Analyzing Massive Data Sets
Summer Semester 2019

Prof. Dr. Peter Fischer

Institut für Informatik

Lehrstuhl für Datenbanken und Informationssysteme

Chapter 9: Data Streams – Part 2

Today's Lecture

- **More algorithms for streams:**
  - **(1)** **Filtering a data stream: Bloom filters**
    - Select elements with property **x** from stream
  - **(2)** **Counting distinct elements: Flajolet-Martin**
    - Number of distinct elements in the last **k** elements of the stream
  - **(3)** **Estimating moments: AMS method**
    - Estimate std. dev. of last **k** elements
  - **(4) Counting frequent items**

(1) Filtering Data Streams

Filtering Data Streams

- **Each element of data stream is a tuple**

- Given a list of keys **S**

- **Determine which tuples of stream are in *S***

- **Obvious solution: Hash table**
  - But suppose we **do not have enough memory** to store all of *S* in a hash table
    - E.g., we might be processing millions of filters
      on the same stream

Applications

- **Example: Email spam filtering**
  - We know 1 billion "good" email addresses
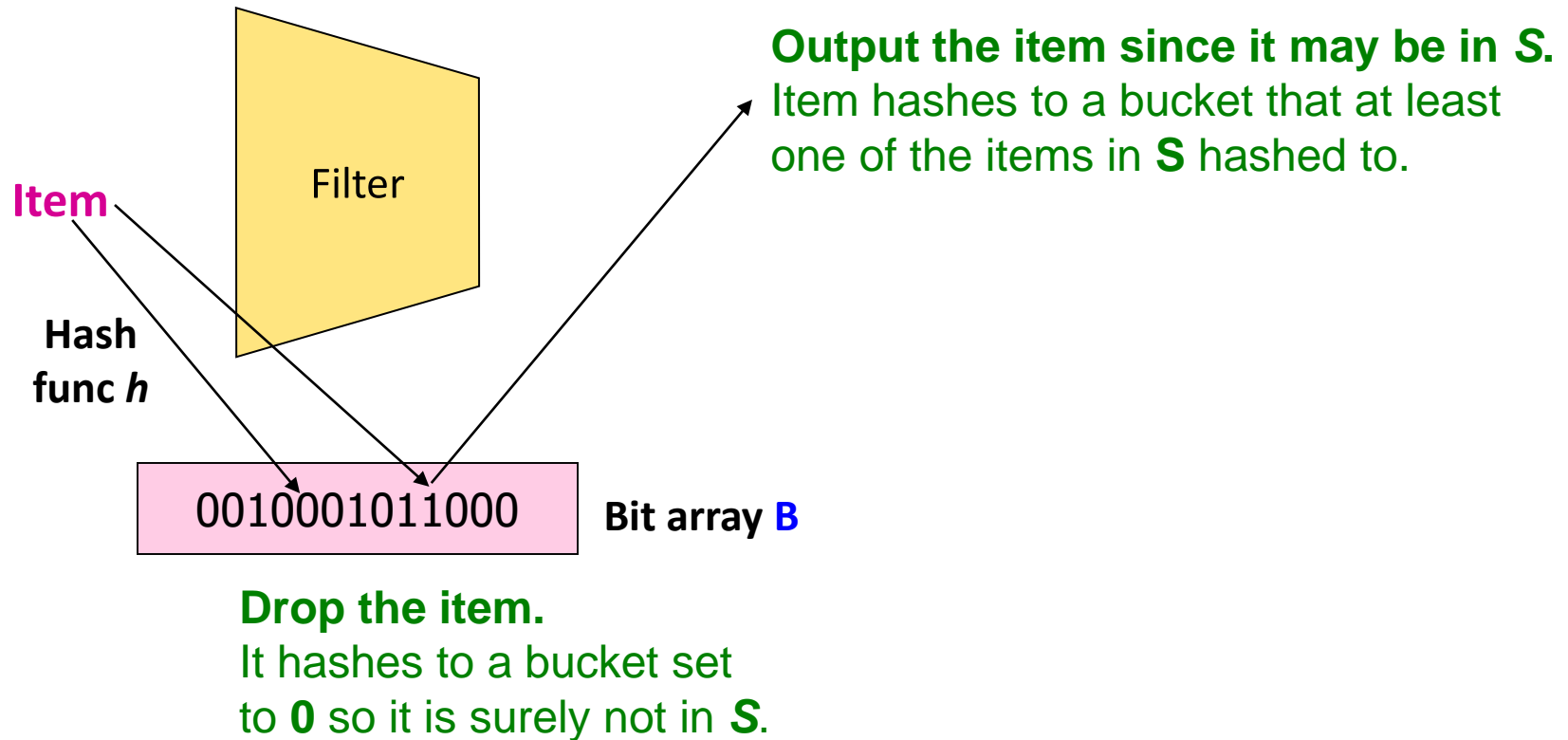 - If an email comes from one of these, it is **NOT** spam

