Homework 6 for Math 1530
Zhen Yao

Problem 55. Prove that the two series

$$\sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad \sum_{n=0}^{\infty} n(\log n) c_n x^{n+3}$$

have the same radius of convergence.

Proof. The radius of convergence for series $\sum_{n=0}^{\infty} c_n x^n$ is $R_1 = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$, and the radius for the second series is

$$\begin{aligned} R_2 &= \limsup_{n\to\infty} \sqrt[n]{|n(\log n) c_n|} \\ &= \limsup_{n\to\infty} \sqrt[n]{|n(\log n)|}\sqrt[n]{|c_n|} \\ &= \limsup_{n\to\infty} \sqrt[n]{|c_n|} = R_1 \end{aligned}$$

The proof is complete. □

Problem 56. Let $f : (-\infty, \infty) \to \mathbb{R}$ be continuous and $\lim_{x\to\infty} f(f(x)) = \infty$. Prove that $\lim_{x\to\infty} |f(x)| = \infty$.

Proof. Suppose $\lim_{x\to\infty} f(x) = a < \infty$ which is finite. This means that for $\forall \varepsilon > 0$, there exists $\delta > 0$, such that $\forall x > \delta$, $|f(x) - a| < \varepsilon$. W e can pick a sequence $\{x_k\}$ such that $\lim_{k\to\infty} x_k = a$, since f is continuous, then we have $\lim_{k\to\infty} f(x_k) = f(a)$.

Also, we can pick a sequence $\{y_k\} \to \infty$ such that $f(y_k) = x_k$. Then we have $f(f(y_k)) \to f(a) \neq \infty$, which is a contradiction. □

Problem 57. Let $f : [0, 1) \to \mathbb{R}$ be a function that is not necessarily continuous. Define

$$g(\delta) = \sup\{|f(y) - f(y')| : y, y' \in (1 - \delta, 1)\}.$$

Prove that $\lim_{x\to 1^-} f(x)$ exists and is finite if and only if $\lim_{\delta\to 0^+} g(\delta) = 0$.

Proof. (1)If $\lim_{\delta\to 0^+} g(\delta) = 0$, then for $\forall \varepsilon > 0$, there exists δ_0, such that $\forall \delta < \delta_0$, $|f(y) - f(y')| \le g(\delta) < \varepsilon$, where $|y - y'| < \delta$. Thus, by definition, f is uniformly continuous on $(1 - \delta, 1)$. Then, we define

$$\lim_{x\to 1^-} f(x) = \lim_{n\to\infty} f\left(1 - \frac{1}{n}\right) = A$$

We can pick a sequence $\{x_k\} \to 1$, then for $\delta < \delta_0$ above, we can find K and N_1 such that for $\forall n > N_1, \forall k > K$, $|x_k - (1 - \frac{1}{n})| < \delta$. Also, we can find N_2 such that $\forall n > N_2$, $|f(1 - 1/n) - A| < \varepsilon$. Thus, for $\forall n > \max\{N_1, N_2\}, \forall k > K$, we have

$$|f(x_k) - A| < \left| f(x_k) - f\left(1 - \frac{1}{n}\right) \right| + \left| f\left(1 - \frac{1}{n}\right) - A \right| < 2\varepsilon$$

Thus, $\lim_{x\to 1^-} f(x)$ exists and is finite.

(2)Suppose $\lim_{x\to 1^-} f(x) = A$ exists and is finite, and we can pick a sequence $\{x_k\} \to 1$ such that $\lim_{k\to\infty} f(x_k) = A$. Then for $\forall \varepsilon > 0$, there exist δ_1, such that $|x_k - 1| < \delta_1$,

$|f(x_k) - A| < \varepsilon$. For this δ_1, we could find x_{k_1} and x_{k_2} satisfying $x_{k_1}, x_{k_2} \in (1-\delta_1, 1)$. Then we have

$$g(\delta_1) = \sup\{|f(x_{k_1}) - f(x_{k_2})|; x_{k_1}, x_{k_1} \in (1-\delta_1, 1)\}$$

and we have

$$|f(x_{k_1}) - f(x_{k_2})| < |f(x_{k_1}) - A| + |A - f(x_{k_2})| < 2\varepsilon$$

and this holds for all ε and all $x_{k_1}, x_{k_2} \in (1-\delta_1, 1)$, then we have $\lim_{\delta \to 0^+} g(\delta_1) = 0$. $\square$

Problem 58. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is α-Hölder continuous with some $\alpha > 1$, then f is constant.

