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Advanced Calculus 2

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Final Exam

Due on May 1, 2020

Problem	Possible points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120	

You need 100 points so 2 problems is a bonus.

Problem 1. Suppose $f : \mathbb{R}^2_+ \to \mathbb{R}$ is a continuous function defined on

$$\mathbb{R}^2_+ = \{(x, y) : x \in \mathbb{R}, \ y > 0\}.$$

Assume also that the limits

$$g(u, v) = \lim_{t \to 0} \frac{f((u + t)\cos v, (u + t)\sin v) - f(u \cos v, u \sin v)}{t},$$

and

$$h(u, v) = \lim_{t \to 0} \frac{f(u\cos(v + t)), u\sin(v + t)) - f(u \cos v, u \sin v)}{t},$$

exist and define continuous functions g, h on the domain

$$D = \{(u, v) : u > 0, \ 0 < v < \pi\}.$$

Prove that the function f is differentiable on $\mathbb{R}^2_+$.

Proof. Let $\Phi : D \to \mathbb{R}^2_+$ defined as $\Phi(u, v) = (u \cos v, u \sin v)$ is diffeomorphism, since $D\Phi(u, v) = \begin{bmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{bmatrix} = u > 0$. Also,

$$g(u, v) = \frac{\partial(f \circ \Phi)}{\partial u}, \quad h(u, v) = \frac{\partial(f \circ \Phi)}{\partial v}$$

and partial derivatives of $f \circ \Phi$ are continuous, and then $f \circ \Phi \in C^1$. Thus, $f = (f \circ \Phi) \circ \Phi^{-1} \in C^1$. □

Problem 2. Let $f \in C^1(\mathbb{R})$ be a continuously differentiable function such that $|f'(x)| \leq 1/2$ for all $x \in \mathbb{R}$. Define $g : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g(x, y) = (x + f(y), y + f(x)).$$

Prove that

(1) g is a diffeomorphism,
(2) $g(\mathbb{R}^2) = \mathbb{R}^2$,
(3) the area $|g([0, 1]^2)|$ of the image of the unit square belongs to the interval $[3/4, 5/4]$.

Hint: Among other tools use the contraction principle.
Proof.

(a) For part (a) and (b), it suffices to show that $\det Dg > 0$ on $\mathbb{R}^2$ and for every (x_0, y_0), there is a unique (x, y) such that $g(x, y) = (x_0, y_0)$. We have

$$\det Dg(x, y) = \begin{vmatrix} 1 & f'(y) \\ f'(x) & 1 \end{vmatrix} = 1 - f'(x)f'(y) \geq \frac{3}{4} > 0.$$

To prove the existence of a unique solution to $g(x, y) = (x_0, y_0)$, use the contraction principle. Let

$$\begin{aligned} T(x, y) &= -g(x, y) + (x, y) + (x_0, y_0) \\ &= (x_0 - f(y), y_0 - f(x)). \end{aligned}$$

Then,

$$\begin{aligned}T(x,y)-T(x',y') &= (x_0-f(y),y_0-f(x))-(x_0-f(y'),y_0-f(x'))\\ &= (f(y')-f(y),f(x')-f(x))\\ &= (f'(\xi)(y'-y),f'(\zeta)(x'-x)),\end{aligned}$$

and then

$$\begin{aligned}|T(x,y)-T(x',y')| &= \sqrt{|f(y')-f(y)|^2+|f(x')-f(x)|^2}\\ &\le \frac{1}{2}\sqrt{|y'-y|^2+|x'-x|^2}.\end{aligned}$$

Thus, T is a contraction. Hence there is a unique fixed point (x,y) of T such that $(x_0-f(y),y_0-f(x))=(x,y)$, and it follows that

$$g(x,y)=(x+f(y),y+f(x))=(x_0,y_0).$$

Thus, g is a diffeomorphism.

(c) Since $\det Dg = 1-f'(x)f'(y)\in\left[\frac{3}{4},\frac{5}{4}\right]$, then

$$\left|g([0,1]^2)\right| = \int_{[0,1]^2}|\det Dg|\in\left[\frac{3}{4},\frac{5}{4}\right].$$

□

Problem 3. Prove that the tangent planes to the surface S defined by $x^2+y^2-z^2=1$ at the points $(x,y,0)\in S$ are parallel to the z-axis.

