Homework 1 for Math 1540

Zhen Yao

Problem 1. Prove that the trigonometric polynomials

$$T(x) = \sum_{k=0}^{n} a_k \cos kx + \sum_{k=0}^{n} b_k \sin kx, \quad a_k, b_k \in \mathbb{R}$$

form an algebra. *Hint:* $\cos x + i \sin x = e^{ix}$.

Proof. With $\cos x + i\sin x = e^{ix}$, then we can have $\cos kx = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$ and $\sin kx = \frac{1}{2}\left(ie^{-ikx} - ie^{ikx}\right)$.

Now it suffices to show that $\cos(kx)\sin(lx), \cos(kx)\cos(lx), \sin(kx)\sin(lx)$ are trigonometric polynomials. We have

$$\begin{aligned}
\cos(kx)\sin(lx) &= \frac{1}{4}\left(e^{ikx} + e^{-ikx}\right)\left(ie^{-ilx} - ie^{ilx}\right) \\
&= \frac{1}{4}\left(ie^{i(k-l)x} - ie^{i(k+l)x} + ie^{-i(k+l)x} - ie^{i(l-k)x}\right) \\
&= -\frac{1}{2}\sin((k-l)x) + \frac{1}{2}\sin((k+l)x)
\end{aligned}$$

which is also a trigonometric polynomial. Similarly, $\cos(kx)\cos(lx), \sin(kx)\sin(lx)$ are also trigonometric polynomial. Thus, trigonometric polynomials form an algebra. $\square$

Problem 2. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane. Let $\mathcal{A}$ be the algebra of functions of the form

$$f\left(e^{i\theta}\right) = \sum_{n=0}^{N} c_n e^{in\theta}, \quad c_n \in \mathbb{C},\ \theta \in \mathbb{R}.$$

It is easy to see that $f \equiv 1$ belongs of $\mathcal{A}$ and $\mathcal{A}$ separates points (do not prove it). Prove that there are complex valued functions on S^1 that cannot be uniformly approximated by functions in $\mathcal{A}$. *Hint:* For $f \in \mathcal{A}$

$$\int_0^{2\pi} f\left(e^{i\theta}\right) e^{i\theta}\, d\theta = 0\,.$$

Proof. The function $f(z) = z \in \mathcal{A}$ separates points in S^1. And we can know that $\frac{1}{z} = e^{-i\theta}$ is not in the closure of $\mathcal{A}$, since

$$\int_0^{2\pi} e^{-i\theta} e^{i\theta} d\theta = 2\pi.$$

$\square$

Problem 3. Prove that complex polynomials

$$p(z) = \sum_{n=0}^{N} c_n z^n, \quad c_n \in \mathbb{C}$$

are not dense in $C(\overline{D}, \mathbb{C})$, where

$$\overline{D} = \{z \in \mathbb{C} : \; |z| \leq 1\}$$

is the unit disc in $\mathbb{C}$. *Hint:* Consider $f(z) = \overline{z}$. Is the previous exercise helpful?

Proof. For $z = e^{i\theta} = \cos x + i \sin x$, we have $\bar{z} = z^{-1}$. Then,

$$\bar{p}(z) = \sum_{n=0}^{N} \bar{c_n} z^{-n}$$

which is not a polynomial, since the exponents are negative. □

Problem 4. We know that if $f : [a, b] \to \mathbb{R}$ is continuous and

$$\int_a^b f(x) x^n \, dx = 0 \tag{0.0.1}$$

for $n = 0, 1, 2, 3, \ldots$, then $f(x) = 0$ for all $a \leq x \leq b$. We proved it using the Weierstrass theorem. Suppose now that $f : [a, b] \to \mathbb{R}$ is continuous and (0.0.1) holds for all $n \geq 2011$. Does it follow that $f(x) = 0$ for all $a \leq x \leq b$?

Proof. Set $g(x) = x^{2011} f(x)$, and then $\int_a^b g(x) x^k dx = 0$ for $k = 0, 1, 2, \cdot$, which implies $g(x) = 0$. Then, we know that $f(x) = 0$ for all $x \neq 0$.

If $a > 0$, then $f(x) = 0$ on $[a, b]$. If $a \neq 0$, then with continuity of f, we have $f(0) = 0$. Thus, $f(x) = 0$ for all $a \leq x \leq b$. □

Problem 5. Prove that if $f : [0, 1] \to \mathbb{R}$ is such that

$$\int_0^1 f(x) e^{nx} \, dx = 0 \quad \text{for all } n = 0, 1, 2, \ldots,$$

then $f(x) = 0$ for all $0 \leq x \leq 1$. Provide two proofs following the methods:

(a) Use the Stone-Weierstrass theorem.

