Stability of Bounded Subsets of Metzler Sparse Matrix Cones ⋆

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Abstract

We study the existence of Hurwitz matrices in a class of Metzler sparse matrix cones, where the non-zero entries are constrained to be in a bounded set. In particular, we provide necessary and sufficient conditions for the existence of one or more Hurwitz matrices in this class of sparse matrix spaces, for scenarios with uniform and non-uniform bounds. Several examples illustrate our results.

Key words: Sparse Matrix Space, Metzler matrices, Hurwitz matrices, Stability

1 Introduction

Stability of sparse matrix spaces, where the interconnections between subsystems is prescribed by non-zero patterns, is fundamental to distributed stabilization of networked control systems, and has recently been studied in [1]. In many practical scenarios, however, the evolution of the dynamics renders the nonnegative orthant invariant. Positive linear systems, which are characterized by dynamics governed by Metzler matrices [2], capture such scenarios and have a variety of applications, including economics [3], compartmental systems [2], population models [2], queueing systems [4], biology [5] and control [6]. In this short note, we consider the overlap of these two problems. In particular, our main objective is to study the class of stable sparse matrix spaces which are Metzler and, more importantly, have entries that are bounded, in contrast to what is considered in [1]. Our work is motivated by [7], where the stability of positive linear systems is characterized using the sign pattern of the underlying Metzler matrix.

Statement of Contributions

The contributions of this short note are twofold. First, we introduce the subset of Metzler sparse matrices and define the notions of absolute stability and sparse stability. This is followed by introducing the subset of bounded Metzler sparse matrix subsets, with uniform and (quasi) non-uniform bounds. As our first contribution, we prove the equivalence of uniform and sparse stability in the uniformly bounded case, and the uniqueness of these concepts in the quasi non-uniform and non-uniform case. As our second contribution, we prove necessary and sufficient conditions for sparse stability of (quasi) non-uniformly bounded Metzler sparse matrix cones. Several examples illustrate our results.

1.1 Mathematical preliminaries and notations

We begin with defining our basic notations. We let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{Z}_{\geq 0}$ denote the reals, the nonnegative reals and the nonnegative integers. We denote the space of vectors by $\mathbb{R}^n$, where $n \in \mathbb{Z}_{\geq 0}$, and by $\mathbb{R}^n_{\geq 0}$ the subset of $\mathbb{R}^n$ with positive components. The space of $n \times m$ matrices is denoted $\mathbb{R}^{n\times m}$, $n, m \in \mathbb{Z}_{\geq 0}$. For a positive-definite matrix $A \in \mathbb{R}^{n\times n}$ we write $A \succeq 0$. For two matrices A and B, we say $A > B$ if $\{a_{ij} > b_{ij}\}$ for all (i, j) pairs. A *Metzler matrix* is a square matrix such that all entries off the diagonal are nonnegative. We denote the spectrum of a matrix $A \in \mathbb{R}^{n\times n}$ by $\mathrm{spec}(A)$ and denote its spectral radius by $\rho(A) = \max |\lambda_i(A)|$, where $\lambda_i(A)$ is the ith eigenvalue of A. We also let $\alpha(A) = \max(\mathrm{Re}(\lambda_i(A)))$. Finally, for a matrix A, we define the matrix $\mathrm{sgn}(A)$ as follows:

$$\mathrm{sgn}(A)_{ij} = \begin{cases} 1 & A_{ij} > 0 \\ 0 & A_{ij} = 0 \\ -1 & A_{ij} < 0. \end{cases}$$

We also need to recall some properties of Metzler matrices; we begin with a result on their stability [6].

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Proposition 1.1 (Stability of Metzler matrices): *Given a Metzler matrix $A \in \mathbb{R}^{n\times n}$, the following are equivalent:*

a) The matrix A is Hurwitz.
b) There exists $\zeta \in \mathbb{R}^n$ such that $\zeta > 0$ and $A\zeta < 0$.
c) There exists $z \in \mathbb{R}^n$ such that $z > 0$ and $z^T A < 0$.
d) There exists a diagonal matrix $P \succeq 0$ such that $A^T P + PA \preceq 0$.
e) The matrix $-A^{-1}$ exists and has nonnegative entries.

