RiSE Seminar – IST Austria

Decomposing Permutation Automata

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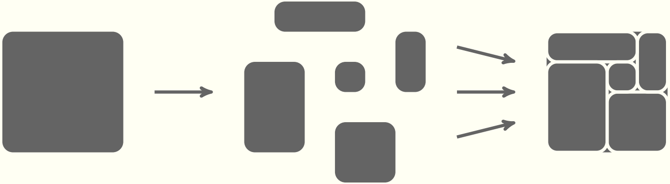
Divide and conquer



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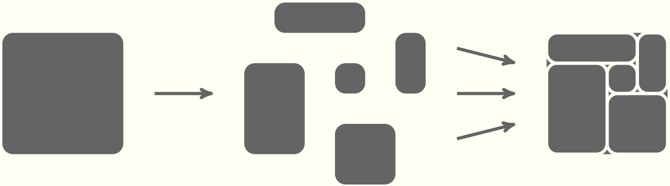

Compositionality

1. decomposition into smaller instances
2. .. that determine the original problem

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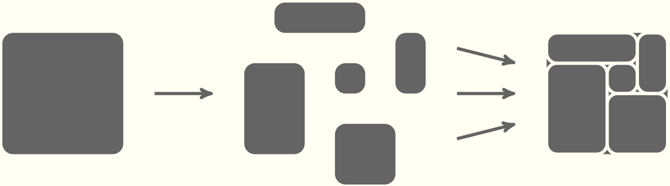

Compositionality

1. decomposition into smaller instances
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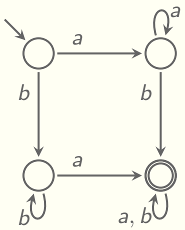
Deterministic Finite state Automata

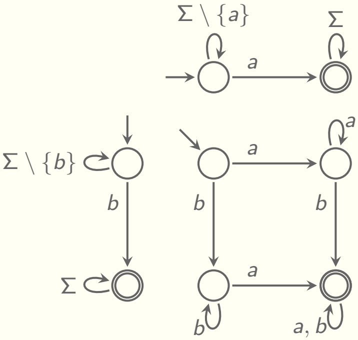
- construct automata with *less states*
- .. which preserve the original *language*

Example of decomposition



$\mathcal{A}$ is a DFA

Example of decomposition



$\mathcal{A} = \mathcal{B}_1 \times \mathcal{B}_2$

Decomposition using 2 DFAs

- $|\mathcal{B}_1| < |\mathcal{A}|$ and $|\mathcal{B}_2| < |\mathcal{A}|$
- $L(\mathcal{A}) = L(\mathcal{B}_1) \cap L(\mathcal{B}_2)$

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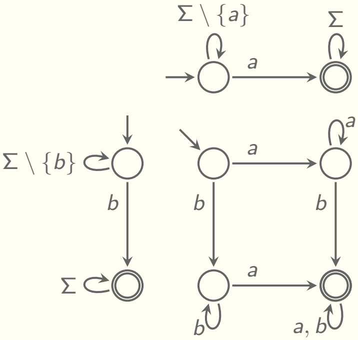

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Applications

- Hardware design of DFAs
- Tractability of Model-checking

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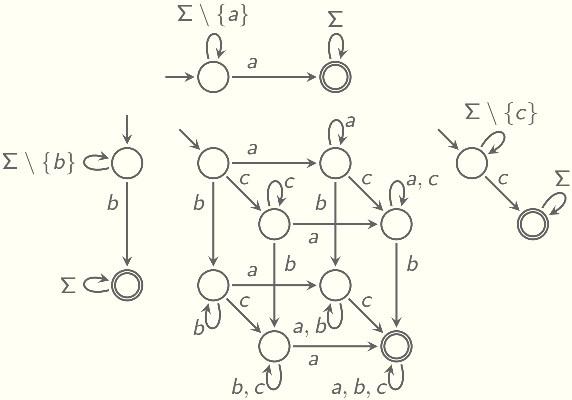

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Decomposition using 3 DFAs

- $|\mathcal{B}_i| < |\mathcal{A}|$ for all i
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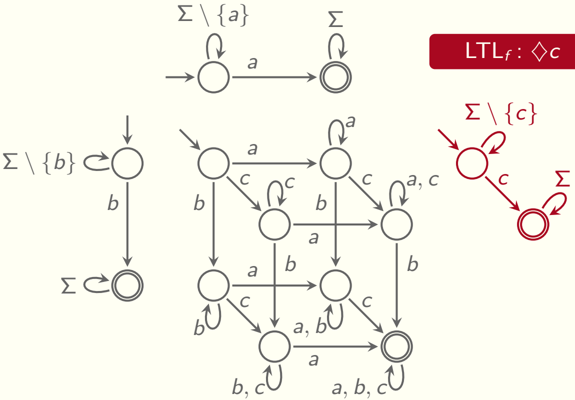

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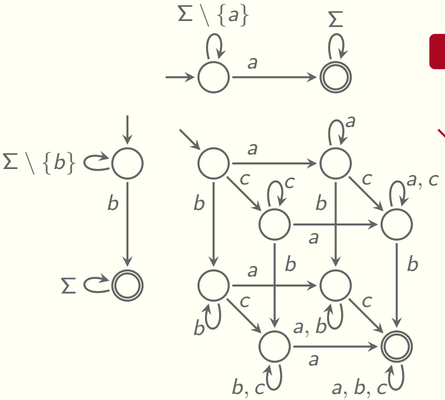
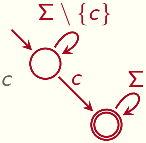


$\text{LTL}_f : \Diamond c$



$\Psi = \bigwedge \Phi_i$

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General Setting

Definitions

Factors of DFA

The DFA $\mathcal{B}$ is a *factor* of the DFA $\mathcal{A}$ when:

- $|\mathcal{B}| < |\mathcal{A}|$ *smaller size* $<$
- $L(\mathcal{A}) \subseteq L(\mathcal{B})$ *coarser language* $\subseteq$

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Compositionality of DFA

The DFA $\mathcal{A}$ is *k-factor composite* if there exists $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ factors such that:

$$L(\mathcal{A}) = \bigcap_{i=1}^{k} L(\mathcal{B}_i)$$

rejectance preservation $=$

It is *composite* when it is k-factor composite for some k.