- **Publish-subscribe systems**
  - You are collecting lots of messages (news articles)
  - People express interest in certain sets of keywords
  - Determine whether each message matches user's interest

First Cut Solution (1)

- **Given a set of keys *S* that we want to filter**

- Create a **bit array *B*** of ***n*** bits, initially all ***0*s**

- Choose a **hash function *h*** with range **[*0,n*)**

- Hash each member of ***s* ∈ *S*** to one of ***n*** buckets, and set that bit to **1**, i.e., ***B[h(s)]=1***

- Hash each element ***a*** of the stream and output only those that hash to bit that was set to **1**
  - **Output *a* if B[h(a)] == 1**

First Cut Solution (2)

Output the item since it may be in *S*.
Item hashes to a bucket that at least
one of the items in **S** hashed to.

Filter

Item

**Hash
func *h***

`0010001011000`   **Bit array B**

Drop the item.
It hashes to a bucket set
to **0** so it is surely not in **S**.

- **Creates false positives but no false negatives**
  - If the item is in *S* we surely output it, if not we may still output it

First Cut Solution (3)

- **|S| = 1 billion email addresses**
  |B|= 1GB = 8 billion bits

- If the email address is in *S*, then it surely hashes to a bucket that has the big set to **1**,
  so it always gets through (*no false negatives*)

- Approximately **1/8** of the bits are set to **1**, so about **1/8th** of the addresses not in *S* get through to the output (*false positives*)
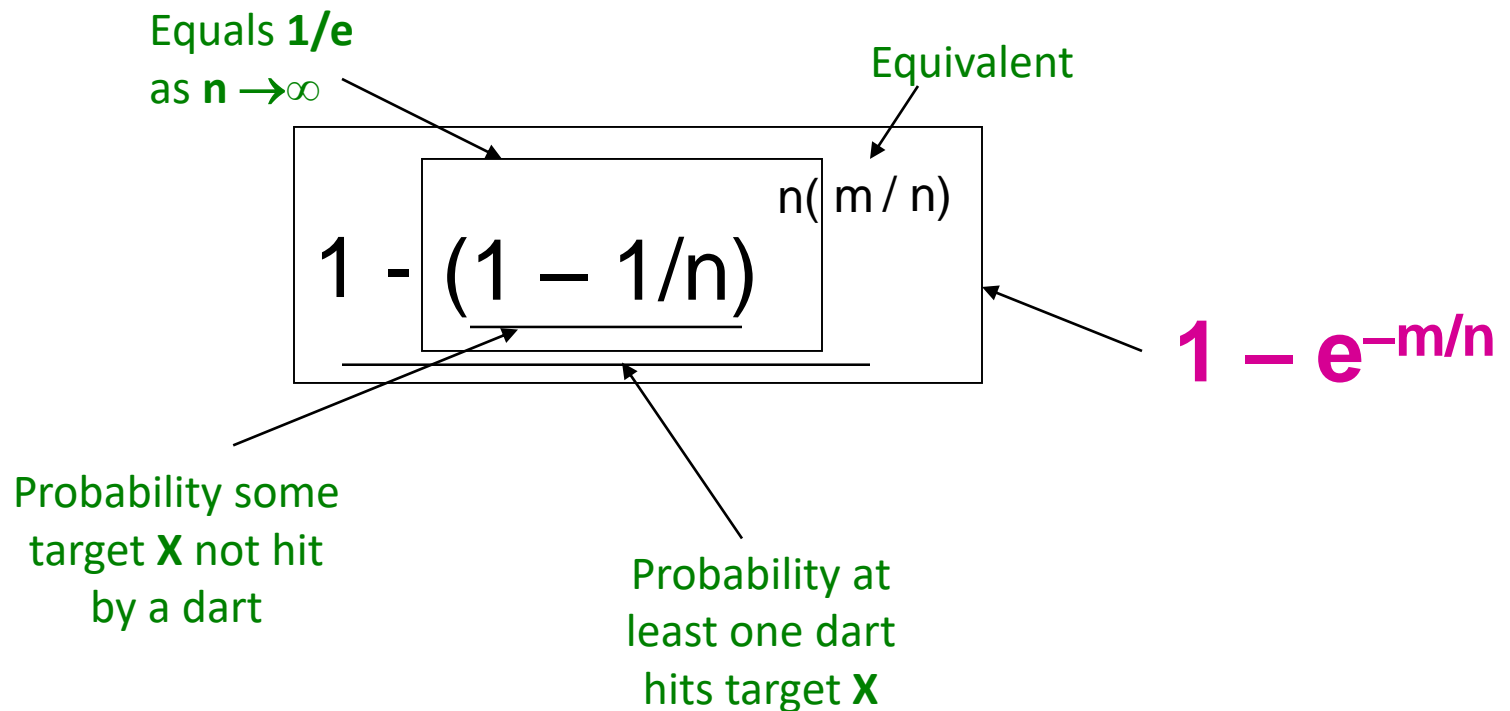 - Actually, less than **1/8th**, because more than one address might hash to the same bit