Finally, we state an important property of Hurwitz Metzler matrices.

Proposition 1.2 (Diagonal elements of Metzler matrices and Hurwitz stability): *Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Then if A is Hurwitz, the diagonal elements must all be strictly negative.*

2 Problem statement

A *sparse matrix space* Σ_α, where $\alpha \subset \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$, is the vector space of matrices with all entries not indexed by an element of α equal to zero. A sparse matrix space Σ_α is *sparse stable* if there exists $A \in \Sigma_\alpha$ that is Hurwitz stable, i.e. the real part of every eigenvalue of A is strictly negative. Necessary, and sufficient, conditions for a sparse matrix system to be stable are established in [1], under the assumption that the free entries, prescribed by α, are chosen from $\mathbb{R}$. However, in many physical situations, there are constraints on the values of such free entries. In this paper, we wish to study the problem of sparse stability, where some of these entries are constrained. We focus our attention on a special class of sparse matrix spaces, called *Metzler sparse matrix cones*, where the free variables are restricted to only include Metzler matrices. We start by providing some preliminaries that are key to our treatment.

2.1 Sparse Matrix Spaces

Let Σ_α in $\mathbb{R}^{n\times n}$, where $\alpha \subset \{1, \ldots, n\} \times \{1, \ldots, n\}$, be a sparse matrix system as before. One can represent Σ_α using a matrix, where a $\star$ represents a free parameter. For example,

$$\Sigma = \begin{bmatrix} \star & 0 & \star & 0 \\ 0 & 0 & \star & \star \\ \star & 0 & \star & 0 \\ 0 & \star & \star & 0 \end{bmatrix} \tag{1}$$

represents a sparse matrix space in $\mathbb{R}^{4\times 4}$. A fruitful way to represent a sparse matrix space is by using a directed graph [1]. A *directed graph*, or simply a *digraph*, is a pair (V, E) such that V is a finite set called the *node set* of the digraph and E is a set of ordered pairs, called the *edge set* of the digraph. A *directed path* in a digraph, or in short *path*, is an ordered sequence of nodes so that any two consecutive nodes in the sequence forms an edge of the digraph. A *cycle* is a path in which the first and last nodes are the same. A *self-loop* is a cycle containing only one node. The representation of a sparse matrix space Σ through a digraph is as follows. The node set is given by $n \in \mathbb{Z}_{\geq 0}$, where n is the dimension of the matrices in Σ. Then if the pair $(i, j) \in \alpha$, the digraph $\mathcal{G}_\Sigma = (V, E)$ has an edge connecting nodes i and j. For the example above, the corresponding digraph is depicted below.

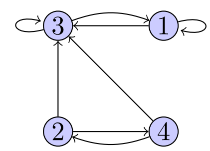

Fig. 1. The digraph representing the sparse matrix space described above.

In this work, we are interested in a subset of a given sparse matrix space Σ formed by restricting the free variables such that the elements in it are Metzler matrices. Since the Metzler property is invariant under multiplication by a positive scalar, this set, which from now on we denote by Σ_M, is clearly a (convex) cone; we hence often refer to it as the *Metzler sparse matrix cone.*

Definition 2.1 *A Metzler sparse matrix cone Σ_M is called* sparse stable *if it contains at least one Hurwitz matrix, and is called* absolutely stable *if all matrices inside Σ_M are Hurwitz.*

Clearly, absolute stability implies sparse stability.

3 Stability of Metzler Sparse Matrix Cones

We start this section by noting that the problem of studying the sparse stability of Metzler sparse matrix cones is trivial, since, according to Proposition 1.2, in order for a Metzler sparse matrix cone to be sparse stable, all the diagonal entries must be free parameters, and hence a Hurwitz choice always exists, see [1]. On the other hand, and as we detail below, the problem of studying absolute stability of Metzler sparse matrix cones is more involved. Our main objective in this paper is to consider scenarios where, additionally, some of the free parameters are restricted to be bounded from below.