(b) Use the change of variables formula and apply the Weierstrass theorem.

Proof.

(a) There exists a sequence of the form $p_n(x) = \sum_{n=0}^{\infty} c_n e^{nx}$ such that converges uniformly to $f(x)$. Since f is continuous on $[0, 1]$, hence bounded. Then $\{p_n(x)\}$ is also bounded, and hence $p_n f$ converges uniformly to f^2. Then we have

$$\int_0^1 f^2(x) \, dx = \lim_{n \to \infty} \int_0^1 p_n(x) f(x) \, dx = 0,$$

which implies $f(x) = 0, x \in [0, 1]$.

(b) Let $e^x = y$, then we have $y = \ln x$, and

$$\int_0^1 f(x)e^{nx}\,dx = \int_1^e f(\ln y)y^{n-1}\,dy = 0.$$

With previous problem, we have $f(\ln y) = 0, y \in [1, e]$, which is equivalent to that $f(x) = 0, x \in [0, 1]$.

□

Problem 6. Prove that if X is a compact metric space and $f : X \to X$ is a continuous mapping such that

$$d(f(x), f(y)) < d(x, y) \quad \text{for all } x, y \in X,$$

then there is a fixed point of f, i.e. $x \in X$ such that $f(x) = x$.

Proof. Choose $x_0 \in X$ and define $x_n = f^n(x_0)$. Since X is compact, then the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ such that converging to x^*. Also, we have $d(f(x), f(y)) < d(x, y) \quad$ for all $x, y \in X$, then there exists $k \in (0, 1)$ such that

$$d(f(x), f(y)) \le kd(x, y), \forall x, y \in X.$$

Then we have

$$\begin{aligned} d(x^*, f(x^*)) &\le d(x^*, x_{n_k}) + d(x_{n_k}, f(x_{n_k})) + d(f(x_{n_k}), f(x^*)) \\ &\le d(x^*, x_{n_k}) + k^{n_k} d(x_0, x_1) + kd(x_{n_k}, x^*) \\ &= k^{n_k} d(x_0, x_1) + (1 + k)d(x^*, x_{n_k}) \xrightarrow[n_k \to \infty]{} 0. \end{aligned}$$

Thus we have $d(x^*, f(x^*)) = 0$, which is a fixed point. □

Problem 7. Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - f(y)| < |x - y| \quad \text{for all } x, y \in \mathbb{R}$$

and f has no fixed point. You can find an explicit formula for f, but you do not have to. It is enough if you find a convincing argument that such a function exists. You do not have to be very precise, but your argument has to be convincing.

Proof. Take $f(x) = \ln(1 + e^x)$. □

Problem 8. Show that there is a unique continuous real valued function $f : [0, 1] \to \mathbb{R}$ such that

$$f(x) = \sin x + \int_0^1 \frac{f(y)}{e^{x+y+1}}\,dy.$$

Proof. Consider the map $T : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by

$$T(f)(x) = \sin x + \int_0^1 \frac{f(y)}{e^{x+y+1}}\, dy = f(x).$$

Clearly f is a solution to the problem if and only if $T(f) = f$. Since $C([0,1],\mathbb{R})$ is compact, then we only need to show that T is a contraction. Indeed, given $f, h \in C([0,1],\mathbb{R})$, we have

$$\begin{aligned} d_\infty(T(f),T(h)) &= \sup_{x\in[0,1]} \left| \int_0^1 \frac{f(y)}{e^{x+y+1}}\, dy - \int_0^1 \frac{h(y)}{e^{x+y+1}}\, dy \right| \\ &\le \sup_{x\in[0,1]} \int_0^1 \frac{|f(y)-h(y)|}{e^{x+y+1}}\, dy \\ &\le d_\infty(f,h) \int_0^1 \frac{1}{e^{x+y+1}}\, dy \\ &= d_\infty(f,h)(e-1)e^{-x-2} \end{aligned}$$

and we know that $(e-1)e^{-x-2} < 1$ for $x \in [0,1]$. Thus, T is a contraction. □

Problem 9. Let (X,d) be a nonempty complete metric space. Let $S : X \to X$ be a given mapping and write S^2 for $S \circ S$ i.e. $S^2(x) = S(S(x))$. Suppose that S^2 is a contraction. Show that S has a unique fixed point.

Proof. Since S^2 is conctraction, then there is a unique fixed point of S^2, denoted by x^*, such that $S^2(x^*) = x^*$. Then we have $S^2(S(x^*)) = S\left(S^2(x^*)\right) = S(x^*)$, which implies that $S(x^*)$ is also a fixed point of S^2. Thus, we have $S(x^*) = x^*$, implying S has a unique fixed point. □

Remark 1. The argument in previous problem also holds if S^n is a contraction.