Proof. For fixed x and $x < y$, and we divide $y - x$ into n small intervals, and denote $x_0 = x, x_1 = x + \frac{y-x}{n}, \cdots, x_n = x + n\frac{y-x}{n} = y$. And we have

$$\begin{aligned}|f(y) - f(x)| &\le \sum_{i=1}^{n} |f(x_{i+1}) - f(x_i)| \le C\sum_{i=1}^{n} |x_{i+1} - x_i|^\alpha \\ &\le cn\left(\frac{y-x}{n}\right)^\alpha = c\frac{(y-x)^\alpha}{n^{\alpha-1}}\end{aligned}$$

Taking $n \to \infty$, and we have $|f(y) - f(x)| \le \lim_{n\to\infty} c\frac{(y-x)^\alpha}{n^{\alpha-1}} = 0$, since $\alpha > 1$. Thus, f is constant. $\square$

Problem 59. Let $f : (1, \infty) \to \mathbb{R}$ be differentiable. Prove that if

$$\lim_{x\to\infty} f'(x) = g, \text{ then } \lim_{x\to\infty} \frac{f(x)}{x} = g.$$

Proof. Since $\lim_{x\to\infty} f'(x) = g$, then for $\forall \varepsilon > 0$, there exists $M > 0$, such that $\forall x > M$, $|f'(x) - g| < \varepsilon$, which means, for fixed $x_0 \in (1, \infty)$, we have

$$\begin{aligned}&\lim_{x\to\infty} \frac{f(x) - f(x_0)}{x - x_0} = g \\ \Rightarrow &\lim_{x\to\infty} f(x) - f(x_0) = gx - gx_0 \\ \Rightarrow &\lim_{x\to\infty} \frac{f(x)}{x} = g + \frac{f(x_0)}{x} - \frac{f(x_0)}{gx}\end{aligned}$$

Taking $x \to \infty$, then we have $\lim_{x\to\infty} \frac{f(x)}{x} = g$. $\square$

Problem 60. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and such that

$$\lim_{x\to\infty} f(x) = g_1 \in \mathbb{R}, \qquad \lim_{x\to\infty} f'(x) = g_2.$$

Prove that $g_2 = 0$.

Proof. Since $\lim_{x\to\infty} f(x) = g_1$, then for $\forall \varepsilon > 0$, there exists $M > 0$, such that $\forall x > M$, $|f(x) - g_1| < \varepsilon$. And for any number $M < x_1 < x_2$, we have

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$$

where $\xi \in (x_1, x_2)$. Then we have

$$\begin{aligned} f'(\xi) &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &\leq \left| \frac{f(x_2) - g - (f(x_1) - g)}{x_2 - x_1} \right| \\ &\leq \frac{|f(x_2) - g|}{x_2 - x_1} + \frac{|f(x_1) - g|}{x_2 - x_1} \\ &\leq \frac{2\varepsilon}{x_2 - x_1} \end{aligned}$$

We can set $x_2 - x_1 = N$ to be fixed and take $x_1, x_2 \to \infty$, we can have $f'(\xi) = 0$. $\square$

Problem 61. Suppose that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ and its derivative f' have no common zeros. Prove that f has only finitely many zeros in $[0, 1]$.

Proof. Set $Z = \{x \in [0, 1]; f(x) = 0\}$ and suppose that Z has infinitely many elements. Since there are infinitely many points in $[0, 1]$, then there exists a point p such that $x_k \to p, x_k \in Z$. Since f is differential on $[0, 1]$, then f is continuous in this interval. Thus, we have $f(p) = \lim_{k\to\infty} f(x_k) = 0$. Also, we have $f'(p) = \lim_{x_k \to\ p} \frac{f(x_k)-f(p)}{x_k-p} = 0$, which is a contradiction. $\square$

Problem 62. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$, $f(0) = 0$, and $\lim\limits_{x\to\infty} f(x) = 0$. Prove that there exists $c \in \mathbb{R}$ such that $f'(c) = 0$.

Proof. Since $\lim_{x\to\infty} f(x) = 0$, then $\forall \varepsilon > 0$, there exists $M > 0$, such that $\forall x > M$, $|f(x)| < \varepsilon$. With mean value theorem, we have $c \in (0, x)$ such that $f(c) = \frac{f(x)-f(0)}{x} < \varepsilon$. This holds for every ε, then we know there exists a c such that $f(c) = 0$. $\square$

Problem 63. Let $f : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$. Suppose that $f(0) < 0 < f(1)$ and $f'(x) \neq 0$ for all $x \in (0, 1)$. Let $S_1 = \{x \in [0, 1] : f(x) > 0\}$ and $S_2 = \{x \in [0, 1] : f(x) < 0\}$. Prove that $\inf(S_1) = \sup(S_2)$.

