Introduction to image processing

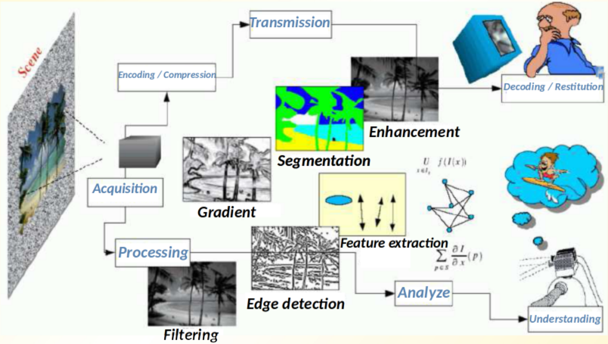
Youssef El Rhabi

Centrale
CASABLANCA

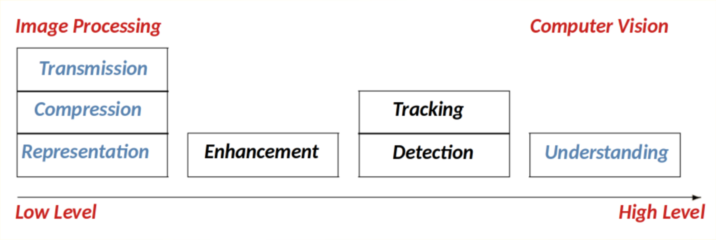
Plan

1. Introduction
2. Linear and non linear filtering
3. Image restoration

A Global Vision



From Image Processing to Computer Vision



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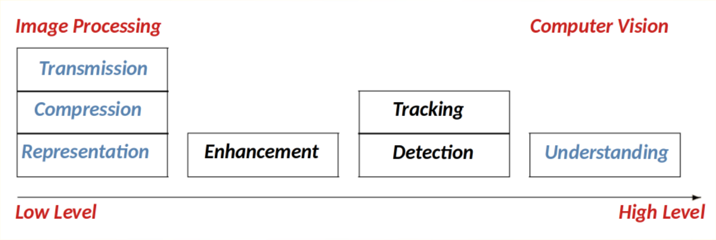

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  - The futur is multimedia : images are everywhere !

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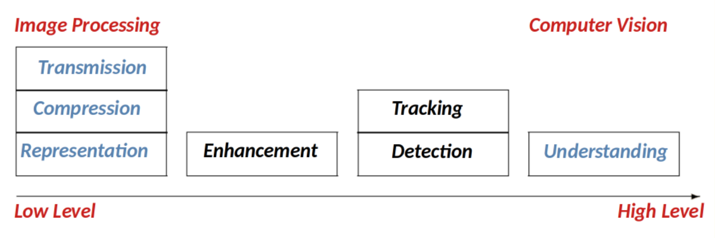

- Why the image processing ?
  - The futur is multimedia : images are everywhere !
  - Possibilities for application are numerous and varied.

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- Security : face detection and identification, fingerprint recognition, watermarking, data hiding
- Entertainment : HD/FHD (4K/1080P), high-quality images, compression (standards JPEG, JPEG 2000, MPEG4 . . .)

Some Examples of image processing

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- Recognition / Understanding : content-based image recognition

Real image : a mathematical definition

Definition

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- An image is an analog plane where the luminance intensities are reals

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  - **Quantization of the luminances** : discretization of the real luminance intensities

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Example

- $m = 1$: 2 possible values (**binary** images)
- $m = 8$: 256 possible values possibles (**grayscale** images)
- $m = 16$: 65535 valeurs possibles (**colors** images)

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- JPEG2000 : an optimal method of compression.

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  - Common convention : black $= 0$, white $= 255$

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- Noise : random variation of brightness or color information in images, and is usually an aspect of electronic noise. Its distribution is generally unknown.
- Geometrical distortions : defects due to the axis difference between the acquisition sensor and the center of the observed scene.

Content

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- Object : region (group of region) bounded by a contour.

