Homework 5 for Math 1530
Due day: Tuesday October 8 recitations.

Problem 43. Prove that there is an increasing sequence of integers $a_1 < a_2 < a_3 < \dots$ such that for every $k \in \mathbb{N}$, the sequence $\{\sin(ka_n)\}_{n=1}^{\infty}$ converges.

Proof. Every sequence in compact space has a convergent subsequence. And since $\sin n$ is dense $[-1,1]$, then $\sin n$ has a convergent subsequence as $\{\sin b_1, \sin b_2, \cdots\}$.Then, we can know that $\{\sin(2b_1), \sin(2b_2), \cdots\}$ has a convergent subsequence $\{\sin(2c_1), \sin(2c_2), \cdots\}$, where $2c_k \in \{2b_n\}$. So we have a sequence $\{c_n\}$ such that both $\{\sin c_n\}$ and $\{\sin(2c_n)\}$ converge as $n \to \infty$. Moreover, $\{\sin(3c_n)\}$ has a convergent subsequence $\{\sin(3d_1), \sin(3d_2), \cdots\}$. And we can continue this process and define $a_1 = b_1, a_2 = c_2, a_3 = d_3, \cdots$, and we can know that $\{\sin(ka_n)\}, \forall k \in \mathbb{N}$ converges. $\square$

Problem 44. Fix $k \in \mathbb{N}$ and define

$$z_n = \frac{1^k + 2^k + \ldots + n^k}{n^{k+1}}, \quad n = 1, 2, 3, \ldots$$

Prove that

$$\lim_{n\to\infty} n\left(z_n - \frac{1}{k+1}\right) = \frac{1}{2}.$$

Proof. We have $\lim_{n\to\infty} n\left(z_n - \frac{1}{k+1}\right) = \lim_{n\to\infty} \frac{(k+1)(1^k+\cdots+n^k)-n^{k+1}}{(k+1)n^k} = \lim_{n\to\infty} \frac{x_n}{y_n}$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} &= \frac{(k+1)((1^k+\cdots+n^k) - n^{k+1} - (k+1)(1^k+\cdots+(n-1)^k) + (n-1)^{k+1}}{n^k(k+1) - (k+1)(n-1)^k} \\
&= \lim_{n\to\infty} \frac{\frac{(k+1)k}{2}n^{k-1} + \binom{k}{2}n^{k-2} + \cdots}{(k+1)(kn^{k-1} - \binom{k}{2}n^{k-2})} \\
&= \frac{1}{2}
\end{aligned}$$

According to Stolz theorem, $\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = 1/2$. Thus, this series converges to $1/2$. $\square$

Problem 45. Let $\beta > 0$ and $\{u_n\}$ be a sequence of positive real numbers such that $\dfrac{u_{n+1}}{u_n} \le \beta$ for every $n \in \mathbb{N}$. Prove that

$$\limsup_{n\to\infty} \sqrt[n]{u_n} \le \limsup_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right).$$

Proof. For $\forall N > 0$, with $\frac{u_{n+1}}{u_n} \le \beta$, we have $u_n \le c_N \beta^{n-N} = c_N \beta^{-N} \beta^n$. Then we can have

$$\begin{aligned}
\sqrt[n]{u_n} &\le \sqrt[n]{c_N \beta^{-N}}\beta \\
\Rightarrow \limsup_{n\to\infty} \sqrt[n]{u_n} &\le \beta
\end{aligned}$$

Since $\frac{u_{n+1}}{u_n} \le \beta$ for all β, then $\limsup_{n\to\infty} \sqrt[n]{u_n} \le \limsup_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right)$ $\square$

Problem 46. Prove that if a sequence (a_n) of real numbers is convergent to a finite limit, $\lim_{n\to\infty} a_n = g \in \mathbb{R}$, then

$$\lim_{x\to\infty} e^{-x}\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = g.$$

Proof. We have

$$\left| e^{-x}\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} - g \right| = \left| e^{-x}\sum_{n=0}^{\infty} (a_n - g)\frac{x^n}{n!} \right|$$

