Homework 4 for Math 1530
Zhen Yao

Problem 35. Prove that if a sequence $\{a_n\}$ is bounded and $a_{n+1} \geq a_n - \dfrac{1}{2^n}$ for all n, then the sequence $\{a_n\}$ is convergent.

Proof. Since $\{a_n\}$ is bounded, then sequence $\{a_n\}$ has a least upper bound, denoted by $M = \sup\{a_n\}$. Then for $\forall \varepsilon > 0$, there exists a_n such that $a_n > M - \varepsilon$. And for fixed ε, suppose a_k is the first element in $\{a_n\}$ such that $a_k > M - \varepsilon$. And for $\forall n > k$, we have

$$a_k - \frac{1}{2^{k+1}} + \frac{1}{2^{n+1}} = a_k - \frac{1}{2^k} - \frac{1}{2^{k+1}} - \cdots - \frac{1}{2^{n-1}} \leq a_n \leq M$$

then there exists N such that for $\forall n > N$, $\frac{1}{2^{k+1}} - \frac{1}{2^{N+1}} \leq \varepsilon$. Then for $\varepsilon > 0$, set $n_0 = \min\{k, N\}$, then for $\forall n, m > n_0$, we have

$$|a_n - a_m| \leq \varepsilon$$

Then $\{a_n\}$ is convergent. □

Problem 36. Find the limit $\lim\limits_{n\to\infty} n \sin(2\pi e n!)$.

Proof. We know that $e = \sum_{n=0}^{\infty} 1/n!$, then we have

$$\begin{aligned}
\lim_{n\to\infty} n\sin(2\pi e n!) &= \lim_{n\to\infty} n \sin\left(2\pi n!\left(\sum_{k=0}^{n}\frac{1}{k!} + \sum_{k=n+1}^{\infty}\frac{1}{k!}\right)\right) \\
&= \lim_{n\to\infty} n \sin\left(2\pi n!\left(\sum_{k=n+1}^{\infty}\frac{1}{k!}\right)\right)
\end{aligned}$$

since $\lim_{n\to\infty} n!\left(\sum_{k=n+1}^{\infty}/k!\right) = 0$, and we denote it by A_n, then we have

$$\begin{aligned}
\lim_{n\to\infty} n\sin(2\pi e n!) &= \lim_{n\to\infty} n \sin(2\pi A_n) \\
&= \lim_{n\to\infty} 2\pi n A_n \frac{\sin(2\pi A_n)}{2\pi A_n} \\
&= \lim_{n\to\infty} 2\pi n A_n \\
&= 2\pi
\end{aligned}$$

□

Problem 37. Find the limit

$$\lim_{n\to\infty} n^3\left(\sqrt{n^2 + \sqrt{n^4+1}} - n\sqrt{2}\right).$$

Proof. Denote the limit by A, and we have

$$\begin{aligned}
A &= n^3\left(\sqrt{n^2+\sqrt{n^4+1}}-n\sqrt{2}\right)\left(\frac{(\sqrt{n^2+\sqrt{n^4+1}}+n\sqrt{2})}{(\sqrt{n^2+\sqrt{n^4+1}}+n\sqrt{2})}\right)\\
&= n^3\frac{\sqrt{n^4+1}-n^2}{(\sqrt{n^2+\sqrt{n^4+1}}+n\sqrt{2})}\\
&= n^3\frac{1}{(\sqrt{n^2+\sqrt{n^4+1}}+n\sqrt{2})(\sqrt{n^2+\sqrt{n^4+1}}+n^2)}\\
&= \frac{1}{(2\sqrt{2})(1+1)}\\
&= \frac{\sqrt{2}}{8}
\end{aligned}$$

□

Problem 38. Suppose that $\{a_n\}_n$ is a sequence such that for every integer $k \geq 2$ the sequence $\{a_{k\cdot n}\}_n$ is convergent. Does it follow that the sequence $\{a_n\}_n$ is convergent?

Proof. No, it is not always convergent. Take $a_n = 1$ when n is prime number, and $a_n = 0$ when n is composite number. Then the sequence $\{a_{kn}\}_n$ is convergent, since $a_{kn} = 0$ when $n > 1$. But $\{a_n\}$ is not convergent since there are infinitely many prime numbers. □

Problem 39. Prove that if $\{a_n\}_n$ is a sequence such that each of the sequences $\{a_{2n}\}_n$, $\{a_{2n+1}\}_n$ and $\{a_{3n}\}_n$ is convergent, then $\{a_n\}_n$ is convergent too.

Proof. (1)Since the sequences $\{a_{2n}\}_n$, $\{a_{2n+1}\}_n$ and $\{a_{3n}\}_n$ are convergent, then they have the same limit. If not, we can assume that they have different limit, say $\lim_{n\to\infty} a_{2n} = M$, $\lim_{n\to\infty} a_{2n+1} = N$, $\lim_{n\to\infty} a_{3n} = P$, then for $\forall\varepsilon > 0$, there exists $N > 0$ such that for $\forall n > N$, we have $|a_{2n} - M| < \varepsilon$, $|a_{2n+1} - N| < \varepsilon$ and $|a_{3n} - P| < \varepsilon$. Then for certain $n = 3k$, where k is an integer, we have $|a_{6k} - M| < \varepsilon$, and $|a_{6k} - P| < \varepsilon$, which means $M = P$. Similarly, we can know $M = N = P$.