Read and display an Image

Code

```
% Load an Image with Octave :
close all, clear, clc
X = imread("lena.jpg");
% The image is loaded in the variable X
% with other images? :
% X = imread('name of your image.entension');

%Visualization : displaying Images in grayscale
imshow(X);
colormap gray;

% Displaying a second figure :
figure(2);
colormap gray;
XX=imread("cameraman.png");
imshow(XX);
```

Function noise - Adding a noise to an image

```
% gaussian noise (standard deviation s) of an image I

function out = gaussian_noise(I,s)

[m,n]=size(I);
J=zeros(m,n);
% create the gaussian noise
J=s*randn(m,n);

% adding the gaussian noise to the image I

out = I+J;
```

An image and a noisy image

Display the image and the noisy image

```
% Adding an artificial noise to an image :
close all , clear , clc
X = double(imread("lena.jpg"));

% the image is loaded in the variable X
% here we prefer the double precision

%Visualization :
imagesc(X);
colormap gray;
% Displaying the image in grayscale

% adding the noise with the function noise
% Try with s = 5, 15, 25 and 30
XX=noise (X,s);

% Displaying the image in grayscale
figure(2);
colormap gray;
imagesc(XX);
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An image and its boundary

Some frameworks to restore a noisy image

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- We will use the variations of the image to restore it $\Longrightarrow$ We need some assumption on the boudaries of Ω, namely :

$$\frac{\partial I}{\partial \vec{n}} = \langle \nabla I, \vec{n} \rangle = 0$$

  where $\vec{n}$ is the normal vector to the boundaries of I $\partial\Omega$ and $\langle ., . \rangle$ is the (euclidian) dot product (inner product),

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- These conditions are the Neumann boundary conditions.

Neumann Conditions

Some details and example

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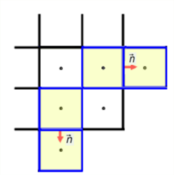
- Any image I could not verify the neumann boundary condition.
- That's why we pose $\widetilde{I}$ an extension image of I constructs such that $\widetilde{I}$ verifies the Neumann boudary conditions, namely :
  - The size of $\widetilde{I}$ is $(N+p)\times(M+r)$, where p is the extension of I among the $x-$axis and r the extension of I among the $x-$axis,
  - p and r depend on the problem to be solved,
  - else the values of $\widetilde{I}$ are obtained by "reflection symmetry" (see below for a simple example) :



At $x=N$ and $y=2$: $\dfrac{\partial \widetilde{I}}{\partial \vec{n}}=\dfrac{\partial \widetilde{I}}{\partial x}=\dfrac{\widetilde{I}_{N+1,2}-\widetilde{I}_{N,2}}{\Delta x}=0$

$$\Longrightarrow \boxed{\widetilde{I}_{N+1,2}=\widetilde{I}_{N,2}=I_{N,2}}.$$

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  - p and r depend on the problem to be solved,
  - on $[1,N]\times[1,M]$, $\widetilde{I}=I$,
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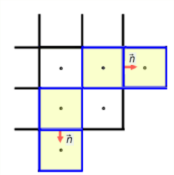

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Function boundary - extension of an image - Neumann conditions

```
function B=boundary(A,d)
%Extension of an A by reflection symmetry (for d pixels)
% Neumann Conditions
%syntaxe : bord(A,d)
[m,n]=size(A);
% Create the matrice B with the right size
M=m+2*d;
N=n+2*d;
B=zeros(M,N);
B(d+1:M-d,d+1:N-d)=A;
% Extension by a reflection symmetry
for i=1:m
    for j=1:d B(i+d,j)=A(i,d-j+1);end;
    for j=N-d+1:N B(i+d,j)=A(i,n+N-j-d);end;
end;
for j=1:N
    for i=1:d B(i,j)=B(2*d-i+1,j);end;
    for i=M-d+1:M B(i,j)=B(2*M-i-2*d,j);end;
end;
```

Display the image and the extended image with the Neumann conditions

```
close all ,clear ,clc
X = double(imread("lena.jpg"));

