

Homework 4 for Math 1530

Zhen Yao

Problem 35. Prove that if a sequence $\{a_n\}$ is bounded and $a_{n+1} \geq a_n - \frac{1}{2^n}$ for all n , then the sequence $\{a_n\}$ is convergent.

Proof. Since $\{a_n\}$ is bounded, then sequence $\{a_n\}$ has a least upper bound, denoted by $M = \sup\{a_n\}$. And for $\forall m \in \mathbb{N}$ and $n > m$, we have

$$M \geq a_n \geq a_m - \frac{1}{2^m} - \frac{1}{2^{m+1}} - \cdots - \frac{1}{2^{n-1}} \geq a_m - \sum_{i=m}^{\infty} \frac{1}{2^i}$$

then there exists N such that for $\forall m > N$, $\sum_{i=m}^{\infty} \frac{1}{2^i} \leq \varepsilon/2$, which implies $\forall n > m > N$, $|a_n - a_m| < \varepsilon/2$.

For $\forall n > N$, we can find the first $a_K, K > N$ such that $a_K > M - \varepsilon/2$. Then for $\varepsilon > 0$ and $\forall n > K$, we have

$$|a_n - M| \leq |a_n - a_K| + |a_K - M| < \varepsilon$$

Then $\{a_n\}$ is convergent. □

Problem 36. Find the limit $\lim_{n \rightarrow \infty} n \sin(2\pi en!)$.

Proof. We know that $e = \sum_{n=0}^{\infty} 1/n!$, then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} n \sin(2\pi en!) &= \lim_{n \rightarrow \infty} n \sin \left(2\pi n! \left(\sum_{k=0}^n \frac{1}{k!} + \sum_{k=n+1}^{\infty} \frac{1}{k!} \right) \right) \\ &= \lim_{n \rightarrow \infty} n \sin \left(2\pi n! \left(\sum_{k=n+1}^{\infty} \frac{1}{k!} \right) \right) \end{aligned}$$

since $\lim_{n \rightarrow \infty} n! \left(\sum_{k=n+1}^{\infty} 1/k! \right) = 0$, and we denote it by A_n , then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} n \sin(2\pi en!) &= \lim_{n \rightarrow \infty} n \sin(2\pi A_n) \\ &= \lim_{n \rightarrow \infty} 2\pi n A_n \frac{\sin(2\pi A_n)}{2\pi A_n} \\ &= \lim_{n \rightarrow \infty} 2\pi n A_n \\ &= 2\pi \end{aligned}$$

□

Problem 37. Find the limit

$$\lim_{n \rightarrow \infty} n^3 \left(\sqrt{n^2 + \sqrt{n^4 + 1}} - n\sqrt{2} \right).$$

Proof. Denote the limit by A , and we have

$$\begin{aligned}
A &= n^3 \left(\sqrt{n^2 + \sqrt{n^4 + 1}} - n\sqrt{2} \right) \left(\frac{(\sqrt{n^2 + \sqrt{n^4 + 1}} + n\sqrt{2})}{(\sqrt{n^2 + \sqrt{n^4 + 1}} + n\sqrt{2})} \right) \\
&= n^3 \frac{\sqrt{n^4 + 1} - n^2}{(\sqrt{n^2 + \sqrt{n^4 + 1}} + n\sqrt{2})} \\
&= n^3 \frac{1}{(\sqrt{n^2 + \sqrt{n^4 + 1}} + n\sqrt{2})(\sqrt{n^2 + \sqrt{n^4 + 1}} + n^2)} \\
&= \frac{1}{(2\sqrt{2})(1 + 1)} \\
&= \frac{\sqrt{2}}{8}
\end{aligned}$$

□

Problem 38. Suppose that $\{a_n\}_n$ is a sequence such that for every integer $k \geq 2$ the sequence $\{a_{k \cdot n}\}_n$ is convergent. Does it follow that the sequence $\{a_n\}_n$ is convergent?

Proof. No, it is not always convergent. Take $a_n = 1$ when n is prime number, and $a_n = 0$ when n is composite number. Then the sequence $\{a_{kn}\}_n$ is convergent, since $a_{kn} = 0$ when $n > 1$. But $\{a_n\}$ is not convergent since there are infinitely many prime numbers. □

Problem 39. Prove that if $\{a_n\}_n$ is a sequence such that each of the sequences $\{a_{2n}\}_n$, $\{a_{2n+1}\}_n$ and $\{a_{3n}\}_n$ is convergent, then $\{a_n\}_n$ is convergent too.

Proof. (1) Since the sequences $\{a_{2n}\}_n$, $\{a_{2n+1}\}_n$ and $\{a_{3n}\}_n$ are convergent, then they have the same limit. If not, we can assume that they have different limit, say $\lim_{n \rightarrow \infty} a_{2n} = M$, $\lim_{n \rightarrow \infty} a_{2n+1} = N$, $\lim_{n \rightarrow \infty} a_{3n} = P$, then for $\forall \varepsilon > 0$, there exists $N > 0$ such that for $\forall n > N$, we have $|a_{2n} - M| < \varepsilon$, $|a_{2n+1} - N| < \varepsilon$ and $|a_{3n} - P| < \varepsilon$. Then for certain $n = 3k$, where k is an integer, we have $|a_{6k} - M| < \varepsilon$, and $|a_{6k} - P| < \varepsilon$, which means $M = P$. Similarly, we can know $M = N = P$.

(2) For any a_m where $m > N$, we can know that a_n belongs to one of $\{a_{2n}\}_n$, $\{a_{2n+1}\}_n$ and $\{a_{3n}\}_n$, such that $|a_m - M| < \varepsilon$. Thus, $\{a_n\}$ is convergent. □

Problem 40. Prove that there is a sequence such that the set of all possible limits of subsequences is the whole interval $[0, 1]$.