Proof. Since $f(0) < 0 < f(1)$ and f is continuous, then there exists a $c \in (0, 1)$ such that $f(c) = 0$. Now consider the interval $[0, c)$, we claim that f is increasing in this interval. If not, then there exists a $x_1 \in (0, c)$ such that $f(x_1) < f(0) < 0$. Also, since f is continuous, then there exists a $x_2 \in (x_2, c)$ such that $f(x_2) = f(0)$. With Rolle Theorem, we can know that there must be a $\xi \in (0, x_2) \subset (0, c)$ such that $f'(\xi) = 0$, which contradicts the fact that $f'(x) \neq 0, \forall x \in (0, 1)$. Similarly, we can know that f is increasing on interval $(c, 1)$. Since f is continuous, then f is increasing on $[0, 1]$.

We have know that $f(c) = 0, c \in (0, 1)$. We claim that $c = \inf(S_1)$ and $c = \sup(S_2)$. First, we consider $x \in S_1$ such that $f(x) > 0$, with f being continuous and increasing, we can know that $c < \forall x \in S_1$. Then, c is a lower bound of S_1. Also, we can find a sequence $\{x_k\} \to c$ where $x_k \in S_1$. For $\forall \varepsilon > 0$, then there exists a $K > 0$, such that $k > K$, $x_k < 0 + \varepsilon$ and $0 < f(x_k) < f(\varepsilon)$. Then we proved that c is a greatest lower bound of S_1. Similarly, we can know c is also a least upper bound of S_2. Thus, $\inf(S_1) = \sup(S_2)$. $\square$

Problem 64. Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function on $[0, \infty)$ such that $f(0) > 0$ and

$$f'(x) = \frac{1}{x^2 + (f(x))^2} \quad \text{for all } x \in [0, \infty).$$

Prove that $\lim_{x\to\infty} f(x)$ exists and is finite.

Proof. Suppose $\lim_{\to\infty} f(x)$ exists and is not finite, then $\lim_{\to\infty} f(x) = \infty$. Also, with $f(0) > 0$, we have $f'(0) = \frac{1}{f^2(0)} > 0$, which means $f > 0$ in a small interval $[0, \delta)$. Then we can know that f is increasing in $[0, \infty)$. Also, we have

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \frac{1}{x^2 + f^2(x)} = 0$$

Since $\lim_{x\to\infty} f'(x) = 0$, then f cannot go to infinity as $x \to \infty$. $\square$

Problem 65. Prove that for $x \in \mathbb{R}$

$$\cos x \geq 1 - \frac{x^2}{2}.$$

Proof. Define $f(x) = \cos x - 1 + \frac{x^2}{2}$, then we have $f'(x) = -\sin x + x$. Then $f''(x) = -\cos x + 1 \geq 0$, which means that $f'(x)$ is increasing. Also we have $f'(0) = 0$. Then we can know that $f(x)$ is decreasing on $(-\infty, 0]$ and increasing on $(0, \infty)$. Thus, $\inf f(x) = f(0) = 0$, which implies $\cos x - 1 + \frac{x^2}{2} \geq 0 \Rightarrow \cos x \geq 1 - \frac{x^2}{2}$. $\square$

Problem 66. Prove that for $x \in [0, 1]$ and $p > 1$ the following inequality is satisfied

$$\frac{1}{2^{p-1}} \leq x^p + (1-x)^p \leq 1.$$

Proof. Since $x \in [0, 1]$ and $p > 1$, then we have $x^p \leq x$ and $(1-x)^p \leq (1-x)$, then we have $x^p + (1-x)^p \leq 1$. On the other hand, we define $f(x) = x^p + (1-x)^p$. Then, $f'(x) = p[x^{p-1} - (1-x)^{p-1}]$, and $f'(x)$ is increasing on $[0, 1]$ with $f'(1/2) = 0$. Then f is decreasing on $[0, 1/2]$ and increasing on $(1/2, 1]$, which means $\min f(x) = f(1/2) = 1/2^{p-1}$. Thus, we have $\frac{1}{2^{p-1}} \leq x^p + (1-x)^p \leq 1$. $\square$

