Homework 2 for Math 1540

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Problem 20. Let $A = (a_{ij})_{m\times n}$ be the matrix of a linear mapping $A \in L(\mathbb{R}^n, \mathbb{R}^m)$. Prove that the norm

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

satisfies the inequality

$$\|A\| \leq \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}.$$

Hint: *You may use the following argument: Write the components of the vector Ax as scalar products of rows on A and x. Then use the Schwarz inequality to estimate the length of the vector Ax.*

Proof. For $\forall x \in \mathbb{R}^n$, we have

$$\begin{aligned}
\|Ax\|^2 &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right)^2 \\
&\leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}^2 \right) \left(\sum_{j=1}^{n} x_j^2 \right) \\
&= \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right) \|x\|.
\end{aligned}$$

Also, we can have

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \leq \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}.$$

□

Problem 21. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = f(xy)$. Prove that

$$x\frac{\partial F}{\partial x} = y\frac{\partial F}{\partial y}.$$

Proof. We have $\frac{\partial F}{\partial x} = f'(xy)y$ and $\frac{\partial F}{\partial y} = f'(xy)x$. Thus, we have

$$x\frac{\partial F}{\partial x} = y\frac{\partial F}{\partial y} = xyf'(xy).$$

□

Problem 22. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree m if $f(tx) = t^m f(x)$ for all $x \in \mathbb{R}^n$ and all $t > 0$. Prove that if f is differentiable on $\mathbb{R}^n$ and homogeneous of degree m, then

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x) = mf(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Proof. Differentiating both sides of the equation $f(tx) = t^m f(x)$ with respect to t and we have

$$x \cdot \nabla f(tx) = mt^{m-1} f(x).$$

Choosing $t = 1$ and we have

$$x \cdot \nabla f(x) = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x) = mf(x).$$

□

Problem 23. We know that a function $f(x, y)$ is differentiable at $(0, 0)$. We also know the directional derivatives

$$\begin{aligned} D_u f(0,0) = 1 \quad &\text{where } u = [1/\sqrt{5}, 2/\sqrt{5}], \\ D_v f(0,0) = 1 \quad &\text{where } v = [1/\sqrt{2}, 1/\sqrt{2}]. \end{aligned}$$

Find the gradient $\nabla f(0, 0)$.

Proof. We have

$$\begin{cases} \frac{1}{\sqrt{5}} \frac{\partial f}{\partial x}(0,0) + \frac{2}{\sqrt{5}} \frac{\partial f}{\partial y}(0,0) = 1 \\ \frac{1}{\sqrt{2}} \frac{\partial f}{\partial x}(0,0) + \frac{1}{\sqrt{2}} \frac{\partial f}{\partial y}(0,0) = 1 \end{cases}$$

Then we have $\nabla f(0,0) = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right) = \left(2\sqrt{2} - \sqrt{5}, \sqrt{5} - \sqrt{2}\right)$. □

Problem 24. Let $f \in C^1(\mathbb{R}^2)$ be such that $f(1,1) = 1$ and $\nabla f(1,1) = (a, b)$. Let $\varphi(x) = f(x, f(x, f(x,x)))$. Find $\varphi(1)$ and $\varphi'(1)$.

Proof. First, we have $\varphi(1) = f(1, f(1, f(1,1))) = f(1, f(1,1)) = f(1,1) = 1$. Second, set $\varphi(x) = f(x, h(x))$, where $h(x) = f(x, f(x,x)) = f(x, g(x))$. Then we have

$$\varphi'(x) = \partial_1 f(x, h(x)) + \partial_2 f(x, h(x)) h'(x),$$

where

$$\begin{aligned} h'(x) &= \partial_1 f(x, g(x)) + \partial_2 f(x, g(x)) g'(x) \\ g'(x) &= \partial_1 f(x,x) + \partial_2 f(x,x). \end{aligned}$$

Then $g'(1) = a + b, g(1) = 1$ and $h'(1) = \partial_1 f(1,1) + \partial_2 f(1,1)(a+b) = a + b(a+b)$. Thus, $\varphi'(1) = a + b(a + b(a+b))$. □

Problem 25. A function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Find the derivative of the function

$$F(t) = (f(t, t^2, \ldots, t^n))^2, \quad t \in \mathbb{R}$$

of one variable.