Analysis: Throwing Darts (1)

- **More accurate analysis for the number of false positives**

- **Consider:** If we throw m darts into n equally likely targets, **what is the probability that a target gets at least one dart?**

- **In our case:**
  - **Targets** = bits/buckets
  - **Darts** = hash values of items

Analysis: Throwing Darts (2)

- We have m darts, n targets
- **What is the probability that a target gets at least one dart?**

Equals **1/e** as $n \rightarrow \infty$

Equivalent

$$1 - (1 - 1/n)^{n(m/n)}$$

$$1 - e^{-m/n}$$

Probability some target **X** not hit by a dart

Probability at least one dart hits target **X**

Analysis: Throwing Darts (3)

- **Fraction of 1s in the array B =**
  = probability of false positive = $1 - e^{-m/n}$

- **Example: 10^9 darts, $8 \cdot 10^9$ targets**
  - Fraction of **1s** in **B** = $1 - e^{-1/8}$ = **0.1175**
    - Compare with our earlier estimate: **1/8 = 0.125**

Bloom Filter

- Consider: **|S|** = **m**, **|B|** = **n**

- Use **k** independent hash functions $h_1, ..., h_k$

- **Initialization:**

  - Set **B** to all **0s**

  - Hash each element **s** ∈ **S** using each hash function h_i, set **B[$h_i(s)$] = 1** (for each **i = 1,.., k**)

    (**note:** we have a single array B!)

- **Run-time:**

  - When a stream element with key **x** arrives

    - If **B[$h_i(x)$] = 1 <u>for all</u> i = 1,..., k** then declare that **x** is in **S**

      - That is, **x** hashes to a bucket set to **1** for every hash function **$h_i(x)$**

    - Otherwise discard the element **x**

Bloom Filter -- Analysis

- **What fraction of the bit vector B are 1s?**
  - Throwing $k \cdot m$ darts at n targets
  - So fraction of **1**s is $(1 - e^{-km/n})$

- But we have k independent hash functions and we only let the element x through **if all** k hash element x to a bucket of value **1**

