How can we transform images?

- **x** is a point in our image where:
  - **x** = (x, y) or in matrix terms

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Say we want new coordinate system

- Map points from one image into another
- Often we can use matrix operations
- Given a point x, map to new point x’ using M

$$\mathbf{x}' = \mathbf{M}\,\mathbf{x}$$

Scaling is just a matrix operation

- Map points from one image into another
- Often we can use matrix operations
- Given a point x, map to new point x’ using M

$$\mathbf{x}' = \mathbf{S}\,\mathbf{x}$$

$$\mathbf{x}' = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \mathbf{x}$$

Translation: add another row

- $\bar{\mathbf{x}}$ is $\mathbf{x}$ but with an added 1
- *Augmented vector*

$$\bar{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translation: add another row

- $\bar{x}$ is $\mathbf{x}$ but with an added 1
- *Augmented vector*
- Now translation is easy

$$\bar{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}}$$

Reminder, $\mathbf{I}$ = Identity

Common to just use $\mathbf{I}$ as a generic, whatever size identity fits here.

$$\mathbf{I}_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\mathbf{I}_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation: add another row

- $\bar{\mathbf{x}}$ is $\mathbf{x}$ but with an added 1
- *Augmented vector*
- Now translation is easy

$$\bar{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}}$$

Translation: add another row

- $\bar{x}$ is **x** but with an added 1
- *Augmented vector*
- Now translation is easy
- x’ = 1*x + 0*y + dx*1
- y’ = 0*x + 1*y + dy*1

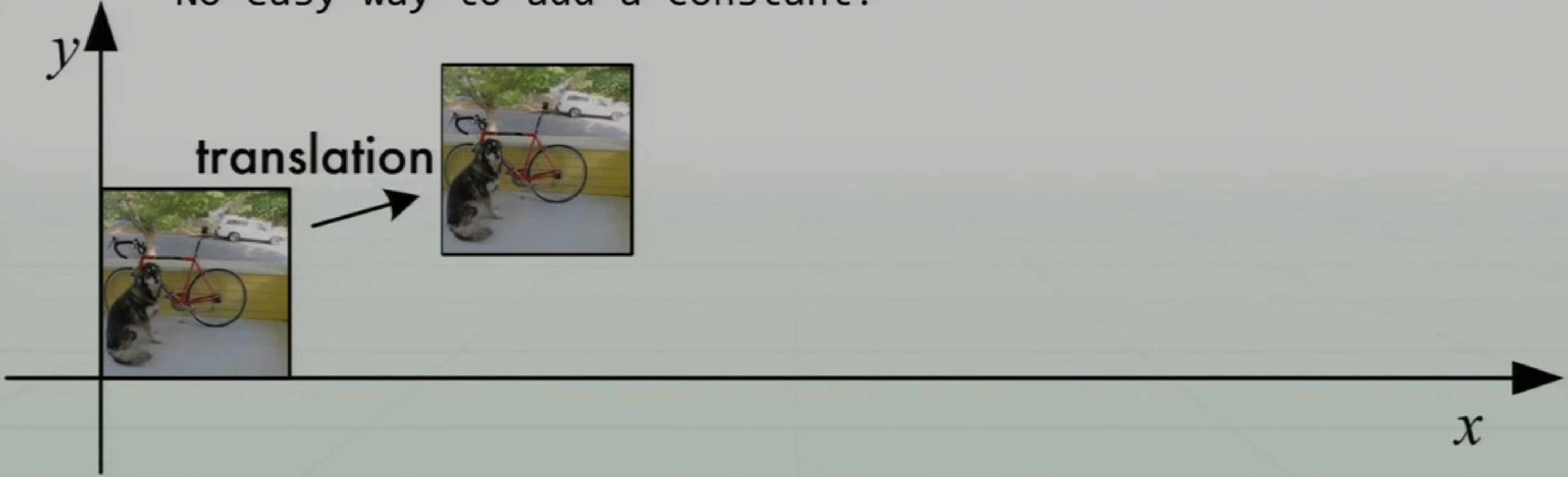
$$\bar{\mathbf{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}}$$

Translation is harder...