Decision algorithms

Decide whether $\mathcal{A}$ is composite (k is not given)

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1. List all automata with less states than $\mathcal{A}$ ensures

$\boxed{\mathcal{B}_1}$ $\boxed{\mathcal{B}_2}$ $\boxed{\mathcal{B}_3}$ $\boxed{\mathcal{B}_4}$ $\boxed{\mathcal{B}_5}$ $\boxed{\mathcal{B}_6}$ $\cdots$ $\boxed{\mathcal{B}_n}$

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Theorem of Kupferman and Mosheiff in MFCS'13 and J. Inf. Comput. 2015 [1]

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Theorems

- *Compositionality is in* $\textsc{ExpSpace}$ *and* $\textsc{NLogSpace}$*-hard for DFAs* — *best known bounds*
- *It is in* $\textsc{PSpace}$ *once parameterized to have at most k-factors*

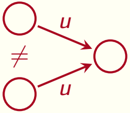
Contributions

Automata subclasses

Permutation automata

The transition function denotes a permutation on states for every input

forbidden pattern



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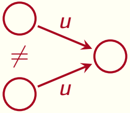

Commutative automata

The transition function does not depends on the input order

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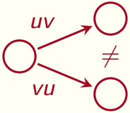

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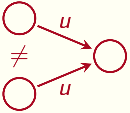

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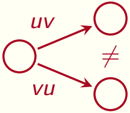

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$\Updownarrow$

a decomposition keeps structural properties

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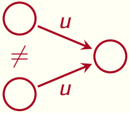

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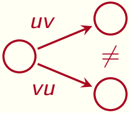

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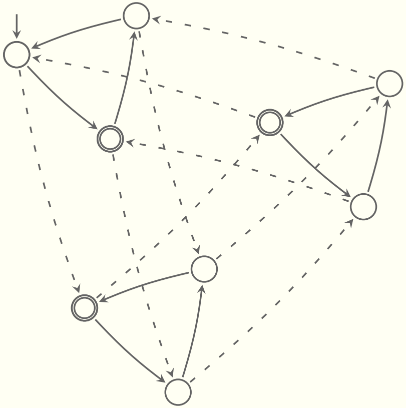
Jecker, Mazzocchi and Wolf

1. Simpler structural characterizations
   - .. implying better complexity results
2. Lower bound on minimal factor numbers

Commutative permutation DFAs

Visiting states using word iteration



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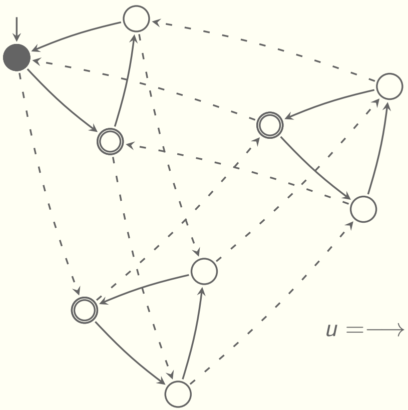

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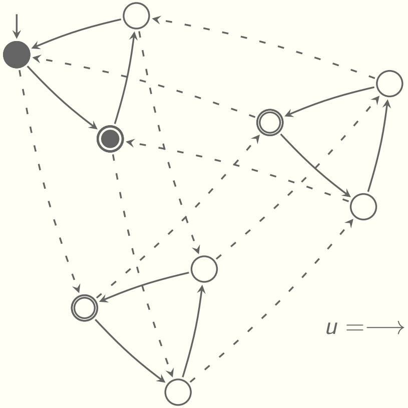

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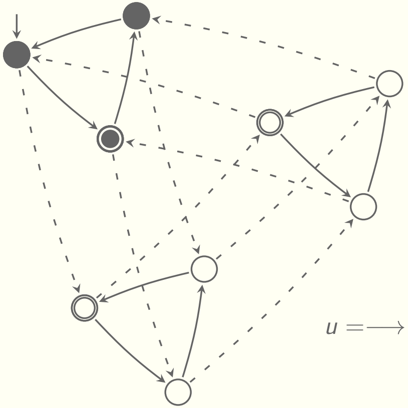

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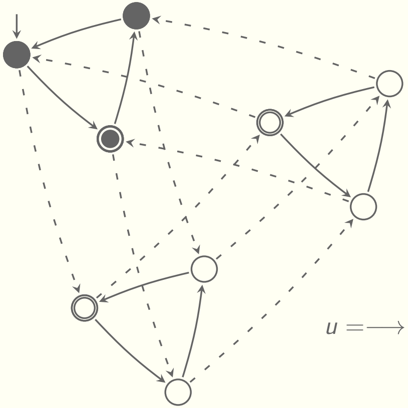

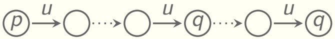

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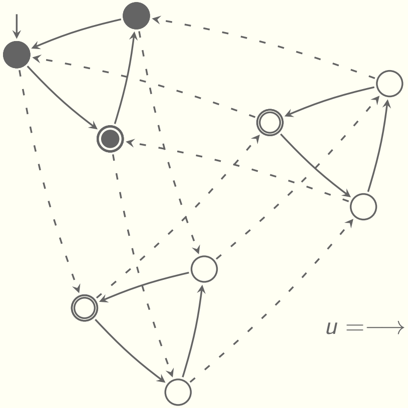

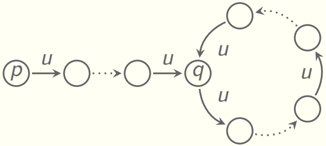

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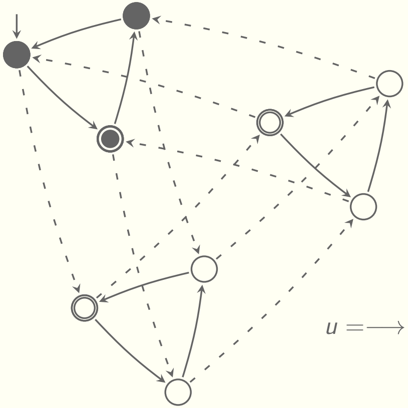