Proof. The gradient vector of S at point $(x,y,0)$ is $(2x,2y,-2z)|_{z=0}=(2x,2y,0)$. And this vector is certainly orthogonal to the tangent planes to S at $(x,y,0)$, and also it is orthogonal to z-axis. Thus, these tangent planes are parallel to the z-axis. □

Problem 4. Prove that if $f:[0,1]\to[0,1]$ is a continuous function then its graph as a subset of $\mathbb{R}^2$ has measure zero.

Proof. Since f is continuous on a compact set, then f is uniformly continuous, i.e., for any $\varepsilon>0$, there exists $\delta>0$, such that if $|x-y|<\delta$, then $|f(x)-f(y)|\le\varepsilon/2$. Now let $\delta=1/n$ for some n such that above condition holds. Denote the set $\left[f\left(\frac{k}{n}\right),f\left(\frac{k+1}{n}\right)\right]\times\left[f\left(\frac{k}{n}\right)-\frac{\varepsilon}{2},f\left(\frac{k}{n}\right)+\frac{\varepsilon}{2}\right]$ by $P_k, k=0,1,\cdots,n-1$ then the volume $|P_k|=\frac{\varepsilon}{n}$. Then, the graph G_f of f satisfies $G_f\subset\bigcup_{k=1}^n P_k$ and then

$$|G_f|=\sum_{i=1}^n|P_k|=n\frac{\varepsilon}{n}=\varepsilon\to 0.$$

Thus, the graph of f has measure zero. □

Problem 5. Let $f:\mathbb{R}^3\to\mathbb{R}^2$ be a continuous function. Let $K\subset\mathbb{R}^3$ be a compact set such that $|f(x)-f(y)|\le 2012|x-y|^2$ for all $x,y\in K$. Prove that the set $f(K)\subset\mathbb{R}^2$ has measure zero as a subset of $\mathbb{R}^2$.

Proof. Since K is compact, then there exists $M > 0$ such that $K \subset [-M, M]^3$ and K can be covered by n^3 closed cubes Q_i of side-length $2M/n$, whose diameter is $2\sqrt{3}M/n$. Each such cube can be covered by a closed ball of radius $2\sqrt{3}M/n$ centered at any point of the cube. Then, K can be covered by no more than n^3 balls $B\left(x_i, 2\sqrt{3}M/n\right), x_i \in K$, such that $Q_i \cap K \neq \varnothing$ and $Q_i \cap K \subset B\left(x_i, 2\sqrt{3}M/n\right)$. Now,

$$f\left(B\left(x_i, 2\sqrt{3}M/n\right)\right) \subset B\left(f(x_i), 2012\left(2\sqrt{3}M/n\right)^2\right),$$

also, $K \subset \bigcup_i^k B\left(x_i, 2\sqrt{3}M/n\right)$, $k \leq n^3$, then,

$$f(K) \subset \bigcup_i^k B\left(f(x_i), 2012\left(2\sqrt{3}M/n\right)^2\right).$$

Denote $2012\left(2\sqrt{3}M/n\right)^2$ by r_i, then

$$\sum_{i=1}^k r_i^2 \leq cn^{-4}n^3 = cn^{-1} \xrightarrow{n\to\infty} 0.$$

Thus, $f(K)$ has measure zero. □

Problem 6. Assume that $f : [0,1] \to [0,1]$ is a continuous function such that the set $\{x \in [0,1] : f(x) = 1\}$ has measure zero. Prove directly (without using any results like monotone or dominated convergence theorem) that

$$\lim_{n\to\infty} \int_0^1 f(x)^n \, dx = 0.$$

Proof. Let $E = \{x \in [0,1] | f(x) = 1\}$. For fixed $\varepsilon > 0$, since E has zero measure, then $E \subset \bigcup_{i=1}^k I_i$, where I_i are open and $\sum_{i=1}^k |I_i| < \varepsilon/2$, i.e., $\bigcup_{i=1}^k I_i$ are a covering of E by open intervals of total lengths less than $\varepsilon/2$.

