) ($\Rightarrow$) Let $a = \inf\{x|x \in A\}$ and $b = \sup\{x|x \in A\}$. Then there exist sequence $\{a_n\} \in A$ and $\{b_n\} \in A$ such that $a_n \to a$ and $b_b \to b$. It follows from Lemma 3.35 that $[a_n, b_n] \subset A$. Hence $(a, b) = \bigcup_{n=1}^{\infty} [a_n, b_n] \subset A$. With the definition of a and b that no number less than a belongs to A and no number bigger than b belongs to A. Thus, A is an interval with endpoints a and b.

□

Theorem 3.37 (Intermediate Value Theorem)**.** *If $f : A \to \mathbb{R}$ is continuous, $A \subset X$ is connected and $f(x) = a < c < b = f(y)$, then there exists a $z \in A$ such that $f(z) = c$.*

Proof. Since f is continuous, then $f(A)$ is an interval that contains a and b, so it must contains c. □

Theorem 3.38. *Let $f : A \to \mathbb{R}^n$, $f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$. Then f is continuous if and only if the functions f_i are continuous for $i = 1, 2, \cdots, n$.*

Proof. If $x_k \to x_0$, then $(f_1(x_k), f_2(x_k), \cdots, f_n(x_k)) \to (f_1(x_0), f_2(x_0), \cdots, f_n(x_0))$ if $f_i(x_k) \to f_i(x_0)$. □

Now we show a new method to prove the Arithmetic-Geometric mean inequality.

Theorem 3.39 (Arithmetic-Geometric mean inequality)**.** *Let $x_1, \cdots, x_n \geq 0$, then*

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}$$

and the equality holds if and only if $x_1 = \cdots = x_n$.

Proof. Let $x_1 + \cdots + x_n = a$. We assume that $a > 0$ or otherwise the inequality is obvious.

Consider the set $A = \{(z_1, \cdots, z_n) \in \mathbb{R}^n | z_1 + \cdots + z_n = a, z_1, \cdots, z_n \geq 0\}$. The set is bounded and closed, and therefore compact. Hence the function $f(z_1, \cdots, z_n) = \sqrt[n]{z_1, \cdots, z_n}$ has maximum in A at some point $(z_1^0, \cdots, z_n^0) \in A$. We will prove that $z_1^0 = \cdots = z_n^0$.

Observe first that $z_1^0, \cdots, z_n^0 \geq 0$ because 0 cannot be maximum of f. Suppose that $z_i^0 \neq z_j^0$ for some $i \neq j$, then

$$\left(\frac{z_i^0 + z_j^0}{2}\right)^2 > z_i^0 z_j^0$$

and therefore

$$\sqrt[n]{z_1^0 \cdots \left(\frac{z_i^0 + z_j^0}{2}\right) \cdots \left(\frac{z_i^0 + z_j^0}{2}\right) \cdots z_n^0} > \sqrt[n]{z_1^0 \cdots z_j^0 \cdots z_j^0 \cdots z_n^0}$$

and $\left(z_1^0, \cdots, \frac{z_i^0+z_j^0}{2}, \cdots, \frac{z_i^0+z_j^0}{2}, \cdots, z_n^0\right) \in A$, this is contradicted with the definition of $(z_1^0, \cdots, z_j^0, \cdots, z_j^0, \cdots, z_n^0)$. Thus $z_1^0 = \cdots = z_n^0 = \frac{a}{n}$, and we have

$$\sqrt[n]{x_1 \cdots x_n} \leq \sqrt[n]{z_1^0 \cdots z_n^0} = \frac{a}{n} = \frac{x_1 + \cdots + x_n}{n}$$

□

3.7. **Uniform continuity.**

Definition 3.23. *$f : A \to Y, A \subset X$ is called uniformly continuous, if for $\forall \varepsilon > 0$, there exists a $\delta > 0$ such that for $\forall x, y \in A$, if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.*

Proposition 3.3. *If $f : (a, b) \to \mathbb{R}$ is uniformly continuous, where (a, b) is a bounded interval, then f is bounded.*

Proof. By the definition of uniformly continuity, we take $\varepsilon = 1$. Then there is a $\delta > 0$ such that if $|x - y| \leq \delta$, then $|f(x) - f(y)| < 1$. Choose N such that $N\delta > \frac{b-a}{2}$.