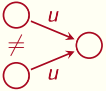

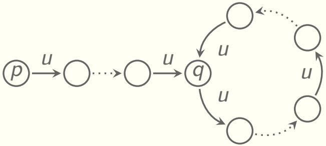

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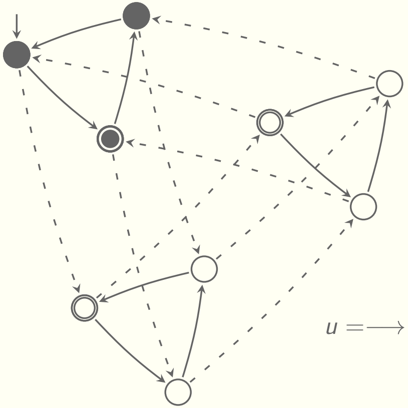

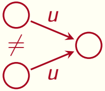

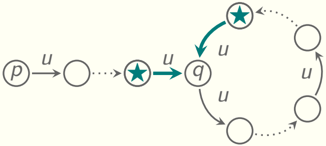

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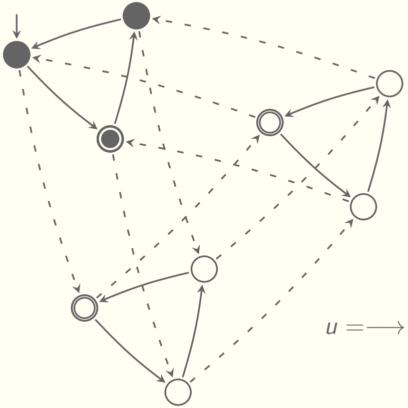

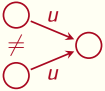

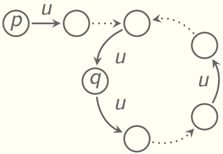

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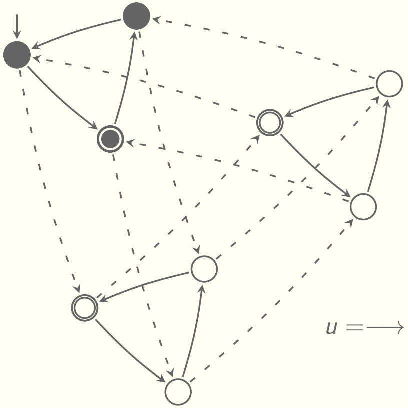

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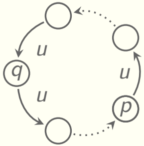

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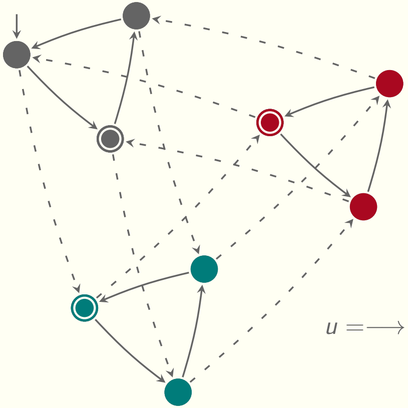

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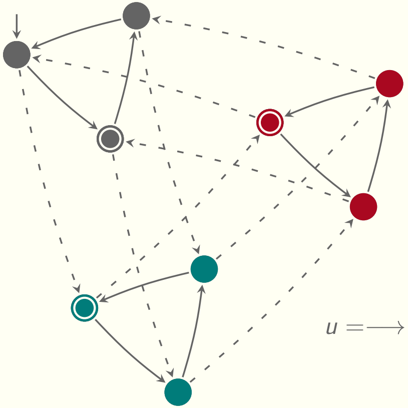

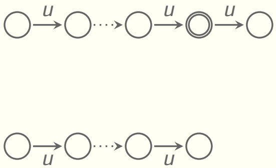

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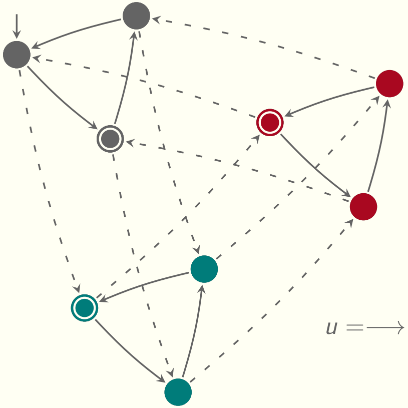

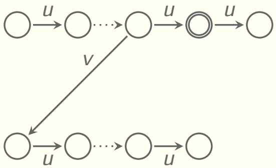

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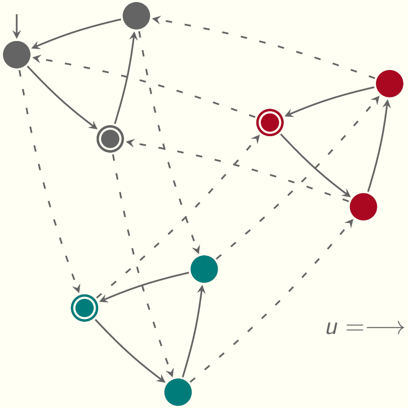

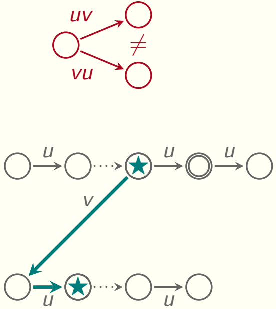

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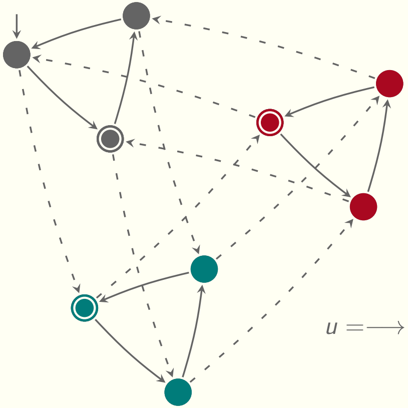

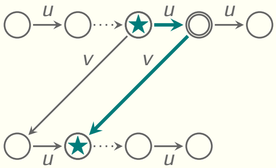

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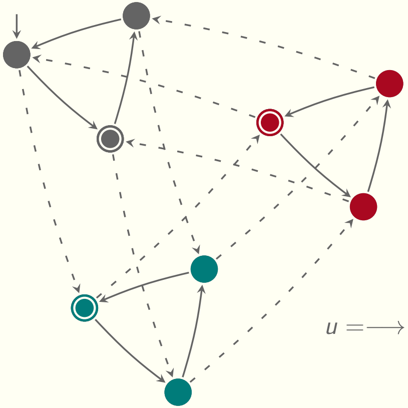