- x’ = M x
  - Want to move x’ by dx and y’ by dy
  - How do we pick M?
  - Can only add up multiples of x or y
  - No easy way to add a constant!



Euclidean: rotation + translation

- Want to translate and rotate at same time
- Still just matrix operation

$$\mathbf{x}' = \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}}$$

Euclidean: rotation + translation

- Want to translate and rotate at same time
- Still just matrix operation
- **R** is rotation matrix, **t** is translation

$$\mathbf{x}' = \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \bar{\mathbf{x}}$$

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Euclidean: rotation + translation

- Want to translate and rotate at same time
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- **R** is rotation matrix, **t** is translation

$$\mathbf{x}' = \begin{bmatrix} \cos\theta & -\sin\theta & dx \\ \sin\theta & \cos\theta & dy \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x}' = [\,\mathbf{R} \quad \mathbf{t}\,]\,\bar{\mathbf{x}}$$

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Euclidean: rotation + translation

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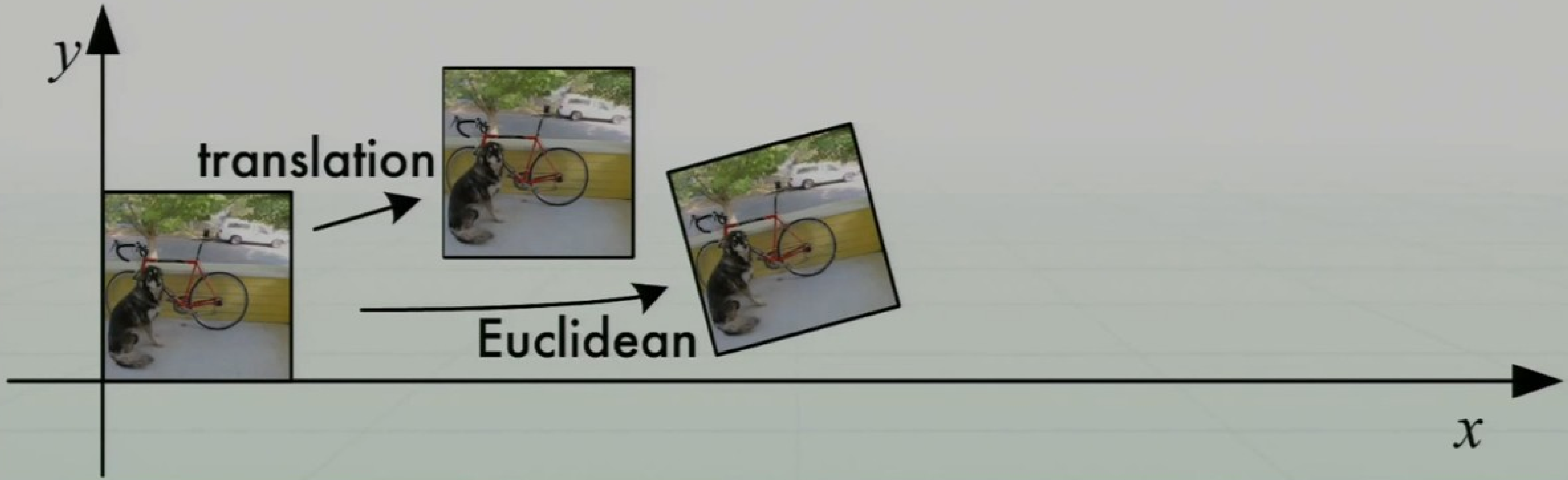