Let $x \geq \frac{a+b}{2}$, consider a sequence $x_k = \frac{a+b}{2} + \delta k, k = 0, 1, 2 \cdots$. Then there is $n < N$ such that $x_n \leq x$ but $x_{n+1} > x$. Thus we have

$$\begin{aligned}
\left|f(x) - f\left(\frac{a+b}{2}\right)\right| &= |(f(x_n) - f(x_{n_1})) + (f(x_{n_1}) - f(x_{n_2})) + \cdots + (f(x_1) - f(x_{n_0}))| \\
&\leq |f(x_n) - f(x_{n_1})| + |f(x_{n_1}) - f(x_{n_2})| + \cdots + |f(x_1) - f(x_{n_0})| \\
&\leq n + 1 \leq N
\end{aligned}$$

which proves f is bounded. □

Exercise 3.2. *Suppose $f : A \to \mathbb{R}$ is uniformly continuous, where $A \subset X$ is a bounded subset of a metric space. Dose it follow that f is bounded?*

No. And we will provide a counterexample.

Proof. If X is a discrete metric space, then any function $f : X \to \mathbb{R}$ is uniformly continuous. Indeed, let arbitrary $\varepsilon > 0$, then take $\delta = 1$, then if $d_X(x, y) < \delta = 1$, it follows that $x = y$ and $|f(x) - f(y)| = 0 < \varepsilon$.

Now take $X = \mathbb{Z}$ with the metric metric. Then X is bounded and and $f : X \to \mathbb{R}, f(n) = n$. Then f is uniformly continuous, but not bounded. $\square$

Definition 3.24. *We say that a function $f : A \to Y$, where $A \subset X$ is $\alpha-$Hölder continuous, $\alpha > 0$, if $\exists c > 0$ such that*

$$d_Y(f(x), f(y)) \leq c d_X(x, y)^\alpha$$

for all $x, y \in A$.

Theorem 3.40. *If $f : A \to Y$ is continuous and $A \subset X$ is compact, then f is uniformly continuous.*

Proof. Prove by contradiction, and suppose $\exists \varepsilon > 0$, $\forall \delta > 0$, there exist $x, y \in A$ such that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$.

In particular, there exist sequences $\{x_n\}, \{y_n\} \in A$ such that $d_X(x_n, y_n) < \frac{1}{n}$ and $d_Y(f(x_n), f(y_n)) \geq \varepsilon$. Since A is compact, there is a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0 \in A$. Since $d_X(x_{n_k}, y_n) < \frac{1}{n_k}$, we also have $y_{n_k} \to x_0$. Hence, with f being continuous, $f(x_{n_k}) \to f(x_0)$ and $f(y_{n_k}) \to f(x_0)$. Therefore, $d_Y(f(x_{n_k}), f(Y_{n_k})) \to 0$, which contradicts the inequality $d_Y(f(x_{n_k}), f(Y_{n_k})) > \varepsilon$. $\square$

Recall that for $f : \mathbb{R}^n \to \mathbb{R}^m$, the graph is defined by

$$G_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m| = \mathbb{R}^{n+m}|y = f(x)\}$$

Exercise 3.3. *Prove that if f is continuous, then G_f is a closed subset of $\mathbb{R}^{n+m}$. Is converse implication true?*
Proof.

