Kalman Filtering with Linear State Constraints

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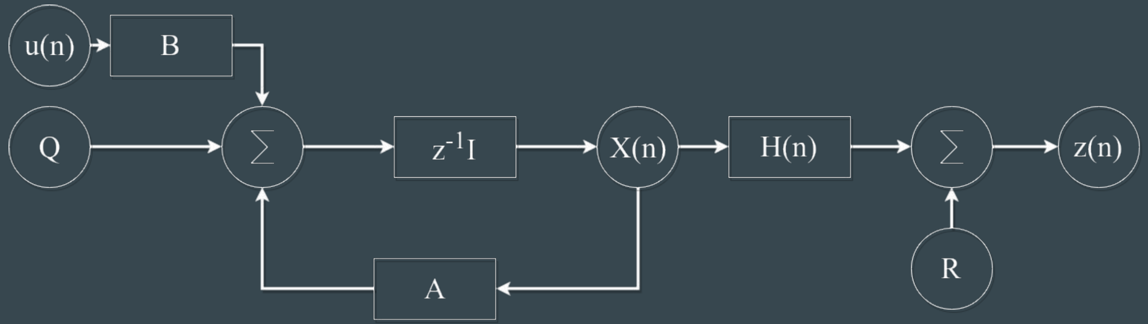
Overview

- Background
- Kalman Filter with Linear Constraints
    - Perfect Measurements
    - State Reduction
    - Estimation Projection
    - PDF Truncation
- Kalman Filter with Nonlinear Constraints
    - Taylor Series Linearization
- Results
- Conclusion

Background: State Space

Process Equation:

$$x_{n+1} = A\,x_n + B\,u_n + Q$$



Measurement Equation:

$$z_n = H\,x_n + R$$

Background: Kalman Filter Equations

Standard Kalman Filter Equations:

$X_{predicted} = A\ x_{n-1} + B\ u_n$ (State Prediction)

$P_{predicted} = A\ P_{n-1}\ A^T + Q$ (Covariance Prediction)

$y = z_n - H\ x_{predicted}$ (Innovation)

$S = H\ P_{predicted}\ H^T + R$ (Innovation Covariance)

$K = P_{predicted}\ H^T S^{-1}$ (Kalman Gain)

$X_n = X_{predicted} + Ky$ (State Update)

$P_n = (I - K\ H)\ P_{predicted}$ (Covariance Update)

Background: Linear Constraints

Time Invariant

Equality Constraint:

$$Dx_n = d$$

Inequality Constraint:

$$Dx_n \leq d$$

Time Variant

Equality Constraint:

$$D(t)\, x_n = d(t)$$

Inequality Constraint:

$$D(t)\, x_n \leq d(t)$$

Background: Nonlinear State Constraints

Time Invariant

Equality Constraint:

$g(x_n) = h$

Inequality Constraint:

$g(x_n) \leq h$

Time Variant

Equality Constraint:

$g(x_n, t) = h(t)$

Inequality Constraint:

$g(x_n, t) \leq h(t)$

Kalman Filter with Constraints

- Sometimes we may know additional information about the system
- Want to incorporate this information into the Kalman Filter equation

- Example: 2-D tracking with a fixed direction (angle)
- Example: 3-D tracking in a room with fixed dimensions

Kalman Filter With Linear Constraints: Perfect Measurement

- Only for equality constraints ($Dx_n = d$)
- Measurement equation augmentation

Implementation:

