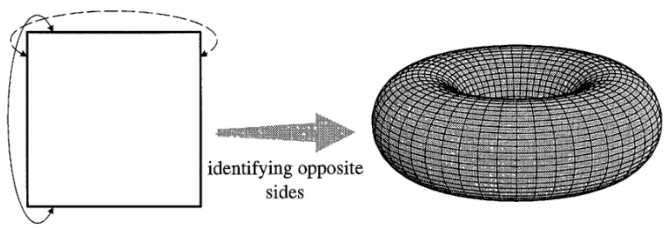
Problem Set 2

General Relativity and Cosmology, Spring 2023, Class No.: PHYS5002P

1. Just because a manifold is topologically nontrivial doesn't necessarily mean it can't be covered with a single chart. In contrast to the circle S^1, show that the infinite cylinder $\mathbb{R} \times S^1$ can be covered with just one chart by explicitly constructing the map.

2. 1) Find an atlas for a circle S^1.
2) Find an atlas for a torus T^2 (a square with opposite sides identified; see the figure below).



3. Prolate spheroidal coordinates can be used to simplify the Kepler problem in celestial mechanics. They are related to the usual Cartesian coordinates (x, y, z) of Euclidean three-space by

$$\begin{aligned} x &= \sinh\chi \sin\theta \cos\phi \\ y &= \sinh\chi \sin\theta \sin\phi \\ z &= \cosh\chi \cos\theta \end{aligned} \tag{1}$$

Restrict your attention to the plane $y = 0$ and answer the following questions:
1) What is the coordinate transformation matrix $\partial x^i/\partial \tilde{x}^j$ relating (x, z) to (χ, θ)?
2) What does the line element $\mathrm{d}s^2$ look like in prolate spheroidal coordinates?
3) A tensor rank (0,2) T in the (x, y, z) coordinates has components

$$T_{ij} = \begin{pmatrix} x^2 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & z^2 \end{pmatrix}, \tag{2}$$

What are its (χ, θ) components in the (χ, θ, ϕ) coordinates in the y plane?

4. Prove the following equations:

$$T_{[\mu_1\mu_2\ldots\mu_{n-1}\mu_n]} = \frac{1}{n}\left(T_{\mu_1[\mu_2\ldots\mu_{n-1}\mu_n]} - T_{\mu_2[\mu_1\ldots\mu_{n-1}\mu_n]} - \ldots - T_{\mu_{n-1}[\mu_2\ldots\mu_1\mu_n]} - T_{\mu_n[\mu_2\ldots\mu_{n-1}\mu_1]}\right)$$

$$S_{(\mu_1\mu_2\ldots\mu_{n-1}\mu_n)} = \frac{1}{n}\left(S_{\mu_1(\mu_2\ldots\mu_{n-1}\mu_n)} + S_{\mu_2(\mu_1\ldots\mu_{n-1}\mu_n)} + \ldots + S_{\mu_{n-1}(\mu_2\ldots\mu_1\mu_n)} + S_{\mu_n(\mu_2\ldots\mu_{n-1}\mu_1)}\right)$$

5. Properties of Lie derivatives:
1) Show that $\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X = \mathcal{L}_{[X,Y]}$ is valid for a scalar, a vector and a dual vector. (It is actually valid for a general tensor.)
2) Show that a Lie derivative commutes with the operation of contraction.
3) Show that a Lie derivative commutes with an exterior derivative, ie, for a p-form Ω show that $\mathcal{L}_X(\mathrm{d}\Omega) = \mathrm{d}\,(\mathcal{L}_X\Omega)$.

6. 1) For a p-form ω and a q-form η, show that $\mathrm{d}(\omega\wedge\eta) = (\mathrm{d}\omega)\wedge\eta + (-1)^p\omega\wedge(\mathrm{d}\eta)$.
2) For a p-form in n-D, show that $(-1)^{s+p(n-p)} **$ is an identity operator.
3) For p-forms ω, η, show that $\omega\wedge *\eta = \eta\wedge *\omega$.

7. In Euclidean three-space, suppose $*F = q\sin\theta\ \mathrm{d}\theta\wedge\mathrm{d}\phi$.
1) Evaluate $\mathrm{d}*F = *J$.
2) What is the 2-form F equal to?
3) What are the electric and magnetic fields equal to for this solution?
4) Evaluate $\int_V \mathrm{d}*F$, where V is a ball of radius R in Euclidean three space.

8. M, N and Q are three manifolds, $\phi: M\to N$ and $\psi: N\to Q$ are two smooth mappings.
1) Show that $(\psi\circ\phi)^*f = (\phi^*\circ\psi^*)f$, f is a function in Q.
2) Show that $(\psi\circ\phi)_*V = \psi_*(\phi_*V)$, where V is a vector in M.
3) Show that $(\psi\circ\phi)^* = \phi^*\circ\psi^*$ for any p-form in Q.

9. Consider a magnetic monopole at the origin of the coordinate system and an S^2 centered at the magnetic monopole with radius r. Denoting U_+ and U_- as the region of S^2 without the south and north pole respectively, the vector potential 1-form of a magnetic monopole in the two regions are given by

$$A^{\pm} = \frac{q}{r}\frac{1}{z\pm r}(x\mathrm{d}y - y\mathrm{d}x), \qquad r = \sqrt{x^2+y^2+z^2} \tag{3}$$

1) Find the field strength F.
2) Is $A^+ - A^-$ exact? Can you find an A such that $F = \mathrm{d}A$ covering the whole S^2?
3) Find $A^{\pm}$ and F in spherical coordinates (r,θ,ϕ).
4) Find $A^+ - A^-$ in spherical coordinates and evaluate it on the equator. What values can charge q have? (Hint: on the equator a gauge transformation must be an integer times ϕ in order for the gauge transformations be single valued).