Since $\lim_{n\to\infty} a_n = g$, then $\forall \varepsilon > 0$, there exists an $N > 0$, such that $\forall n > N$, $|a_n - g| < \varepsilon$. Then,

$$\begin{aligned}
\left| e^{-x}\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} - g \right| &= \left| e^{-x}\sum_{n=0}^{N-1} (a_n - g)\frac{x^n}{n!} + e^{-x}\sum_{n=N}^{\infty} (a_n - g)\frac{x^n}{n!} \right| \\
&\le \left| e^{-x}\sum_{n=0}^{N-1} (a_n - g - \varepsilon)\frac{x^n}{n!} \right| + \left| e^{-x}\sum_{n=0}^{N-1} \varepsilon\frac{x^n}{n!} \right| + \left| e^{-x}\sum_{n=N}^{\infty} \varepsilon\frac{x^n}{n!} \right| \\
&= \left| e^{-x}\sum_{n=0}^{N-1} (a_n - g - \varepsilon)\frac{x^n}{n!} \right| + \left| e^{-x}\sum_{n=0}^{\infty} \varepsilon\frac{x^n}{n!} \right| \\
&= \left| e^{-x}\sum_{n=0}^{N-1} (a_n - g - \varepsilon)\frac{x^n}{n!} \right| + \varepsilon
\end{aligned}$$

Also, for the first term on the right hand, we have

$$\lim_{x\to\infty} \left| e^{-x}\sum_{n=0}^{N-1} (a_n - g - \varepsilon)\frac{x^n}{n!} \right| = \sum_{n=0}^{N-1} \lim_{x\to\infty} (a_n - g - \varepsilon)\frac{x^n}{e^x n!} = 0$$

Then we have $\left| e^{-x}\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} - g \right| \le \varepsilon$, which means the limit goes to g. ☐

Problem 47. Suppose that a sequence of functions $f_1, f_2, \ldots$ converges uniformly on $[0,1]$ to some function f. Suppose also that there is a constant M such that $|f_i(x)| \le M$ for all i and x. Prove that the sequence of squares $f_1^2, f_2^2, \ldots$ converges uniformly to f^2.

Proof. Since $f_1, f_2, \cdots$ converges uniformly on $[0,1]$, then for $\forall \varepsilon > 0$, $\exists \delta > 0$, then $\forall x, y \in [0,1]$, if $|x - y| < \delta$, then $|f_i(x) - f_i(y)| < \frac{\varepsilon}{2M}$. Also, $f_1, f_2, \cdots$ converges to function f, then for the same ε as before, there exists an $N > 0$ such that $\forall n > N$, we have $|f_n(x) - f(x)| < \frac{\varepsilon}{2M}, \forall x \in [0,1]$. Then, if $|x - y| < \delta$ and $n > N$, we have

$$|f_i^2(x) - f_i^2(y)| \le |f_i(x) - f_i(y)||f_i(x) + f_i(y)| \le 2M\frac{\varepsilon}{2M} = \varepsilon$$
$$|f_n^2(x) - f^2(x)| \le |f_n(x) - f(x)||f_n(x) + f(x)| \le 2M\frac{\varepsilon}{2M} = \varepsilon$$

Then we know that $f_1^2, f_2^2, \cdots$ converges uniformly to f^2. ☐

Problem 48. Prove that if $f : (0,1) \to \mathbb{R}$ is uniformly continuous, then there is a continuous function $F : [0,1] \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in (0,1)$.

Proof. Consider the sequence $a_n = 1/n, n \in \mathbb{N}$ and $b_n = 1 - 1/n, n \in \mathbb{N}$, and we define

$$F(0) = \lim_{n\to\infty} f(a_n) = A$$
$$F(1) = \lim_{n\to\infty} f(b_n) = B$$
$$F(x) = f(x), x \in (0,1)$$

Now we prove that $F(x)$ defined above is continuous. It is obvious that F is continuous on $(0,1)$. Now consider a sequence $\{x_n\}$ converges to 0, and we need to prove that $\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} f(x_n) = A$. For $\forall\varepsilon > 0$, there exists an $N_1 > 0$, such that for $n > N_1$, we have $|f(a_n) - A| < \varepsilon/2$. Since $f(x)$ is uniformly continuous, then for the same ε, there exists $\delta > 0$, such that $\forall x, y \in (0,1)$ if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon/2$. Since $\{x_n\}$ converges to 0, and $a_n = 1/n$, then there exists an N_2 such that $x_n < \delta$ and $a_n < \delta$, then we have $|x_n - a_n| < \delta$, and we have

$$|f(x_n) - A| \le |f(a_n) - A| + |f(x_n) - f(a_n)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then we proved that $F(x)$ is continuous at $x = 0$, similarly, $F(x)$ is also continuous at $x = 1$. Then F is continuous. □