(1) Let f be continuous. We need to prove that if $G_f \ni (x_k, y_k) \to (x_0, y_0)$, then $(x_0, y_0) \in G_f$. We have $(x_k, y_k) \in G_f$, then $y_k = f(x_k)$. Now we have $x_k \to x_0 \in \mathbb{R}$, then $f(x_k) \to y_0$. Since f is continuous, then $f(x_k) \to f(x_0)$. Hence $y_0 = f(x_0)$ and therefore $(x_0, y_0) = (x_0, f(x_0)) \in G_f$.
(2) The converse implication is false. For example, consider function

$$f(x) = \begin{cases} \dfrac{1}{x}, \text{if} \quad x \neq 0 \\ 0, \text{if} \quad x = 0 \end{cases}$$

which is discontinuous. But its graph is a closed subset of $\mathbb{R}^3$, which is shown as below.

$\square$

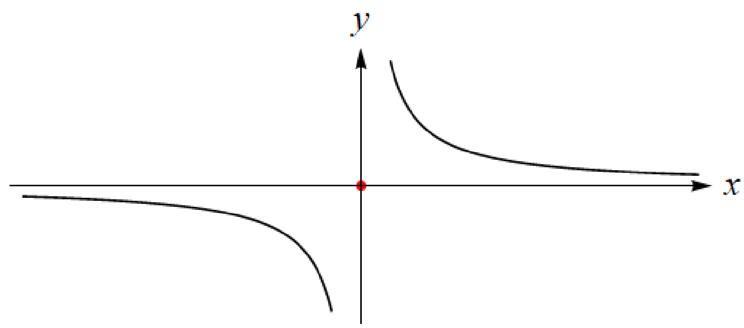

FIGURE 6. $y = \sin(1/x)$

Exercise 3.4. *Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is bounded, then f is continuous if and only if G_f is a closed subset of $\mathbb{R}^{n+m}$.*
Proof.

(1) ($\Rightarrow$) is proved in the previous exercise.
(2) ($\Leftarrow$) Suppose G_f is closed, then we want to prove that if $x_n \to x_0$, then $f(x_n) \to f(x_0)$. We prove it by contradiction and assume that there is a sequence $x_n \to x_0$ such that $\lim_{x_n \to x_0} f(x_n) \neq f(x_0)$.

Then there is a $\varepsilon > 0$, and for $\forall N > 0$ such that if $n > N$, i.e., $\forall n > 0$, we will have $|f(x_n) - f(x_0)| \geq \varepsilon$. Then we can find a convergent subsequence $\{x_{n_k}\}$ such that $\forall k$, $|f(x_{n_k}) - f(x_0)| \geq \varepsilon$. The sequence $f(x_{n_k})$ is bounded in $\mathbb{R}^m$, then it has a convergent subsequence $\{f(x_{n_{k_l}})\}$ converging to y_0. And now we have

$$G_f \ni \left(x_{n_{k_l}}, f(x_{n_{k_l}})\right) \to (x_0, y_0)$$

and since G_f is closed, then $(x_0, y_0) \in G_f$ and $y_0 = f(x_0)$. Therefore $f(x_{n_{k_l}}) \to f(x_0)$, which is a contradiction.

□

Next we give some important examples and theorems about continuity and metric space.

Exercise 3.5. *Let $\{f_n\}_{n=1}^{\infty}$, $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions such that:*

(1) $f_n \geq 0$.
(2) $f_{n+1} \leq f_n$.
(3) $\forall x \in [0,1]$, $f_n(x) \to 0$.

Then f_n uniformly converges to 0.

Proof. Given $\varepsilon > 0$, we need to find $N > 0$ such that if $\forall n \geq N$ and $\forall x \in [0,1]$, then $0 < f_n(x) < \varepsilon$.

For $x \in [0,1]$, let N_x be the least integer such that $f_{N_x} < \varepsilon$. Then for $\forall n \geq N_x$, $f_n(x) < \varepsilon$. Since f_{N_x} is continuous function, so there exists an open neighborhood U_x of x in $[0,1]$ such that for $z \in U_x$, we have $f_{N_x}(x) < \varepsilon$.