% the image is loaded in the variable X
% here we prefer the double precision

%Visualization :
imagesc(X);
colormap gray;
% Displaying the image in grayscale

% Extend the image with s pixel
% Try with s = 5, 15, 25 and 30

XX=boundary(X,s);
% Displaying the image in grayscale
figure(2);
colormap gray;
imagesc(XX);
```

Reminder

First order's derivatives

$f : D \subset \mathbb{R}^{\mathbf{2}} \to \mathbb{R}$ a $\mathscr{C}^1(D)$ function then for all $(x,y) \in D$, we have :

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \text{ and } \frac{\partial f(x,y)}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

In other words :

$$\frac{\partial f(x,y)}{\partial x} \simeq \frac{f(x+h,y) - f(x,y)}{h} \text{ and } \frac{\partial f(x,y)}{\partial y} \simeq \frac{f(x,y+h) - f(x,y)}{h}.$$

For an image I, $h = 1$ (we can not see under 1 pixel) :

$$\frac{\partial I(x,y)}{\partial x} \simeq I(x+1,y) - I(x,y) \text{ and } \frac{\partial I(x,y)}{\partial y} \simeq I(x,y+1) - I(x,y).$$

Reminder

The forward difference

Now in its discrete form, if $(x, y) = (i, j)$ with $I(i, j) = I_{i,j}$, we can rewrite the previous result as the following :

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$$\frac{\partial I_{i,j}}{\partial x} \simeq I_{i,j} - I_{i-1,j} \text{ and } \frac{\partial I_{i,j}}{\partial y} \simeq I_{i,j} - I_{i,j-1}.$$

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$$\frac{\partial I_{i,j}}{\partial x} \simeq I_{i,j} - I_{i-1,j} \text{ and } \frac{\partial I_{i,j}}{\partial y} \simeq I_{i,j} - I_{i,j-1}.$$

The central difference

$$\frac{\partial I_{i,j}}{\partial x} \simeq \frac{I_{i+1,j} - I_{i-1,j}}{2} \text{ and } \frac{\partial I_{i,j}}{\partial y} \simeq \frac{I_{i,j+1} - I_{i,j-1}}{2}.$$

First order's image Gradient computation & Octave

The central difference

```
function [I_x,I_y] = compute_derivatives(I)

    [ny,nx,s]=size(I);
    % estimate first derivatives - center derivative scheme
    I_x = (I(:,[2:nx nx],:)-I(:,[1 1:nx-1],:))/2;
    I_y = (I([2:ny ny],:,:)-I([1 1:ny-1],:,:))/2;
```

First order's image Gradient computation & Octave

Display the image and its first derivatives

```
% Adding an artificial noise to an image :
close all ,clear ,clc
X = double(imread("lena.jpg"));
% the image is loaded in the variable X in double precision
%Visualization in grayscale
imagesc(X);
colormap gray;
% Compute the derivatives of the image I
[X_x,X_y]=compute_derivatives(X);
% Displaying the x-derivative of the image in grayscale
figure(2);
colormap gray;
imagesc(X_x);
% Displaying the y-derivative of the image in grayscale
figure(3);
colormap gray;
imagesc(X_y);
```

Reminder

Second order's derivatives

$f : D \subset \mathbb{R}^{\mathbf{2}} \to \mathbb{R}$ a $\mathscr{C}^2(D)$ function then for all $(x,y) \in D$, we have :

$$\frac{\partial^2 f(x,y)}{\partial x^2} = \lim_{h\to 0} \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2},$$
$$\frac{\partial^2 f(x,y)}{\partial y^2} = \lim_{h\to 0} \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}.$$

And the cross partial derivatives :

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \lim_{h\to 0} \frac{f(x+h,y+h) - f(x+h,y-h) - f(x-h,y+h) + f(x-h,y-h)}{4h^2}.$$

Reminder

Second order's derivatives

In other words :

$$\frac{\partial^2 f(x,y)}{\partial x^2} \simeq \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2},$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} \simeq \frac{f(x,y+h) - 2f(x,y) f(x,y-h)}{h^2}.$$

And the cross partial derivatives :

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} \simeq \frac{f(x+h,y+h) - f(x+h,y-h) - f(x-h,y+h) + f(x-h,y-h)}{4h^2}.$$

Reminder

Second order's derivatives

For an image I, $h = 1$ (we can not see under 1 pixel) :

$$\frac{\partial^2 I(x,y)}{\partial x^2} \simeq I(x+1,y) - 2\,I(x,y) + I(x-1,y),$$

$$\frac{\partial^2 I(x,y)}{\partial y^2} \simeq I(x,y+1) - 2\,I(x,y) + I(x,y-1).$$

And the cross partial derivatives :

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} \simeq \frac{f(x+1,y+1) - f(x+1,y-1) - f(x-1,y+1) + f(x-1,y-1)}{4}.$$

The second order derivatives of an image

Now in its discrete form, if $(x, y) = (i, j)$ with $I(i, j) = I_{i,j}$, we can rewrite the previous result as the following :

$$\frac{\partial^2 I_{i,j}}{\partial x^2} \simeq I_{i+1,j} - 2\, I_{i,j} + I_{i,j-1},$$

$$\frac{\partial^2 I_{i,j}}{\partial y^2} \simeq I_{i,j+1} - 2\, I_{i,j} + I_{i,j-1}.$$

And the cross partial derivative :

$$\frac{\partial^2 I_{i,j}}{\partial x\, \partial y} \simeq \frac{I_{i+1,j+1} - I_{i+1,j-1} - I_{i-1,j+1} + I_{i-1,j-1}}{4}.$$

second order's image Gradient computation & Octave

Second order finite difference