3.1 Bounded Metzler Sparse Matrix Subsets

Before stating our definitions and results for the bounded scenario, we state a theorem, adapted from [7] to match the notion of absolute stability, which provides necessary and sufficient conditions for absolute stability when the free parameters are not bounded.

Theorem 3.1 (Necessary and sufficient conditions for absolute stability): *Let Σ_M be a Metzler sparse matrix cone and let $A \in \Sigma_M$. Then the following statements are equivalent:*

a) Σ_M is absolutely stable.
b) The matrix $\mathrm{sgn}(A)$ is Hurwitz.
c) There exists $v \in \mathbb{R}^n_{>0}$ such that $v^T \mathrm{sgn}(A) < 0$.
d) The digraph corresponding to the matrix A is acyclic (with the exception of self-loops) and all diagonal

entries of A are negative.

Note that this result provides two simple tests for determining the stability of an unbounded Metzler sparse matrix cone, one by checking the eigenvalues of the matrix $\mathrm{sgn}(A)$ using (b), and another a graph-theoretic test using (d). We are now in a position to state the classes into which we divide bounded subsets of Metzler sparse matrix cones.

Definition 3.2 *We say a subset S of a Metzler sparse matrix cone Σ_M has:*

- uniform bounds *if for all $A \in S$, there exists $x \in \mathbb{R}_{\geq 0}$ such that*

$$a_{ij} \begin{cases} \geq -x & \text{if } i = j \text{ and } \Sigma_{M_{ij}} = \star \\ \geq x & \text{if } i \neq j \text{ and } \Sigma_{M_{ij}} = \star \\ = 0 & \text{if } \Sigma_{M_{ij}} = 0 \end{cases}$$

  We denote the subset S by (Σ_M, x).
- quasi non-uniform bounds *if for all $A \in S$, there exist $x, y \in \mathbb{R}_{\geq 0}$ such that*

$$a_{ij} \begin{cases} \geq -x & \text{if } i = j \text{ and } \Sigma_{M_{ij}} = \star \\ \geq y & \text{if } i \neq j \text{ and } \Sigma_{M_{ij}} = \star \\ = 0 & \text{if } \Sigma_{M_{ij}} = 0 \end{cases}$$

  We denote the subset S by (Σ_M, x, y).
- non-uniform bounds *if for all $A \in S$, there exist $x_{ij} \in \mathbb{R}_{\geq 0}$, $(i,j) \in E$, where $\mathcal{G}_{\Sigma_M} = (\{1, \ldots, n\}, E)$, such that*

$$a_{ij} \begin{cases} \geq -x_{ij} & \text{if } i = j \text{ and } \Sigma_{M_{ij}} = \star \\ \geq x_{ij} & \text{if } i \neq j \text{ and } \Sigma_{M_{ij}} = \star \\ = 0 & \text{if } \Sigma_{M_{ij}} = 0 \end{cases}$$

  We denote the subset S by $(\Sigma_M, \{x_{ij}\})$.

We also need the following definition.

Definition 3.3 *Let S be a subset of a Metzler sparse matrix cone Σ_M, with non-uniform bounds given by the set $\{x_{ij} \in \mathbb{R}_{\geq 0}\}$, $(i,j) \in E$, where $\mathcal{G}_{\Sigma_M} = (\{1, \ldots, n\}, E)$. We then define the matrix $\min(S)$ as:*

$$\min(S)_{ij} = \begin{cases} -x_{ij} & i = j \\ x_{ij} & i \neq j \text{ and } \Sigma_{M_{ij}} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

In other words, $\min(S)$ corresponds to the matrix in S with entries taking their minimum value. The following result, which appears in [8] without explicitly stating the proof, provides information on the spectrum of Metzler matrices based on component-wise comparison of two matrices. Since this result plays a key role in some of the upcoming results, we have provided a complete proof in the Appendix A.

Theorem 3.4 (Ordering of the real parts of the eigenvalues of a Metzler matrix): *Let $M_{\min}, M, M_{\max}$ be Metzler matrices. Then if $M_{\min} \leq M \leq M_{\max}$, $\alpha(M_{\min}) \leq \alpha(M) \leq \alpha(M_{\max})$.*

As our first result, we prove that for a subset of a Metzler sparse matrix cone which is uniformly bounded, absolute stability and sparse stability are equivalent.