Proof. We have

$$F'(t) = 2f(t, t^2, \ldots, t^n)(1 + 2t + \cdots + nt^{n-1}) \frac{\partial f}{\partial t}.$$

□

Problem 26. Verify by a direct computation that the vector field $F(x) = x|x|^{-n}$ defined on $\mathbb{R}^n \setminus \{0\}$ is divergence free, i.e.

$$\operatorname{div} F(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{x_i}{|x|^n}\right) = 0 \quad \text{for all } x \neq 0.$$

Proof.

$$\begin{aligned}\operatorname{div} F(x) &= \sum_{i=1}^{n}\left(|x|^{-n} - n|x|^{-n-1}x_i^2\right)\\ &= n|x|^{-n} - n|x|^{-n-2}|x|^2 = 0.\end{aligned}$$

□

Problem 27. Prove that for $\alpha > 0$ the function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$,

$$\Phi(x) = x|x|^\alpha$$

is of class C^1. Find $D\Phi(x)$.

Proof. We need to prove the continuity of $\partial\Phi_i/\partial x_j$, and we have

$$\Phi_i(x) = x_j|x|^\alpha \in C^\infty\left(\mathbb{R}^n \setminus \{0\}\right).$$

For $x \neq 0$, we have

$$\begin{aligned}\frac{\partial\Phi_i(x)}{\partial x_j} &= \frac{\partial x_i}{\partial x_j}|x|^\alpha + x_i\frac{\partial|x|^\alpha}{\partial x_j}\\ &= \delta_{ij}|x|^\alpha + \alpha x_i x_j|x|^{\alpha-2}\\ &= |x|^\alpha\left(\delta_{ij} + \alpha\frac{x_i}{|x|}\frac{x_j}{|x|}\right).\end{aligned}$$

Some notation are defined as below

$$\begin{aligned}a &= [a_1, \cdots, a_n]\\ b &= [b_1, \cdots, b_n]\\ a \otimes b &= \begin{pmatrix} a_1b_1 & \cdots & a_1b_n\\ \vdots & \ddots & \vdots\\ a_nb_1 & \cdots & a_nb_n\end{pmatrix}\end{aligned}$$

then fr $x \neq 0$, we have

$$D\Phi(x) = |x|^\alpha\left(\delta_{ij} + \alpha\frac{x}{|x|} \otimes \frac{x}{|x|}\right),$$

and $D\Phi(x) \to 0$ as $x \to 0$. It remains to show that $D\Phi(0) = 0$.

We prove it directly by definition and we have

$$\frac{\Phi(x) - \Phi(0) - 0 \cdot x}{|x|} = \frac{x|x|^\alpha}{|x|} = \underbrace{\frac{x}{|x|}}_{\text{bounded}} \cdot|x|^\alpha \xrightarrow{x\to 0} 0.$$

Thus, the derivative of $\Phi(x)$ is

$$D\Phi(x) = \begin{cases} |x|^\alpha \left(\delta_{ij} + \alpha \dfrac{x}{|x|} \otimes \dfrac{x}{|x|} \right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

□

Problem 28. Find all the points $(x, y) \in \mathbb{R}^2$ where the function

$$f(x, y) = |e^x - e^y| \cdot (x + y - 2)$$

is differentiable.

Proof.

(1) f is C^∞ at $\{(x, y) \in \mathbb{R}^2 | x \neq y\}$. Indeed,

$$f(x, y) = \begin{cases} (e^x - e^y)(x + y - 2), & x > y, \\ (e^y - e^x)(x + y - 2), & x < y, \end{cases}$$

is C^∞.

(2) It remains to show the differentiability at point (x, x). We prove it by the definition and we have

$$\begin{aligned} \frac{\partial^+ f}{\partial x}(x, x) &= \lim_{h \to 0^+} \frac{f(x + h, x) - f(x, x)}{h} \\ &= \lim_{h \to 0^+} \frac{\left(e^{x+h} - e^x\right)(2x + h - 2)}{h} \\ &= \lim_{h \to 0^+} \frac{e^{x+h} - e^x}{h}(2x + h - 2) \\ &= e^x(2x - 2), \\ \frac{\partial^- f}{\partial x}(x, x) &= \lim_{h \to 0^-} \frac{f(x, x + h) - f(x, x)}{h} \\ &= -e^x(2x - 2). \end{aligned}$$

Then, when $x \neq 1$,

$$\frac{\partial^+ f}{\partial x}(x, x) \neq \frac{\partial^- f}{\partial x}(x, x),$$

hence, f is not differentiable at $\{(x, x)\} \setminus \{(1, 1)\}$.