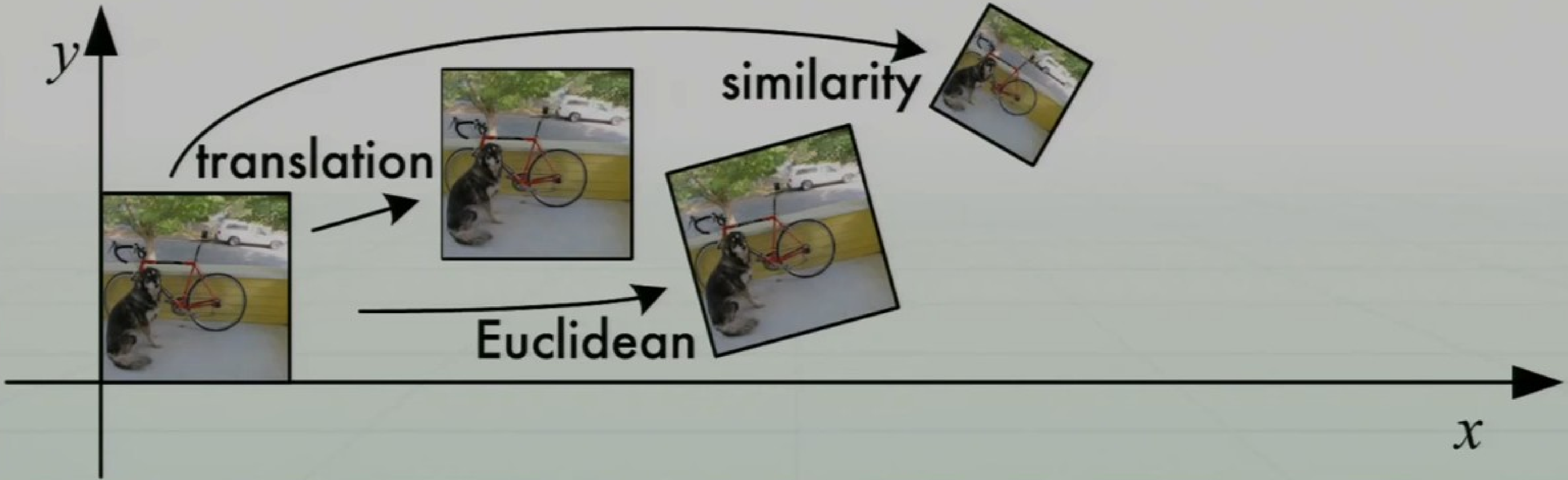
Similarity: scale, rotate, translate

$$\mathbf{x}' = [\, s\mathbf{R} \quad \mathbf{t} \,]\, \bar{\mathbf{x}}$$

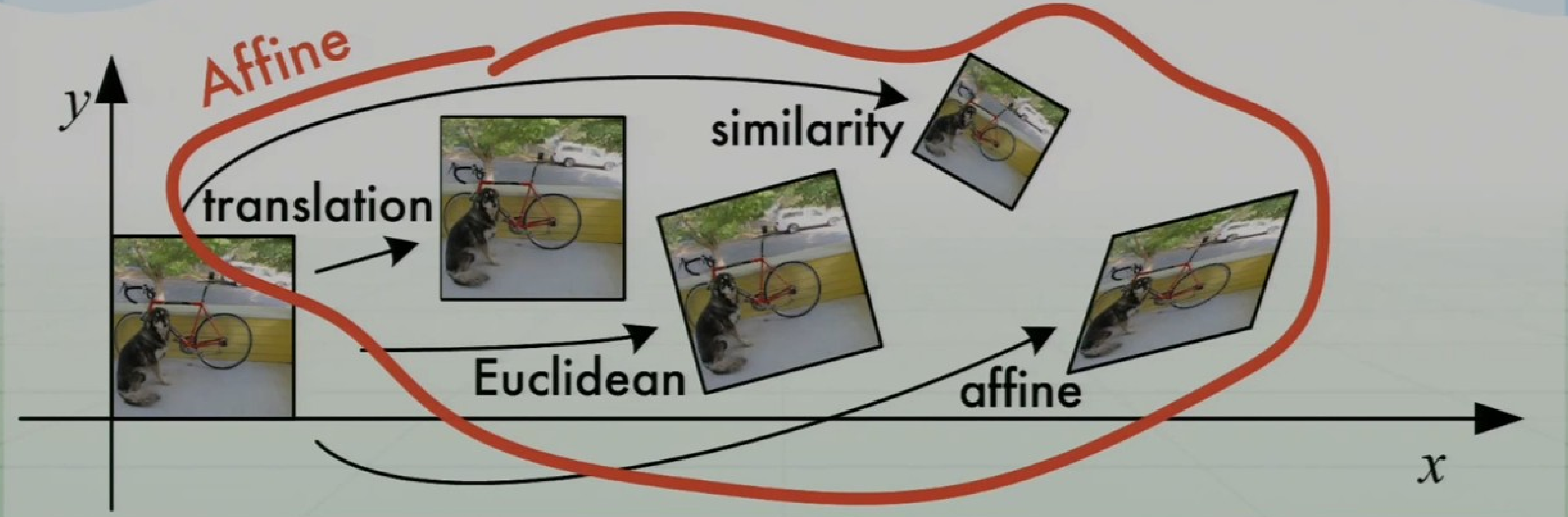
$$\mathbf{x}' = \begin{bmatrix} a & -b & dx \\ b & a & dy \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Similarity: scale, rotate, translate