Proposition 3.5 (Equivalence of stability concepts in uniformly bounded scenarios): *Let Σ_M be a Metzler sparse matrix cone and $x \in \mathbb{R}_{\geq 0}$ and suppose that (Σ_M, x) is sparse stable. Then both Σ_M and (Σ_M, x) are absolutely stable.*

PROOF. We start by proving the first statement. Suppose that Σ_M is not absolutely stable. Then, using Theorem 3.1, the matrix $\mathrm{sgn}(M)$ is not Hurwitz for all matrices M in Σ_M. Thus the matrix $x \times \mathrm{sgn}(M)$ is not Hurwitz. By definition, all matrices in (Σ_M, x) satisfy $M \geq x \times \mathrm{sgn}(M)$. Then, by Theorem 3.4, no matrix in (Σ_M, x) can be Hurwitz, meaning (Σ_M, x) is not sparse stable, a contradiction. This proves that Σ_M is absolutely stable. Now, since $(\Sigma_M, x) \subset \Sigma_M$, (Σ_M, x) is also absolutely stable. □

Next, we show that for a subset of a Metzler sparse matrix cone that is quasi non-uniformly bounded, absolute stability and sparse stability are not equivalent.

Proposition 3.6 (Uniqueness of stability concepts in quasi non-uniformly bounded scenarios): *Let Σ_M be a Metzler sparse matrix cone and let $x, y \in \mathbb{R}_{\geq 0}$. Suppose that the digraph corresponding to Σ_M has one or more cycles that are not self-loops. Then, if $y < x$, there exists at least one unstable matrix in (Σ_M, x, y). Moreover, if $y > x$ then (Σ_M, x, y) contains no stable matrix.*

PROOF. Suppose that $y < x$. Let $M \in (\Sigma_M, x, y)$. Since the digraph associated with Σ_M is not acyclic, by Theorem 3.1, the matrix $\mathrm{sgn}(M)$ is not Hurwitz (as $M \in \Sigma_M$). Since x is the lower bound on the diagonal elements of matrices in (Σ_M, x, y), we can choose all of the diagonal elements in M to equal $-x$. Now, since $y < x$ by assumption, we can let the non-zero off-diagonal entries of M all have a value of x. Then, we have $M = x \times \mathrm{sgn}(M)$. Since $\mathrm{sgn}(M)$ is not Hurwitz, M is not Hurwitz and thus (Σ_M, x, y) contains an unstable matrix. This concludes the proof for this part.

Suppose now that $y > x$. Let $M \in (\Sigma_M, x, y)$. Since $y > x$, it follows that any matrix $M \in (\Sigma_M, x, y)$ satisfies $M > (x \times \mathrm{sgn}(M))$, which we know is not Hurwitz (note that this matrix is not in (Σ_M, x, y)). Then by invoking Theorem 3.4, M is also unstable. Since the choice of M in this argument was arbitrary, the argument holds for all $M \in (\Sigma_M, x, y)$ and we conclude that (Σ_M, x, y) contains no stable matrices. □

Theorem 3.7 (Necessary and sufficient condition for sparse stability in quasi non-uniformly

bounded scenarios): *Let Σ_M be a Metzler sparse matrix cone and let $x, y \in \mathbb{R}_{\geq 0}$. Then (Σ_M, x, y) is sparse stable if and only if the matrix $\min((\Sigma_M, x, y))$ is Hurwitz.*

PROOF. It is clear that if $\min((\Sigma_M, x, y))$ is Hurwitz, we have that (Σ_M, x, y) is sparse stable. To prove the other direction, suppose that (Σ_M, x, y) is sparse stable and, by contradiction, let us assume that $\min((\Sigma_M, x, y))$ is not Hurwitz.

