Homework 1 for MATH 1530

Zhen Yao

Problem 1. Use the equivalence

$$p \wedge (q \vee r) \quad \equiv \quad (p \wedge q) \vee (p \wedge r) \tag{1}$$

to prove

$$p \vee (q \wedge r) \quad \equiv \quad (p \vee q) \wedge (p \vee r).$$

To this end apply (1) to $\neg p$, $\neg q$, $\neg r$ in place of p, q, r, and negate the statement using De Morgan's Laws.

Proof. Place $\neg p$, $\neg q$, $\neg r$ into the first equation, and we can have

$$(\neg p) \wedge ((\neg q) \vee (\neg r)) \equiv (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$$

and we negate the statement by using De Morgan's Laws several times, which yields

$$\begin{aligned} \neg\left((\neg p) \wedge ((\neg q) \vee (\neg r))\right) &\equiv \neg(\neg p \wedge \neg q) \wedge \neg(\neg p \wedge \neg r) \\ \Rightarrow p \vee \neg\left((\neg q) \vee (\neg r)\right) &\equiv (p \vee q) \wedge (p \vee r) \\ \Rightarrow p \vee (q \wedge r) &\equiv (p \vee q) \wedge (p \vee r) \end{aligned}$$

□

Problem 2. Negate the statement[1]

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in \mathbb{R} \;\; \forall y \in \mathbb{R} \;\; (|x - y| < \delta \;\; \Rightarrow \;\; |\sin x - \sin y| < \varepsilon).$$

Proof.

$$\exists \varepsilon > 0 \;\; \forall \delta > 0 \;\; \exists x \in \mathbb{R} \;\; \exists y \in \mathbb{R} \;\; (|x - y| < \delta \;\; \wedge \;\; |\sin x - \sin y| \geq \varepsilon).$$

□

Problem 3. Negate the statement: *For all real numbers x, y satisfying $x < y$, there is a rational number q such that $x < q < y$.* Formulate the negation as a sentence and not as a formula involving quantifiers.

Proof. Formulate the statement into formula is that

$$\forall x \in \mathbb{R} \;\; \forall y \in \mathbb{R} \; \exists q \in \mathbb{Q} \;\; (x < y \Rightarrow x < q < y)$$

The negation of this formula is

$$\exists x \in \mathbb{R} \;\; \exists y \in \mathbb{R} \; \forall q \in \mathbb{Q} \;\; (x < y \wedge ((q \leq x) \vee (q \geq y)))$$

So the sentence is: *There exist real numbers x, y satisfying $x < y$, then for any rational number $q \in \mathbb{Q}$, q satisfies the condition $q \leq x$ or $y \leq q$.* □

Problem 4. Use an argument by contradiction prove that $\sqrt{3}$ is irrational.

[1]This is a true statement known as uniform continuity of the function $\sin x$. However, you are not asked to prove the statement only to negate it.

Proof. Let's assume that $\sqrt{3}$ is a rational number. Then $\sqrt{3} = p/q$ for some positive integers and we assume p and q have no common factors. Then we have

$$3 = \frac{p^2}{q^2}, \qquad p^2 = 3q^2$$

(1)Firstly, if p is an even number, and the square of an even number is still even, so $3q^2$ is an even number. The q^2 is an even number since 3 is odd, then we have q is also an even number. Thus, p and q are all even number and have common factor 2, which contradicts our assumption.
(2)Secondly, if p is an odd number, and the square of an odd number is still odd, so $3q^2$ is an odd number. The q^2 is an odd number since 3 is odd, then we have q is also an odd number. Now both p and q are odd number, then we can set $p = 2n + 1$ and $q = 2m + 1$ where n and m are some positive integers. And we have

$$4n^2 + 4n + 1 = 3(4m^2 + 4m + 1)$$
$$\Rightarrow 2n^2 + 2n = 6m^2 + 6m + 1$$

The left side is even and the right side is odd, which is impossible. Also this condition contradicts our assumption. So $\sqrt{3}$ is not a rational number. □