$$x = \begin{bmatrix} X \\ V_x \\ Y \\ V_y \end{bmatrix}$$

$$V_x = (3/4)V_y$$

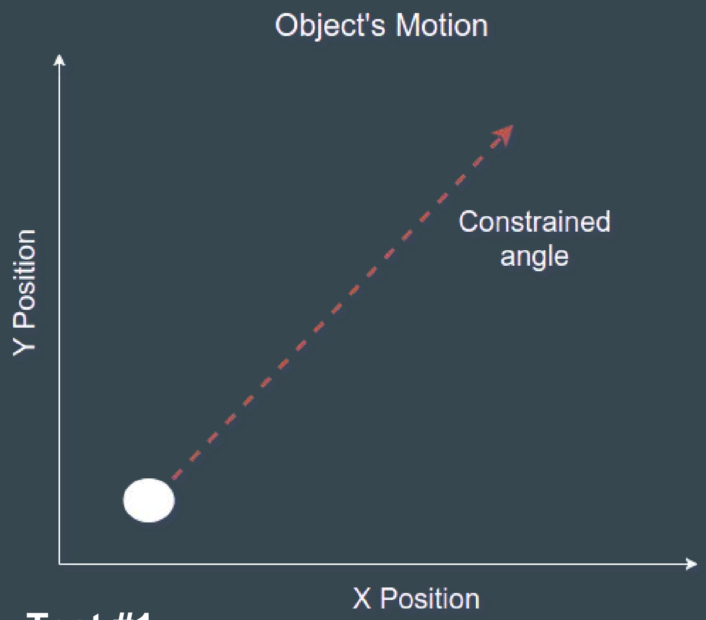
$$D = \begin{bmatrix} 0 & 1 & 0 & -3/4 \end{bmatrix}$$

$$d = 0$$

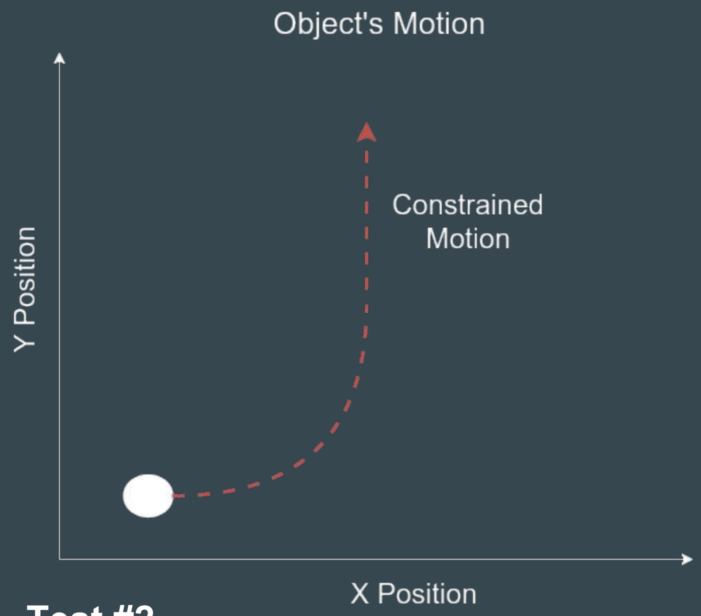
$$\begin{bmatrix} z_n \\ d \end{bmatrix} = \begin{bmatrix} H \\ D \end{bmatrix} x_n + \begin{bmatrix} R \\ 0 \end{bmatrix}$$

Kalman Filter With Linear Constraints: Perfect Measurement

Matlab Simulation:



Test #1



Test #2

Kalman Filter With Linear Constraints: State Reduction

- Only for Equality Constraints ($Dx_n = d$)

<u>**Implementation:**</u>

$$A = \begin{bmatrix} 1 & dt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} X \\ V_x \\ Y \\ V_y \end{bmatrix}$$