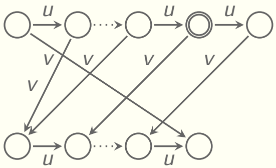

- permutation automaton
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- $[p]_u = \{\delta(p, u^n) : n \in \mathbb{N}\}$
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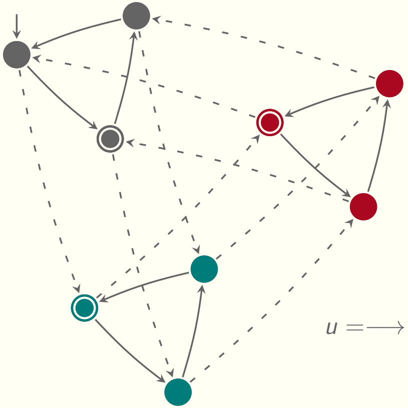

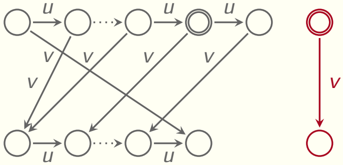

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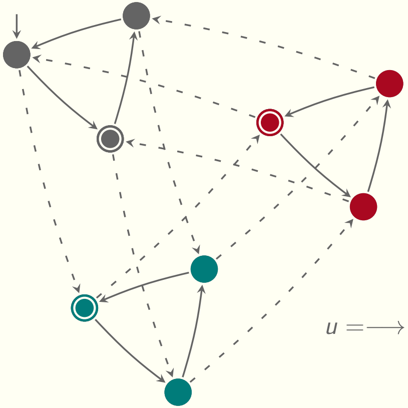

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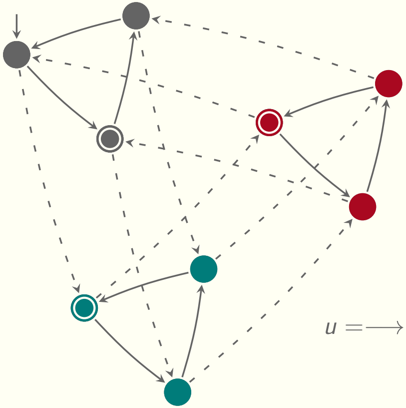

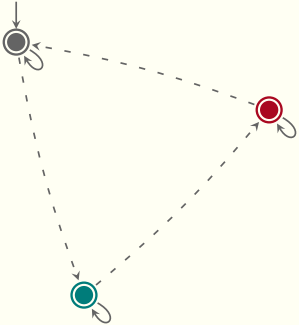

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- **effective** merge of states $\Rightarrow$ ensures $<$
- relaxed acceptance $\Rightarrow$ ensures $\subset$

Characterization by word coverage

Word u that covers $p \in \neg F$

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a commutative permutation DFA is k-factor composite

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k words suffice to cover all non-finals states

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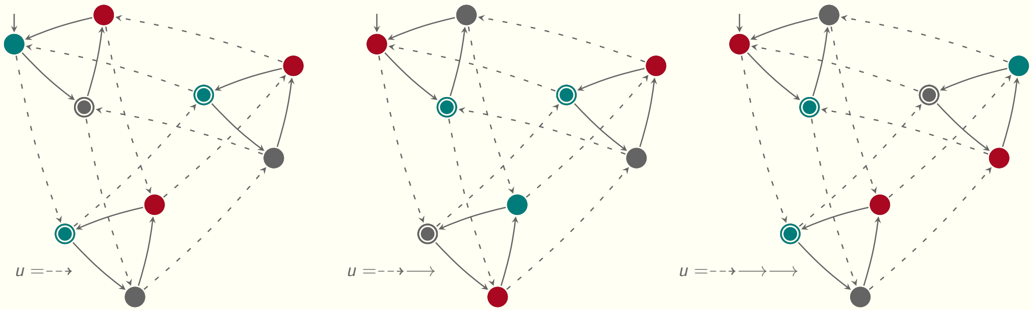

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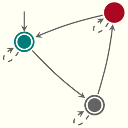

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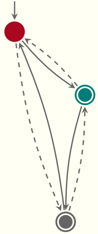

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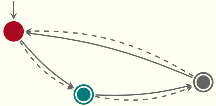

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Deciding compositionality of commutative permutation DFAs (k is not given)

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Theorem

- *Compositionality is in* $\textsc{NLogSpace}$ *for commutative permutation DFAs*

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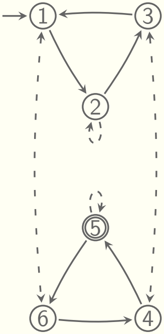
Theorems

- *Compositionality is in* NLOGSPACE *for commutative permutation DFAs*
- *It is* NP-COMPLETE *once parameterized to have at most k-factors*

Permutation DFAs

Generalizing word coverage



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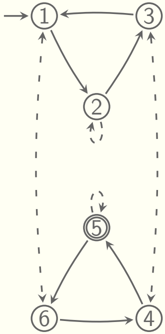

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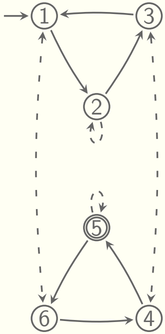

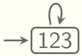

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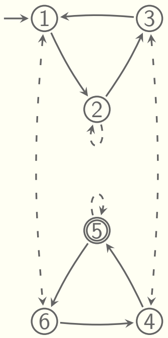

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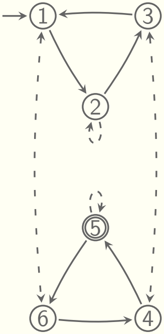

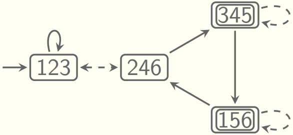

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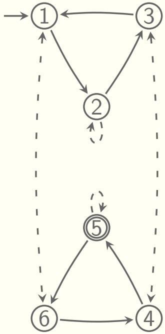

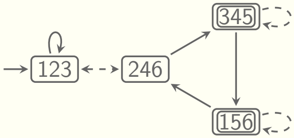

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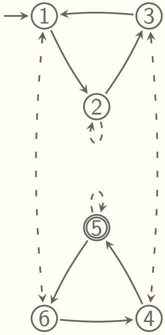

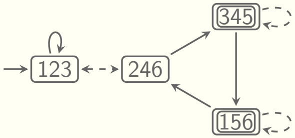

Factor? Usefull?

