Topic 9: Continuity Equation

Preface

The continuity equation is the last piece of theory necessary before we can dive into the fun devices like diodes, BJTs, and MOSFETs. It is an incredibly fundamental equation which describes how current passes through a piece of semiconductor.

Background

Imagine we have a piece of semiconductor with some length, L, and cross-sectional area, A. We want to zoom into a small slice of the semiconductor and see if we can make some observations. The length of this small slice is dx, implying the overall volume of the slice is Adx.

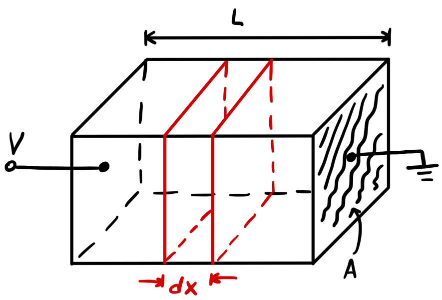

Figure 1: Semiconductor setup visualization

The goal is to derive an expression for the overall rate of change in the number of electrons within a slice; $\left(\frac{\#\text{ of e}^-}{s}\right)$. The main quantity we know is the doping concentration and the physical dimensions of the semiconductor. If the concentration varies with time the number of carriers within the slice will also vary with time. Starting with a change in concentration with time, $\frac{dn}{dt}$, we can try and use physical parameters express this rate in terms of known quantities. The square brackets below denote units of the enclosed quantity.

$$\left[\frac{dn}{dt}\right] = \frac{\#\text{ of e}^-}{\text{cm}^3 - s} \Rightarrow \left[\frac{dn}{dt}Adx\right] = \frac{\#\text{ of e}^-}{s} \tag{1}$$

If we change our perspective to two dimensions, we can see that the semiconductor is no different from what we have already studied. In the most general form, current can either enter or leave the slice. Current in the slice is also composed of carrier generation and recombination since they vary the number of carriers within a slice. Therefore, we have four terms which account for varying charge within a slice of semiconductor.

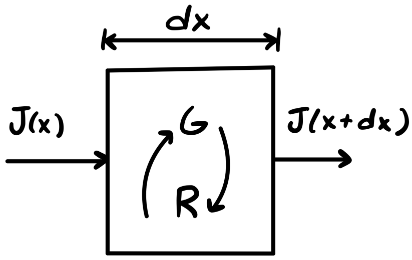

Figure 2: Red cross section of semiconductor from Fig. 1.

Note that the J terms, the current density, in Fig. 2 do not explicitly state drift or diffusion. We want to keep the assumption of what composes J to be as general as possible. The four components composing the current within the slice are:

1. Current density entering the slice, $J(x)$.
2. Current density leaving the device, $J(x+dx)$.
3. Charge increase due to generation, G.
4. Charge reduction due to recombination, R.

We cannot just add up each of the four terms above because the units are different. Luckily, we can convert them each to units of $\left(\frac{\#\text{ of e}^-}{s}\right)$.

$$[J(x)] = \frac{C}{\text{cm}^2 - s} \Rightarrow \left[\frac{J(x)A}{q}\right] = \frac{\#\text{ of e}^-}{s} \tag{2}$$

$$[G] = [R] = \frac{\#\text{ of e}^-}{\text{cm}^3\text{-s}} \Rightarrow [GAdx] = [RAdx] = \frac{\#\text{ of e}^-}{s} \tag{3}$$

The unit of amps in the current density was simplified to coulombs per second since A also represents cross sectional area in our notation. Converting from Coulombs to number of electrons involves dividing by the fundamental charge, q. With each of the four terms having identical units we can combine them into one expression. However, we need to account for the polarity of the carrier charge when dividing by q.

$$\frac{\partial n}{\partial t}Adx = \frac{J_n(x)A}{-q} - \frac{J_n(x+dx)A}{-q} + (G_n - R_n)\,Adx \tag{4}$$

$$\frac{\partial p}{\partial t}Adx = \frac{J_p(x)A}{q} - \frac{J_p(x+dx)A}{q} + (G_p - R_p)\,Adx \tag{5}$$

As mentioned, the minus q is needed for electron current densities. The negative on the $J(x+dx)$ term is due to current leaving the slice, which is independent of what carrier is present. The $(G-R)$ term is needed because generation and recombination constantly fight against one another to increase and decrease the number of carriers respectively.