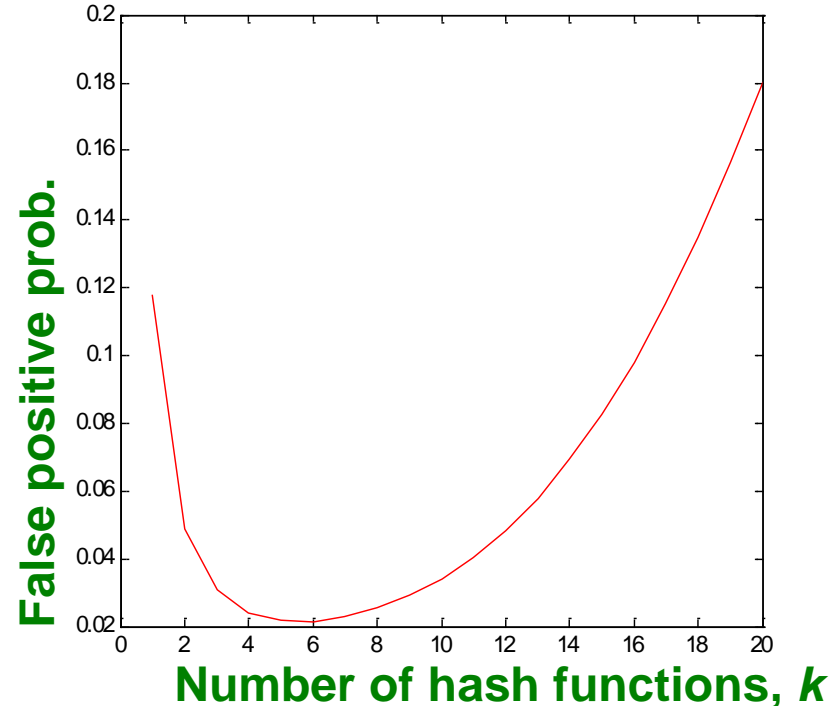- So, false **positive probability** $= (1 - e^{-km/n})^k$

Bloom Filter – Analysis (2)

- **m = 1 billion, n = 8 billion**
  - **k = 1**: $(1 - e^{-1/8}) =$ **0.1175**
  - **k = 2**: $(1 - e^{-1/4})^2 =$ **0.0493**

- **What happens as we keep increasing k?**

Number of hash functions, k

- "Optimal" value of k: $n/m \ln(2)$
  - **In our case:** Optimal **k = 8 ln(2) = 5.54 ≈ 6**
    - **Error at k = 6**: $(1 - e^{-1/6})^2 =$ **0.0235**

Bloom Filter: Wrap-up

- **Bloom filters guarantee no false negatives, and use limited memory**
  - Great for pre-processing before more expensive checks
- **Suitable for hardware implementation**
  - Hash function computations can be parallelized

- Is it better to have **1** big **B** or *k* small **B**s?
  - **It is the same:** $(1 - e^{-km/n})^k$ vs. $(1 - e^{-m/(n/k)})^k$
  - But keeping **1 big B** is simpler

(2) Counting Distinct Elements

Counting Distinct Elements

- **Problem:**
  - Data stream consists of a universe of elements chosen from a set of size N
  - Maintain a count of the number of distinct elements seen so far

- **Obvious approach:**
  Maintain the set of elements seen so far
  - That is, keep a hash table of all the distinct elements seen so far

Applications

- **How many different words are found among the Web pages being crawled at a site?**
  - Unusually low or high numbers could indicate artificial pages (spam?)

- **How many different Web pages does each customer request in a week?**

- **How many distinct products have we sold in the last week?**

Using Small Storage

- **Real problem: What if we do not have space to maintain the set of elements seen so far?**

- **Estimate the count in an unbiased way**

- **Accept that the count may have a little error, but limit the probability that the error is large**

Flajolet-Martin Approach

- Pick a hash function h that maps each of the N elements to at least $\log_2 N$ bits

- For each stream element a, let $r(a)$ be the number of trailing **0s** in $h(a)$
  - $r(a)$ = position of first 1 counting from the right
    - E.g., say $h(a) = 12$, then *12* is *1100* in binary, so $r(a) = 2$

- Record R = **the maximum** $r(a)$ **seen**
  - $R = \max_a r(a)$, over all the items a seen so far

- **Estimated number of distinct elements = 2^R**

Why It Works: Intuition

- **Very very rough and heuristic intuition why Flajolet-Martin works:**
  - *h(a)* hashes *a* with **equal prob.** to any of *N* values
  - Then *h(a)* is a sequence of **log$_2$ N** bits, where *2^{-r}* fraction of all *a*s have a tail of *r* zeros
    - About 50% of *a*s hash to ***0**
    - About 25% of *a*s hash to ****00**
    - So, if we saw the longest tail of *r=2* (i.e., item hash ending ***100**) then we have probably seen **about 4** distinct items so far
  - **So, it takes to hash about *2^r* items before we see one with zero-suffix of length *r***

Why It Works: More formally

- **Now we show why Flajolet-Martin works**

- **Formally, we will show that probability of finding a tail of *r* zeros:**
  - **Goes to 1 if $m \gg 2^r$**
  - **Goes to 0 if $m \ll 2^r$**

    where m is the number of distinct elements seen so far in the stream

- **Thus, 2^R will almost always be around *m!***

Why It Works: More formally

- **What is the probability that a given $h(a)$ ends in at least r zeros is 2^{-r}**
  - **h(a)** hashes elements uniformly at random
  - Probability that a random number ends in at least r zeros is **2^{-r}**
- Then, the probability of **NOT** seeing a tail of length r among m elements:

$$(1 - 2^{-r})^m$$

Prob. all end in fewer than r zeros.