We then have that

$$\alpha(\min((\Sigma_M, x, y))) \geq 0.$$

By the definition of $\min((\Sigma_M, x, y))$, any matrix $M \in (\Sigma_M, x, y)$ satisfies $M \geq \min((\Sigma_M, x, y))$. Then, by Theorem 3.4, $\alpha(M) \geq \alpha(\min((\Sigma_M, x, y))) \geq 0$. Thus (Σ_M, x, y) is not sparse stable. This completes the proof. $\square$

We now generalize the prior results to scenarios with non-uniform bounds. We again are only interested in cases where the digraph corresponding to the Metzler sparse matrix cone is cyclic, as otherwise Theorem 3.1 guarantees stability. We show that both Proposition 3.6 and Theorem 3.7 generalize into the non-uniform case.

Proposition 3.8 (Uniqueness of stability concepts in non-uniformly bounded scenarios): *Let Σ_M be a Metzler sparse matrix cone and consider the set $\{x_{ij} \in \mathbb{R}_{\geq 0}\}$, $(i, j) \in E$, where $\mathcal{G}_{\Sigma_M} = (\{1, \ldots, n\}, E)$. Suppose that the digraph associated with Σ_M has one or more cycles of size 2 or greater. Then $(\Sigma_M, \{x_{ij}\})$ contains at least one unstable matrix.*

PROOF. Let us define $x^* = \min\{x_{ij} | i = j\}$ and $y^* = \max\{x_{ij} | i \neq j\}$, and consider the Metzler sparse matrix subset (Σ_M, x^*, y^*). Since by construction $(\Sigma_M, x^*, y^*) \subset (\Sigma_M, \{x_{ij}\})$, the result follows from Proposition 3.6 as (Σ_M, x^*, y^*) contains one or more unstable matrices. $\square$

We now give a necessary and sufficient condition for the stability of a non-uniformly bounded Metzler sparse matrix subset; the proof is identical to that of Theorem 3.7 and is omitted.

Theorem 3.9 (Necessary and sufficient condition for sparse stability in non-uniformly bounded scenarios): *Let Σ_M be a Metzler sparse matrix cone and let $\{x_{ij} \in \mathbb{R}_{\geq 0}\}$, $(i, j) \in E$, where $\mathcal{G}_{\Sigma_M} = (\{1, \ldots, n\}, E)$. Then $(\Sigma_M, \{x_{ij}\})$ is sparse stable if and only if the matrix $\min((\Sigma_M, \{x_{ij}\}))$ is Hurwitz.*

We finish by providing two examples that illustrate our results. In our first example the conditions of Theorem 3.7 are satisfied, whereas in our second example they are not. For simplicity, we use quasi non-uniform bounds for the Metzler sparse matrix subsets. We ensure that the examples are not trivial in the sense that the underlying Metzler sparse matrix cone is absolutely stable, by Theorem 3.1, or is sparse stable by results in [1]; in particular, we use examples with cyclic digraphs and choose subsets which do not contain diagonally dominant matrices.

Example 3.10 An example of a Metzler sparse matrix subset that is sparse stable is $(\Sigma_{MS}, 1, 0.5)$, where Σ_{MS} is given in Figure 2.

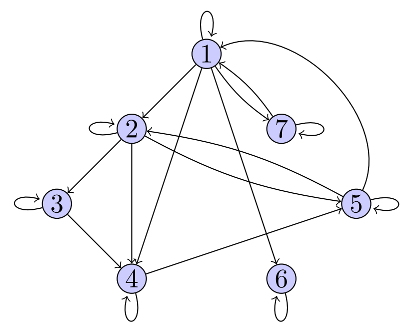

Fig. 2. The digraph representation of Σ_{MS}.

Recall that the matrix $\min((\Sigma_{MS}, 1, 0.5))$ is the matrix in $(\Sigma_{MS}, 1, 0.5)$ that has -1 on the diagonals and its non-zero off diagonal entries are 0.5. We have that $\mathrm{spec}(\min((\Sigma_{MS}, 1, 0.5))) = \{-0.0323, -1.4739 \pm 0.3112i, -0.7206, -1.1496 \pm 0.3968i, -1\}$, and hence $(\Sigma_{MS}, 1, 0.5)$ is sparse stable, which is in line with Theorem 3.7.