Since $[0,1]$ is compact, then there exists a finite open covering such that $[0,1] \in U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_k}$. Now we pick $N = \max\{N_{x_1}, N_{x_2}, \cdots, N_{x_k}\}$, where N_{x_j} is the least number such that $f_{N_{x_j}}(x_j) < \varepsilon$. If $n \geq N$ and for $x \in [0,1]$, then $x \in U_{x_i}$ for some $i \in \{1, 2, \cdots, k\}$, which implies $0 \leq f_n(x) \leq f_N(x) \leq f_{N_i}(x) < \varepsilon$. □

Exercise 3.6. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a norm, i.e., for $\forall x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we have*

(1) $F(x) \geq 0$ *and* $F(x) = 0$ *if and only if* $x = 0$.
(2) $F(x + y) \leq F(x) + F(y)$.
(3) $F(tx) = |t|F(x)$

Then there exist $A, B > 0$ such that $A\|x\| \leq F(x) \leq B\|x\|$ for all $x\mathbb{R}$.

Proof.

(1) We claim that F is bounded on unit sphere $\{\|x\| = 1\}$.
Let $\{e_1, e_2, \cdots, e_n\}$ be orthonormal basis for $\mathbb{R}^n$, then any $x \in \mathbb{R}^n$ has a representation

$$x = \sum_{i=1}^{n} c_i e_i$$

If $\|x\| = 1$, then we have $|c_i| \leq 1$. And we have

$$F(x) = F\left(\sum_{i=1}^{n} c_i e_i\right) \leq \sum_{i=1}^{n} |c_i| F(e_i) \leq \sum_{i=1}^{n} F(e_i) = B$$

Then there exists a $B > 0$.

(2) Now we claim F is continuous.
If $x \neq y$, then we have $y = x + \|y - x\| \cdot \frac{y-x}{\|y-x\|}$. Thus, we have

$$\begin{aligned} F(y) &\leq F(x) + \|y - x\| F\left(\frac{y - x}{\|y - x\|}\right) \\ \Rightarrow F(y) - F(x) &\leq B\|y - x\| \end{aligned}$$

Now we switch x and y, then we have $F(x) - F(y) \leq B\|y - x\|$. Thus we have $|F(x) - F(y)| \leq B\|y - x\|$, which implies F is continuous.

Now we complete the proof of theorem. Since F is continuous, so it obtains its minimum A on the compact unit sphere, i.e.,

$$\begin{aligned} A &= \inf_{\|x\|=1} F(x) = F(x_0) > 0 \\ \Rightarrow A &\leq F(x) \leq B, \|x\| = 1 \end{aligned}$$

Now if $\|x\| \neq 0$ is any point in $\mathbb{R}^n$, then

$$\begin{aligned} F(x) &= F\left(\|x\| \cdot \frac{x}{\|x\|}\right) = \|x\| \cdot F\left(\frac{x}{\|x\|}\right) \\ \Rightarrow A\|x\| &\leq F(x) \leq B\|x\| \end{aligned}$$

□

Exercise 3.7. *Prove that if X is a metric space, and $f : X \times [0, 1] \to \mathbb{R}$ is continuous, then $g : X \to \mathbb{R}$, which is defined by*

$$g(x) = \sup_{t \in [0,1]} f(x, t)$$

is continuous.

Proof. Prove by contradiction, and suppose g is not continuous, i.e., there exists $\varepsilon > 0$, for $\forall\delta > 0$, $\exists x$ such that if $d_X(x, x_0) < \delta$, then $|g(x) - g(x_0)| \geq \varepsilon$.

Fix such $\varepsilon > 0$, we pick $\delta = \frac{1}{n}$, then exists x_n such that $d_X(x_n, x_0) < \frac{1}{n}$, then $|g(x_n) - g(x_0)| \geq \varepsilon$, which implies

$$\left|\sup_t f(x_n, t) - \sup_t f(x_0, t)\right| \geq \varepsilon$$

then there exist $t_n, t_0 \in [0, 1]$ such that

$$|f(x_n, t_n) - f(x_0, t_0)| \geq \varepsilon$$

where $x_n \to x_0$. Then there exists a subsequence $t_{n_k} \to s$, such that $f(x_{n_k}, t_{n_k}) \to f(x_0, s)$. Then we have

$$f(x_{n_k}, t_{n_k}) = \sup_t f(x_{n_k}, t) \geq f(x_{n_k}, t_0)$$
$$f(x_0, t_0) = \sup_t f(x_0, t) \geq f(x_0, s)$$

