Polinomi $\boxed{\sum_{i=0}^{n} a_i x^i}$ x variabile o indeterminata

$a_i \in K$ campo $K = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{F}_2$ K Körper

field

$K[x]$ = anello dei polinomi nella variabile a coefficienti in K

$(K[x], +, 0_A, *, 1_A)$ $\qquad \sum a_i x^i + \sum b_i x^i = \sum (a_i + b_i) x^i$

$A = K[x]$ $\quad 0_A = 0_K \quad 1_A = 1_K$ $\qquad \sum a_i x^i * \sum b_j x^j = \sum c_k x^k$

$$c_k = \sum_{i+j=k} a_i b_j$$

$$x^i x^j = x^{i+j}$$

$0_A = 0 \cdot x^0 + 0 \cdot x^1 + 0 \cdot x^2 + \dots$

$1_A = 1 \cdot x^0 + 0 \cdot x^1 + 0 \cdot x^2 + \dots$

Più in generale $k \in K$ $\quad k_A = k \cdot x^0 + 0 \cdot x + 0 \cdot x^2 +$

$k_A = k$

Notate che $K[x]$ è anello <u>commutativo</u>

$a(x) = \sum a_i x^i \qquad b(x) = \sum b_j x^j$

$a(x) * b(x) \qquad c_k = \sum_{i+j=k} a_i b_j = \sum_{j+i=k} b_j a_i = d_k$

dove $c(x) = a(x) * b(x)$ e $d(x) = b(x) * a(x)$

$$c(x) = d(x)$$

ossia $\forall\, 0 \leq k \quad c_k = d_k$

FUNZIONE POLINOMIALE $\quad a(x) \in K[x] \quad (\mathbb{Q}[x], \mathbb{R}[x])$

$k \in K \quad \hat{a} : k \longmapsto a(k) = \sum_{i=0}^{n} a_i k^i \quad$ dove $a(x) = \sum_{i=0}^{n} a_i x^i$

VALUTAZIONE di $a(x)$ in $k \in K$

Esempio: $K = \mathbb{Q} \quad a(x) = 1 \cdot x^0 + 0 \cdot x + 1 \cdot x^2 = 1 + x^2$

$\hat{a} : q \longmapsto 1 + q^2 \qquad 1 = 1_A \qquad \hat{a} : \mathbb{Q} \to \mathbb{Q}$

Cosa accade nell'altra direzione $a(x) \in K[x]$ $\hat{a}$

a è unico? Sì se K ha infiniti elementi

Esempio: $K = \mathbb{F}_2 = \{0,1\}$ $a(x) = 1+x$ $\hat{a}: k \mapsto 1+k$ $\hat{a}: \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}$

$b(x) = x^2+1 \neq a(x)$ $\hat{b}: \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}$ $\hat{a} = \hat{b}$

Notate $E = End(\mathbb{F}_2) = \{f : \mathbb{F}_2 \to \mathbb{F}_2\}$ $|End(\mathbb{F}_2)| = 2^2 = 4$

$|\{f : \underset{a}{A} \to \underset{b}{B}\}|$ $\begin{array}{c} v_1, \dots, v_a \\ \downarrow f \\ w_1, \dots, w_a \\ b, \dots, b \end{array}$ $\{f : A \to B\} =: B^A$

$|B^A| = |B|^{|A|}$

FATTO: $End(\mathbb{F}_2) = \{\widehat{a+bx} : a, b \in \mathbb{F}_2\}$

$\hat{0}_A = 0_E$

$\hat{1}_A = 1_E$

$\hat{x} = id : a \mapsto a \ \forall a \in \mathbb{F}_2$

$\widehat{1+x}$ = scambio, negazione

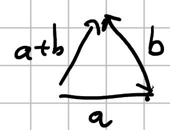
Def: Sia K campo $a(x) \in K[x]$ si chiama GRADO di $a(x)$

$$m = \max\{k \in \mathbb{N} : a_k \neq 0\} \leftarrow \text{è finito}$$

Ha senso poiché $\underset{\sum a_i x^i}{a(x)} \longleftrightarrow (a_0\, a_1 \quad a_m\, 0 \dots 0 \dots)$

$x \longrightarrow (0, 1, 0, 0 \dots)$

$\deg(a(x)) = m$ $\deg : K[x] \longrightarrow \mathbb{N}$



Teorema: Siano $a, b \in K[x]$ i) $\deg(a+b) \leq \max(\deg a, \deg b)$ $\underset{\geq}{\deg a + \deg b}$

$|a+b| \leq |a| + |b|$

ii) $\deg(a * b) = \deg a + \deg b$

Dim: i) $a(x) = \sum_{i=0}^{m} a_i x^i$ $a_m \neq 0$ $a_{m+k} = 0$ $k > 0$

$b(x) = \sum_{j=0}^{n} b_j x^j$ $b_n \neq 0$ $b_{n+k} = 0$ $k > 0$

$\sum_{i=0}^{\infty} (a_i + b_i) x^i$ $\max\{i : a_i + b_i \neq 0\} \leq n = \max\{m, n\}$

assumo $m \leq n$ $a_{n+k} + b_{n+k} = 0$ $k > 0$

ii) $a_0 + a_1 x \quad + a_m x^m \longleftrightarrow (a_0 \quad a_m\ 0 \dots)$

$b_0 + \quad + b_n x^n \longleftrightarrow (b_0 \quad b_n\ 0 \dots)$

$c = a * b \qquad c_k = \sum_{i+j=k} a_i b_j \qquad$ Se $k > n+m$

$$\sum_{i+j=k} a_i b_j = \underbrace{\sum_{i>m} a_i b_j}_{=0} + \sum_{i \le m} a_i b_j = \sum a_i \underbrace{b_{k-i}}_{=0}$$

$k - i \ge k - m > n+m-m = n$ quindi $b_{k-i} = 0$

$k = n+m \qquad k-i \ge k-m \ge n \qquad c_{n+m} = a_m b_n \neq 0$

poiché $a_m, b_n \neq 0_K \Rightarrow a_m b_n \neq 0_K$ in K

Proposizione: Sia K campo $a, b \in K$ $a, b \neq 0_K \Rightarrow ab \neq 0_K$

Dim: Per assurdo $ab = 0_K$ $\exists\, a^{-1} \in K$ $a^{-1}(ab) = a^{-1} \cdot 0_K$

$b = 1_K b = a^{-1} a b = 0_K$

MORALE: Ne segue che $\deg : K[x] \to \mathbb{N}$ si comporta come un logaritmo $\log(ab) = \log a + \log b$

$e = 2.718281828$?

Domanda quanto vale $\deg(0_A)$?

$$\deg(0_A) = \max \underbrace{\{k \in \mathbb{N} : (0_A)_k \neq 0\}}_{\emptyset}$$

$$\deg a + \deg 0_A \overset{\downarrow}{=} \deg(a \cdot 0_A) = \deg(0_A) := -\infty$$

FATTO: Più in generale $\deg(ab) = \deg a + \deg b$ è vera se $D[x]$ D anello commutativo $\alpha, \beta \neq 0_D \Rightarrow \alpha\beta \neq 0_D$