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Example 3.11 We consider the following Metzler sparse matrix cone, Σ_{MU} for an example which admits a non-trivial unstable Metzler sparse matrix subset, $(\Sigma_{MU}, 1, 0.5)$, where Σ_{MU} is given in Figure 3.

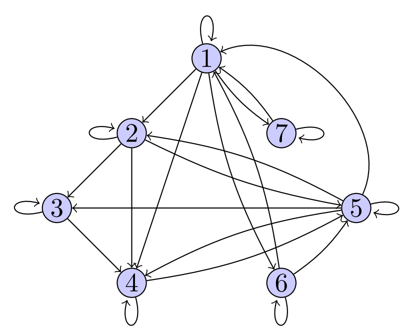

Fig. 3. The digraph representation of Σ_{MU}.

We have $\mathrm{spec}(\min((\Sigma_{MU}, 1, 0.5))) = \{0.1695, -1.6302 \pm 0.4019i, -0.53, -1.1896 \pm 0.02565i, -1\}$. This matrix is not Hurwitz, so by Theorem 3.7 $(\Sigma_{MU}, 1, 0.5)$ is not sparse stable.

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We finish by pointing out that investigating a graph-theoretic characterization of our necessary and sufficient conditions for the sparse stability of a Metzler sparse matrix subset is an interesting avenue for future work.

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A Appendix

We provide a proof for Theorem 3.4. Before that, we need the following two preliminary results, which we recall from [9] and [10], respectively.

Proposition A.1 (Perron-Frobenius Theorem [9]): *For a real matrix $A \geq 0$, $\exists \lambda_{\max} \in \mathbb{R}_{\geq 0}$ such that $\lambda_{\max} = \rho(A)$.*

Proposition A.2 *Let A be a nonnegative matrix and C be a complex matrix such that $|C| \leq A$ (component-wise), with the component-wise absolute value norm on C. Then for all $s \in \lambda(C)$, $|s| \leq \rho(A)$.*

Note that, since $|C|$ is a nonnegative real-valued matrix, the latter result can be extended in the following manner. For any nonnegative matrix $B \leq A$, $\forall b \in \lambda(B)$, $|b| \leq \rho(A)$. Since $\rho(B) \in \lambda(B)$ by Proposition A.1, this yields that, for all $B \leq A$, $\rho(B) \leq \rho(A)$. We are now ready to state the proof of Theorem 3.4.

PROOF. [Theorem 3.4]. For any Metzler matrix, M_0, there exists a positive real number ζ, such that $\zeta I + M_0 = A \geq 0$. Then, for $s \in \lambda(A)$, the value $s - \zeta \in \lambda(M_0)$. More specifically, since $\rho(A) \in \lambda(A)$ by Proposition A.1, we have $\rho(A) - \zeta \in \lambda(M_0)$. Then, since for a nonnegative matrix $\rho(A) = \alpha(A)$ (this follows from Proposition A.1), we have that $\rho(A) - \zeta = \alpha(M_0)$.

Finally, consider Metzler matrices $M_{\min}$, M and $M_{\max}$. Let $\zeta = \max\{\max |m_{\min_{ii}}|,\ \max |m_{ii}|,\ \max |m_{\max_{ii}}|\}$. Then the matrices $N_{\min} = M_{\min} + \zeta I$, $N = M + \zeta I$, and $N_{\max} = M_{\max} + \zeta I$ are all nonnegative. Additionally, we still have that $N_{\min} \leq N \leq N_{\max}$. Then, by Proposition A.2, we have that $\rho(N_{\min}) \leq \rho(N) \leq \rho(N_{\max})$. Finally, from the result just discussed, we have:

$$\alpha(M_{\min}) = \rho(N_{\min}) - \zeta,$$
$$\alpha(M) = \rho(N) - \zeta,$$
$$\alpha(M_{\max}) = \rho(N_{\max}) - \zeta$$

This then gives $\alpha(M_{\min}) \leq \alpha(M) \leq \alpha(M_{\max})$, thus completing the proof. $\square$