Thus, we have

$$f(x_0, t_0) \leftarrow f(x_{n_k}, t_0) \leq f(x_{n_k}, t_{n_k}) \to f(x_0, s) \leq f(x_0, t_0)$$

which means $f(x_{n_k}, t_{n_k}) \to f(x_0, t_0)$, which is a contradiction. □

Theorem 3.41. *If $A \subset U$ is a dense subset of a metric space, i.e., cl(A) $= X$, and $f : A \to \mathbb{R}$ is uniformly continuous, then there is a unique continuous function $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in A$. Moreover, F is uniformly continuous.*

Before we prove the theorem, we first introduce an exercise, then we provide the proof of theorem.

Exercise 3.8. *If $f : (a, b) \to \mathbb{R}$ is uniformly continuous, and $-\infty < a < b < \infty$, then f extends to a continuous function $F : [a, b) \to \mathbb{R}$. In particular, F is bounded.*

Proof. For $x \in X$, we choose a unique $A \ni a_k^x \to x$. Since f is uniformly continuous, then $\forall\varepsilon > 0$, there exists $\delta > 0$, and for $\forall x, y \in A$, if $d_X(x, y) < \delta$, then $|f(x) - f(y)| < \varepsilon$. For such $\delta > 0$, there exists a $N > 0$ such that $\forall k, l$, then $d_X(a_k^x, a_l^x) < \delta$, and hence

$$|f(a_k^x) - f(a_l^x)| < \varepsilon$$

then we can know that $\{f(a_k^x)\}_{k=1}^{\infty}$ is a Cauchy sequence. Thus it is convergent.

We now define

$$F(x) = \lim_{k\to\infty} f(a_k^x)$$

If $d_x(x, y) < \delta$, then $d_X(a_k^x, a_k^y) < \delta$. Then for some $K > 0$, if $\forall k \geq K$, then

$$|f(a_k^x) - f(a_k^y)| < \varepsilon \Rightarrow |F(x) - F(y)| < \varepsilon$$

Thus, F is uniformly continuous. □

Now we define a metric space with $l^\infty = \{x = (x_1, x_2, \cdots) | \|x\|_\infty = \sup_i |x_i| < \infty\}$ and metric $d_\infty(x, y) = \sup_i(x_i - y_i)$. Then we discuss some exercises.

Exercise 3.9. *Show that the Hilbert cube*

$$H = \{x = (x_1, x_2, \cdots) | 0 \leq x_i \leq 2^{-i}, i = 1, 2, \cdots\}$$

is a compact subset of l^∞.

Proof. Let $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, \cdots\right) \in H$, and using diagonal method, we can find a sequence $\{x^{(n_k)}\}$ such that for $\forall i$, $x_i^{(n_k)}$ converges, such that $x_i^{(n_k)} \to x_i$, with $0 \le x_i^{(n_k)} \le 2^{-i}$. Then we have $0 \le x_i \le 2^{-i}$, which implies that $x = (x_1, x_2, \cdots) \in H$.

It remains to show that $x^{(n_k)} \xrightarrow{l^\infty} x$. Given $\varepsilon > 0$, let $N > 0$ be such that $2^{-N} < \varepsilon/2$. Also, there exists K, for $\forall k \ge K$, such that $\left|x_i^{(n_k)} - x_i\right| < \varepsilon/2, i = 1, 2, \cdots, N$. Then for $i > N$, we have $\left|x_i^{(n_k)} - x_i\right| < \left|x_i^{(n_k)}\right| + |x_i| < \varepsilon$ Thus, for $\forall k \ge K$ and $\forall n \in \mathbb{N}$, we have

$$\begin{aligned} &\left|x_i^{(n_k)} - x_i\right| < \varepsilon \\ \Rightarrow &\|x_i^{(n_k)} - x_i\|_\infty < \varepsilon \\ \Rightarrow &x^{(n_k)} \xrightarrow{l^\infty} x \end{aligned}$$

The proof is complete. □

Theorem 3.42 (Weierstrass)**.** *If $f : [a, b] \to \mathbb{R}$ is a continuous function, then for every $\varepsilon > 0$, there is a polynomial p_ε such that*

$$|f(x) - p_\varepsilon| < \varepsilon$$

for all $x \in [a, b]$.