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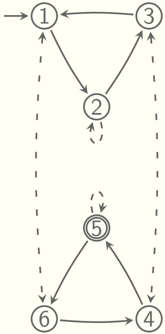

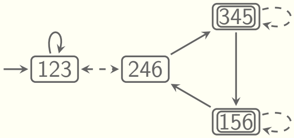

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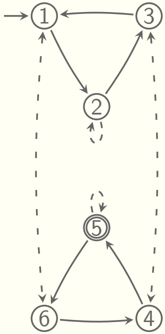

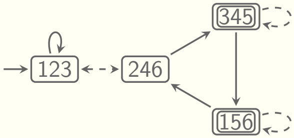

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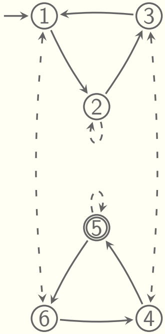

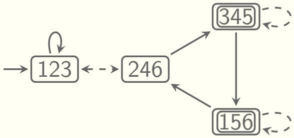

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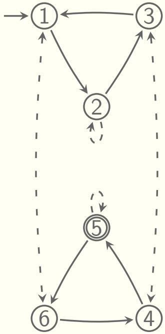

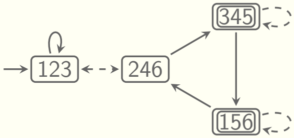

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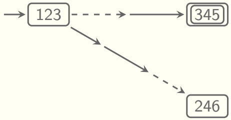

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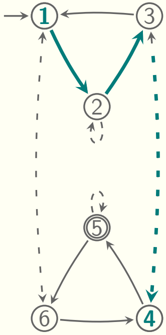

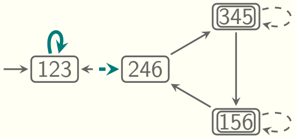

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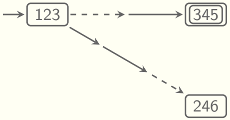

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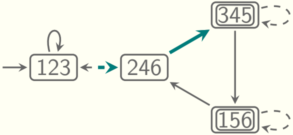

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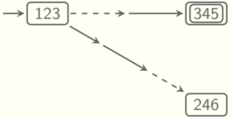

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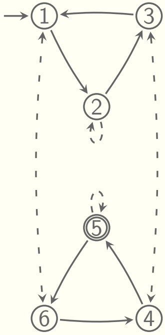

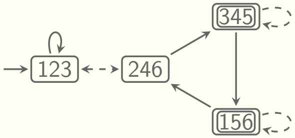

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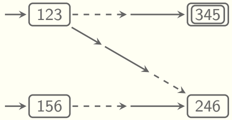

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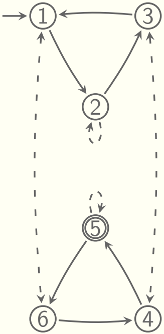

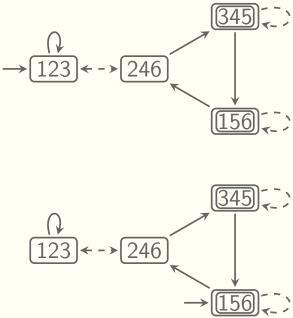

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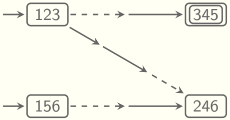

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some sets suffice to cover all non-finals states

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Theorem

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Width matters

Minimizing number of factors

Width of $\mathcal{A}$ $\quad = \quad$ minimal k such that $\mathcal{A}$ is k-composite

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Decompositions of a natural $n \in \mathbb{N}$

Prime decomposition: $n = n_1 \times n_2 \times \cdots \times n_k$ where all $n_i \in \mathbb{N}$ are prime.

2-Decomposition: $n = n' \times n''$ with $n' < n$ and $n'' < n$.

Minimizing number of factors

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2-Decomposition: $n = n' \times n''$ with $n' < n$ and $n'' < n$.

DFA are set of naturals (alphabet size is the representation base)

- *DFA of width 3 by Kupferman and Mosheiff in [1]*

Minimizing number of factors

Width of $\mathcal{A}$ $=$ minimal k such that $\mathcal{A}$ is k-composite

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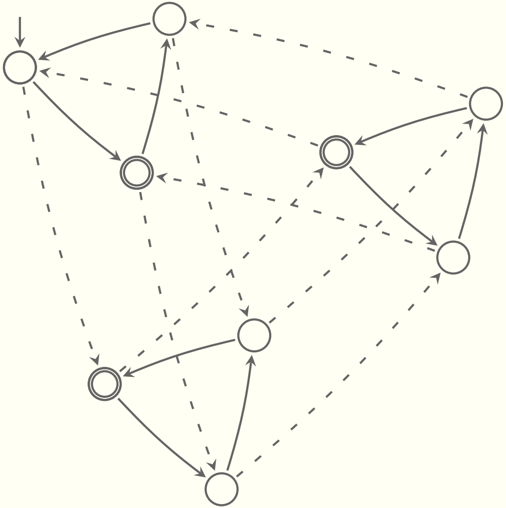
- *Family of unary DFAs of unbounded width by Jecker, Mazzocchi and Kupferman in MFCS'20 [2]*

The special case of Torus DFAs



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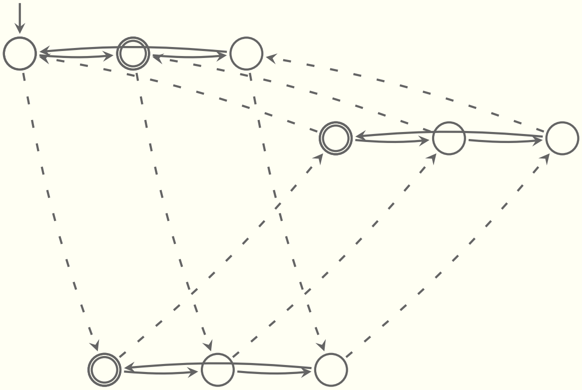