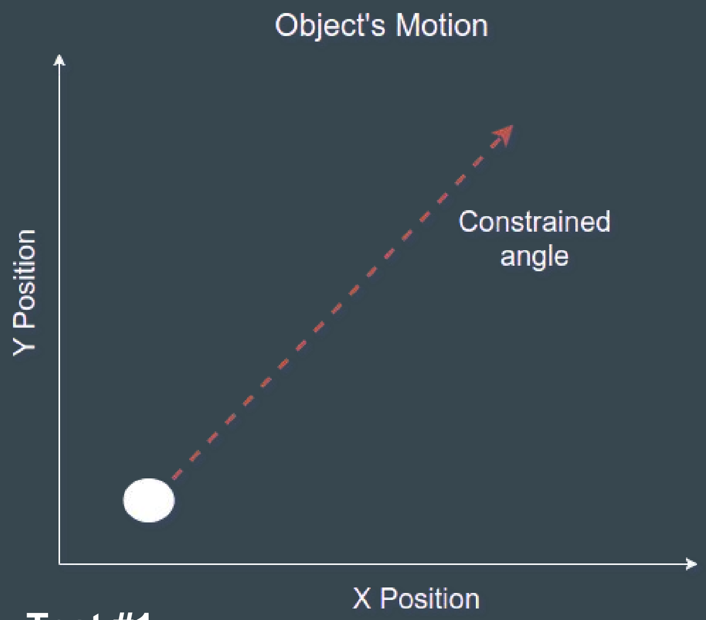
$$V_x = (3/4)V_y$$

$$D = \begin{bmatrix} 0 & 1 & 0 & -3/4 \end{bmatrix} \qquad d = 0$$

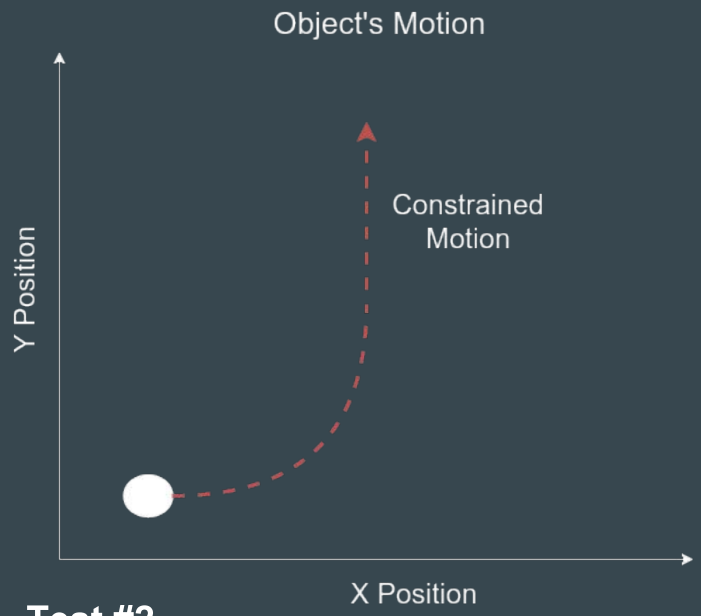
$$A' = \begin{bmatrix} 1 & 0 & 3/4 \\ 0 & 1 & dt \\ 0 & 0 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} X \\ Y \\ V_y \end{bmatrix}$$

Kalman Filter With Linear Constraints: State Reduction

Matlab Simulation:



Test #1



Test #2

Kalman Filter With Linear Constraints: Estimation Projection

- Both Equality Constraints ($Dx_n = d$) and inequality* constraints ($Dx_n \leq d$)
- *Require a priori knowledge of active inequality constraint ($Dx_n \leq d \rightarrow Dx_n = d$)

