Homework 1 for Math 1540

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Problem 1. Prove that the trigonometric polynomials

$$T(x) = \sum_{k=0}^{n} a_k \cos kx + \sum_{k=0}^{n} b_k \sin kx, \quad a_k, b_k \in \mathbb{R}$$

form an algebra. *Hint:* $\cos x + i \sin x = e^{ix}$.

Proof. With $\cos x + i \sin x = e^{ix}$, then we can have $\cos kx = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$ and $\sin kx = \frac{1}{2}\left(ie^{-ikx} - ie^{ikx}\right)$.

Now it suffices to show that $\cos(kx)\sin(lx), \cos(kx)\cos(lx), \sin(kx)\sin(lx)$ are trigonometric polynomials. We have

$$\begin{aligned}
\cos(kx)\sin(lx) &= \frac{1}{4}\left(e^{ikx} + e^{-ikx}\right)\left(ie^{-ilx} - ie^{ilx}\right) \\
&= \frac{1}{4}\left(ie^{i(k-l)x} - ie^{i(k+l)x} + ie^{-i(k+l)x} - ie^{i(l-k)x}\right) \\
&= -\frac{1}{2}\sin((k-l)x) + \frac{1}{2}\sin((k+l)x)
\end{aligned}$$

which is also a trigonometric polynomial. Similarly, $\cos(kx)\cos(lx), \sin(kx)\sin(lx)$ are also trigonometric polynomial. Thus, trigonometric polynomials form an algebra. □

Problem 2. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane. Let $\mathcal{A}$ be the algebra of functions of the form

$$f\left(e^{i\theta}\right) = \sum_{k=0}^{N} c_k e^{in\theta}, \quad c_k \in \mathbb{C},\ \theta \in \mathbb{R}.$$

It is easy to see that $f \equiv 1$ belongs of $\mathcal{A}$ and $\mathcal{A}$ separates points (do not prove it). Prove that there are complex valued functions on S^1 that cannot be uniformly approximated by functions in $\mathcal{A}$. *Hint:* For $f \in \mathcal{A}$

$$\int_0^{2\pi} f\left(e^{i\theta}\right) e^{i\theta}\, d\theta = 0\,.$$

Proof. The function $f(z) = z \in \mathcal{A}$ separates points in S^1. And we can know that $\frac{1}{z} = e^{-i\theta}$ is not in the closure of $\mathcal{A}$, since

$$\int_0^{2\pi} e^{-i\theta} e^{i\theta} d\theta = 2\pi.$$

□

Remark 2.1. This show that the Stone-Weiestrass theorem holds for real valued functions, but does not hold for complex valued functions.

Second Proof of Problem 2. Let $f : S^1 \to \mathbb{C}$ and $f\left(e^{i\theta}\right) = \cos\theta = \mathrm{Re}\left(e^{i\theta}\right)$, then we have

$$\int_0^{2\pi} f\left(e^{i\theta}\right) e^{i\theta} d\,\theta = \int_0^{2\pi} \cos\theta(\cos\theta + i\sin\theta) d\,\theta = \pi \neq 0.$$

If $g \in \mathcal{A}$, then we have

$$\begin{aligned}\int_0^{2\pi} g\left(e^{i\theta}\right) e^{i\theta}\, d\theta &= \sum_{k=0}^{N} c_n \int_0^{2\pi} e^{ik\theta} e^{i\theta}\, d\theta \\ &= \sum_{k=0}^{N} c_k \left(\int_0^{2\pi} \cos(k+1)\theta\, d\theta + i\int_0^{2\pi} \sin(k+1)\theta\, d\theta\right) \\ &= 0.\end{aligned}$$

Now we suppose $\mathcal{A} \ni g_k \rightrightarrows f$, then we have $g_k\left(e^{i\theta}\right) e^{i\theta} \rightrightarrows f\left(e^{i\theta}\right) e^{i\theta}$ and

$$0 = \int_0^{2\pi} g_k\left(e^{i\theta}\right) e^{i\theta}\, d\theta \to \int_0^{2\pi} f\left(e^{i\theta}\right) e^{i\theta}\, d\theta \neq 0$$

which is a contradiction. □

Problem 3. Prove that complex polynomials

$$p(z) = \sum_{n=0}^{N} c_n z^n, \quad c_n \in \mathbb{C}$$

are not dense in $C(\overline{D}, \mathbb{C})$, where

$$\overline{D} = \{z \in \mathbb{C} :\ |z| \leq 1\}$$

is the unit disc in $\mathbb{C}$. *Hint:* Consider $f(z) = \overline{z}$. Is the previous exercise helpful?