Before we prove this theorem, we need to consider some material about metric space. Now recall that

$$C([a, b], \mathbb{R}) = \{f : [a, b] \to \mathbb{R} | f \text{ is continuous}\}$$

is a metric space with the metric $d_\infty(f, g) = \|f - g\|_\infty = \sup_{x\in[a,b]} |f(x) - g(x)|$. We use compactness of $[a, b]$ to ensure that $d_\infty(f, g) < \infty$ for every $f, g \in C([a, b], \mathbb{R})$. The compactness of $[a, b]$ is needed. For example, $f(x) = x$ and $g(x) = 0$ for $x \in \mathbb{R}$ belong to to $C(\mathbb{R}, \mathbb{R})$ but $d_\infty(f, g) = \sup_{x\in\mathbb{R}} |x| = \infty$, which means d_∞ is not a metric in $C(\mathbb{R}, \mathbb{R})$.

Lemma 3.43. *Suppose $f_n, f \in C([a, b])$. Then f_n uniformly converges to f, i.e., $f_n \rightrightarrows f$ if and only if $d_\infty(f_n, f) \to 0$.*

Proof.

$$\begin{aligned} &f_n \rightrightarrows f \text{ in } d_\infty \Leftrightarrow d_\infty(f_n, f) \to 0 \\ \Leftrightarrow &\forall \varepsilon > 0, \exists N, \forall n \ge N, d_\infty(f_n, f) < \varepsilon \\ \Leftrightarrow &\forall \varepsilon > 0, \exists N, \forall n \ge N, \forall x \in [a, b], |f_n(x) - f(x)| < \varepsilon \\ \Leftrightarrow &f_n \rightrightarrows f \end{aligned}$$

□

Remark 3.6. *Similarly, if $E \subset X$ is a compact subset, then*

$$C(E, \mathbb{R}) = \{f : E \to \mathbb{R} | f \textit{ is continuous}\}$$

is a metric space, where $d_\infty(f, g) = \sup_{x\in E} (f(x) - g(x))$.

The following lemma can be proved by the exact the same argument mentioned above.

Lemma 3.44. *Suppose $E \subset X$ is a compact subset and $f_n, f \in C(E, \mathbb{R}), n = 1, 2, \cdots$. Then $f_n \to f$ in $C(E, \mathbb{R})$ if and only if $f_n \rightrightarrows f$.*

Theorem 3.45. *Suppose $E \subset X$ is a compact subset. Then $(C(E), d_\infty)$ is a complete metric space.*

Proof. Let $\{f_n\}_n \in C(E)$ be Cauchy sequence, i.e., for $\forall \varepsilon > 0$, $\exists N > 0$, such that $\forall n, m \geq N$, $d_\infty(f_n, f_m) < \varepsilon$, and this is equivalent to the statement that for $\forall \varepsilon > 0$, $\exists N > 0$, such that $\forall n, m \geq N$ and $\forall x \in E$, $|f_n(x) - f_m(x)| < \varepsilon$.

For $\forall x \in E$, $\{f_n(x)\}_n$ is a Cauchy sequence, then convergent. Let $f(x) = \lim_{n\to\infty} f_n(x)$, and for $n, m \geq N$, we have $|f_n(x) - f_m(x)| < \varepsilon$. Now we fix n and let $m \to \infty$, then we have $|f_n(x) - f(x)| < \varepsilon$. Then for $\forall \varepsilon > 0$, there exists $N > 0$, such that for $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$, which implies $f_n \rightrightarrows f$ in d_∞. Then f is continuous, which means $f \in C(E)$. Thus, $(C(E), d_\infty)$ is a complete metric space. □

We have another definition of dense subset.