$$\mathbf{x}' = [\, s\mathbf{R} \quad \mathbf{t} \,]\, \bar{\mathbf{x}}$$



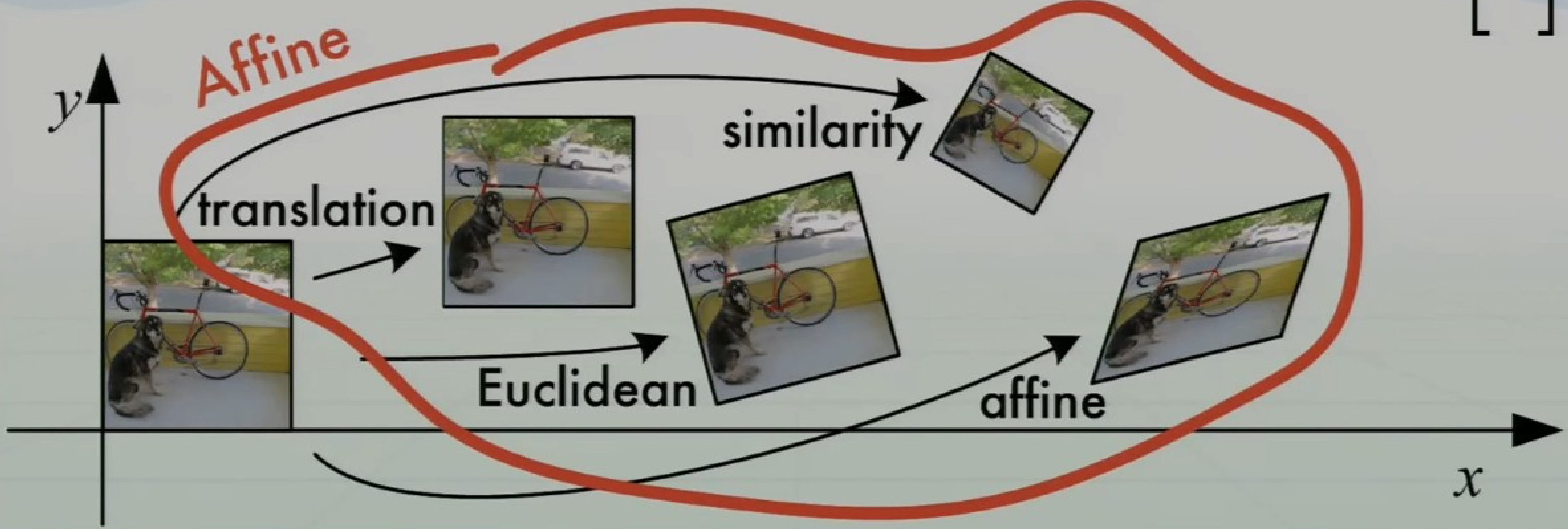
Affine: scale, rotate, translate, shear



Affine: scale, rotate, translate, shear

General case of 2x3 matrix

$$\mathbf{x}' = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Combinations are still affine

Say you want to translate, then rotate, then translate back, then scale.