Prob. that given **h(a)** ends in fewer than r zeros

Why It Works: More formally

- **Note:** $(1 - 2^{-r})^m = (1 - 2^{-r})^{2^r (m 2^{-r})} \approx e^{-m 2^{-r}}$

- **Prob. of NOT finding a tail of length _r_ is:**
  - If $m \ll 2^r$, then prob. tends to **1**
    - $(1 - 2^{-r})^m \approx e^{-m 2^{-r}} = 1$ as $\mathbf{m/2^r \to 0}$
      - So, the probability of finding a tail of length _r_ tends to **0**
  - If $m \gg 2^r$, then prob. tends to **0**
    - $(1 - 2^{-r})^m \approx e^{-m 2^{-r}} = 0$ as $\mathbf{m/2^r \to \infty}$
      - So, the probability of finding a tail of length _r_ tends to **1**

- **Thus, 2^R will almost always be around _m!_**

Why It Doesn't Work

- **E[2^R] is actually infinite**
  - Probability halves when **$R \rightarrow R+1$**, but value doubles
- **Workaround involves using many hash functions h_i and getting many samples of R_i**
- **How are samples R_i combined?**
  - **Average?** What if one very large value 2^{R_i}?
  - **Median?** All estimates are a power of **2**
  - **Solution:**
    - Partition your samples into small groups
    - Take the median of groups
    - Then take the average of the medians

(3) Computing Moments

Generalization: Moments

- **Suppose a stream has elements chosen from a set *A* of *N* values**

- **Let m_i be the number of times value *i* occurs in the stream**

- **The kth *moment* is**

$$\sum_{i \in A} (m_i)^k$$

Special Cases

$$\sum_{i \in A} (m_i)^k$$

- **0th moment =** number of distinct elements
  - The problem just considered

- **1st moment =** count of the numbers of elements = length of the stream
  - Easy to compute

- **2nd moment =** *surprise number S =* a measure of how uneven the distribution is

Example: Surprise Number

- **Stream of length 100**
- **11 distinct values**

- Item counts: **10, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9**
  Surprise S = 910

- Item counts: **90, 1, 1, 1, 1, 1, 1, 1 ,1, 1, 1**
 **Surprise S = 8,110**

AMS Method

- **AMS method works for all moments**
- **Gives an unbiased estimate**
- **We will just concentrate on the 2nd moment S**
- **We pick and keep track of many variables X:**
  - For each variable X we store *X.el* and *X.val*
    - *X.el* corresponds to the item i
    - *X.val* corresponds to the **count** of item i
  - Note this requires a count in main memory, so number of *X*s is limited
- **Our goal is to compute $S = \sum_i m_i^2$**

One Random Variable (X)

- **How to set *X.val* and *X.el*?**
  - Assume stream has length *n* (we relax this later)
  - Pick some random time *t* (*t<n*) to start, so that any time is equally likely
  - Let at time *t* the stream have item *i*. *We set X.el = i*
  - Then we maintain count *c (X.val = c)* of the number of *i*s in the stream starting from the chosen time *t*
  - **Then the estimate of the 2ⁿᵈ moment ($\sum_i m_i^2$) is:**
    $$S = f(X) = n(2 \cdot c - 1)$$
    - Note, we will keep track of multiple **X**s, (**X₁, X₂,… Xₖ**) and our final estimate will be $S = 1/k \sum_j^k f(X_j)$