Problem 10. Let E be a compact set and let $\mathcal{F} \subset C(E,\mathbb{R})$ be an equicontinuous family of functions. Does it imply that the family $\mathcal{F}$ is bounded in $C(E,\mathbb{R})$?

Proof. No. We can define $f_n(x) = n, \forall n \in \mathbb{N}, x \in E$. Thus, each $f_n(x), x \in E$ is equicontinuous function, hence $f_n(x) \in \mathcal{F}$. But, $\mathcal{F}$ is not bounded. □

Problem 11. Let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded and uniformly continuous. Prove that the family of functions $\{g_z\}_{z\in\mathbb{R}^n}$, $g_z(x) = f(x)f(x-z)$ is equicontinuous.

Proof. Since f is uniformly continuous, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}^n$, if $d(x,y) < \delta$, $|f(x), f(y)| < \varepsilon$. Then, for any $z \in \mathbb{R}^n$ we have $d(x-z, y-z) < \delta$, and $|f(x-z), f(y-z)| < \varepsilon$. Now we have

$$\begin{aligned} |f(x)f(x-z) - f(y)f(y-z)| =& |f(x)f(x-z) - f(y)f(x-z) + f(y)f(x-z) \\ & - f(y)f(y-z)| \\ \le& |f(x)-f(y)| \cdot |f(x-z)| + |f(x-z) - f(y-z)| \cdot |f(y)| \\ \le& \varepsilon |f(x-z)| + \varepsilon |f(y)|. \end{aligned}$$

Hence, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for $\forall x, y$ and $\forall z$, if $d(x,y) < \delta$, then $d(g(x), g(y)) < K\varepsilon$. Thus $\{g_z\}_{z\in\mathbb{R}^n}$ is equicontinuous. □

Problem 12. Suppose E is a compact metric space and $f_n : E \to \mathbb{R}$, $n = 1, 2, \ldots$ is a bounded and equicontinuous sequence of functions. Suppose that f_n converges pointwise to a continuous function $f : E \to \mathbb{R}$ (i.e. $f_n(x) \to f(x)$ for every $x \in E$). Prove directly (i.e. without using Arzela-Ascoli theorem) that $f_n \rightrightarrows f$ uniformly on E.

Proof.

(1) Definition of pointwise convergence is that f_n pointwise converges to f if and only if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$, i.e., for every $\varepsilon > 0$, there exists $N > 0$, such that for all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

(2) Definition of uniformly convergence is that for every $\varepsilon > 0$, there exists $N > 0$ such that for all $n \geq N$ and all $x \in E$, $|f_n(x) - f(x)| < \varepsilon$.

As $\{f_n\}$ is equicontinuous, there exists a $\delta > 0$, such that for $\forall n \in \mathbb{N}$ and $\forall x, y \in E$:

$$d(x,y) < \delta \Longrightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}.$$

and letting $n \to \infty$, then the above equation implies

$$d(x,y) < \delta \Longrightarrow |f(x) - f(y)| < \frac{\varepsilon}{3}.$$

Since E is compact, then it can be covered by finite many open balls of radius δ, i.e., there exists a $K > 0$ and $x_1, \cdots, x_K \in E$ such that

$$E \subset \bigcup_{j=1}^{K} B\left(x_j, \delta\right).$$

As f_n converges pointwise to f, then there exists $N > 0$ such that for $\forall n \geq N$,

$$|f_n(x_j) - f(x_j)| < \frac{\varepsilon}{3},$$

for all $1 \leq j \leq K$.

Now for any $x \in E$, and $\forall n \geq N$, then there exists a $j \in \{1, 2, \cdots, K\}$, for which $x \in B(x_j, \delta)$, and hence

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(x_j)| + |f_n(x_j) - f(x_j)| + |f(x_j) - f(x)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus, $f_n \rightrightarrows f$ uniformly on E. □

Problem 13. Suppose $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$ is a bounded and equicontinuous sequence of functions. Suppose that f_n converges pointwise to a continuous function $f : \mathbb{R} \to \mathbb{R}$. Does it imply that $f_n \rightrightarrows f$ uniformly on $\mathbb{R}$?

Proof. No. We can take $f_n(x) = \frac{x}{n}$, which is converges pointwise to $f(x) = 0$, but it does not converges uniformly to 0. Since for $\varepsilon = 1$, then we cannot find $N > 0$ such that for all $x \in \mathbb{R}$, $\frac{x}{N} < \varepsilon = 1$. □

Problem 14. Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of increasing functions that is pointwise convergent to a continuous function $f : [a,b] \to \mathbb{R}$. Prove that $f_n \rightrightarrows f$ uniformly on $[a,b]$.