Proof. Take $a_n = f(n) = |\sin n|$. Since $|\sin n|$ is dense in the interval $[0, 1]$, so all the possible limits of subsequences is the whole interval. □

Problem 41. Prove that if $\lim_{n \rightarrow \infty} a_n = a \in \mathbb{R}$, then for any sequence $\{b_n\}$

$$\limsup_{n \rightarrow \infty} (a_n + b_n) = a + \limsup_{n \rightarrow \infty} b_n.$$

Proof. (1) If $\limsup_{n \rightarrow \infty} b_n = \infty$, then the equation holds.

(2) Consider the case $\limsup_{n \rightarrow \infty} b_n$ is finite. Suppose $A_k = \sup\{a_n, n \geq k\}$, $B_k =$

$\sup\{b_n, n \geq k\}$ and $C_k = \sup\{a_n + b_n, n \geq k\}$. For $\forall n \geq k$, we have $a_n + b_n \leq A_k + B_k$. Then we have

$$\begin{aligned} \sup_{n \geq k} (a_n + b_n) &\leq \sup_{n \geq k} a_n + \sup_{n \geq k} b_n \\ \Rightarrow \lim_{k \rightarrow \infty} \sup_{n \geq k} (a_n + b_n) &\leq \lim_{k \rightarrow \infty} \sup_{n \geq k} a_n + \lim_{k \rightarrow \infty} \sup_{n \geq k} b_n \\ \Rightarrow \limsup_{n \rightarrow \infty} (a_n + b_n) &\leq a + \limsup_{n \rightarrow \infty} b_n \end{aligned}$$

On the other hand, we write $b_n = (b_n + a_n) + (-a_n)$, and with the result above, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} b_n &\leq \limsup_{n \rightarrow \infty} (a_n + b_n) + \limsup_{n \rightarrow \infty} (-a_n) \\ \Rightarrow \limsup_{n \rightarrow \infty} b_n &\leq \limsup_{n \rightarrow \infty} (a_n + b_n) - a \\ \Rightarrow \limsup_{n \rightarrow \infty} (a_n + b_n) &\geq a + \limsup_{n \rightarrow \infty} b_n \end{aligned}$$

Then we proved that $\limsup_{n \rightarrow \infty} (a_n + b_n) = a + \limsup_{n \rightarrow \infty} b_n$. \square

Problem 42. Let c_0 be the class of all sequences $\{x_n\}$ such that $\lim_{n \rightarrow \infty} x_n = 0$. Prove that if $\{a_n\}$ is a bounded sequence, then

$$\inf_{(x_n) \in c_0} \left(\sup_{n \in \mathbb{N}} \{a_n + x_n\} \right) = \limsup_{n \rightarrow \infty} a_n.$$

Proof. For $\forall \varepsilon > 0$, let c_k be the class of all sequences $\{x_n\}$ such that $\forall n \geq k, |x_n| < \varepsilon/2$. Then we know $c_1 \subset c_2 \subset \dots \subset c_0$. Then we have $\inf_{(x_n) \in c_0} (\sup_{n \in \mathbb{N}} \{a_n + x_n\}) = \lim_{k \rightarrow \infty} \inf_{(x_n) \in c_k} (\sup_{n \in \mathbb{N}} \{a_n + x_n\})$.

For a certain k , we have $\inf_{(x_n) \in c_k} (\sup_{n \in \mathbb{N}} \{a_n + x_n\}) = \inf_{(x_n) \in c_k} (\sup_{n \geq k} \{a_n + x_n\})$, since we can find a sequence $\{x_n\}$ such that makes $a_i + x_i, 1 \leq i \leq k-1$ small enough and the supremum of $\{a_n + x_n\}$ will be after k th terms. For $\forall n \geq k, |x_n| < \varepsilon/2$. Then we have

$$\begin{aligned} \sup_{n \geq k} (a_n - \varepsilon/2) &< \sup_{n \geq k} (a_n + x_n) < \sup_{n \geq k} (a_n + \varepsilon/2) \\ \Rightarrow \sup_{n \geq k} a_n - \varepsilon/2 &< \sup_{n \geq k} (a_n + x_n) < \sup_{n \geq k} a_n + \varepsilon/2 \\ \Rightarrow \left| \sup_{n \geq k} (a_n + x_n) - \sup_{n \geq k} a_n \right| &< \frac{\varepsilon}{2} \end{aligned}$$

which implies

$$\left| \inf_{(x_n)_k} \left(\sup_{n \geq k} (a_n + x_n) \right) - \sup_{n \geq k} a_n \right| < \frac{\varepsilon}{2}$$

Also, there exists a $k_0 \in \mathbb{N}$ such that $|\sup_{n \geq k} a_n - \limsup_{n \rightarrow \infty} a_n| < \varepsilon/2$. Then for $\forall k \geq k_0$, we have

$$\begin{aligned} &\left| \inf_{(x_n)_k} \left(\sup_{n \geq k} (a_n + x_n) \right) - \limsup_{n \rightarrow \infty} a_n \right| \\ &\leq \left| \inf_{(x_n)_k} \left(\sup_{n \geq k} (a_n + x_n) \right) - \sup_{n \geq k_2} a_n \right| + \left| \sup_{n \geq k} a_n - \limsup_{n \rightarrow \infty} a_n \right| = \varepsilon \end{aligned}$$

Then

$$\inf_{(x_n) \in c_0} \left(\sup_{n \in \mathbb{N}} \{a_n + x_n\} \right) = \lim_{k \rightarrow \infty} \inf_{(x_n)_k} \left(\sup_{n \geq k} (a_n + x_n) \right) = \limsup_{n \rightarrow \infty} a_n$$

□