Proof. For $z = e^{i\theta} = \cos x + i\sin x$, we have $\bar{z} = z^{-1}$. Then,

$$\bar{p}(z) = \sum_{n=0}^{N} \bar{c_n} z^{-n}$$

which is not a polynomial, since the exponents are negative. □

Second Proof of Problem 3. Suppose $g_k \rightrightarrows f$, then $g_k\left(e^{i\theta}\right) \rightrightarrows f\left(e^{i\theta}\right)$, where g_k restricted to $S^1 = \{z \in \mathbb{C} :\ |z| = 1\}$ converges uniformly to f which is also restricted to S^1. Now we take $f\left(e^{i\theta}\right) = \mathrm{Re}\left(e^{i\theta}\right) = \cos\theta$ and $g_k\left(e^{i\theta}\right) = \sum_{n=0}^{N} c_n e^{in\theta}$, then we can use the similar argument as second proof of Problem 2. □

Problem 4. We know that if $f : [a,b] \to \mathbb{R}$ is continuous and

$$\int_a^b f(x) x^n\, dx = 0 \tag{1}$$

for $n = 0, 1, 2, 3, \ldots$, then $f(x) = 0$ for all $a \leq x \leq b$. We proved it using the Weierstrass theorem. Suppose now that $f : [a,b] \to \mathbb{R}$ is continuous and (1) holds for all $n \geq 2011$. Does it follow that $f(x) = 0$ for all $a \leq x \leq b$?

Proof. Set $g(x) = x^{2011} f(x)$, and then $\int_a^b g(x)x^k dx = 0$ for $k = 0, 1, 2, \cdot$, which implies $g(x) = 0$. Then, we know that $f(x) = 0$ for all $x \neq 0$.

If $a > 0$, then $f(x) = 0$ on $[a, b]$. If $a \neq 0$, then with continuity of f, we have $f(0) = 0$. Thus, $f(x) = 0$ for all $a \leq x \leq b$. □

Problem 5. Prove that if $f : [0, 1] \to \mathbb{R}$ is such that

$$\int_0^1 f(x)e^{nx}\, dx = 0 \quad \text{for all } n = 0, 1, 2, \ldots,$$

then $f(x) = 0$ for all $0 \leq x \leq 1$. Provide two proofs following the methods:

(a) Use the Stone-Weierstrass theorem.

(b) Use the change of variables formula and apply the Weierstrass theorem.

Proof.

(a) There exists a sequence of the form $p_n(x) = \sum_{n=0}^{\infty} c_n e^{nx}$ such that converges uniformly to $f(x)$. Since f is continuous on $[0, 1]$, hence bounded. Then $\{p_n(x)\}$ is also bounded, and hence $p_n f$ converges uniformly to f^2. Then we have

$$\int_0^1 f^2(x)\, dx = \lim_{n\to\infty} \int_0^1 p_n(x)f(x)\, dx = 0,$$

which implies $f(x) = 0, x \in [0, 1]$.

(b) Let $e^x = y$, then we have $y = \ln x$, and

$$\int_0^1 f(x)e^{nx}\, dx = \int_1^e f(\ln y)y^{n-1}\, dy = 0.$$

By **Problem 4**, we have $f(\ln y) = 0, y \in [1, e]$, which is equivalent to that $f(x) = 0, x \in [0, 1]$.

□

Problem 6. Prove that if X is a compact metric space and $f : X \to X$ is a continuous mapping such that

$$d(f(x), f(y)) < d(x, y) \quad \text{for all } x, y \in X,$$

then there is a fixed point of f, i.e. $x \in X$ such that $f(x) = x$.