Proof. We want to show that for $\forall \varepsilon > 0$, there exists $N > 0$ such that for $\forall n \geq N$ and $\forall x \in [a,b]$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Let $g_n(x) = f(x) - f_n(x)$, then $g_n(x)$ is increasing sequences of continuous functions. Let $\varepsilon > 0$, and

$$E_n = \{x \in [a,b] : g_n(x) = f(x) - f_n(x) < \varepsilon\}.$$

Then E_n is open since it is the inverse image of continuous function. And we can know that $\{E_n\}$ is an ascending sequence of open sets, since $g_n(x)$ is decreasing and if $x \in E_n$, then $g_n(x) < \varepsilon$ and of course $g_{n+1} < \varepsilon$, which implies $x \in E_{n+1}$. Then we have

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$$

Now letting $n \to 0$, and we have $g_n(x) \to 0$. Then we will have the sets of $x \in [a,b]$ such that $g_n(x) < \varepsilon$ will be the set $[a,b]$. Then,

$$[a,b] \subset \bigcup_{n=1}^{\infty} E_n,$$

and since $[a,b]$ is compact, then there exists a finite subcover, such that

$$[a,b] \subset \bigcup_{n=1}^{N} E_n.$$

Then for $\varepsilon > 0$ defined above, there exists $N > 0$ such that for all $\forall n \geq N$ and $\forall x \in [a,b]$, $|f_n(x) - f(x)| < \varepsilon$. $\square$

Problem 15. Let $\{f_n\}$ be a sequence of real valued C^1 functions on $[0,1]$ such that, for all n,

$$|f_n'(x)| \leq \frac{1}{\sqrt{x}} \quad (0 < x \leq 1),$$

$$\int_0^1 f_n(x)\,dx = 0.$$

Prove that the sequence has a subsequence that converges uniformly on $[0,1]$.

Proof. Let $\mathcal{F}$ be the set of all f_n that satisfy the above conditions. And we can prove that $\mathcal{F}$ is bounded. Indeed, if there exists f_n that is not bounded, then there is a point x_0 such that $\lim_{x \to x_0} f(x) = \infty$. Since $\int_0^1 f_n(x)\,dx = 0$, then there exists another point x_1 such that $\lim_{x \to x_1} f(x) = -\infty$. With mean value theorem, we have

$$|f'(c)| = \left| \frac{f(x_0) - f(x_1)}{x_0 - x_1} \right| = \infty > \frac{1}{\sqrt{c}},$$

where $c \in (x_0, x_1)$, or $c \in (x_1, x_0)$. Then this is a contradiction. Thus, $\mathcal{F}$ is bounded.

Now we prove that $\mathcal{F}$ is equicontinuous. Indeed, let $\varepsilon > 0$ and $x \in [0, 1)$, there exists $M > 0$ such that $|f_n'(x)| \leq M$ for all n and $x \in [0, 1]$. Pick $y \in (x, x + \delta)$, where $\delta = \varepsilon/M$. Then we have

$$|f_n(x) - f_n(y)| \leq f'(c')|x - y| \leq \varepsilon$$

where $c' \in (x, y)$. Thus we proved that $\mathcal{F}$ is equicontinuous. Hence, $\mathcal{F}$ has uniformly convergent subsequence. □

Problem 16. We know that every continuous function $f : [a, b] \to \mathbb{R}$ can be uniformly approximated by polynomials (Weierstrass' theorem). Prove that if a continuous function $f : \mathbb{R} \to \mathbb{R}$ can be uniformly approximated by polynomials on all of $\mathbb{R}$, then f is a polynomial.

Proof. Suppose that f can be uniformly approximated by polynomial $\{P_n(x)\}$ of degree at most d, and the polynomial $P_n(x)$ has the form

$$P_n(x) = a_n^d x^d + \cdots + a_n^1 x + a_n^0$$

such that

$$|f(x) - P_n(x)| < \frac{1}{n}, \forall x \in \mathbb{R}.$$

Then for any m, n, we have

$$|P_n(x) - P_m(x)| \leq |P_n(x) - f(x)| + |f(x) - P_m(x)| < \frac{1}{n} + \frac{1}{m}$$

which implies that $P_n(x) - P_m(x)$ is a polynomial which is bounded on $\mathbb{R}$ and hence a constant. Therefore, there exists some polynomial $P(x)$ and some $c_n \in \mathbb{R}$ such that

$$P_n(x) = P(x) + c_n.$$

Also, we can have $|c_n - c_m| < \frac{1}{n} + \frac{1}{m}$, which implies c_n is a Cauchy sequence, hence converging to a constant c.