Problem 5. Prove the following statement[2]

$$\forall \varepsilon > 0 \;\; \exists n_0 \in \mathbb{N} \;\; \forall n \in \mathbb{N} \quad (n \geq n_0 \;\Rightarrow\; n^{-1} \leq \varepsilon).$$

Proof. Suppose to the contrary that the statement is false, then its negation is true

$$\exists \varepsilon > 0 \;\; \forall n_0 \in \mathbb{N} \;\; \exists n \in \mathbb{N} \quad (n \geq n_0 \;\wedge\; n^{-1} > \varepsilon)$$

There exist a $\varepsilon > 0$ such that for every n_0, there is n such that $n \geq n_0$ and $n^{-1} > \varepsilon$, and it is true for $n_0 = \frac{2}{\varepsilon}$. This means that for $n_0 = \frac{2}{\varepsilon}$, there is a n such that

$$n \geq \frac{2}{\varepsilon} \qquad \textbf{and} \qquad \frac{1}{n} > \varepsilon \Rightarrow n < \frac{1}{\varepsilon}$$

These two inequalities contradict each other. The proof is complete. □

Problem 6. Find a mistake in the solution to Problem 9 provided on page 19 in my notes and write a correct solution.

Proof. (1)The mistake is that $|f(x) - f(y)| < \varepsilon$ for all $\varepsilon > 0$ does not imply that $|f(x) - f(y)| < 0$, we cannot have $|f(x) - f(y)| = 0$. So the proof is not correct.

(2) The continuous functions satisfy the condition. clearly the continuous functions satisfy this conditions since it is defined as this way. Suppose that a function f satisfies this condition, then we have for any $\varepsilon > 0$, and for any $x, y \in \mathbb{R}$, there exists a $\delta > 0$, such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

then we have

$$\lim_{x \to y} f(x) = f(y)$$

[2]Compare with Example 1.12.

since $y \in \mathbb{R}$ is arbitrary, so function f is continuous in every point $y \in \mathbb{R}$. Thus, the function f that satisfies the condition is a continuous function. $\square$

Problem 7. Prove that for any set A and any family of sets $\{A_i\}_{i\in I}$

$$A \setminus \bigcup_{i\in I} A_i = \bigcap_{i\in I} (A \setminus A_i)\,,$$

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Proof. (1)We have

$$\begin{aligned} x \in A \setminus \bigcup_{i\in I} A_i &= x \in A \wedge \neg \left(x \in \bigcup_{i\in I} A_i \right) \\ &= x \in A \wedge \neg\,(x \in A_1 \vee \cdots \vee x \in A_i \cdots, i \in I) \\ &= x \in A \wedge (x \notin A_1 \wedge \cdots \wedge x \notin A_i \cdots) \\ &= (x \in A \wedge x \notin A_1) \wedge \cdots \wedge (x \in A \wedge x \notin A_i) \cdots \\ &= (x \in A \setminus A_1) \wedge \cdots \wedge (x \in A \setminus A_i) \cdots \\ &= x \in (A \setminus A_1) \wedge \cdots \wedge (A \setminus A_i) \\ &= x \in \bigcap_{i\in I} (A \setminus A_i) \end{aligned}$$

(2)We have

$$\begin{aligned} x \in A \setminus \bigcap_{i\in I} A_i &= x \in A \wedge \neg \left(x \in \bigcap_{i\in I} A_i \right) \\ &= x \in A \wedge \neg(x \in A_1 \wedge \cdots \wedge x \in A_i \cdots) \\ &= x \in A \wedge (x \notin A_1 \vee \cdots \vee x \notin A_i \cdots) \\ &= (x \in A \wedge x \notin A_1) \vee \cdots \vee (x \in A \wedge x \notin A_i) \cdots \\ &= x \in (A \setminus A_1) \vee \cdots \vee x \in (A \setminus A_i) \cdots \\ &= x \in \bigcup_{i\in I} (A \setminus A_i) \end{aligned}$$