Proof. Define $\alpha = \inf_{x\in X} d(x, f(x))$, since X is compact and $x \mapsto d(x, f(x))$ is continuous, then α is attained. Let $\alpha = d(x_0, f(x_0))$, and we need to prove that $f(x_0) = x_0$.

Suppose if not, i.e., $f(x_0) \neq x_0$, then we have

$$\alpha \leq d(f(f(x_0)), f(x_0)) < d(f(x_0), x_0) = \alpha$$

where we used the fact that $d(f(x), f(y)) < d(x, y)$, for all$x, y \in X$. Then this is a contradiction.

It remains to show that there is at most one fixed point. Indeed, if $x_1 \neq x_2$ are two distinct fixed point, then we have $d(x_1, x_2) < d(f(x_1), f(x_2)) = d(x_1, x_2)$, which is a contradiction. □

Problem 7. Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - f(y)| < |x - y| \quad \text{for all } x, y \in \mathbb{R}$$

and f has no fixed point. You can find an explicit formula for f, but you do not have to. It is enough if you find a convincing argument that such a function exists. You do not have to be very precise, but your argument has to be convincing.

Proof. Take $f(x) = \ln(1 + e^x)$. $\square$

Problem 8. Show that there is a unique continuous real valued function $f : [0, 1] \to \mathbb{R}$ such that

$$f(x) = \sin x + \int_0^1 \frac{f(y)}{e^{x+y+1}}\, dy.$$

Proof. Consider the map $T : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ defined by

$$T(f)(x) = \sin x + \int_0^1 \frac{f(y)}{e^{x+y+1}}\, dy = f(x).$$

Clearly f is a solution to the problem if and only if $T(f) = f$. Since $C([0, 1], \mathbb{R})$ is compact, then we only need to show that T is a contraction. Indeed, given $f, h \in C([0, 1], \mathbb{R})$, we have

$$\begin{aligned} d_\infty(T(f), T(h)) &= \sup_{x\in[0,1]} \left| \int_0^1 \frac{f(y)}{e^{x+y+1}}\, dy - \int_0^1 \frac{h(y)}{e^{x+y+1}}\, dy \right| \\ &\le \sup_{x\in[0,1]} \int_0^1 \frac{|f(y) - h(y)|}{e^{x+y+1}}\, dy \\ &\le d_\infty(f, h) \int_0^1 \frac{1}{e^{x+y+1}}\, dy \\ &= d_\infty(f, h)(e - 1)e^{-x-2} \end{aligned}$$

and we know that $(e - 1)e^{-x-2} < 1$ for $x \in [0, 1]$. Thus, T is a contraction. $\square$

Problem 9. Let (X, d) be a nonempty complete metric space. Let $S : X \to X$ be a given mapping and write S^2 for $S \circ S$ i.e. $S^2(x) = S(S(x))$. Suppose that S^2 is a contraction. Show that S has a unique fixed point.

Proof. Since S^2 is conctraction, then there is a unique fixed point of S^2, denoted by x^*, such that $S^2(x^*) = x^*$. Then we have $S^2(S(x^*)) = S\left(S^2(x^*)\right) = S(x^*)$, which implies that $S(x^*)$ is also a fixed point of S^2. Thus, we have $S(x^*) = x^*$, implying S has a unique fixed point. $\square$

Remark 9.1. The statement also holds if S^n is a contraction.

Problem 10. Let E be a compact set and let $\mathcal{F} \subset C(E, \mathbb{R})$ be an equicontinuous family of functions. Does it imply that the family $\mathcal{F}$ is bounded in $C(E, \mathbb{R})$?

Proof. No. We can define $f_n(x) = n, \forall n \in \mathbb{N}, x \in E$. Thus, each $f_n(x), x \in E$ is equicontinuous function, hence $f_n(x) \in \mathcal{F}$. But, $\mathcal{F}$ is not bounded. $\square$

Problem 11. Let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded and uniformly continuous. Prove that the family of functions $\{g_z\}_{z\in\mathbb{R}^n}$, $g_z(x) = f(x)f(x-z)$ is equicontinuous.