By compactness of $[0,1]$, $[0,1] \setminus \bigcup_{i=1}^k I_i$ is closed and hence compact. Then f attains maximum M on this set, which is strictly less than 1, and hence $|f(x)| \leq M < 1$ for all $x \in [0,1] \setminus \bigcup_{i=1}^k I_i$. Then,

$$f(x) \leq \begin{cases} 1, & x \in \bigcup_{i=1}^k I_i, \\ M^n, & x \notin \bigcup_{i=1}^k I_i. \end{cases}$$

and then we have

$$\begin{aligned} \int_0^1 f^n \, dx &\leq 1 \cdot \left|\sum_{i=1}^k I_i\right| + M^n \cdot 1, \\ &\leq \frac{\varepsilon}{2} + M^n. \end{aligned}$$

Since $M < 1$, then there exists $N > 0$ such that for all $n \geq N$, $M^n < \varepsilon/2$. Thus, for all $n \geq N$, $\int_0^1 f^n \, dx \leq \varepsilon \to 0$. □

Problem 7. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ and $\mathbf{v}_i : \mathbb{R} \to \mathbb{R}^n$, $i = 1, 2, \ldots, n-1$, be C^∞ smooth functions such that for any $t \in \mathbb{R}$ the vectors

$$\gamma'(t), \mathbf{v}_1(t), \ldots, \mathbf{v}_{n-1}(t)$$

form an orthonormal basis of $\mathbb{R}^n$ (here we differentiate γ but **do not** differentiate $\mathbf{v}_i$, $i = 1, 2, \ldots, n-1$).

Consider the mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi(x_1, \ldots, x_n) = \gamma(x_n) + \sum_{i=1}^{n-1} x_i \mathbf{v}_i(x_n).$$

(a) Find the derivative $D\Phi(x_1, \ldots, x_n)$;

(b) Prove that Φ is a diffeomorphism in a neighborhood of and point of the form $(0, \ldots, 0, x_n)$;

(c) Find the limit

$$\lim_{r \to 0} \frac{|\Phi(B^n(0,r))|}{|B^n(0,r)|},$$

where $B^n(0,r)$ denotes the ball of radius r centered at the origin and $|A|$ stands for the volume of the set A.

Proof.

(a)

$$D\Phi(x_1, \cdots, x_n) = \begin{pmatrix} \mathbf{v}_1(x_n) & & & & \\ & \mathbf{v}_2(x_n) & & & \\ & & \ddots & & \\ & & & \mathbf{v}_{n-1}(x_n) & \\ & & & & \gamma'(x_n) + \sum_{i=1}^{n-1} x_i \mathbf{v}_i'(x_n) \end{pmatrix}.$$

(b)

$$D\Phi(0, \ldots, 0, x_n) = \begin{pmatrix} \mathbf{v}_1(x_n) & & & & \\ & \mathbf{v}_2(x_n) & & & \\ & & \ddots & & \\ & & & \mathbf{v}_{n-1}(x_n) & \\ & & & & \gamma'(x_n) \end{pmatrix}.$$

is an orthogonal matrix and then $\det D\Phi(0, \ldots, 0, x_n) = \pm 1 \neq 0$. Then it remains to show that Φ is bijection. And this is obvious. Thus, Φ is a diffeomorphism in a neighborhood of a point of the form $(0, \ldots, 0, x_n)$.

(c) By the change of variable, we have

$$|\Phi(B^n(0,r))| = \int_{B^n(0,r)} |J_\Phi| \, dA$$

and $|J_\Phi| \to 1$ as $r \to 0$, Then for any $\varepsilon > 0$, there exists r such that $||J_\Phi| - 1| < \varepsilon$, and then

$$\left| \frac{|\Phi(B^n(0,r))|}{|B^n(0,r)|} - 1 \right| = \left| \frac{\int_{B^n(0,r)} |J_\Phi| \, dA}{|B^n(0,r)|} - 1 \right| < \varepsilon.$$

Thus, the limit is equal to 1.