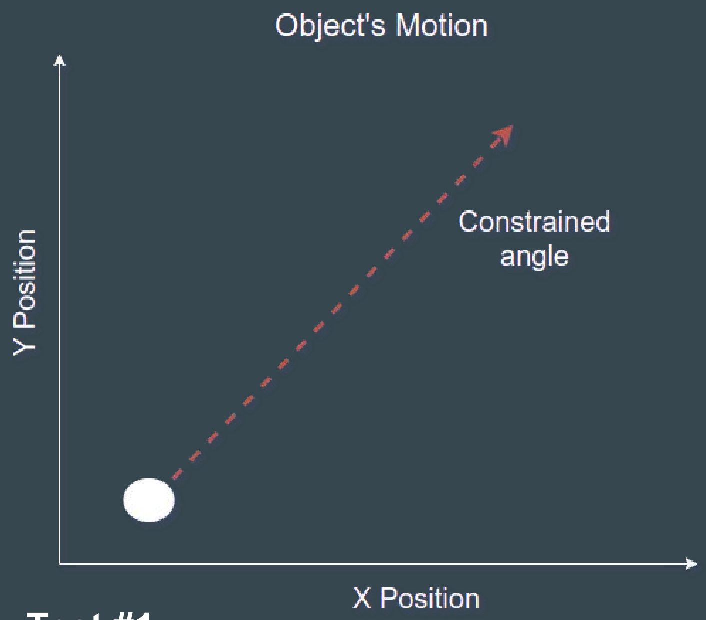
Implementation:

Standard Kalman Filter Algorithm + Projection:

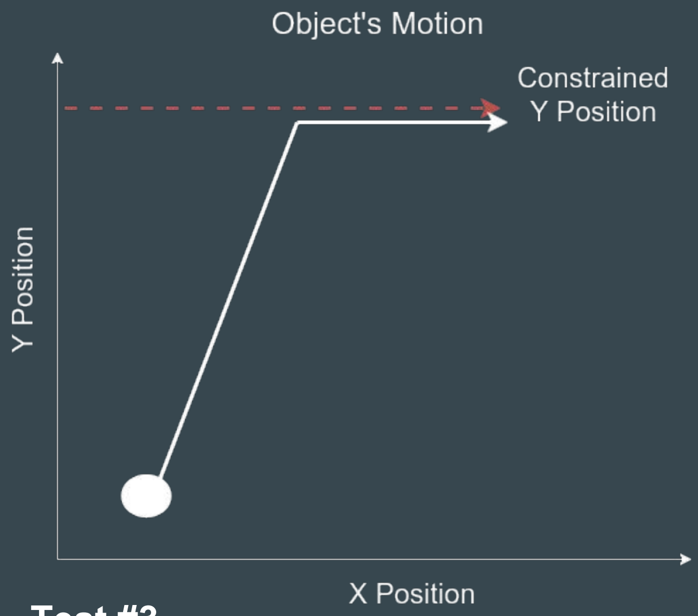
$$X_{projected} = X_n - PDD^{|} (DPD)^{-1} (DX_n - d)$$

Kalman Filter With Linear Constraints: State Projection

Matlab Simulation:



Test #1



Test #3

Kalman Filter With Linear Constraints: PDF Truncation

- Both equality constraints ($Dx_n = d$) and inequality constraints ($Dx_n \leq d$)
- Do <u>not</u> need a priori knowledge of active constraints

- Computationally expensive
- Introduces a bias (no longer the optimal solution)