Problem 67. Let $W(x)$ be a polynomial such that $W(x) \geq 0$ for $x \in \mathbb{R}$. Prove that

$$u(x) = W(x) + W'(x) + W''(x) + \ldots \geq 0.$$

Proof. Since $u(x) = W(x) + W'(x) + W''(x) + \cdots + W^{(n)}(x) + \cdots$. Then we have

$$u'(x) = W'(x) + W''(x) + \cdots + W^{(n)}(x) + \cdots$$

Then we have

$$u(x) = W(x) + u'(x)$$

And $u(x)$ will obtains its minimum at some point c such that $u'(c) = 0$, then we have

$$u(x) \geq u(c) = W(c) + u'(c) \geq 0$$

The proof is complete. $\square$

Problem 68. Prove that the polynomial

$$W_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

has no multiple roots.

Proof. We have $W_n'(x) = 1 + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}$, and then we get $W_n'(x) = W_n(x) - \frac{x^n}{n!}$. If r is a root of $W_n(x)$, then we have $W_n'(r) = 0$, and it follows

$$W_n'(r) = W_n(r) - \frac{r^n}{n!} = 0$$
$$\Rightarrow \frac{r^n}{n!} = 0$$

so we have r must be 0. Also, we can know that $W_n(0) = 1 \neq 0$, then we know $W_n(x)$ has no multiple roots. $\square$

Problem 69. Suppose that $f \in C^\infty(\mathbb{R})$ and $f(a) = 0$. Prove that there is $g \in C^\infty(\mathbb{R})$ such that $f(x) = (x-a)g(x)$ for all $x \in \mathbb{R}$.

Proof. Since $f(x) \in C^\infty(\mathbb{R})$, we can know that $f(x)$ can be expressed as polynomial, with $f(a) = 0$

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \\ &= (x-a)\sum_{n=2}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^{n-1} \end{aligned}$$

Then we can define $g(x) = \sum_{n=2}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^{n-1}$ and it is easy to see that $g(x) \in C^\infty(\mathbb{R})$. $\square$

Problem 70. Let $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Prove that $f \in C^\infty(\mathbb{R})$ and $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$

Hint: *Use induction to prove that f is n-times differentiable, $f^{(n)}(0) = 0$ and $f^{(n)}(x) = W_n(1/x)e^{-1/x^2}$ for $x \neq 0$, where W_n is a polynomial.*

Remark. This is a very important example. Since all derivatives at 0 are equal zero, Maclaurin's series of f equals zero. However, $f(x) > 0$ for $x \neq 0$ so it is not equal to the Maclaurin series at any point except $x = 0$. Another reason why this is so important is that it allows to construct compactly supported smooth functions, see Problem 71.

Proof. First, we have $f'(x) = 2\left(\frac{1}{x}\right)^3 e^{-1/x^2}, x \neq 0$. Then $f'(x) = W_1(1/x)e^{-1/x^2}$ with $W_1(1/x) = 2\left(\frac{1}{x}\right)^3$ being a polynomial of $1/x$. Suppose that for $k > 1$, $f^{(k)}(x) = W_k(1/x)e^{-1/x^2}$, we need to prove that $f^{(k+1)}(x)$ still has the form of $W_{k+1}(1/x)e^{-1/x^2}$. We can know

$$\begin{aligned} f^{(k+1)}(x) &= (f^{(k)}(x))' \\ &= -\left(\frac{1}{x}\right)^2 W_k'(1/x)e^{-1/x^2} + 2\left(\frac{1}{x}\right)^3 W_k(1/x)e^{-1/x^2} \\ &= \left(-\left(\frac{1}{x}\right)^2 W_k'(1/x) + 2\left(\frac{1}{x}\right)^3 W_k(1/x)\right) e^{-1/x^2} \end{aligned}$$

Since the derivative of polynomial is still a polynomial, then we can know $f^{(k+1)}(x)$ is indeed of form of $W_{k+1}(1/x)e^{-1/x^2}$ with

$$W_{k+1}(1/x) = \left(-(1/x)^2 W_k'(1/x) + 2(1/x)^3 W_k(1/x)\right)$$

Then, we concluded that $f(x) \in C^\infty(\mathbb{R} \setminus \{0\})$.