Now we claim $f(x) = P(x) + c$. Let $\varepsilon > 0$, and pick $N = \frac{2}{\varepsilon}$, then for $\forall n > N$, we have $|c_n - c| < \frac{\varepsilon}{2}$. Now for $\forall n > N$, we have

$$|f(x) - P(x) - c| \leq |f(x) - P_n(x)| + |P_n(x) - P(x) - c| \leq \frac{1}{n} + |c_n - c| = \varepsilon.$$

□

Problem 17. Prove that if $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, 3, \ldots$ are differentiable functions such that

(a) $f_n(0) = 0$ for all n,

(b) $|f'_n(x)| \leq e^x$ for all n and all x,

then there is a subsequence of f_n that converges pointwise to a continuous function $f : \mathbb{R} \to \mathbb{R}$.
Hint. *Show that the family satisfies the assumptions of the Arzela-Ascoli theorem on every interval* $[-n, n]$ *and then apply the diagonal method.*

Proof. For any $x \in [-n, n]$, we have

$$\frac{|f_n(x) - f_n(0)|}{|x - 0|} = |f'(c)| \leq e^c < e^n,$$

where $c \in (0, x)$ or $c \in (x, 0)$. Then we have $|f_n(x)| < e^n n$ for all n and $x \in [-n, n]$. Then $\{f_n\}$ is the set of bounded functions. Now we prove that $\{f_n\}$ is equicontinuous on interval $[-n, n]$. For any $\varepsilon > 0$, we pick $\delta = \frac{\varepsilon}{e^n}$. Then for any $x, y \in [-n, n]$, if $|x - y| < \delta$, then

$$|f(x) - f(y)| = f'(c')|x - y| \leq e^n \frac{\varepsilon}{e^n} = \varepsilon,$$

where $c' \in (x, y)$ or $c' \in (y, x)$. Thus, $\{f_n\}$ is equicontinuous.

For interval $[-1, 1]$, $\{f_n\}$ has a convergent subsequence, denoted by $f_{11}, f_{12}, \cdots$. Now the sequence $\{f_{1n}(x)\}$ is bounded on the interval $[-2, 2]$, so it has convergent subsequence, denoted by $f_{21}, f_{22}, \cdots$. Contiune this process and we can have subsequences

$$\begin{gathered} f_{11}, f_{12}, f_{13}, \cdots \\ f_{21}, f_{22}, f_{23}, \cdots \\ f_{31}, f_{32}, f_{33}, \cdots \end{gathered}$$

Sequence in each line is a subsequence of the previous one. We now select $f_{11}, f_{22}, f_{33}, \cdots$.

We claim that $\{f_{nn}\}$ is uniformly convergent at every point of $\mathbb{R}$. Let $\varepsilon > 0$, and for any $x \in \mathbb{R}$, there exists $N > 0$, such that $x_i \in [-N, N]$ and $|x - x_i| < \delta$, where $\delta > 0$ as in the definition of equicontinuity. For $n, m \geq N$, we have

$$|f_{nn}(x_i) - f_{mm}(x_i)| < \frac{\varepsilon}{3}, x \in [-N, N].$$

Then we have]

$$|f_{nn}(x) - f_{mm}(x)| \leq |f_{nn}(x) - f_{nn}(x_i)| + |f_{nn}(x_i) - f_{mm}(x_i)| + |f_{mm}(x_i) - f_{mm}(x)| < \varepsilon.$$

Hence $\{f_{nn}(x)\}$ is a Cauchy sequence. Now we define

$$f(x) = \lim_{n \to \infty} f_{nn}(x),$$

and we can have

$$|f(x) - f_{nn}(x)| < \varepsilon$$

as $m \to \infty$. $\square$

Problem 18. Let the functions $f_n : [0,1] \to [0,1]$, $n = 1, 2, \ldots$, satisfy $|f_n(x) - f_n(y)| \leq |x - y|$ whenever $|x - y| \geq 1/n$. Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 19. If $f = (f_1, \ldots, f_n) : [a,b] \to \mathbb{R}^n$ is a continuous function, then we define

$$\int_a^b f(t)\,dt = \left\langle \int_a^b f_1(t)\,dt, \ldots, \int_a^b f_n(t)\,dt \right\rangle .$$

Prove that

$$\left\| \int_a^b f(t)\,dt \right\| \leq \int_a^b \|f(t)\|\,dt.$$

Proof. WRITE YOUR SOLUTION HERE. □