<u>**Implementation:**</u>

Standard Kalman Filtering Algorithm + Truncation

Kalman Filter With Linear Constraints: PDF Truncation

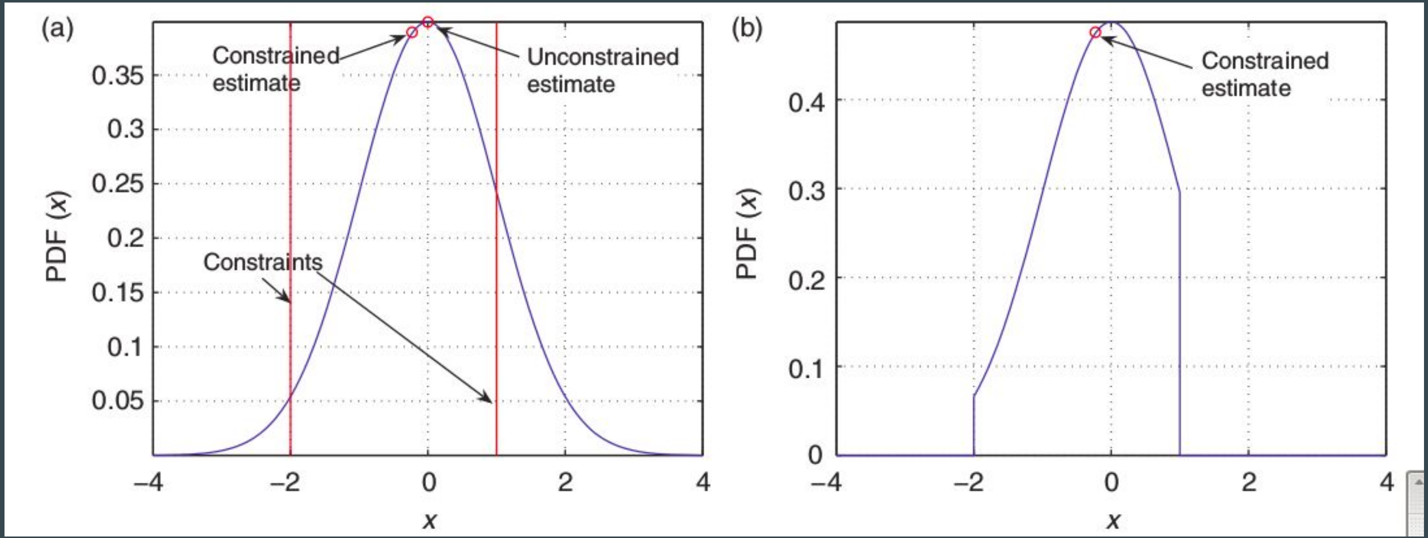
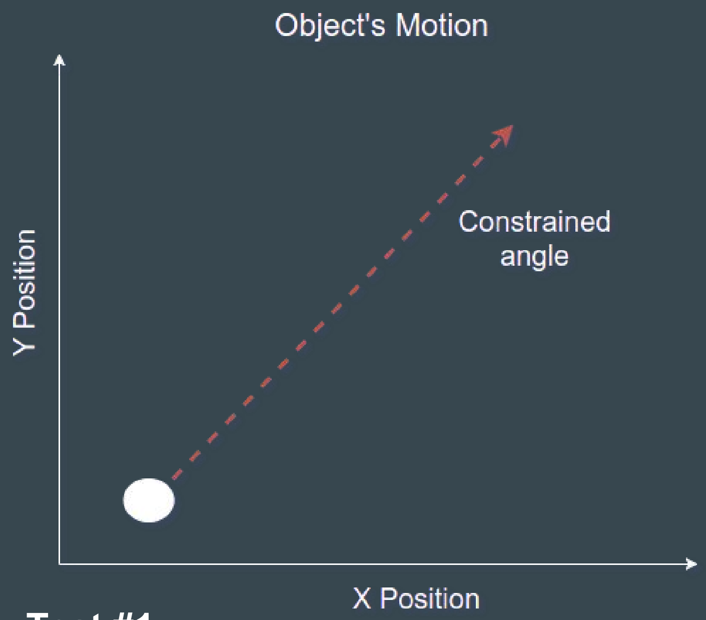