$\mathbf{x'} = \mathbf{S}\ \mathbf{t}\ \mathbf{R}\ \mathbf{t}\ \bar{\mathbf{x}} = \mathbf{M}\ \bar{\mathbf{x}},$

If $\mathbf{M} = (\mathbf{S}\ \mathbf{t}\ \mathbf{R}\ \mathbf{t})$

$\mathbf{M}$ is still affine transformation

Wait, but these are all 2x3, how to we multiply them together?

Added row to transforms

$$\bar{\mathbf{x}}' = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

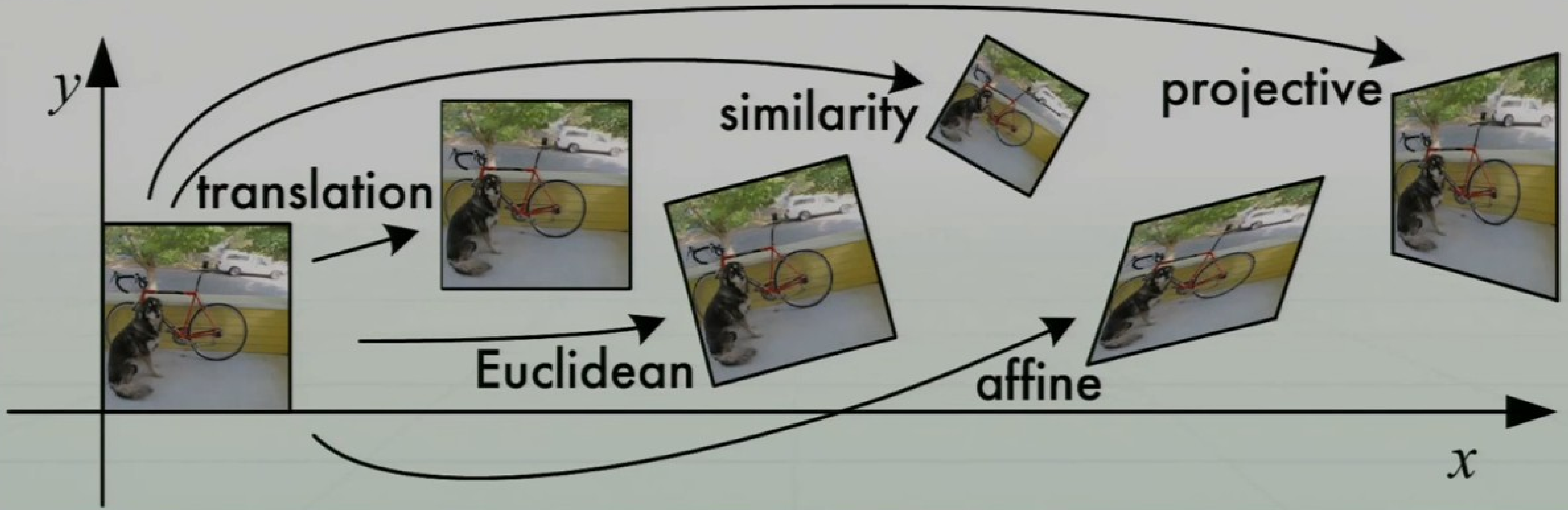
$$\bar{\mathbf{x}}' = \begin{bmatrix} \cos\theta & -\sin\theta & dx \\ \sin\theta & \cos\theta & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\bar{\mathbf{x}}' = \begin{bmatrix} a & -b & dx \\ b & a & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\bar{\mathbf{x}}' = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Projective transform

- AKA perspective transform or homography
- Wait but affine was any 2x3 matrix...