The proof is complete. $\square$

Problem 8. Prove that if $f : X \to Y$ is a function and $A_1, A_2, A_3, \ldots$ are subsets of X, then

$$f\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f(A_i)\,,$$

and

$$f\left(\bigcap_{i=1}^{\infty} A_i\right) \subset \bigcap_{i=1}^{\infty} f(A_i). \tag{2}$$

Provide an example to show that we do not necessarily have equality in (2)

Proof. (1)First, if $y \in f\left(\bigcup_{i=1}^{\infty} A_i\right)$, then we can have $y = f(x)$ for some $x \in \bigcup_{i=1}^{\infty} A_i$. Then we can know there exist one subset A_k such that $x \in A_k$, then we have $y = f(x) \in f(A_k) \subset \bigcup_{i=1}^{\infty} A_i$. We proved that

$$f\left(\bigcup_{i=1}^{\infty} A_i\right) \Rightarrow \bigcup_{i=1}^{\infty} f(A_i)$$
$$f\left(\bigcup_{i=1}^{\infty} A_i\right) \subset \bigcup_{i=1}^{\infty} f(A_i)$$

If $y \in \bigcup_{i=1}^{\infty} f(A_i)$, then there exist one A_m such that $y \in f(A_m)$. Then we have $y = f(x)$ for some $x \in A_m$. Since $A_m \subset \bigcup_{i=1}^{\infty} A_i$, then $y = f(x) \in f\left(\bigcup_{i=1}^{\infty} A_i\right)$. We proved that

$$\bigcup_{i=1}^{\infty} f(A_i) \subset f\left(\bigcup_{i=1}^{\infty} A_i\right)$$

i.e.,

$$f\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f(A_i)$$

(2)If $y \in f\left(\bigcap_{i=1}^{\infty} A_i\right)$, then $y = f(x)$ for some $x \in \bigcap_{i=1}^{\infty} A_i$. Thus, $x \in A_k$ for every $A_k, k = 1, 2, 3, \cdots$, so we have $y = f(x) \in f(A_k)$ for every A_k, and then $y = f(x) \in \bigcap f(A_i)$ which implies

$$y \in f\left(\bigcap_{i=1}^{\infty} A_i\right) \Rightarrow \bigcap_{i=1}^{\infty} f(A_i)$$
$$y \in f\left(\bigcap_{i=1}^{\infty} A_i\right) \subset \bigcap_{i=1}^{\infty} f(A_i)$$

Example: Let $f(x) = x^2, x \in \mathbb{R}$, and $A_1 = [-4, 0], A_2 = [0, 4]$. Then we have $f(A_1) = [0, 16]$ and $f(A_2) = [0, 16]$, so $f(A_1) \bigcap f(A_2) = [0, 16]$. However, $A_1 \bigcap A_2 = \emptyset$, then $f(A_1 \bigcap A_2) = \emptyset$. Then we have, in this case, $f(A_1) \bigcap f(A_2) \not\Rightarrow f(A_1 \bigcap A_2)$.

□

Problem 9. Prove that if $f : X \to Y$ is one-to-one and $A_1, A_2, A_3, \ldots$ are subsets of X, then

$$f\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} f(A_i).$$

Proof. If $y \in \bigcap_{i=1}^{\infty} f(A_i)$, then there exist one $x \in X$ such that $y = f(x) \in \bigcap_{i=1}^{\infty} f(A_i)$. Then we have $f(x) \in f(A_k)$ for every k, then we get $x \in \bigcap_{i=1}^{\infty} A_i$. Hence, $y = f(x) \in$

$f\left(\bigcap_{i=1}^{\infty} A_i\right)$, which implies

$$\bigcap_{i=1}^{\infty} f(A_i) \Rightarrow f\left(\bigcap_{i=1}^{\infty} A_i\right)$$
$$\bigcap_{i=1}^{\infty} f(A_i) \subset f\left(\bigcap_{i=1}^{\infty} A_i\right)$$

Since we already know that $f\left(\bigcap_{i=1}^{\infty} A_i\right) \subset \bigcap_{i=1}^{\infty} f(A_i)$, then we proved that, if f is one-to-one function, then

$$\bigcap_{i=1}^{\infty} f(A_i) = f\left(\bigcap_{i=1}^{\infty} A_i\right)$$

□

Problem 10. Prove that $5^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$.