Expectation Analysis

Count: 1   2   3 m_a

Stream: a   a   b   b   b   a   b   a
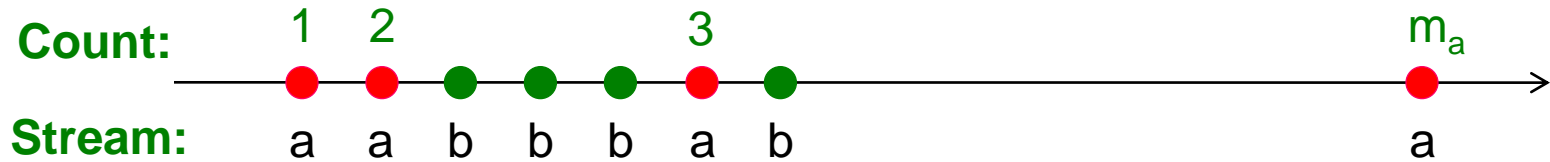
- **2ⁿᵈ moment is** $S = \sum_i m_i^2$

- c_t … number of times item at time t appears from time **t** onwards ($c_1 = m_a$, $c_2 = m_a - 1$, $c_3 = m_b$)

- $E[f(X)] = \frac{1}{n} \sum_{t=1}^{n} n(2c_t - 1)$

$$= \frac{1}{n} \sum_i n \left(1 + 3 + 5 + \cdots + 2m_i - 1\right)$$

m_i … total count of item i in the stream (we are assuming stream has length **n**)
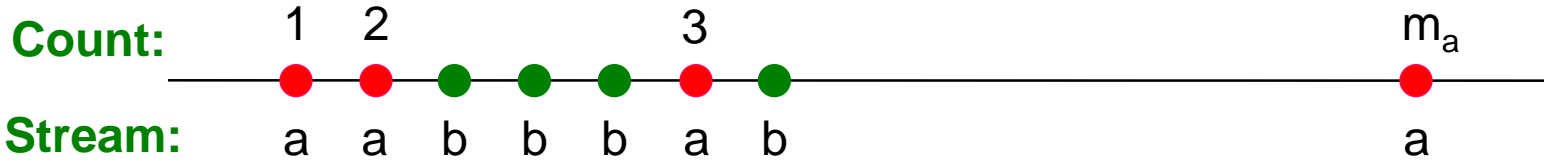
Group times by the value seen

Time t when the last i is seen ($c_t = 1$)

Time t when the penultimate i is seen ($c_t = 2$)

Time t when the first i is seen ($c_t = m_i$)

Expectation Analysis

- $E[f(X)] = \frac{1}{n} \sum_i n \, (1 + 3 + 5 + \cdots + 2m_i - 1)$
  - Little side calculation: $(1 + 3 + 5 + \cdots + 2m_i - 1) = \sum_{i=1}^{m_i} (2i - 1) = 2 \frac{m_i(m_i+1)}{2} - m_i = (m_i)^2$
- **Then $E[f(X)] = \frac{1}{n} \sum_i n \, (m_i)^2$**

- **So, $E[f(X)] = \sum_i (m_i)^2 = S$**
- **We have the second moment (in expectation)!**

Higher-Order Moments

- **For estimating k^{th} moment we essentially use the same algorithm but change the estimate:**
  - For **k=2** we used *n (2·c − 1)*
  - For **k=3** we use: *n (3·c² − 3c + 1)*   (where **c=X.val**)

- **Why?**
  - **For k=2:** Remember we had $(1 + 3 + 5 + \cdots + 2m_i - 1)$ and we showed terms *2c-1* (for **c=1,…,m**) sum to *m^2*
    - $\sum_{c=1}^{m} 2c - 1 = \sum_{c=1}^{m} c^2 - \sum_{c=1}^{m}(c-1)^2 = m^2$
    - **So: $2c - 1 = c^2 - (c-1)^2$**
  - **For k=3: c³ - (c-1)³ = 3c² - 3c + 1**

- **Generally:** Estimate $= n\ (c^k - (c-1)^k)$

Combining Samples

- **In practice:**
  - Compute $f(X) = n(2\,c - 1)$ for
    as many variables X as you can fit in memory
  - Average them in groups
  - Take median of averages

- **Problem: Streams never end**
  - We assumed there was a number n,
    the number of positions in the stream
  - But real streams go on forever, so n is
   a variable – the number of inputs seen so far

Streams Never End: Fixups

- **(1)** The variables *X* have *n* as a factor – keep *n* separately; just hold the count in *X*

- **(2)** Suppose we can only store *k* counts. We must throw some *X*s out as time goes on:
  - **Objective:** Each starting time *t* is selected with probability *k/n*
  - **Solution: (fixed-size sampling!)**
    - Choose the first *k* times for *k* variables
    - When the nth element arrives (*n > k*), choose it with probability *k/n*
    - If you choose it, throw one of the previously stored variables **X** out, with equal probability