Proof. Since f is bounded, then there exists a $M > 0$ such that for all $x \in \mathbb{R}^n$, $|f(x)|\leq M$. Since f is uniformly continuous, then for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}^n$, if $d(x, y) < \delta$, $|f(x) - f(y)|< \frac{\varepsilon}{2M}$. Then, for any $z \in \mathbb{R}^n$ we have $d(x - z, y - z) < \delta$, and $|f(x - z) - f(y - z)|< \frac{\varepsilon}{2M}$. Now we have

$$\begin{aligned}|f(x)f(x-z) - f(y)f(y-z)| =& |f(x)f(x-z) - f(y)f(x-z) + f(y)f(x-z) \\ & - f(y)f(y-z)| \\ \leq& |f(x) - f(y)|\cdot|f(x-z)| + |f(x-z) - f(y-z)|\cdot|f(y)| \\ \leq& |f(x-z)|\frac{\varepsilon}{2M} + |f(y)|\frac{\varepsilon}{2M} \leq \varepsilon.\end{aligned}$$

Hence, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for $\forall x, y$ and $\forall z$, if $d(x, y) < \delta$, then $d(g(x), g(y)) < \varepsilon$. Thus, $\{g_z\}_{z\in\mathbb{R}^n}$ is equicontinuous. $\square$

Problem 12. Suppose E is a compact metric space and $f_n : E \to \mathbb{R}$, $n = 1, 2, \dots$ is a bounded and equicontinuous sequence of functions. Suppose that f_n converges pointwise to a continuous function $f : E \to \mathbb{R}$ (i.e. $f_n(x) \to f(x)$ for every $x \in E$). Prove directly (i.e. without using Arzela-Ascoli theorem) that $f_n \rightrightarrows f$ uniformly on E.

Proof.

(a) Definition of pointwise convergence is that f_n pointwise converges to f if and only if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$, i.e., for any $x \in E$ and $\forall\varepsilon > 0$, there exists $N > 0$, such that for all $n \geq N$, $|f_n(x) - f(x)|< \varepsilon$. Here, the choice of N depends on x and ε.

(b) Definition of uniformly convergence is that for every $\varepsilon > 0$, there exists $N > 0$ such that for all $n \geq N$ and all $x \in E$, $|f_n(x) - f(x)|< \varepsilon$. Here the choice of N works for all $x \in E$.

As $\{f_n\}$ is equicontinuous, there exists a $\delta > 0$, such that for $\forall n \in \mathbb{N}$ and $\forall x, y \in E$:

$$d(x, y) < \delta \Longrightarrow |f_n(x) - f_n(y)|< \frac{\varepsilon}{3}.$$

and letting $n \to \infty$, then the above equation implies

$$d(x, y) < \delta \Longrightarrow |f(x) - f(y)|< \frac{\varepsilon}{3}.$$

Since E is compact, then it can be covered by finite many open balls of radius δ, i.e., there exists a $K > 0$ and $x_1, \cdots, x_K \in E$ such that

$$E \subset \bigcup_{j=1}^{K} B(x_j, \delta).$$

As f_n converges pointwise to f, then there exists $N > 0$ such that for $\forall n \geq N$,

$$|f_n(x_j) - f(x_j)|< \frac{\varepsilon}{3},$$

for all $1 \leq j \leq K$.

Now for any $x \in E$, and $\forall n \geq N$, then there exists a $j \in \{1, 2, \cdots, K\}$, for which $x \in B(x_j, \delta)$, and hence

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(x_j)| + |f_n(x_j) - f(x_j)| + |f(x_j) - f(x)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus, $f_n \rightrightarrows f$ uniformly on E. □

Problem 13. Suppose $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$ is a bounded and equicontinuous sequence of functions. Suppose that f_n converges pointwise to a continuous function $f : \mathbb{R} \to \mathbb{R}$. Does it imply that $f_n \rightrightarrows f$ uniformly on $\mathbb{R}$?

Proof. No. We can take $f_n(x) = \frac{x}{n}$, which is converges pointwise to $f(x) = 0$, but it does not converges uniformly to 0. Since for $\varepsilon = 1$, then we cannot find $N > 0$ such that for all $x \in \mathbb{R}$, $\frac{x}{N} < \varepsilon = 1$. □

Problem 14. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of increasing functions that is pointwise convergent to a continuous function $f : [a, b] \to \mathbb{R}$. Prove that $f_n \rightrightarrows f$ uniformly on $[a, b]$.