Return of the Taylor Series:

Hopefully, the $J(x)$ and $J(x+dx)$ terms reminds you of diffusion and our use of Taylor series to expand them. Using a first order approximation for $J(x+dx)$ we have:

$$J(x+dx) = J(x) + \frac{J(x)}{dx}dx$$

Plugging in the Taylor series expansion into equations 4 and 5 cancels out the $J(x)$ term of each equation.

$$\frac{\partial n}{\partial t}Adx = \frac{dJ_n}{dx}\frac{Adx}{q} + (G_n - R_n)\,Adx \tag{6}$$

$$\frac{\partial p}{\partial t}Adx = -\frac{dJ_p}{dx}\frac{Adx}{q} + (G_p - R_p)\,Adx \tag{7}$$

Each of the terms in Eq. 6 and 7 have an Adx term, canceling them results in,

$$\frac{\partial n}{\partial t} = \frac{1}{q}\frac{dJ_n}{dx} + (G_n - R_n) \tag{8}$$

$$\frac{\partial p}{\partial t} = -\frac{1}{q}\frac{dJ_p}{dx} + (G_p - R_p) \tag{9}$$

Eq. 8 and 9 represent the most general form of the continuity equation. No assumptions have been made about the components of J_n and J_p, but we can simplify the equation by substituting the drift and diffusion current expressions. If we were to differentiate the following drift and diffusion currents we get,

$$J_n = qn\mu_n\mathscr{E} + qD_n\frac{dn}{dx} \tag{10}$$

$$J_p = qn\mu_p\mathscr{E} - qD_p\frac{dp}{dx} \tag{11}$$

$$\frac{dJ_n}{dx} = q\mu_n(n\frac{d\mathscr{E}}{dx} + \frac{dn}{dx}\mathscr{E}) + qD_n\frac{d^2n}{dx^2} \tag{12}$$

$$\frac{dJ_p}{dx} = q\mu_p(p\frac{d\mathscr{E}}{dx} + \frac{dp}{dx}\mathscr{E}) - qD_p\frac{d^2p}{dx^2} \tag{13}$$

The product rule was necessary to reach equations 12 and 13. We are assuming that both the concentration and the electric field vary with position to keep the expression as general as possible. Charge, mobility, and diffusion constants are each constants which factor out of the differentiation. Lastly, the diffusion current term becomes a second derivative. Plugging Eq. 12 and 13 into Eq. 8 and 9 we arrive at the final continuity equation.

$$\frac{\partial n}{\partial t} = \frac{1}{q}\left(q\mu_n(n\frac{d\mathscr{E}}{dx} + \frac{dn}{dx}\mathscr{E}) + qD_n\frac{d^2n}{dx^2}\right) + (G_n - R_n)$$

$$\Rightarrow \frac{\partial n}{\partial t} = \mu_n(n\frac{d\mathscr{E}}{dx} + \frac{dn}{dx}\mathscr{E}) + D_n\frac{d^2n}{dx^2} + (G_n - R_n) \tag{14}$$

$$\frac{\partial p}{\partial t} = -\frac{1}{q}\left(q\mu_p(p\frac{d\mathscr{E}}{dx} + \frac{dp}{dx}\mathscr{E}) - qD_p\frac{d^2p}{dx^2}\right) + (G_p - R_p)$$

$$\Rightarrow \frac{\partial p}{\partial t} = -\mu_p(p\frac{d\mathscr{E}}{dx} + \frac{dp}{dx}\mathscr{E}) + D_p\frac{d^2p}{dx^2} + (G_p - R_p) \tag{15}$$

Equations 14 and 15 are quite lengthy, and with practice you will gradually memorize some of components. Please do not make it your mission to memorize the equation expecting to plug in numbers, it is a waste of time. The better approach is to rely on equations 8 and 9 and plug in the expressions for the current density and simplify each component. Depending on the problem, there are many assumptions which can be made. For example, a semiconductor with uniform doping does not experience any diffusion; $\frac{dp}{dx} = 0$ and $\frac{dn}{dx} = 0$. Each of the terms with derivatives on concentration become zero. Likewise, a semiconductor experiencing zero drift results in bot $\mathscr{E} = 0$ and $\frac{d\mathscr{E}}{dx} = 0$, so both of the terms drop.

Conclusion

This section of notes has been kept short to emphasize the importance of the continuity equation. We have derived a partial differential equation describing how carrier concentration varies over time due to drift, diffusion, generation, and recombination. The next set of notes will focus on the PN junction, and the continuity equation will be pivotal in developing expressions for the junction's behavior. I encourage you to experiment with equations 14 and 15 to see how each equation simplifies without one of the mentioned transport mechanisms.