Counting Itemsets

Counting Itemsets

- **New Problem:** **Given a stream, which items appear more than *s* times in the window?**

- **Possible solution:** Think of the stream of baskets as one binary stream per item
  - **1** = item present; **0** = not present
  - Use **DGIM** to estimate counts of **1**s for all items

6   10   4   3   2   2   1   1   0
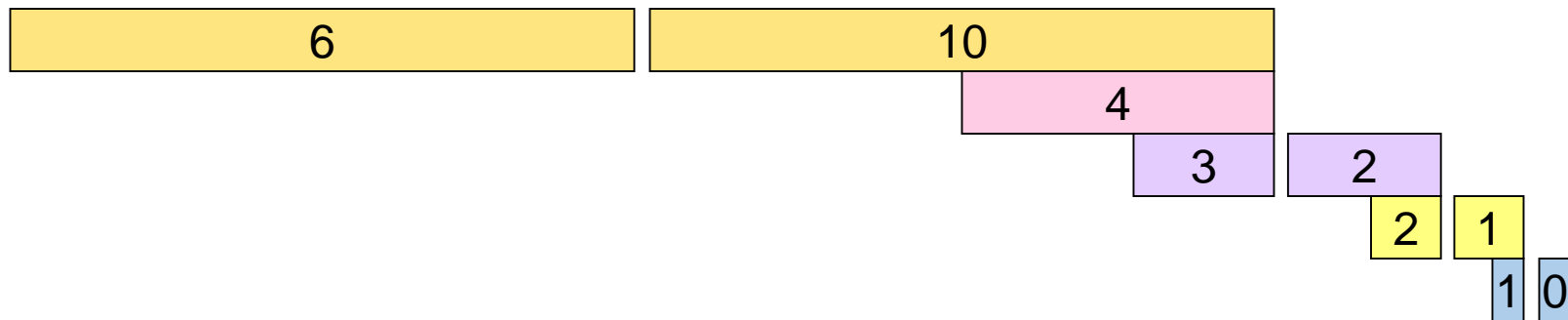
0 1 0 0 1 1 1 0 0 0 1 0 1 0 0 1 0 0 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1 0 1 0 1 1 0 0 1 1 0 1 0

N

Extensions

- **In principle, you could count frequent pairs or even larger sets the same way**
  - **One stream per itemset**

- **Drawbacks:**
  - Only approximate
  - **Number of itemsets is way too big**

Exponentially Decaying Windows

- **Exponentially decaying windows: A heuristic for selecting likely frequent item(sets)**
  - **What are "currently" most popular movies?**
    - Instead of computing the raw count in last **N** elements
    - Compute a **smooth aggregation** over the whole stream
- If stream is $a_1, a_2,...$ and we are taking the sum of the stream, take the answer at time t to be: $=$
  $$\sum_{i=1}^{t} a_i (1-c)^{t-i}$$
    - c is a constant, presumably tiny, like 10^{-6} or 10^{-9}
- **When new a_{t+1} arrives:**
  Multiply current sum by **(1-c)** and add a_{t+1}

Example: Counting Items

- If each a_i is an "item" we can compute the **characteristic function** of each possible item x as an Exponentially Decaying Window
  - That is: $\sum_{i=1}^{t} \delta_i \cdot (1-c)^{t-i}$ where $\delta_i = 1$ if $a_i = x$, and 0 otherwise
  - Imagine that for each item x we have a binary stream (1 if x appears, 0 if x does not appear)
  - New item x arrives:
    - Multiply all counts by **(1-c)**
    - Add **+1** to count for element x