For the point $(1, 1)$, we have

$$\frac{f(1 + s, 1 + t) - f(1, 1) - 0 \cdot (s, t)}{\sqrt{s^2 + t^2}} = \left| e^{1+s} - e^{1+t} \right| \underbrace{\frac{(s + t)}{\sqrt{s^2 + t^2}}}_{\text{bounded}} \xrightarrow{(s,t) \to (0,0)} 0.$$

Thus, f is differentiable at point $(1, 1)$.

□

Problem 29. Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x, y) = x^{2/3} y^{2/3}, \text{ for all } (x, y) \in \mathbb{R}^2.$$

Prove that g is *differentiable* at $(0, 0)$.

Proof. We have $\frac{\partial g}{\partial x}(0,0) = 0$ and $\frac{\partial g}{\partial y}(0,0) = 0$. Also, we have

$$\lim_{h\to 0} \frac{h^{2/3}h^{2/3}}{h} = 0 = f(0,0).$$

Thus, f is continuous at $(0,0)$, hence differentiable at $(0,0)$. $\square$

Problem 30. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is differentiable at each point, but whose partial derivatives are not continuous at $(0,0)$.

Proof. Take

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right), \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0) \end{cases}$$

Then we have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} h\sin(1/|h|) = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} h\sin(1/|h|) = 0$$

Also, we have

$$\frac{\partial f}{\partial x}(x,y) = 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}}$$
$$\frac{\partial f}{\partial y}(x,y) = 2y\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{y\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}}$$

which oscillate rapidly near the origin. Thus, the partial derivatives are not continuous at $(0,0)$. $\square$

Problem 31. Prove that the partial derivatives (of first order) of a function $f : \mathbb{R}^n \to \mathbb{R}$ exist everywhere and they are bounded, then the function f is continuous.

Proof. Let $x_0 = (x_{01}, \cdots, x_{0n}) \in \mathbb{R}^n$ be arbitray, and define $f_i = \frac{\partial f}{\partial x_i}$. Since partial derivatives exist everywhere and bounded, then we define $M = \sum_{i=1}^n \sup|f_i(x)|, x \in \mathbb{R}^n$. Then, for $x = (x_1, \cdots, x_n)$, we have

$$\begin{aligned}|f(x_0) - f(x)| \leq & |f(x_{01}, \cdots, x_{0n}) - f(x_1, x_{02}, \cdots, x_{0n})| + \\ & |f(x_1, x_{02}, \cdots, x_{0n}) - f(x_1, x_2, x_{03}, \cdots, x_{0n})| + \cdots \\ & + |f(x_1, \cdots, x_{n-1}, x_{0n}) - f(x_1, \cdots, x_n)| \\ \leq & M|(x_{01}, \cdots, x_{0n}) - (x_1, x_{02}, \cdots, x_{0n})| + \\ & M|(x_1, x_{02}, \cdots, x_{0n}) - (x_1, x_2, x_{03}, \cdots, x_{0n})| + \cdots \\ & M|(x_1, \cdots, x_{n-1}, x_{0n}) - (x_1, \cdots, x_n)|,\end{aligned}$$

where in the last step we used Mean Value theorem. Thus, we can know that for any $x \in \mathbb{R}^n$, $f(x)$ is bounded. $\square$

Problem 32. Prove that if $f, g \in C^k(\Omega)$, $\Omega \subset \mathbb{R}^n$, then for any multiindex α with $|\alpha| \leq k$ we have

$$D^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g \,,$$

where $\beta \leq \alpha$ means that $\beta_i \leq \alpha_i$ for $i = 1, 2, \ldots, n$, $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ and

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\,(\alpha - \beta)!} \,.$$

Proof. We have

$$\begin{aligned}
D^1(fg) &= D^1 fg + fD^1 g \\
D^2(fg) &= D^2 fg + D^1 fD^1 g + D^1 fD^1 g + fD^2 g \\
&\cdots \\
D^\alpha(fg) &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g,
\end{aligned}$$

where it is like the Binomial theorem. □