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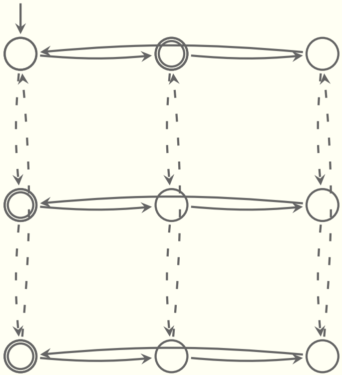

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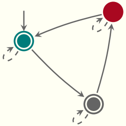
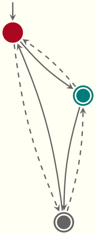
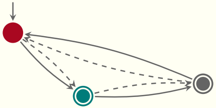


$u = \dashrightarrow$



$u = \dashrightarrow\longrightarrow$



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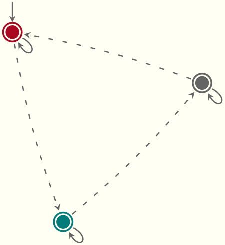

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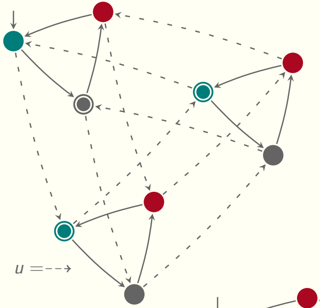

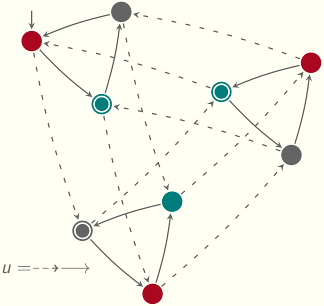

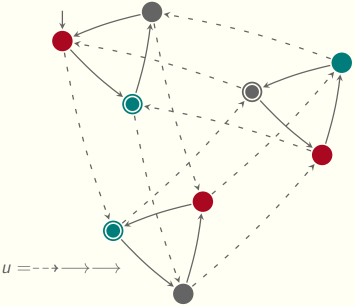

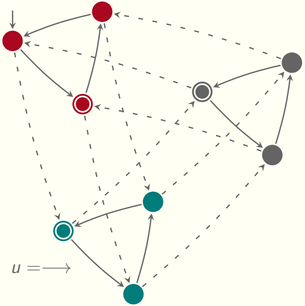

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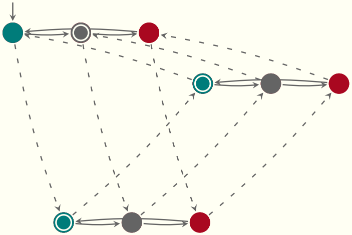

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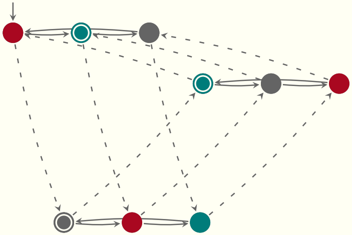

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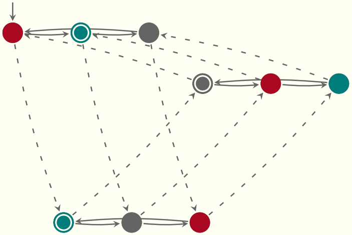

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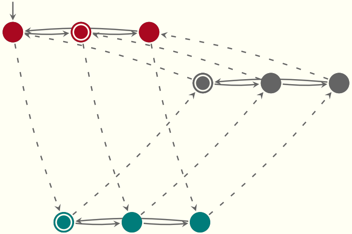

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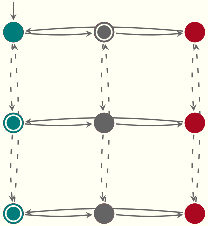

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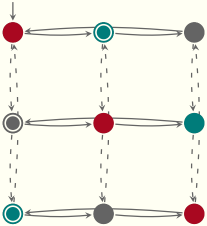

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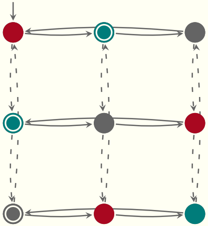

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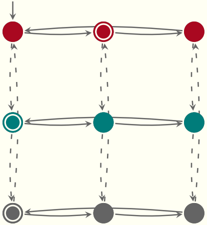

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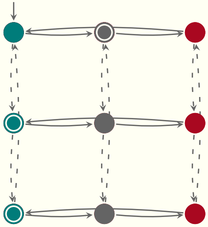

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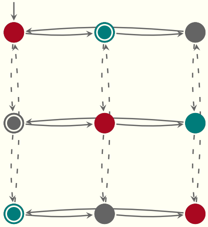

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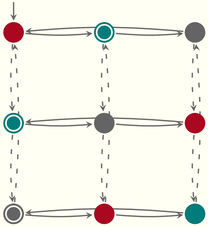

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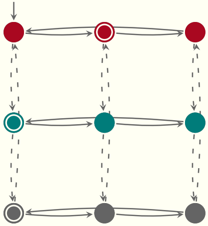


$u = \longrightarrow$

Torus with odd prime size

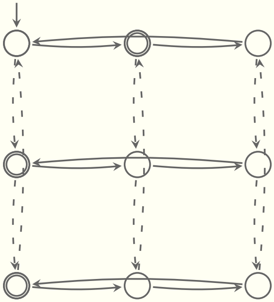
States visited by a word iteration

$\Updownarrow$

Line of the torus

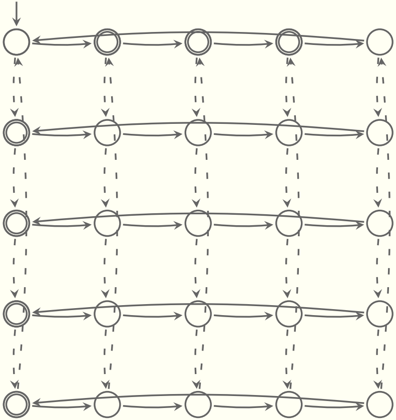
(horizontal, vertical, diagonal i.e. visits **once** every raws & colums)