Definition 3.25. *A subset $A \subset X$ is dense if cl$(A) = X$.*

Proposition 3.4. *Let $A \subset X$ be a subset. Then the following conditions are equivalent:*

(1) *A is dense in X.*
(2) *cl$(A) = X$.*
(3) *For $\forall x \in A$, there is a sequence $x_n \in A$ such that $x_n \to x$.*
(4) *Every ball of x contains elements of A, i.e., $\forall x \in A$, there exists $r > 0$, such that $B(x, r) \cap A \neq \varnothing$.*

Theorem 3.46 (Weieratrass)**.** *The class of polynomials on $[a, b]$ form a dense subset of $C([a, b], \mathbb{R})$.*

The Weieratrass theorem is a direct result from the following theorem.

Theorem 3.47 (Stone-Weieratrass)**.** *Let X be a metric space and $E \subset X$ a compact subset. Let $\mathcal{A} \subset C(E, \mathbb{R})$ satisfies the following properties:*

(1) *$\mathcal{A}$ is an algebra, i.e., if $f, g \in \mathcal{A}, \alpha \in \mathbb{R}$, then $f + g, fg, \alpha f \in \mathcal{A}$.*
(2) *The constant function $f(x) \equiv 1$ for $x \in E$ belongs to $\mathcal{A}$.*
(3) *$\mathcal{A}$ separates points, i.e., for every $x, y \in E, x \neq y$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.*

Then $\mathcal{A}$ is a dense subset of $C(E, \mathbb{R})$, i.e., cl$(A) = C(E, \mathbb{R})$.

Observe that the set $\mathcal{A} = \{f : [a, b] \to \mathbb{R} | f \text{ is a polynomial}\}$ has all three properties above and hence the Weierstrass theorem follows the Stone-Weieratrass theorem.

Proof. We want to prove that $\text{cl}(A) = C(E, \mathbb{R})$. The set $\text{cl}(A)$ satisfies the properties $(1), (2)$ and (3) as well. Indeed, properties (2) and (3) are obvious because they are already satisfied by elements of $\mathcal{A}$, which is a subset of $\text{cl}(A)$, and the property (1) can be proved as follows:

Let $f, g \in \text{cl}(A), \alpha \in \mathbb{R}$. Then there are sequences $\{f_n\}, \{g_n\} \in \mathcal{A}$ such that $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$. And hence

$$f_n + g_n \rightrightarrows f + g, f_n g_n \rightrightarrows fg, \alpha f_n \rightrightarrows \alpha f$$

we have $f+g, fg, \alpha f \in \mathrm{cl}(A)$. It follows from Weierstrass theorem 3.42 that for every $n \in \mathbb{N}$, there is a polynomial p_n such that $||t| - p_n(t)| < 1/n$ for $-n \leq t \leq n$. Thus, we have

$$||f(x)| - p_n(f(x))| < \frac{1}{n}$$

provided $-n \leq f(x) \leq n$. If $f \in \mathrm{cl}(A)$, then f is continuous and hence bounded (since E is bounded), so if n is sufficiently large enough such that $-n \leq f(x) \leq n$ for all $x \in E$ and hence

$$||f(x)| - p_n(f(x))| < \frac{1}{n}, \forall x \in E \tag{3.7.1}$$

Since $\mathrm{cl}(A)$ is an algebra, it follows that $p_n(f(x)) \in \mathrm{cl}(A)$ and hence (3.7.1) implies that $|f|$ is a limit of an uniformly convergent sequence of functions in $\mathrm{cl}(A)$, i.e., the limit of a sequence in $\mathrm{cl}(A)$ that converge in the metric space. Therefore, we have

$$|f| \in \mathrm{cl}(A)$$

We proved that if $f \in \mathrm{cl}(A)$, then $|f| \in \mathrm{cl}(A)$. Now we introduce notation

$$(f \wedge g)(x) = \max\{f(x), g(x)\}$$
$$(f \vee g)(x) = \min\{f(x), g(x)\}$$

and since

$$f \wedge g = \frac{f+g}{2} + \frac{|f-g|}{2}$$
$$f \vee g = \frac{f+g}{2} - \frac{|f-g|}{2}$$

we can conclude that if $f+g \in \mathrm{cl}(A)$, then $f \wedge g, f \vee g \in \mathrm{cl}(A)$.