$\square$

Problem 8. Prove that if $K \in C^1(\mathbb{R}^2 \setminus \{(0,0)\})$ satisfies the estimate

$$|\nabla K(x)| \leq \frac{1}{|x|^3} \quad \text{for all } x \neq (0,0)$$

then there is a constant $C > 0$ such that

$$\iint_{\{x \in \mathbb{R}^2 : |x| > 2|y|\}} |K(x-y) - K(x)|\, dx \leq C$$

for all $y \in \mathbb{R}^2$.
Hint: *Use the mean value theorem to estimate $|K(x-y) - K(x)|$ and then integrate in polar coordinates.*

Proof. Denote the left hand side by I. If $y = 0$, then $I = 0$. We can assume $|y| > 0$, and the point of interval connecting x to $x - y$ are of form $x - ty, t \in [0,1]$. Then, with $|x| > 2|y|$,

$$|x - ty| > |x| - |y| > \frac{|x|}{2}.$$

With mean value theorem,

$$\begin{aligned}
|K(x-y) - K(x)| &\leq |\nabla K(x - ty)| \cdot |(x-y) - x| \\
&\leq \frac{|y|}{|x - ty|^3} \\
&\leq \frac{8|y|}{|x|^3}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\iint_{|x|>2|y|} |K(x-y) - K(x)|\, dx &\leq \iint_{|x|>2|y|} \frac{8|y|}{|x|^3}\, dx \\
&= 8|y| \int_0^{2\pi} \int_{2|y|}^{\infty} \frac{1}{r^3} r\, dr d\theta \\
&= 8|y| 2\pi \left(-r^{-1}\right) \Big|_{2|y|}^{\infty} = 8\pi.
\end{aligned}$$

□

Problem 9. Use Green's theorem to prove the following result: If the vertices of a polygon, in counterclockwise order, are (x_1, y_1), (x_2, y_2), $\ldots$,(x_n, y_n), then the area of the polygon is

$$A = \frac{1}{2} \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i),$$

where we use notation $x_{n+1} = x_1$, $y_{n+1} = y_1$.

Proof. By Green's theorem, the area is

$$A = \frac{1}{2} \int_C x\, dy - y\, dx = \frac{1}{2} \sum_{i=1}^{n} \int_{(x_i, y_i)}^{(x_{i+1}, y_{i+1})} x\, dy - y\, dx.$$

Let $\alpha(t) = (x_i + t(x_{i+1} - x_i), y_i + t(y_{i+1} - y_i))\,, t \in [0,1]$ is a parameterization of the segment connecting (x_i, y_i) to (x_{i+1}, y_{i+1}). Let $\gamma(t) = (x(t), y(t)) \to \mathbb{R}^2, t \in [a,b]$, then

$$\int_\gamma P\,dx + Q\,dy = \int_a^b \left(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\right)\,dt$$

Then,

$$\begin{aligned}
\int_{(x_i,y_i)}^{(x_{i+1},y_{i+1})} x\,dy - y\,dx &= \int_0^1 (x_i + t(x_{i+1}-x_i))(y_{i+1}-y_i) - (y_i + t(y_{i+1}-y_i))(x_{i+1}-x_i)\,dt \\
&= \left(tx_i + \frac{t^2}{2}(x_{i+1}-x_i)\right)(y_{i+1}-y_i)\Big|_0^1 \\
&- \left(ty_i + \frac{t^2}{2}(y_{i+1}-y_i)\right)(x_{i+1}-x_i)\Big|_0^1 \\
&= x_i y_{i+1} - x_{i+1} y_i,
\end{aligned}$$

and hence,

$$A = \frac{1}{2}\sum_{i=1}^{n}(x_i y_{i+1} - x_{i+1} y_i).$$

$\square$

Problem 10. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^1 boundary and let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, $\Phi(x,y) = (u(x,y), v(x,y))$, be a C^2 diffeomorphism. Prove that

$$\int_{\partial\Omega} uv_x\,dx + uv_y\,dy = \pm|\Phi(\Omega)|\,,$$

where $|\Phi(\Omega)|$ denotes the area of $\Phi(\Omega)$ and $\partial\Omega$ has positive orientation. Show on examples that both cases $+|\Phi(\Omega)|$ and $-|\Phi(\Omega)|$ are possible.