Proof. We want to show that for $\forall \varepsilon > 0$, there exists $N > 0$ such that for $\forall n \geq N$ and $\forall x \in [a, b]$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Let $g_n(x) = f(x) - f_n(x)$, then $g_n(x)$ is decreasing sequences of continuous functions. Let $\varepsilon > 0$, and

$$E_n = \{x \in [a, b] : g_n(x) = f(x) - f_n(x) < \varepsilon\}.$$

Then E_n is open since it is the inverse image of continuous function. And we can know that $\{E_n\}$ is an ascending sequence of open sets, since $g_n(x)$ is decreasing and if $x \in E_n$, then $g_n(x) < \varepsilon$ and of course $g_{n+1} < \varepsilon$, which implies $x \in E_{n+1}$. Then we have

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$$

Now letting $n \to 0$, and we have $g_n(x) \to 0$. Then we will have the sets of $x \in [a, b]$ such that $g_n(x) < \varepsilon$ will be the set $[a, b]$. Then,

$$[a, b] \subset \bigcup_{n=1}^{\infty} E_n,$$

and since $[a, b]$ is compact, then there exists a finite subcover, such that

$$[a, b] \subset \bigcup_{n=1}^{N} E_n.$$

Then for $\varepsilon > 0$ defined above, there exists $N > 0$ such that for all $\forall n \geq N$ and $\forall x \in [a, b]$, $|f_n(x) - f(x)| < \varepsilon$. □

Next we present another similar problem regarding the relations between pointwise convergence and uniform convergence.

Exercise 14.1. Let $\{f_n\}$ be an equicontinuous family of functions $f_n : E \to \mathbb{R}$ defined on a compact metric space. Prove that if f_n converges pointewise to a continuous function $f : E \to \mathbb{R}$, then f_n converges uniformly to f. Provide a direct argument without using Arzela-Ascoli theorem.

Proof. Since f is continuous and E is compact, then f is uniformly continuous. Equicontinuity of the family $\{f_n\}$ and uniform continuity of f implies that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if for all $x, y \in E$, $d(x, y) < \delta$, then we have

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}, \quad |f(x) - f(y)| < \frac{\varepsilon}{3}.$$

Since E is compact, then for open covering $B(x_i, \delta), x_i \in E$, there exists finite subcovering such that $E \subset \bigcup_{i=1}^{K} B(x_i, \delta)$. Then, for any $x \in E$, there exists some $i = 1, 2, \cdots, K$, such that $x \in B(x_i, \delta)$. Since f_n converges pointwise to f, then there exists $N_i > 0$ such that for all $n > N_i$, we have

$$|f_n(x_i) - f(x_i)| < \frac{\varepsilon}{3}.$$

For the same ε and δ above, let $N = \max\{N_i\}$, then for all $n > N$ and any $x \in E$, then there exists i such that $d(x, x_i) < \delta$,

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

Problem 15. Let $\{f_n\}$ be a sequence of real valued C^1 functions on $[0, 1]$ such that, for all n,

$$|f_n'(x)| \leq \frac{1}{\sqrt{x}} \quad (0 < x \leq 1),$$

$$\int_0^1 f_n(x)\, dx = 0.$$

Prove that the sequence has a subsequence that converges uniformly on $[0, 1]$.

Proof.

(a) The second equation implies that there exists $x_n \in [0, 1]$ such that $f_n(x_n) = 0$. Also, the first inequality implies

$$|f_n(x)| = \left| \int_{x_0}^{x} f_n'(t)\, dt \right| \leq \left| \int_{x_0}^{x} \frac{1}{\sqrt{t}}\, dt \right| = \left| 2\sqrt{t} \Big|_{x_0}^{x} \right| = 2 \left| \sqrt{x} - \sqrt{x_0} \right| \leq 2,$$

since $x \in [0, 1]$. Thus, f_n is bounded.