Let $h \in C(E, \mathbb{R})$ be arbitrary. We need to prove that for every $\varepsilon > 0$, there exists $f \in \mathrm{cl}(A)$ such that $|f(z) - h(z)| < \varepsilon$, for all $z \in E$. Indeed, this claim implies that $h \in \mathrm{cl}(A)$.

Let $x_1, x_2 \in E, x_1 \neq x_2$. Then there exists a function in $\mathcal{A}$, denote it by $f_{x_1x_2}$ such that $f_{x_1x_2}(x_1) = h(x_1)$ and $f_{x_1x_2}(x_2) = h(x_2)$. Indeed, choose $g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$ and define $f_{x_1x_2} = \alpha g + \beta$ for a suitable choice of $\alpha, \beta \in \mathbb{R}$.

Fix $\varepsilon > 0$. Let $x \in E$, and for $y \in E$, we have $f_{yx}(y) = h(y)$ and hence there is a neighborhood $U(y)$ of y such that

$$f_{yx}(z) > h(z) - \varepsilon$$

The sets $\{U(y)\}_{y \in E}$ form an open covering of E. Thus, there is a finite subcovering

$$E = \bigcup_{i=1}^{n} U(y_i)$$

Denote $f_x = f_{y_1x} \vee \cdots f_{y_nx}$. Clearly, $f_x \in \mathrm{cl}(A)$. We have if $z \in E$, then $z \in U(y_i)$ for some $i \in \{1, 2, \cdots, n\}$, then we have

$$f_{y_ix}(z) > h(z) - \varepsilon$$

which implies

$$f_x(z) = \max\{f_{y_1x}(z), \cdots, f_{y_nx}(z)\} \geq f_{y_ix}(z) > h(z) - \varepsilon$$

i.e., $f_x(z) > h(z) - \varepsilon$ for all $z \in E$. Also, we have

$$f_x(x) = \max\{\underbrace{f_{y_1x}(x)}_{h(x)}, \cdots, \underbrace{f_{y_nx}(x)}_{h(x)}\} = h(x)$$

which implies $f_x(x) = h(x)$.

Hence there is a neighborhood $V(x)$ of x such that

$$f_x(z) < h(z) + \varepsilon$$

for all $z \in V(x)$. The sets $\{V(x)\}_{x\in E}$ form an open covering of E, so there is a finite subcoveirng

$$E = \bigcup_{i=1}^{m} V(x_i)$$

We define $f = f_{x_1} \vee \cdots \vee f(x_m)$ and clearly $f \in \mathrm{cl}(A)$. Also, we have

$$f(z) = \min\{\underbrace{f_{x_1}(z)}_{>h(z)-\varepsilon}, \cdots, \underbrace{f_{x_m}(z)}_{>h(z)-\varepsilon}\} > h(z) - \varepsilon$$

i.e.,

$$f(z) > h(z) - \varepsilon, \forall z \in E \tag{3.7.2}$$

On the other hand, we have if $z \in E$, then $z \in V(x_i)$ for some i, and then $f_{x_i}(z) < h(z) + \varepsilon$. Thus, we have

$$f(z) = \min\{f_{x_1}(z), \cdots, f_{x_m}(z)\} \leq f_{x_i}(z) < h(z) + \varepsilon \tag{3.7.3}$$

The inequality (3.7.2) and (3.7.3) gives

$$\begin{aligned} &h(z) - \varepsilon < f(z) < h(z) + \varepsilon, \forall z \in E \\ &\Rightarrow |f(z) - h(z)| < \varepsilon, \forall z \in E \end{aligned}$$

The proof is complete. $\square$