Problem 49. Prove that if $f : [0,\infty) \to \mathbb{R}$ is continuous and the limit $\lim_{x\to\infty} f(x)$ exists and is finite, then f is uniformly continuous.

Proof. Suppose $f(x)$ is not uniformly continuous, then $\exists\varepsilon > 0$, for $\forall\delta > 0$ and $\exists x, y \in (0,\infty)$, such that if $|x - y| < \delta$, then $|f(x) - f(y)| \ge \varepsilon$. Set ε be such that the above statement is true, and set $\delta = 1/n$. So we can find $x_n, y_n \in [0,\infty)$ such that $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \ge \varepsilon$. For x_n, we can find a subsequence $\{x_{n_k}\}$ such that $\lim_{\to\infty} x_{n_k} = x_0$. And we have $|x_{n_k} - y_{n_k}| < 1/n_k$, then we have $\lim_{\to\infty} y_{n_k} = x_0$. Hence, we have $\lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| = |f(x_0) - f(x_0)| = 0$, which is a contradiction. The proof is complete. □

Problem 50. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then there exist constants $a \ge 0$, $b \ge 0$ such that $|f(x)| \le a|x| + b$ for all $x \in \mathbb{R}$.

Proof. Since $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then $\forall\varepsilon > 0$, there exists $\delta > 0$, for $\forall x, y$, such that $|x - y| < \delta$, $|f(x) - f(y)| \le \varepsilon$. This holds for $\varepsilon = 1$.

We can present x as $x = m\delta + r$, where $0 \le r < \delta$, then we have

$$\begin{aligned}
|f(x) - f(0)| =& |f(m\delta + r) - f(0)| \\
=& |f(m\delta + r) - f((m-1)\delta + r) + f((m-1)\delta + r) - f((m-2)\delta + r) + \cdots \\
& + f(2\delta + r) - f(\delta + r) + f(\delta + r) - f(r) + f(r) - f(0)| \\
\le& m + 1 \\
\le& \frac{1}{\delta}x - \frac{r}{\delta} + 1 \\
\le& \frac{1}{\delta}|x| - \frac{r}{\delta} + 1
\end{aligned}$$

Then we have $|f(x)| \le \frac{1}{\delta}|x| + f(0) + 2$. Set $a = \frac{1}{\delta}$ and $b = f(0) + 2$, and the proof is complete. □

Problem 51. Prove that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = -x$ for all $x \in \mathbb{R}$.

Proof. We can know that f is invertible, since $f(f(f(f(x)))) = f(f(-x)) = x$. Any invertible function is either increasing or decreasing. If f is increasing, we take $x < y$, then we have $f(x) < f(y)$ and $f(f(x)) = -x < f(f(y)) = -y$, which is a contradiction. If f is decreasing, we take $x > y$, then we have $f(x) > f(y)$ and $f(f(x)) = -x < f(f(y)) = -y$, which is a contradiction. Thus, f cannot be a continuous function. $\square$

Problem 52. Suppose that $f : (0, 1) \to \mathbb{R}$ is continuous. Suppose also that there are two sequences $x_n, y_n \in (0, 1)$ both convergent to 0 such that $f(x_n) \to 0$, $f(y_n) \to 1$. Prove that there is a sequence $z_n \in (0, 1)$ convergent to 0 such that $f(z_n) \to 1/2$.

Proof. Since $f(x)$ is continuous and $\lim_{n\to\infty} f(x_n) = 0, \lim_{n\to\infty} f(y_n) = y$, then there for $\forall\varepsilon > 0$, there exists a $\delta > 0$, such that $|x_n - 0| < \delta, |y_n - 0| < \delta$, then we have $|f(x_n) - 0| < \varepsilon$ and $|f(y_n) - 1| < \varepsilon$. Also $\{x_n\}, \{y_n\} \to 0$, then for the same $\delta > 0$, there exists an $N > 0$, such that $\forall n > N$, we have $|x_n - 0| < \delta$ and $|y_n - 1| < \delta$.