Proof. (1)For $n = 1$, $5^2 - 1 = 24 = 3 \cdot 8$, which is divisible by 8.

(2)For $n > 1$, suppose that $5^{2n} - 1$ is divisible by 8. We need to prove that $5^{2(n+1)} - 1$ is divisible by 8. By assumption, that $5^{2n} - 1 = 8k$ for some $k \in \mathbb{N}$. We have

$$\begin{aligned} 5^{2(n+1)} - 1 &= 5^{2n}5^2 - 1 = 25 \cdot (8k+1) - 1 = 25 \cdot 8k + 3 \cdot 8 \\ &= 8 \cdot (25k + 3) \end{aligned}$$

which is divisible by 8. □

Problem 11. Prove that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$.

Proof. (1)For $n = 1$, we have $1 \geq 1$, which is true.

(2)Suppose the inequality is true for $n = k$, then we need to prove it for $k + 1$

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

Now all we need to prove is that $\sqrt{k} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$, and we have

$$\begin{aligned} \sqrt{k} + \frac{1}{\sqrt{k+1}} - \sqrt{k+1} &= \frac{1}{\sqrt{k+1}}\left(\sqrt{k(k+1)} + 1 - (k+1)\right) \\ &= \frac{1}{\sqrt{k+1}}\left(\sqrt{k^2+k} - k\right) \\ &> 0 \end{aligned}$$

It is easy to see that $\sqrt{k^2+k} - k > 0$, then the proof is complete. □

Problem 12. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive numbers. Prove that

$$\prod_{i=1}^{n}(a_i + b_i)^{1/n} \geq \prod_{i=1}^{n} a_i^{1/n} + \prod_{i=1}^{n} b_i^{1/n}.$$

Hint: Divide both sides by the expression on the left hand side and use the arithmetic-geometric mean inequality.

Proof. Divide the right side of the equation by $\prod_{i=1}^{n}(a_i+b_i)$ and apply arithmetic-geometric mean inequality, then we have

$$\prod_{i=1}^{n}\left(\frac{a_i}{a_i+b_i}\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n}\frac{a_i}{a_i+b_i}$$
$$\text{and} \quad \prod_{i=1}^{n}\left(\frac{b_i}{a_i+b_i}\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n}\frac{b_i}{a_i+b_i}$$

Adding these two inequalities and we have

$$\prod_{i=1}^{n}\left(\frac{1}{a_i+b_i}\right)^{\frac{1}{n}}\left(\prod_{i=1}^{n}a_i^{1/n}+\prod_{i=1}^{n}b_i^{1/n}\right) \leq \frac{1}{n}\sum_{i=1}^{n}1 = \frac{1}{n}n = 1$$

And multiplying both sides with $\prod_{i=1}^{n}(a_i+b_i)^{\frac{1}{n}}$, we can have final result

$$\prod_{i=1}^{n}a_i^{1/n}+\prod_{i=1}^{n}b_i^{1/n} \leq \prod_{i=1}^{n}(a_i+b_i)^{\frac{1}{n}}$$

□

Problem 13. Prove that Schwartz inequality

$$\left|\sum_{i=1}^{n}a_i\,b_i\right| \leq \left(\sum_{i=1}^{n}a_i^2\right)^{1/2}\left(\sum_{i=1}^{n}b_i^2\right)^{1/2}.$$

Proof. (1)For $n=1$, we have $\|a_1b_1\| \leq a_1b_1$. It is true.