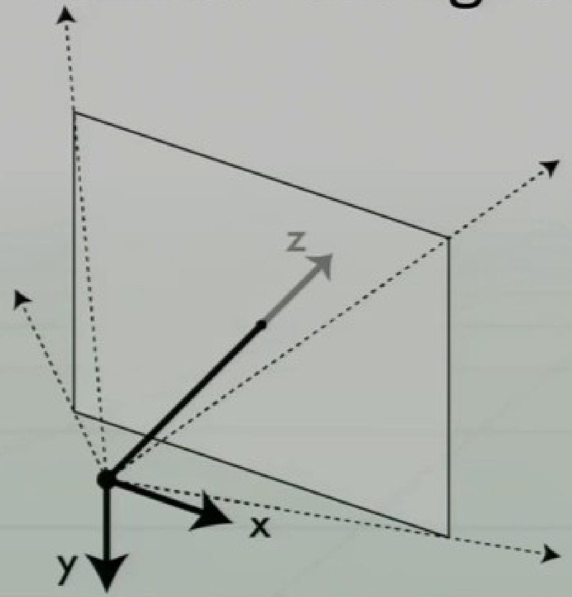
Need some new coordinates!

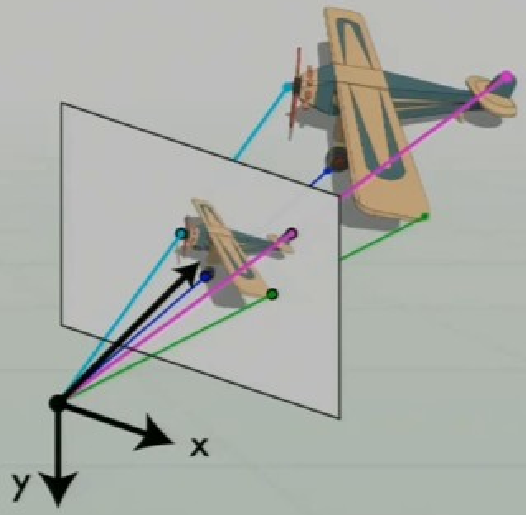
- Homogeneous coordinate system
  - Useful because we can do this kind of transform
- Each point in 2d is actually a vector in 3d
- Equivalent up to scaling factor
- Have to normalize to get back to 2d

$$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} \qquad \bar{\mathbf{x}} = \tilde{\mathbf{x}} / \tilde{w}$$

Why does this make sense?

- Remember our pinhole camera model
- Every point in 3d projects onto our viewing plane through our aperture
- Points along a vector are indistinguishable





Projective transform

- AKA perspective transform or homography
- Wait but affine was any 2x3 matrix…
- Homography is general 3x3 matrix
- Multiplication by scalar is equivalent

$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\,\tilde{\mathbf{x}}$$

Projective transform

- AKA perspective transform or homography
- Wait but affine was any 2x3 matrix…
- Homography is general 3x3 matrix
- Multiplication by scalar: equivalent projection
  - 3*H ~ H

$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\,\tilde{\mathbf{x}}$$

$$\tilde{\mathbf{x}}' = \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$$

Using homography to project point

- Multiply **x~** by **H~** to get **x'~**
- Convert to **x'** by dividing by **w'~**