Toward a large decomposition



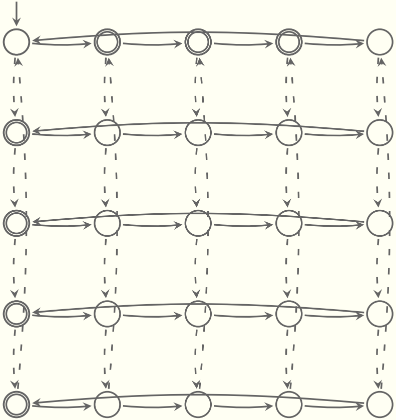
◁· size of dimensions = 3 ··▷

Toward a large decomposition



◁·· size of dimensions $= n$ (odd prime) ··▷

Toward a large decomposition



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Compositionality criteron

k-factor composite

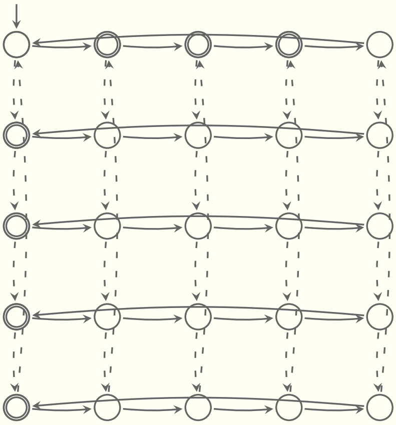
$\Updownarrow$

k words suffice to cover all non-finals

$\Updownarrow$

k lines of non-final suffice to cover them all

Toward a large decomposition



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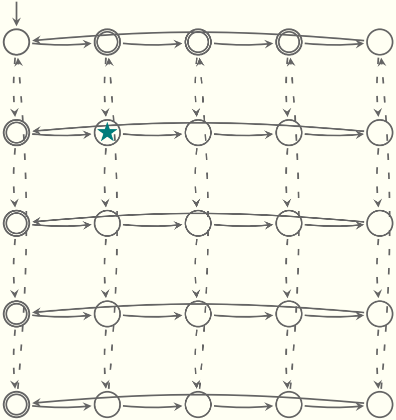

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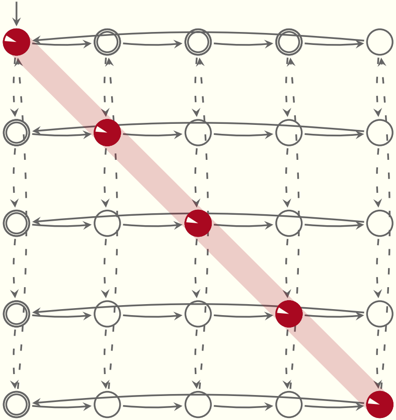

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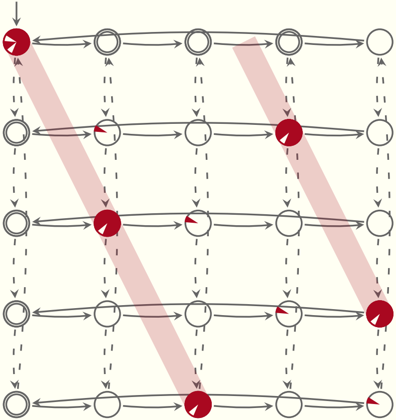

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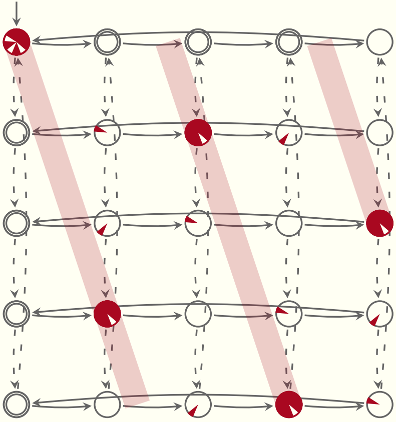

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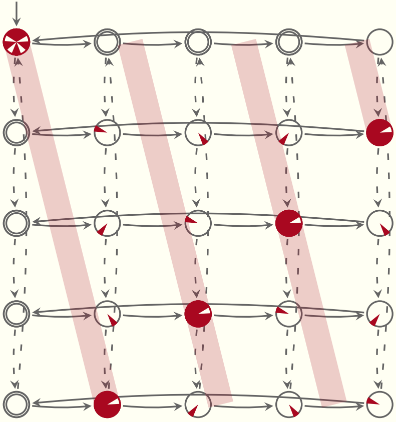

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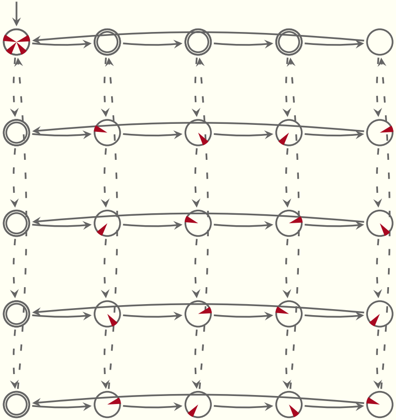

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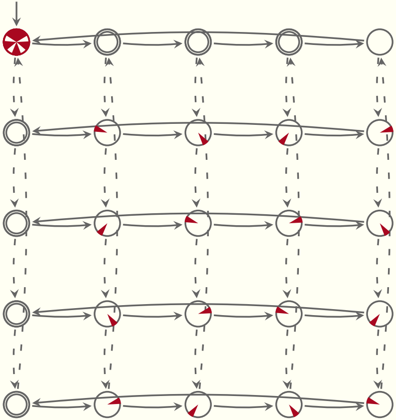

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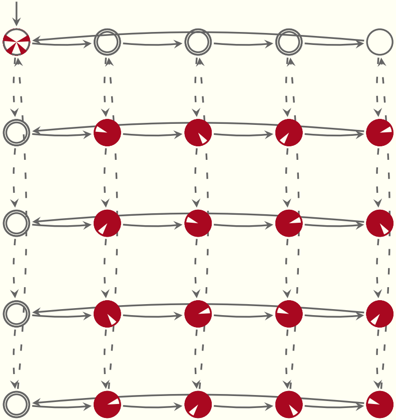