Proof. By Green's theorem, $\int_{\partial\Omega} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, then

$$\begin{aligned}
\int_{\partial\Omega} uv_x\,dx + uv_y\,dy &= \iint_\Omega \frac{\partial}{\partial x}(uv_y) - \frac{\partial}{\partial y}(uv_x) \\
&= \iint_\Omega u_x v_y + uv_{yx} - u_y v_x - uv_{xy} \\
&= \iint_\Omega u_x v_y - u_y v_x,
\end{aligned}$$

where in the last step we used the fact that $\Phi \in C^2$. Moreover,

$$\iint_\Omega u_x v_y - u_y v_x = \iint_\Omega \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \iint_\Omega \det D\Phi.$$

In particular, with change of variable, we have

$$\int_{\Phi(\Omega)} f = \int_\Omega (f \circ \Phi)|\det D\Phi|,$$

setting $f = 1$ gives

$$\int_\Omega |\det D\Phi| = |\Phi(\Omega)|.$$

Thus,

$$\int_{\partial\Omega} uv_x\, dx + uv_y\, dy = \begin{cases} |\Phi(\Omega)|, & \det D\Phi > 0, \\ -|\Phi(\Omega)|, & \det D\Phi < 0. \end{cases}$$

Taking $\Phi(x, y) = (x, y)$, then the integral equals to $|\Phi(\Omega)|$, if we take $\Phi(x, y) = (-x, y)$, then the integral equals to $-|\Phi(\Omega)|$. □

Problem 11. Let f be a polynomial of total degree at most three in $(x, y, z) \in \mathbb{R}^3$. Prove that:

$$\int_{x^2+y^2+z^2\leq 1} f(x, y, z)\, dx\, dy\, dz = \frac{4\pi f((0, 0, 0))}{3} + \frac{2\pi\,(\Delta f)\,((0, 0, 0))}{15}.$$

Here $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is the Laplacian operator on $\mathbb{R}^3$.

Proof. With Taylor's formula,

$$f(x) = f(0) + \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(0)x_i + \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_i}(0)x_i x_i + \frac{1}{6}\sum_{i,j,k=1}^{3} \frac{\partial^3 f}{\partial x_i \partial x_i \partial x_k}(0)x_i x_i x_k + R.$$

and since f is a polynomial of degree at most 3, then the remainder $R = 0$. Also, with the property of odd function,

$$\int_B x_i\, dx = 0, \quad \int_B x_i x_i x_k\, dx = 0,$$

and

$$\int_B x_i x_i\, dx = 0,\ i \neq j.$$

Then,

$$\int_B f(x, y, z)\, dx\, dy\, dz = f(0)\underbrace{\int_B dx}_{4\pi/3} + \frac{1}{2}\sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}(0)\int_B x_i^2\, dx,$$

also, with

$$\begin{aligned} \int_B x_1^2\, dx = \int_B x_2^2\, dx = \int_B x_3^2\, dx &= \frac{1}{3}\int_B |x|^2\, dx \\ &= \int_0^1 \int_{S(r)} r^2\, d\sigma\, dr \\ &= \int_0^1 r^2 \cdot 4\pi r^2\, dr = \frac{4\pi}{5}, \end{aligned}$$

the proof is completed. □

Problem 12. Suppose that $K \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ and

$$K(x) = \frac{K(x/\|x\|)}{\|x\|} \quad \text{for all } x \in \mathbb{R}^2 \setminus \{0\}.$$

(a) Prove that $\nabla K(tx) = t^{-2} \nabla K(x)$ for $x \neq 0$ and $t > 0$.
(b) Use the divergence theorem to prove that (on both sides we integrate vector valued functions)

$$\int_{\{1 \leq \|x\| \leq 2019\}} \nabla K(x)\, dx = \int_{\partial\{1 \leq \|x\| \leq 2019\}} K(x) \vec{\mathbf{n}}\, d\sigma(x).$$

(c) Prove that

$$\int_{\{1 \leq \|x\| \leq 2019\}} \nabla K(x)\, dx = 0.$$

Hint. *Show first that $K(tx) = t^{-1}K(x)$ for $x \neq 0$, $t > 0$. In (a) differentiate $K(tx)$. Part (a) is not needed for parts (b) and (c).*
Proof.

