Epsilon Space and Time Change

⬛ Suggest a time and space change in the epsilon of a transmission line with length of your choosing to fit a certain filter for the reflection coefficient gamma.

Spatial Reconstruction
Temporal Reconstruction

- given L
- given Γ
- find space and time change in ϵ to fit a certain filter

Spatial Reconstruction

$$\frac{d\Gamma(z)}{dz} = 2j\beta(z)\Gamma(z) - \frac{1}{2}(1-\Gamma^2(z))\frac{d\ln Z}{dz}$$
$$\frac{d\ln Z}{dz} = \left[4j\beta(z)\Gamma(z) - 2\frac{d\Gamma(z)}{dz}\right]\frac{1}{1-\Gamma^2(z)}$$

Now, we have $\begin{cases}\beta(z) = w\sqrt{\mu(z)\epsilon(z)} \\ Z(z) = \sqrt{\frac{\mu(z)}{\epsilon(z)}}\end{cases}$ where $\mu(z) = \mu_0$

We solve for $\xi(z) = \sqrt{\epsilon(z)}$

$$\frac{d}{dz}\ln\left(\frac{\sqrt{\mu_0}}{\xi(z)}\right) = \left(4jw\sqrt{\mu_0}\xi(z)\Gamma(z) - 2\frac{d\Gamma(z)}{dz}\right)\frac{1}{1-\Gamma^2(z)}$$
$$-\frac{d}{dz}\ln(\xi(z)) = \left(4jw\sqrt{\mu_0}\xi(z)\Gamma(z) - 2\frac{d\Gamma(z)}{dz}\right)\frac{1}{1-\Gamma^2(z)}$$
$$-\frac{\xi'(z)}{\xi(z)} = \left(4jw\sqrt{\mu_0}\xi(z)\Gamma(z) - 2\frac{d\Gamma(z)}{dz}\right)\frac{1}{1-\Gamma^2(z)}$$
$$-\frac{d\xi}{dz} = \left(4jw\sqrt{\mu_0}\Gamma(z)\xi^2(z) - 2\frac{d\Gamma(z)}{dz}\xi(z)\right)\frac{1}{1-\Gamma^2(z)}$$
$$-\frac{d\xi}{dz} = \underbrace{\left(4jw\sqrt{\mu_0}\Gamma(z)\frac{1}{1-\Gamma^2(z)}\right)}_{b(z)}\xi^2(z) + \underbrace{\left(-2\frac{d\Gamma(z)}{dz}\frac{1}{1-\Gamma^2(z)}\right)}_{a(z)}\xi(z)$$

$a(z), b(z)$ are known. We get

$$\xi'(z) + a(z)\xi(z) + b(z)\xi^2(z) = 0$$

Solving the Bernoulli ODE

$$\frac{\xi'(z)}{\xi^2(z)} + \frac{a(z)}{\xi(z)} = -b(z)$$

Put $v(z) = \frac{1}{\xi(z)} \rightarrow v'(z) = -\frac{1}{\xi^2(z)}\xi'(z)$

$$-v'(z) + a(z)v(z) = -b(z)$$
$$v'(z) - a(z)v(z) = b(z)$$

This is linear ODE with integrating factor $I(z) = e^{-A(z)}$

$$I(z)v(z)\Big|_{z=L}^{z} = \int_L^z I(x)b(x)dx$$
$$I(z)v(z) = I(L)v(L) + \int_L^z I(x)b(x)dx$$
$$v(z) = I(L)v(L)I^{-1}(z) + I^{-1}(z)\int_L^z I(x)b(x)dx$$
$$\frac{1}{\xi(z)} = I(L)v(L)I^{-1}(z) + I^{-1}(z)\int_L^z I(x)b(x)dx$$
$$\xi(z) = \frac{1}{I(L)\frac{1}{\xi(L)}I^{-1}(z) + I^{-1}(z)\int_L^z I(x)b(x)dx}$$

We know that $Z(z=L) = Z_L \to Z_L = \frac{\sqrt{\mu_0}}{\xi(L)} \to \xi(L) = \frac{\sqrt{\mu_0}}{Z_L}$

Then, we have a closed expression for $\epsilon(z)$

$$\boxed{\epsilon(z) = \left(\frac{1}{I(L)\frac{Z_L}{\sqrt{\mu_0}}I^{-1}(z) + I^{-1}(z)\int_L^z I(x)b(x)dx}\right)^2}$$

Temporal Reconstruction

$$\frac{d\Gamma(t)}{dt} = 2j\beta(t)\Gamma(t) + \frac{1}{2}(1-\Gamma^2(t))\frac{d\ln Z}{dt}$$
$$\frac{d\ln Z}{dt} = \left[-4j\beta(t)\Gamma(t) + 2\frac{d\Gamma(t)}{dt}\right]\frac{1}{1-\Gamma^2(t)}$$

Now, we have $\begin{cases}\beta(t) = w\sqrt{\mu(t)\epsilon(t)} \\ Z(t) = \sqrt{\frac{\mu(t)}{\epsilon(t)}}\end{cases}$ where $\mu(t) = \mu_0$

We solve for $\xi(t) = \sqrt{\epsilon(t)}$. Using same $a(t), b(t)$ (with $z \to t$), we get the following ODE

$$\xi'(t) - a(t)\xi(t) - b(t)\xi^2(t) = 0$$

Using same substitution

$$v'(t) + a(t)v(t) = -b(t)$$

This is linear ODE with integrating factor $I(t) = e^{A(t)}$

Using the same calculation, we have

$$\xi(t) = \frac{1}{I(0)\frac{1}{\xi(0)}I^{-1}(t) + I^{-1}(t)\int_0^t -I(\tau)b(\tau)d\tau}$$

We know that at $t = 0$, we must have $\Gamma(0) = 0$ (cause not back-propagation at $t = 0$). Plugging in Ricatti equation wrt t at $t = 0$

$$\frac{d\Gamma}{dt}\Big|_{t=0} = \frac{1}{2}\frac{d\ln Z}{dt}\Big|_{t=0}$$
$$\frac{d\Gamma}{dt}\Big|_{t=0} = \frac{1}{2}\frac{d}{dt}\ln\frac{\sqrt{\mu_0}}{\xi(t)}\Big|_{t=0}$$
$$\frac{d\Gamma}{dt}\Big|_{t=0} = -\frac{1}{2}\frac{d\ln\xi(t)}{dt}\Big|_{t=0}$$
$$\frac{d\Gamma}{dt}\Big|_{t=0} = -\frac{\xi'(t)}{2\xi(t)}\Big|_{t=0}$$
$$\frac{d\Gamma}{dt}\Big|_{t=0} = \frac{-a(t) - b(t)\xi(t)}{2}\Big|_{t=0}$$
$$\xi(0) = \frac{1}{b(t)}\left(-a(t) - 2\frac{d\Gamma}{dt}\right)\Big|_{t=0} \to \frac{0}{0}$$

Thus, we get

$$\boxed{\epsilon(t) = \left(\frac{1}{I(0)\left(\frac{1}{b(t)}\left(-a(t) - 2\frac{d\Gamma}{dt}\right)^{-1}\right)\Big|_{t=0} I^{-1}(t) + I^{-1}(t)\int_0^t -I(\tau)b(\tau)d\tau}\right)^2}$$