(2)For any a_m where $m > N$, we can know that a_n belongs to one of $\{a_{2n}\}_n$, $\{a_{2n+1}\}_n$ and $\{a_{3n}\}_n$, such that $|a_m - M| < \varepsilon$. Thus, $\{a_n\}$ is convergent. □

Problem 40. Prove that there is a sequence such that the set of all possible limits of subsequences is the whole interval $[0, 1]$.

Proof. Take $a_n = f(n) = |\sin n|$. Since $|\sin n|$ is dense in the interval $[0, 1]$, so all the possible limits of subsequences is the whole interval. □

Problem 41. Prove that if $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$, then for any sequence $\{b_n\}$

$$\limsup_{n\to\infty}(a_n + b_n) = a + \limsup_{n\to\infty} b_n.$$

Proof. (1)If $\limsup_{n\to\infty} b_n = \infty$, then the equation holds.

(2)Consider the case $\limsup_{n\to\infty} b_n$ is finite. Suppose $A_k = \sup\{a_n, n \geq k\}$, $B_k =$

$\sup\{b_n, n \geq k\}$ and $C_k = \sup\{a_n + b_n, n \geq k\}$. For $\forall n \geq k$, we have $a_n + b_n \leq A_k + B_k$. Then we have

$$\begin{aligned}
&\sup_{n\geq k}(a_n + b_n) \leq \sup_{n\geq k} A_k + \sup_{n\geq k} B_k \\
\Rightarrow &\lim_{k\to\infty} \sup_{n\geq k}(a_n + b_n) \leq \lim_{k\to\infty} \sup_{n\geq k} A_k + \lim_{k\to\infty} \sup_{n\geq k} B_k \\
\Rightarrow &\limsup_{n\to\infty}(a_n + b_n) \leq a + \limsup_{n\to\infty} b_n
\end{aligned}$$

On the other hand, we write $b_n = (b_n + a_n) + (-a_n)$, and with the result above, we have

$$\begin{aligned}
&\limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty}(a_n + b_n) + \limsup_{n\to\infty}(-a_n) \\
\Rightarrow &\limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty}(a_n + b_n) - a \\
\Rightarrow &\limsup_{n\to\infty}(a_n + b_n) \geq a + \limsup_{n\to\infty} b_n
\end{aligned}$$

Then we proved that $\limsup_{n\to\infty}(a_n + b_n) = a + \limsup_{n\to\infty} b_n$. $\square$

Problem 42. Let c_0 be the class of all sequences $\{x_n\}$ such that $\lim\limits_{n\to\infty} x_n = 0$. Prove that if $\{a_n\}$ is a bounded sequence, then

$$\inf_{(x_n)\in c_0} \left(\sup_{n\in\mathbb{N}} \{a_n + x_n\} \right) = \limsup_{n\to\infty} a_n.$$

Proof. For $\forall \varepsilon > 0$, let c_k be the class of all sequences $\{x_n\}$ such that $\forall n \geq k$, $|x_n| < \varepsilon/2$. Then we know $c_1 \subset c_2 \subset \cdots \subset c_0$. Then we have $\inf_{(x_n)\in c_0} (\sup_{n\in\mathbb{N}}\{a_n + x_n\}) = \lim_{k\to\infty} \inf_{(x_n)\in c_k} (\sup_{n\in\mathbb{N}}\{a_n + x_n\})$.

For a certain k, we have $\inf_{(x_n)\in c_k} (\sup_{n\in\mathbb{N}}\{a_n + x_n\}) = \inf_{(x_n)\in c_k} \left(\sup_{n\geq k}\{a_n + x_n\}\right)$, since we can find a sequence $\{_n\}$ such that makes $a_i + x_i, 1 \leq i \leq k-1$ small enough and the supremum of $\{a_n + x_n\}$ will be after kth terms. For $\forall n \geq k$, $|x_n| < \varepsilon/2$. Then we have

$$\begin{aligned}
&\sup_{n\geq k} (a_n - \varepsilon) < \sup_{n\geq k} (a_n + x_n) < \sup_{n\geq k} (a_n + \varepsilon) \\
\Rightarrow &\sup_{n\geq k} a_n - \varepsilon < \sup_{n\geq k} (a_n + x_n) < \sup_{n\geq k} a_n + \varepsilon \\
\Rightarrow &\left| \sup_{n\geq k}(a_n + x_n) - \sup_{n\geq k} a_n \right| < \frac{\varepsilon}{2}
\end{aligned}$$

which implies

$$\left| \inf_{(x_n)_k} \left(\sup_{n\geq k}(a_n + x_n) \right) - \sup_{n\geq k} a_n \right| < \frac{\varepsilon}{2}$$

Also, there exists a $k_0 \in \mathbb{N}$ such that $\left|\sup_{n\geq k} a_n - \limsup_{n\to\infty} a_n\right| < \varepsilon/2$. Then for $\forall k \geq k_0$, we have

$$\begin{aligned}
&\left| \inf_{(x_n)_k} \left(\sup_{n\geq k}(a_n + x_n) \right) - \limsup_{n\to\infty} a_n \right| \\
\leq &\left| \inf_{(x_n)_k} \left(\sup_{n\geq k}(a_n + x_n) \right) - \sup_{n\geq k_2} a_n \right| + \left| \sup_{n\geq k} a_n - \limsup_{n\to\infty} a_n \right| = \varepsilon
\end{aligned}$$

Then

$$\inf_{(x_n)\in c_0}\left(\sup_{n\in\mathbb{N}}\{a_n+x_n\}\right)=\lim_{k\to\infty}\inf_{(x_n)_k}\left(\sup_{n\geq k}(a_n+x_n)\right)=\limsup_{n\to\infty}a_n$$

$\square$