(a) Note that

$$K(tx) = \frac{K(tx/\|tx\|)}{\|tx\|} = t^{-1} \frac{K(x/\|x\|)}{\|x\|} = t^{-1} K(x).$$

Then,

$$t^{-1} \frac{\partial K}{\partial x_i}(x) = \frac{\partial}{\partial x_i}(t^{-1} K(x)) = \frac{\partial}{\partial x_i}(K(tx)) = \frac{\partial K}{\partial x_i}(tx) \cdot t,$$

which gives,

$$\frac{\partial K}{\partial x_i}(tx) = t^{-2} \frac{\partial K}{\partial x_i}(x).$$

Thus,

$$\nabla K(tx) = t^{-2} \nabla K(x).$$

(b) With divergence theorem, $\int_\Omega \boldsymbol{\nabla} \cdot \vec{F} = \int_{\partial\Omega} \vec{F} \cdot \vec{\mathbf{n}}$, if $F(x) = (K(x), 0)$, then $\boldsymbol{\nabla} \cdot F = \frac{\partial K}{\partial x_1}$ and then

$$\int_{\{1 \leq \|x\| \leq 2019\}} \frac{\partial K}{\partial x_1} = \int_{\partial\{1 \leq \|x\| \leq 2019\}} F \cdot \vec{\mathbf{n}}\, d\sigma(x) = \int_{\partial\{1 \leq \|x\| \leq 2019\}} K(x) n_1\, d\sigma(x),$$

where in the last step we used $F \cdot \vec{\mathbf{n}} = (K(x), 0) \cdot (n_1, n_2) = K(x) n_1$.
Similarly, if $F(x) = (0, K(x))$, then $\boldsymbol{\nabla} \cdot F = \frac{\partial K}{\partial x_2}$ and then

$$\int_{\{1 \leq \|x\| \leq 2019\}} \frac{\partial K}{\partial x_2} = \int_{\partial\{1 \leq \|x\| \leq 2019\}} K(x) n_2\, d\sigma(x).$$

Thus,

$$\int_{\{1 \leq \|x\| \leq 2019\}} \nabla K(x)\, dx = \int_{\partial\{1 \leq \|x\| \leq 2019\}} K(x) \vec{\mathbf{n}}\, d\sigma(x).$$

(c)

$$\begin{aligned}\int_{\{1\le\|x\|\le 2019\}} \nabla K(x)\,dx &= \int_{\partial\{1\le\|x\|\le 2019\}} K(x)\vec{\mathbf{n}}\,d\sigma(x)\\ &= -\int_{\|x\|=1} K(x)\frac{x}{\|x\|}\,d\sigma(x) + \int_{\|x\|=2019} K(x)\frac{x}{\|x\|}\,d\sigma(x).\end{aligned}$$

Note that $K(2019x) = 2019^{-1}K(x)$. Use standard parameterization of the circle, we have

$$\int_\gamma f\,dxdy = \int_a^b f(x(t),y(t))\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}\,dt,$$

where $\gamma : [a,b] \to \mathbb{R}$ and $|\dot{\gamma}(t)|= \sqrt{\dot{x}(t)^2+\dot{y}(t)^2}$. Take $\gamma(t) = 2019(\cos t, \sin t)$, then

$$\begin{aligned}\int_{\|x\|=2019} K(x)\frac{x}{\|x\|}\,d\sigma(x) &= \int_0^{2\pi} K(2019\cos t, 2019\sin t)\frac{(2019\cos t, 2019\sin t)}{2019}2019\,dt\\ &= \int_0^{2\pi} K(\cos t,\sin t)\frac{(\cos t,\sin t)}{1}\,dt\\ &= \int_{\|x\|=1} K(x)\frac{x}{\|x\|}\,d\sigma(x).\end{aligned}$$

Thus,

$$\int_{\{1\le\|x\|\le 2019\}} \nabla K(x)\,dx = 0.$$

$\square$