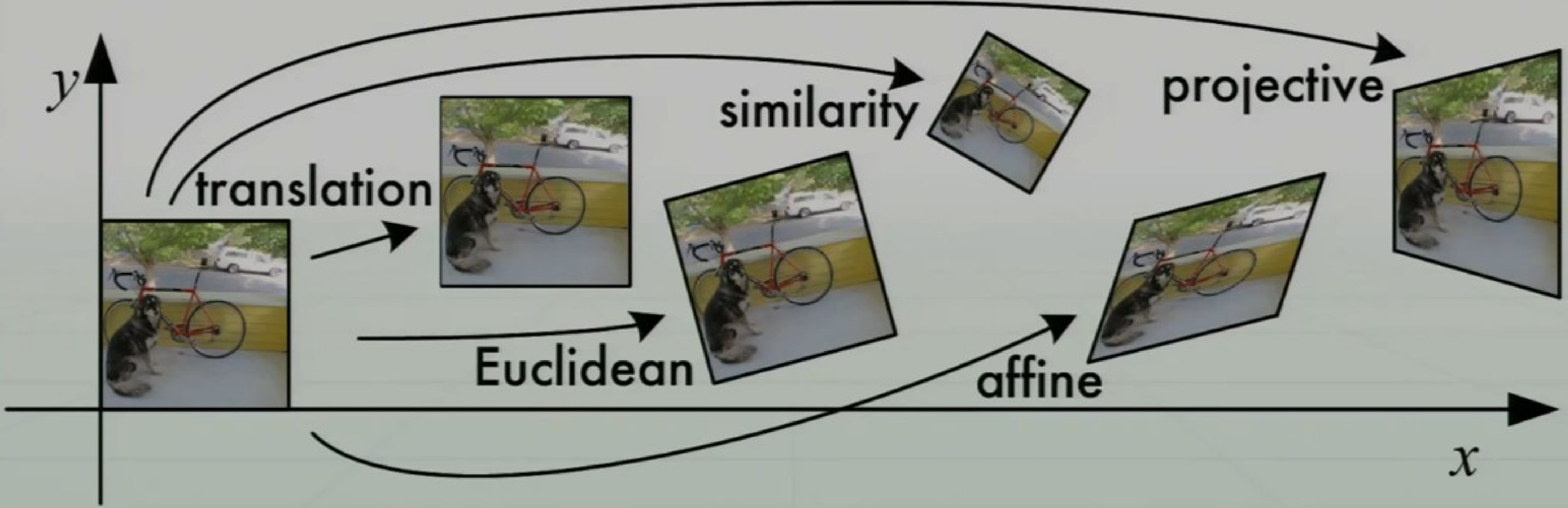
$$\tilde{\mathbf{x}}' = \tilde{\mathbf{H}}\,\tilde{\mathbf{x}}$$

$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{w}' \end{bmatrix} = \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$$

$$\bar{\mathbf{x}} = \tilde{\mathbf{x}} \,/\, \tilde{w}$$

Lots to choose from

- What do each of them do?
- Which is right for panorama stitching?



How hard are they to recover?

$$\bar{\mathbf{x}}' = \begin{bmatrix} 1 & 0 & dx \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\bar{\mathbf{x}}' = \begin{bmatrix} \cos\theta & -\sin\theta & dx \\ \sin\theta & \cos\theta & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\bar{\mathbf{x}}' = \begin{bmatrix} a & -b & dx \\ b & a & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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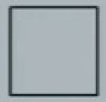
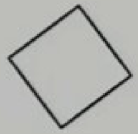
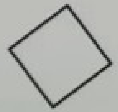
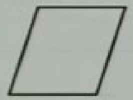
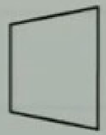
$$\tilde{\mathbf{x}}' = \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$$

Lots to choose from

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\left[\begin{array}{c\|c} \mathbf{I} & \mathbf{t} \end{array}\right]_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\left[\begin{array}{c\|c} \mathbf{R} & \mathbf{t} \end{array}\right]_{2\times 3}$	3	lengths	
similarity	$\left[\begin{array}{c\|c} s\mathbf{R} & \mathbf{t} \end{array}\right]_{2\times 3}$	4	angles	
affine	$\left[\begin{array}{c} \mathbf{A} \end{array}\right]_{2\times 3}$	6	parallelism	
projective	$\left[\begin{array}{c} \tilde{\mathbf{H}} \end{array}\right]_{3\times 3}$	8	straight lines	

Say we want affite transformation

- Have our matched points
- Want to estimate **A** that maps from **x** to **x'**
- **xA = x'**
- How many degrees of freedom?
  - 6
- How many knowns do we get with one match? **mA = n**
  - 2
  - $n_x = a_{00}*m_x + a_{01}*m_y + a_{02}*1$
  - $n_y = a_{10}*m_x + a_{11}*m_y + a_{12}*1$

$$\mathbf{x'} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$