Second, we prove the derivative of $f(x)$ at point 0 exists, we have

$$f'(0) = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}} - 0}{x - 0} = \lim_{x\to 0} \frac{1}{xe^{1/x^2}} = \lim_{t\to\infty} \frac{t}{e^{t^2}} = \lim_{t\to\infty} \frac{1}{2te^{t^2}} = 0$$

Then $f(x)$ is differential at point $x = 0$, thus, $f \in C^1(\mathbb{R})$. Then, we assume that $f^{(K)}(0) = 0$, and we want to prove that $f^{(k+1)}(x) = 0$. By definition of derivative, we have

$$\begin{aligned}
f^{(K+1)}(0) &= \lim_{x\to 0} \frac{f^{(K)}(x) - f^{(K)}(0)}{x - 0} \\
&= \lim_{x\to 0} \frac{f^{(K)}(x)}{x} \\
&= \lim_{x\to 0} \frac{W_K(1/x)e^{-1/x^2}}{x} \\
&= \lim_{x\to 0} \frac{\left(W_K(1/x)e^{-1/x^2}\right)'}{1} \\
&= \lim_{x\to 0} -\frac{1}{x^2} W_K'(1/x)e^{-1/x^2} + \frac{2}{x^4} W_K(1/x)e^{-1/x^2} \\
&= \lim_{x\to 0} -\frac{W_K'(1/x)}{x^2 e^{1/x^2}} + \frac{2W_K(1/x)}{x^4 e^{1/x^2}} \\
&= 0
\end{aligned}$$

In the last step, we could use L'Hospital's rule to determine the limit, and in limit steps, saying there exists k_1 and k_2 such that $(W_K'(1/x))^{(k_1)}$ and $(W_K(1/x))^{(k_2)}$ are constants, while the denominator always has the term e^{1/x^2}, and we already know that $\lim_{x\to 0} e^{1/x^2} = \infty$. Thus, we can know that $f^{(K+1)}(0) = 0$.

$\square$

Problem 71. Use the function from Problem 70 to construct $f \in C^\infty(a, b)$ such that $f(x) = 0$ for $x \in \mathbb{R} \setminus (a, b)$.

Proof. Set the function in Problem 63 as $g(x) : [0, 1] \to \mathbb{R}$, then g is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Also, $g(0) < 0 < g(1)$ and $g'(x) \neq 0, \forall x \in (0, 1)$.

Now consider $f(x) = g\left(\frac{x-a}{b-a}\right), x \in [a, b]$ and $f(x) = 0, x \in \mathbb{R} \setminus [a, b]$. And for $f(x)$, using Taylor Theorem, we have

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
&= \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} \left(\frac{x-a}{b-a}\right)^k
\end{aligned}$$

Then, we define $f(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} \left(\frac{x-a}{b-a}\right)^k$, and $f^\infty(a, b)$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus (a, b)$. $\square$

Problem 72. Let $n \geq 3$. Consider an n-times continuously differentiable function $f \in C^n(\mathbb{R})$ such that $f^{(k)}(0) = 0$, for $k = 2, 3, \ldots, n-1$ and $f^{(n)}(0) \neq 0$. Clearly, by the mean value theorem for any $h > 0$ there is $0 < \theta(h) < h$ such that

$$f(h) - f(0) = hf'(\theta(h)).$$

Prove that

$$\lim_{h \to 0} \frac{\theta(h)}{h} = \left(\frac{1}{n}\right)^{\frac{1}{n-1}}.$$

Hint: Expand f and f' using Taylor's formula.

Proof. With Taylor Theorem, we have

$$f(h) = f(0) + f'(0)h + \frac{f''(0)}{2!}h^2 + \cdots + \frac{f^{(n)}(0)}{n!}h^n + h - [^{n+1}\psi(h)]$$
$$f'(\theta) = f'(0) + f''(0)h + \frac{f^{(3)}(0)}{2!}h^2 + \cdots + \frac{f^{(n)}(0)}{(n-1)!}\theta^{n-1}$$

With $f(h) - f(0) = hf'(\theta(h))$, $f^{(k)}(0) = 0, k = 2, 3, \ldots, n-1$ and $f^{(n)}(0) \neq 0$, we have

$$f(h) - f(0) = \left(f'(0) + \frac{f^{(n)}(0)}{(n-1)!}\right) h^n = h\left(f'(0) + \frac{f^{(n)}(0)}{(n-1)!}\theta^{n-1}\right)$$
$$\Rightarrow \frac{h^{n-1}}{n!} = \frac{\theta^{n-1}}{(n-1)!}$$
$$\Rightarrow \left(\frac{\theta}{h}\right)^{n-1} = \frac{1}{n}$$

Thus we have $\lim \theta/h = (1/n)^{\frac{1}{n-1}}$.

$\square$