(b) For $x, y \in [0, 1]$ and $|x - y| < \delta = \frac{\varepsilon^2}{4}$, we have

$$|f_n(x) - f_n(y)| = \left| \int_y^x f_n'(t)\, dt \right| \leq \left| \int_y^x \frac{1}{\sqrt{t}}\, dt \right| = 2\left(\sqrt{y} - \sqrt{x}\right) \leq 2\sqrt{y - x} < \varepsilon,$$

where in the last step we used $\sqrt{y} - \sqrt{x} \leq \sqrt{y - x}$, since $\sqrt{y} = \sqrt{y - x + x} \leq \sqrt{y - x} + \sqrt{x}$. Thus, $\{f_n\}$ is equicontinuous.

Hence, the family $\{f_n\}$ is bounded, closed and equicontinuous, then the sequence has a uniformly convergent subsequence. $\square$

Problem 16. We know that every continuous function $f : [a, b] \to \mathbb{R}$ can be uniformly approximated by polynomials (Weierstrass' theorem). Prove that if a continuous function $f : \mathbb{R} \to \mathbb{R}$ can be uniformly approximated by polynomials on all of $\mathbb{R}$, then f is a polynomial.

Proof. Suppose that f can be uniformly approximated by polynomial $\{P_n(x)\}$ of degree at most d, and the polynomial $P_n(x)$ has the form

$$P_n(x) = a_n^d x^d + \cdots + a_n^1 x + a_n^0$$

such that

$$|f(x) - P_n(x)| < \frac{1}{n}, \forall x \in \mathbb{R}.$$

Then for any m, n, we have

$$|P_n(x) - P_m(x)| \leq |P_n(x) - f(x)| + |f(x) - P_m(x)| < \frac{1}{n} + \frac{1}{m}$$

which implies that $P_n(x) - P_m(x)$ is a polynomial which is bounded on $\mathbb{R}$ and hence a constant. Therefore, there exists some polynomial $P(x)$ and some $c_n \in \mathbb{R}$ such that

$$P_n(x) = P(x) + c_n.$$

Also, we can have $|c_n - c_m| < \frac{1}{n} + \frac{1}{m}$, which implies c_n is a Cauchy sequence, hence converging to a constant c.

Now we claim $f(x) = P(x) + c$. Let $\varepsilon > 0$, and pick $N = \frac{2}{\varepsilon}$, then for $\forall n > N$, we have $|c_n - c| < \frac{\varepsilon}{2}$. Now for $\forall n > N$, we have

$$|f(x) - P(x) - c| \leq |f(x) - P_n(x)| + |P_n(x) - P(x) - c| \leq \frac{1}{n} + |c_n - c| = \varepsilon.$$

$\square$

Problem 17. Prove that if $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, 3, \ldots$ are differentiable functions such that

(a) $f_n(0) = 0$ for all n,

(b) $|f_n'(x)| \leq e^x$ for all n and all x,

then there is a subsequence of f_n that converges pointwise to a continuous function $f : \mathbb{R} \to \mathbb{R}$. *Hint:* Show that the family satisfies the assumptions of the Arzela-Ascoli theorem on every interval $[-n, n]$ and then apply the diagonal method.

Proof. For any $x \in [-n, n]$, we have

$$\frac{|f_n(x) - f_n(0)|}{|x - 0|} = |f'(c)| \leq e^c < e^n,$$

where $c \in (0, x)$ or $c \in (x, 0)$. Then we have $|f_n(x)| < e^n n$ for all n and $x \in [-n, n]$. Then $\{f_n\}$ is the set of bounded functions. Now we prove that $\{f_n\}$ is equicontinuous on interval $[-n, n]$. For any $\varepsilon > 0$, we pick $\delta = \frac{\varepsilon}{e^n}$. Then for any $x, y \in [-n, n]$, if $|x - y| < \delta$, then

$$|f(x) - f(y)| = f'(c')|x - y| \leq e^n \frac{\varepsilon}{e^n} = \varepsilon,$$

where $c' \in (x, y)$ or $c' \in (y, x)$. Thus, $\{f_n\}$ is equicontinuous.

