Exercise 1: Logistic equation

We have the basic logistic equation for population growth, which we will call f_R:

$$f_R = \frac{df_R}{dt} = rR(1 - {}^{R}/_{K})$$

where r is the population growth rate and K is carrying capacity.

First, we want to solve for the equilibria. We do this by setting $f_R = 0$ and solving for R* (we use * to denote equilibrium).

It turns out that R* = 0 and R* = K are both equilibria. R* = 0 is what we often call a trivial equilibrium (a population cannot grow if there are 0 individuals).

We can visualize these equilibria in our state space, which in this case is simply a line (we only have one dimension):

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____________|________________________|_____________________
          R* = 0                   R* = K
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To determine whether our equilibria are stable or not, we want to determine whether small perturbations away from an equilibrium return to that equilibrium or grow away from them. We will call these perturbations ε, where ε(t) follows the distance from R* over time after an initial perturbation, ε(0).

Using Taylor expansion we can show that

$$\frac{d\,\varepsilon(t)}{dt} = \frac{df_R}{dt}|_{R=R*} + h.o.t.$$

because ε(0) is small, higher order terms (i.e., h.o.t. in the above equation) are negligible.

Now we can simply take the derivative of f_R (with R = R*) to evaluate the behaviour of ε(t) – this is the linear term hence why we call it the linearization.

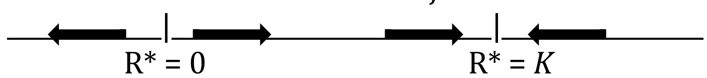
$$\frac{df_R}{dR} = r - {}^{2rR}/_{K}$$

Note that this derivative is equivalent to our Jacobian (in the form of a 1 x 1 matrix). This is a linearization of our model around our equilibria when we set R = R*.

At R* = 0, $\frac{df_R}{dR} = r$ (so ε(t) is positive).
At R* = K, $\frac{df_R}{dR} = r - 2r = -r$ (so ε(t) is negative).

Since ε(t) is positive near R = 0, it grows away from the equilibrium, and since ε(t) is negative near R = K, the perturbation is shrinking back towards the equilibrium.

We can re-draw our line with the trajectories as follows:



What we have done here is known as local stability analysis, however we can begin to get a global picture of our phase space by visualizing our equilibria this way.