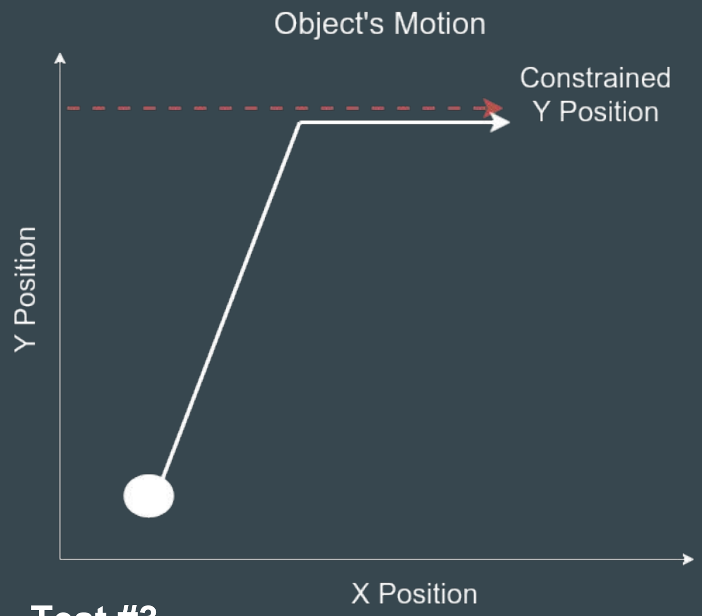
Fig. 2: a) PDF prior to truncation.
b) PDF after truncation with noticeable bias. [3]

Kalman Filter With Linear Constraints: PDF Truncation

Matlab Simulation:



Test #1



Test #3

Kalman Filter With Nonlinear Constraints: Taylor Series Linearization

- Change nonlinear constraint ($g(x_n) = d$) into linear constraint ($Dx_n = h$)
- Neglect second-order and above terms
- Use any of the previously discussed constrained methods

$$g(x) = \sum_{n=0}^{\infty} \frac{g^n(0)}{n!} x^n = g(0) + g'(0)x + \ldots$$

Kalman Filter With Nonlinear Constraints: Taylor Series Linearization

Implementation:

Taylor series expansion evaluated about some point $\hat{x_n}$

$$D = g'(\hat{x_n})$$

$$d = h - g(\hat{x_n}) + g'(\hat{x_n})\hat{x_n}$$

Results:

Algorithm Performance Comparison:

- Unconstrained Kalman filtering method Vs constrained methods
- Mean Squared Error (MSE) of the estimated states

Results:

X Position MSE:

	Standard:	Perfect Measurement:	State Reduction:	Estimation Projection:	PDF Truncation:
Test #1:	11.51 dB	11.37 dB	10.75 dB	11.36 dB	11.37 dB
Test #2:	11.51 dB	11.42 dB	10.77 dB	N/A	N/A
Test #3:	11.51 dB	N/A	N/A	11.51 dB	11.51 dB

Table 1: MSE of the estimated state error for the X position.

Results:

Y Position MSE:

	Standard:	Perfect Measurement:	State Reduction:	Estimation Projection:	PDF Truncation:
Test #1:	4.432 dB	3.937 dB	5.029 dB	3.880 dB	3.937 dB
Test #2:	4.726 dB	4.072 dB	4.774 dB	N/A	N/A
Test #3:	6.057 dB	N/A	N/A	4.007 dB	4.237 dB

Table 2: MSE of the estimated state error for the Y position.

Conclusion:

- Equality Constraint Only: Perfect Measurement, State Reduction
- Equality/Inequality Constraint: State Projection, PDF Truncation

- First three methods produce similar performance
- PDF truncation introduces a suboptimal bias

- Nonlinear constraints can be linearized at the expense of performance

Thank you

Matlab Code & Presentation Available:

https://github.com/McCarthyMadison/Kalman-Filter-Constrained.git

References:

[1] D. Simon and Tien Li Chia, "Kalman filtering with state equality constraints," in IEEE Transactions on Aerospace and Electronic Systems, vol. 38, no. 1, pp. 128-136, Jan 2002.

[2] D. Simon, "Kalman filtering with state constraints: a survey of linear and nonlinear algorithms," in IET Control Theory & Applications, vol. 4, no. 8, pp. 1303-1318, Aug 2010.

[3] D. Simon and D. L. Simon, "Constrained Kalman filtering via density function truncation for turbofan engine health estimation," in International Journal of Systems Science, vol. 41, no. 2, pp. 159-171, Nov 2009.