Now we pick the $n = N + 1$ and $\varepsilon < /2$, and we have $f(x_n) < \delta = 1/2$ and $1/2 < f(y_n) < 3/2$. Since f is continuous, there exists a z between x_n and y_n such that $f(x_n) < f(z) = 1/2 < f(y_n)$. We denote this z by z_1. Then we set $n = N + 2$, then we can find z_2. Continue this process and we can get a sequence $\{z_n\}$ between sequence $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} z_n = 0$ and $\lim_{n\to\infty} f(z_n) = 1/2$. $\square$

Problem 53. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that the function

$$g(x) = \sup_{t\in[a,x]} f(t)$$

is continuous.

Proof. Since f is continuous, then $\forall\varepsilon > 0$, there exists a $\delta > 0$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Now we fix $y \in [a, b]$ and we assume $|x - y| < \delta$.

Firstly, we consider $x \in (y, y + \delta)$. If $\sup_{t\in[a,x]} f(t) = f(x_0)$ for some $x_0 \in (y, x)$, then we have two conditions. One is that $\sup_{t\in[a,y]} f(t) = f(y)$, then we have $|g(x) - g(y)| < \varepsilon$. The second one is that $\sup_{t\in[a,y]} f(t) = f(t_0)$ for some $t_0 \in [a, y]$, and since $\sup_{t\in[a,x]} f(t) = f(x_0)$ for some $x_0 \in (y, x)$, we have $f(y) < f(t_0) \leq f(x_0)$. Thus, we have $|g(x) - g(y)| = |f(x_0) - f(t_0)| < |f(x_0) - f(y)| < \varepsilon$.

Similarly, we can know that $|g(x) - g(y)| < \varepsilon$ for $x \in (y - \delta, y)$. Then we proved that g is continuous. $\square$

Problem 54. A function $f : [0, 1] \to \mathbb{R}$ is continuous and has the property that

$$\lim_{x\to 0^+} \frac{f(x + 1/3) + f(x + 2/3)}{x} = 1.$$

Prove that there is $x_0 \in [0, 1]$ such that $f(x_0) = 0$.

Proof. Denote $x = 1/n, n \geq 3, n \in \mathbb{N}$, then we have

$$\lim_{n\to\infty} f\left(\frac{1}{n} + \frac{1}{3}\right) + f\left(\frac{1}{n} + \frac{2}{3}\right) = \frac{1}{n} = 0$$

Take $n \to \infty$, and we have $f(1/3) + f(2/3) = 0$. If both $f(1/3)$ and $f(2/3)$ are equal to zero, the the proof is complete. If not, then one of $f(1/3)$ and $f(2/3)$ is greater than zero, and the other is less than zero. Since f is continuous, there must exist a x_0 such that $f(x_0) = 0$. □

Method 2.

Proof. We want to prove that $f(1/3)+f(2/3) = 0$, and we only need to prove $f(1/3)f(2/3) < 0$. Now we prove it by contradiction.

Suppose $f(1/3)f(2/3) > 0$, and we assume $f(1/3) > 0$ and $f(2/3) > 0$. Without losing generality, we assume $f(1/3) < f(2/3)$. Since f is continuous, then for $\forall \varepsilon > 0$, there exists a $\delta > 0$, such that if $x \in (0, \delta)$, $|f(1/3)-f(x+1/3)| < \varepsilon$ and $|f(2/3)-f(x+2/3)| < \varepsilon$. Now we set $\varepsilon = \frac{f(1/3)}{2}$, then we have

$$f(x + 1/3) > f(1/3) - \varepsilon = \frac{f(1/3)}{2} > 0$$

$$f(x + 2/3) > f(2/3) - \varepsilon = \frac{f(1/3)}{2} > 0$$

which implies that, when $x < f(1/3)$, we have

$$\frac{f(x + 1/3) + f(x + 2/3)}{x} > \frac{f(1/3)}{x} > 1$$

And it is contradiction. Then $f(1/3)f(2/3) < 0$. □