For interval $[-1, 1]$, $\{f_n\}$ has a convergent subsequence, denoted by $f_{11}, f_{12}, \cdots$. Now the sequence $\{f_{1n}(x)\}$ is bounded on the interval $[-2, 2]$, so it has convergent subsequence, denoted by $f_{21}, f_{22}, \cdots$. Contiune this process and we can have subsequences

$$\begin{gathered} f_{11}, f_{12}, f_{13}, \cdots \\ f_{21}, f_{22}, f_{23}, \cdots \\ f_{31}, f_{32}, f_{33}, \cdots \end{gathered}$$

Sequence in each line is a subsequence of the previous one. We now select $f_{11}, f_{22}, f_{33}, \cdots$.

We claim that $\{f_{nn}\}$ is uniformly convergent at every point of $\mathbb{R}$. Let $\varepsilon > 0$, and for any $x \in \mathbb{R}$, there exists $N > 0$, such that $x_i \in [-N, N]$ and $|x - x_i| < \delta$, where $\delta > 0$ as in the definition of equicontinuity. For $n, m \geq N$, we have

$$|f_{nn}(x_i) - f_{mm}(x_i)| < \frac{\varepsilon}{3}, x \in [-N, N].$$

Then we have

$$|f_{nn}(x) - f_{mm}(x)| \leq |f_{nn}(x) - f_{nn}(x_i)| + |f_{nn}(x_i) - f_{mm}(x_i)| + |f_{mm}(x_i) - f_{mm}(x)| < \varepsilon.$$

Hence $\{f_{nn}(x)\}$ is a Cauchy sequence. Now we define

$$f(x) = \lim_{n \to \infty} f_{nn}(x),$$

and we can have

$$|f(x) - f_{nn}(x)| < \varepsilon$$

as $m \to \infty$. □

Problem 18. Let the functions $f_n : [0, 1] \to [0, 1]$, $n = 1, 2, \ldots$, satisfy $|f_n(x) - f_n(y)| \leq |x - y|$ whenever $|x - y| \geq 1/n$. Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence.

Proof. Although functions f_n are not equicontinuous (and not even continuous), one can still apply the Arzela-Ascoli diagonalization argument[1].

Let $n \in \mathbb{N}$ be fixed. Take any $x_1 \in (0, 1)$ and consider the set U_{x_1} defined by

$$U_{x_1} = \left(x_1 - \frac{2}{n}, x_1 - \frac{1}{n}\right) \bigcup \left(x_1 + \frac{1}{n}, x_1 + \frac{2}{n}\right),$$

then U_{x_1} is open and for any $x \in U_{x_1}$, we have $|x - x_1| > 1/n$, and by the definition of f_n, we have

$$|f_n(x) - f_n(x_1)| \leq \frac{1}{n}.$$

Then, for any $x, y \in [0,1]$ such that $|x - x_1| \le 1/n, |y - x_1| \le 1/n$, then

$$|f_n(x) - f_n(y)| \le |f_n(x) - f_n(x_1)| + |f_n(x-1) - f_n(y)| \le \frac{2}{n}.$$

Let x_1 be arbitrary over $\mathbb{Q}$, then $\{(x_1 - 1/n, x_1 + 1/n)\}$ will be an open covering of $[0,1]$. Since $[0,1]$ is compact, then there exists a finite subcovering. Then apply above argument on the open set U_{x_1}, for any $x, y \in U_x$ or $[0,1] \setminus (x_1 - 2/n, x_1 + 2/n)$, we have

$$|f_n(x) - f_n(y)| \le \max\left\{\frac{2}{n}, |x-y|\right\},$$

which shows "asymptotic equicontinuity", the oscillations of f_n decay uniformly as $n \to \infty$. □

Problem 19. If $f = (f_1, \dots, f_n) : [a,b] \to \mathbb{R}^n$ is a continuous function, then we define

$$\int_a^b f(t)\,dt = \left\langle \int_a^b f_1(t)\,dt, \dots, \int_a^b f_n(t)\,dt \right\rangle.$$

Prove that

$$\left\| \int_a^b f(t)\,dt \right\| \le \int_a^b \|f(t)\|\,dt.$$

Proof. WRITE YOUR SOLUTION HERE. □

References

[1] Hayk (https://math.stackexchange.com/users/558859/hayk). Show that $\{f_n\}$ has a uniformly convergent subsequence. Mathematics Stack Exchange. https://math.stackexchange.com/q/2855807 (version: 2018-07-18).