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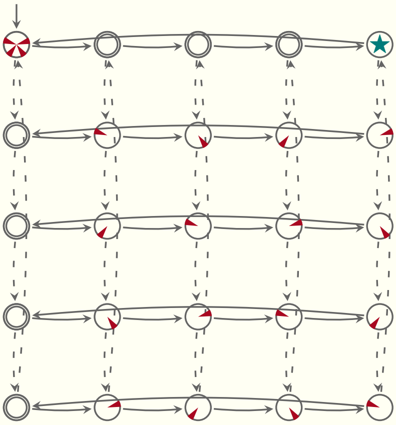

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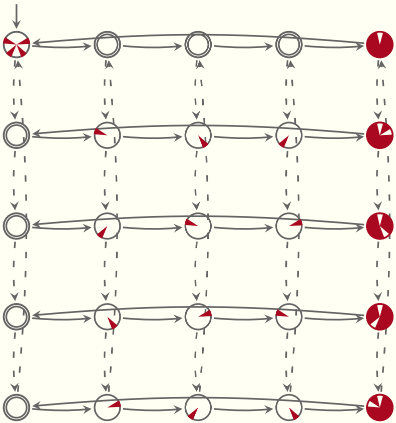

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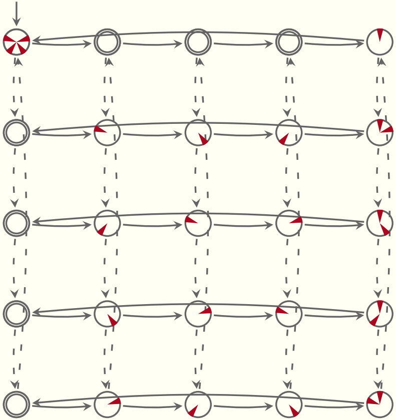

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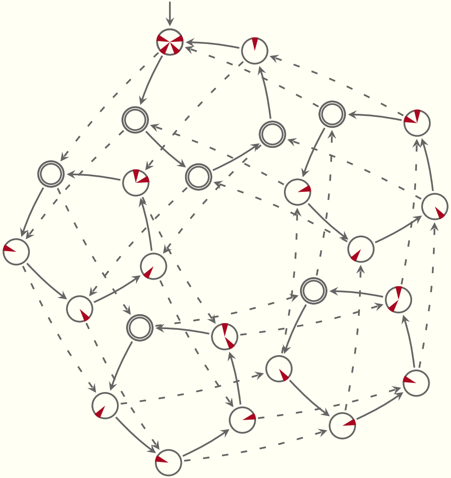

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Theorem

The Torus DFA with odd prime size n requires $\sqrt{n}$ factors to be decomposed

In a nutshell

	composite?	k-composite?
DFAs	ExpSpace [1]	PSpace
permutation DFAs	~~PSpace [1]~~ NP/FPT	
commutative permutation DFAs	~~PTime [1]~~ NLogSpace	NP-complete
unary (i.e. singleton alphabet) DFAs	LogSpace [2]	LogSpace

[1] O. Kupferman and J. Mosheiff
Prime Languages
In *J. of Information and Computation* 240, 2015

[2] I. Jecker, O. Kupferman and N. Mazzocchi
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commutative permutation DFAs	~~PTime [1]~~ NLogSpace	NP-complete
unary (i.e. singleton alphabet) DFAs	LogSpace [2]	LogSpace

Large decompositions (n states and m letters)

- unary commutative permutation DFAs requiring $\ln(n)/\ln(\ln(n))$ factors to be decomposed [2]
- commutative permutation DFAs requiring $(\sqrt[m]{n} - 1)^{m-1}$ factors to be decomposed

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appendix

Sketch: Characterization by coverage

back

k-covered implies k-composite

1. $\forall p \in \neg F$ u_p covers p
2. $p_1 \xrightarrow{a} p_2 \Leftrightarrow \delta(p_1, u_p^i) \xrightarrow{a} \delta(p_2, u_p^i)$

$$\begin{aligned} w \notin L(\mathcal{A}) &\iff \exists p \in \neg F,\ \mathcal{A}\colon p_I \xrightarrow{w} p \\ &\overset{2}{\iff} \exists p \in \neg F,\ \mathcal{B}_p\colon [p_I] \xrightarrow{w} [p] \\ &\overset{1}{\iff} \exists p \in \neg F,\ w \notin L(\mathcal{B}_p) \\ &\iff w \notin \bigcap_p L(\mathcal{B}_p) \end{aligned}$$

- $L(\mathcal{A}) = \bigcap_p L(\mathcal{B}_p)$

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- $L(\mathcal{A}) = \bigcap_p L(\mathcal{B}_p)$

k-composite implies k-covered

1. $L(\mathcal{A}) = \bigcap_i L(\mathcal{B}_i)$
2. $\forall \mathcal{B}_i \begin{cases} w \notin L(\mathcal{B}_i) \implies w \notin L(\mathcal{A}) \\ |\mathcal{B}_i| < |\mathcal{A}| \end{cases}$
3. $\exists \lambda \; \delta(p, u^\lambda) = p$

$$\exists v_i \begin{cases} \mathcal{B}_i\colon q \xrightarrow{v_i} q \\ \mathcal{A}\colon p \xrightarrow{v_i} p' \neq p \end{cases}$$

$$\begin{aligned} w \notin L(\mathcal{A}) &\overset{1}{\implies} \exists i,\ w \notin L(\mathcal{B}_i) \\ &\overset{3}{\implies} \exists i,\ wv_i^* \notin L(\mathcal{B}_i) \\ &\overset{2}{\implies} \exists i,\ wv_i^* \notin L(\mathcal{A}) \end{aligned}$$

- All v_i cover together rejecting states of $\mathcal{A}$

Sketch: Characterization by coverage

back

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&\overset{1}{\iff} \exists p \in \neg F,\ w \notin L(\mathcal{B}_p) \\
&\iff w \notin \bigcap_p L(\mathcal{B}_p)
\end{aligned}$$

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k-composite implies k-covered

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