(2)For $n=2$, by arithmetic-geometric mean inequality, we have

$$\begin{aligned}(a_1b_1+a_2b_2)^2 &= a_1^2b_1^2+2a_1a_2b_1b_2+a_2^2b_2^2\\ &\leq a_1^2b_1^2+a_2^2b_2^2+(a_1^2b_2^2+a_2^2b_1^2)\\ &= (a_1^2+a_2^2)(b_1^2+b_2^2)\end{aligned}$$

Then the inequality holds for $n=2$.

(3)Suppose the inequality holds for $n=k>2$, then we need to show that it still holds for $n=k+1$. We have

$$\begin{aligned}\left(\sum_{i=1}^{k+1}a_ib_i\right)^2 &= \left(\sum_{i=1}^{k}a_ib_i\right)^2+2a_{k+1}b_{k+1}\left(\sum_{i=1}^{k}a_ib_i\right)+(a_{k+1}b_{k+1})^2\\ &\leq \left(\sum_{i=1}^{k}a_i^2\right)\left(\sum_{i=1}^{k}b_i^2\right)+2a_{k+1}b_{k+1}\left(\sum_{i=1}^{k}a_ib_i\right)+(a_{k+1}b_{k+1})^2\end{aligned}$$

Meanwhile, we also have

$$\left(\sum_{i=1}^{k+1}a_i^2\right)\left(\sum_{i=1}^{k+1}b_i^2\right) = \left(\sum_{i=1}^{k}a_i^2\right)\left(\sum_{i=1}^{k}b_i^2\right)+b_{k+1}^2\left(\sum_{i=1}^{k}a_i^2\right)+a_{k+1}^2\left(\sum_{i=1}^{k}b_i^2\right)+(a_{k+1}b_{k+1})^2$$

Then we only have to prove that

$$b_{k+1}^2\left(\sum_{i=1}^k a_i^2\right)+a_{k+1}^2\left(\sum_{i=1}^k b_i^2\right)\geq 2a_{k+1}b_{k+1}\left(\sum_{i=1}^k a_ib_i\right)$$

By by arithmetic-geometric mean inequality, we have

$$\frac{b_{k+1}}{a_{k+1}}\left(\sum_{i=1}^k a_i^2\right)+\frac{a_{k+1}}{b_{k+1}}\left(\sum_{i=1}^k b_i^2\right)\geq 2\sqrt{\left(\sum_{i=1}^k a_i^2\right)\left(\sum_{i=1}^k b_i^2\right)}$$

multiplying both sides with $b_{k+1}a_{k+1}$ and we have

$$\begin{aligned}b_{k+1}^2\left(\sum_{i=1}^k a_i^2\right)+a_{k+1}^2\left(\sum_{i=1}^k b_i^2\right)&\geq 2a_{k+1}b_{k+1}\sqrt{\left(\sum_{i=1}^k a_i^2\right)\left(\sum_{i=1}^k b_i^2\right)}\\&\geq 2a_{k+1}b_{k+1}\left(\sum_{i=1}^k a_ib_i\right)\end{aligned}$$

where the the last step is from the assumption that the inequality holds for $n=k$. The proof is complete. □

Problem 14. Use the Schwarz inequality to prove that if $a_1,\ldots,a_n>0$, then

$$\frac{n}{\frac{1}{a_1}+\ldots+\frac{1}{a_n}}\leq\frac{a_1+\ldots+a_n}{n}.$$

Proof. By Schwarz inequality, we have

$$\begin{aligned}(a_1+\cdots+a_n)\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right)&\geq\left(\sum_{i=1}^n a_i\frac{1}{a_i}\right)^2\\&\geq n^2\end{aligned}$$

then we rearrange the inequality, which implies

$$\frac{a_1+\cdots+a_n}{n}\geq\frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}$$

The proof is complete. □