- **Call this sum the "weight" of item x**

Sliding Versus Decaying Windows

1/c
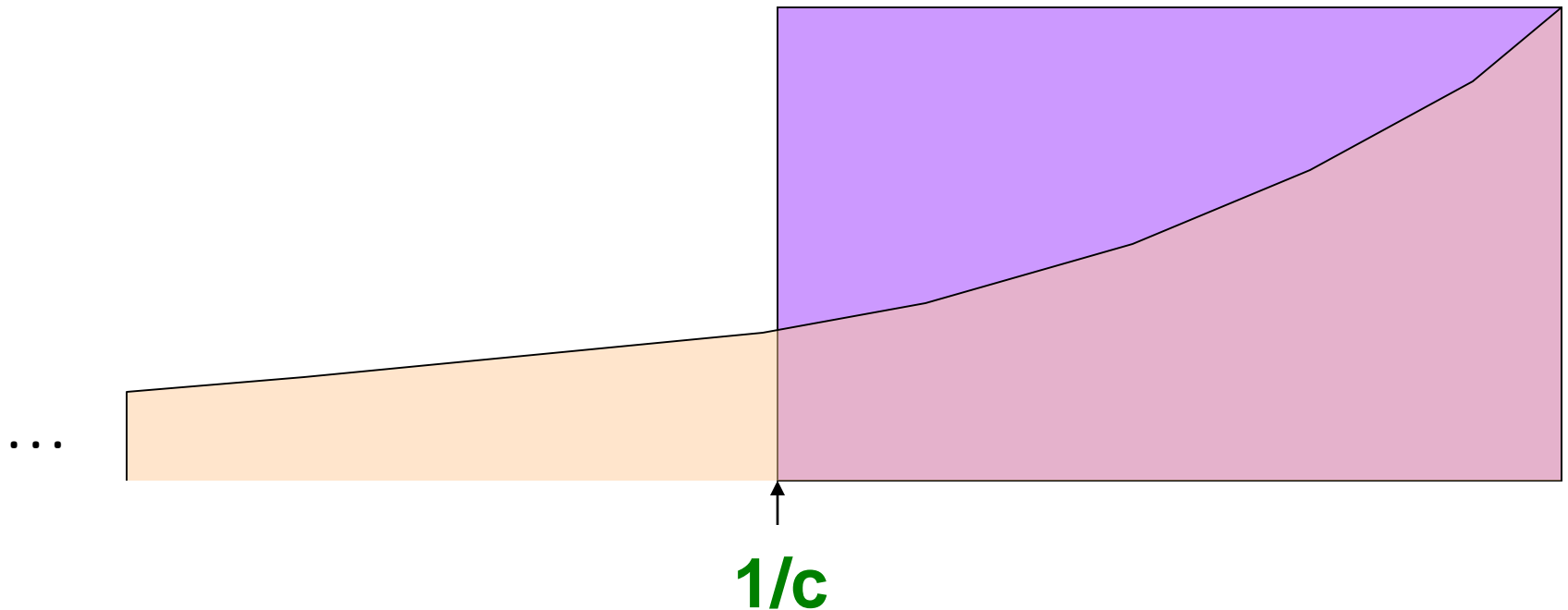
- **Important property:** Sum over all weights $\sum_t (1-c)^t$ is 1/[1 − (1 − c)] = 1/c

Example: Counting Items

- **What are "currently" most popular movies?**
- **Suppose we want to find movies of weight > ½**
  - **Important property:** Sum over all weights $\sum_t (1 - c)^t$ is 1/[1 − (1 − c)] = **1/c**
- **Thus:**
  - There cannot be more than **2/c** movies with weight of **½** or more
- So, **2/c** is a limit on the number of movies being counted at any time

Extension to Itemsets

- **Count (some) itemsets in an E.D.W.**
  - **What are currently "hot" itemsets?**
    - **Problem:** Too many itemsets to keep counts of all of them in memory

- **When a basket B comes in:**
  - Multiply all counts by **(1-c)**
  - For uncounted items in **B**, create new count
  - Add **1** to count of any item in **B** and to any **itemset** contained in **B** that is already being counted
  - **Drop counts < ½**
  - Initiate new counts (next slide)

Initiation of New Counts

- Start a count for an itemset $S \subseteq B$ if every proper subset of S had a count prior to arrival of basket B

  - **Intuitively:** If all subsets of **S** are being counted this means they are "**frequent/hot**" and thus **S** has a potential to be "**hot**"

- **Example:**

  - Start counting S={i, j} iff both **i** and **j** were counted prior to seeing B

  - Start counting S={i, j, k} iff {i, j}, {i, k}, and {j, k} were all counted prior to seeing B

How many counts do we need?

- Counts for single items **< (2/c)·(avg. number of items in a basket)**

- **Counts for larger itemsets = ??**

- **But we are conservative about starting counts of large sets**
  - If we counted every set we saw, one basket of **20** items would initiate **1M** counts