```
function [I_xx,I_yy,I_xy] = compute_2derivatives(I)

    [ny,nx,s]=size(I);
    % second order's derivatives estimates
    I_xx = I(:,[2:nx nx],:)+I(:,[1 1:nx-1],:)-2*I;
    I_yy = I([2:ny ny],:,:)+I([1 1:ny-1],:,:)-2*I;
    Dp = I([2:ny ny],[2:nx nx],:)+I([1 1:ny-1],[1 1:nx-1],:);
    Dm = I([1 1:ny-1],[2:nx nx],:)+I([2:ny ny],[1 1:nx-1],:);
    I_xy = (Dp-Dm)/4;
```

Display the image and its second order's derivatives

```
% Adding an artificial noise to an image :
close all ,clear ,clc
X = double(imread("lena.jpg"));
% the image is loaded in the variable X
% here we prefer the double precision

%Visualization in grayscale
imagesc(X);
colormap gray;
%
% Compute the derivatives of the image I
[X_xx,X_yy,X_xy]=compute_2derivatives(X);

% Displaying the x-derivative of the image in grayscale
figure(2);colormap gray;imagesc(X_xx);
% Displaying the y-derivative of the image in grayscale
figure(3);colormap gray;imagesc(X_yy);
% Displaying the cross partial derivative of the image in grayscale
figure(4);colormap gray;imagesc(X_xy);
```

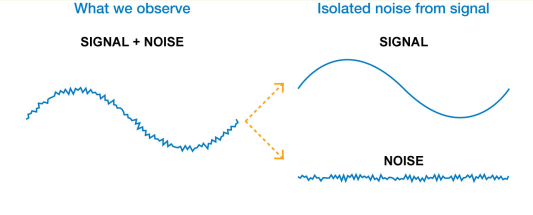
Image denoising - a model

A model

If I denotes the "ideal" image and n, the gaussian noise (assumption), it is usual to suppose that this noise is added by the sensor to the "ideal" image :

$$I^{noise} = I + n,$$

To fix our idea, let us consider an simple 1D-example with a signal :



Our objective is to extract the signal from the noise (or isolated noise from the signal). This noise "increase the variations" of the signal, a first path forward : if we could rationally "control or reduce" these variations . . . Toward regularization methods ?

Image denoising - heat equation

The heat equation

The first Partial Differential Equation (PDE) used in image processing was the heat equation. The main idea is compute the solution (the denoised image) of the heat equation where the initial condition is the observed image I^{noise} (the noisy image !) :

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} = \Delta u(x,y,t) \text{ on } \Omega\times]0,T[\\ u(x,y,0) = I^{noise}(x,y) \text{ on } \Omega \\ \dfrac{\partial u(x,y,t)}{\partial \vec{n}} = 0 \text{ on } \partial\Omega\times]0,T[. \end{cases}$$

where u is function defined on $\Omega \times [0,T]$, $\vec{n}$ the normal vector, Ω is the domain of the image I^{noise}, and T is the maximum time.

Image denoising - Heat equation - The discrete form of the laplacian

Reminder : the operator Laplacian Δu

$u : \Omega \subset \mathbb{R}^{\mathbf{2}} \to \mathbb{R}$ is a $\mathscr{C}^2(\Omega)$ function :

$$\Delta u(x,y) = \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2}$$

For an image I, if we use the discrete form of the second order derivatives of u, we have :

$$\Delta I_{i,j} = I_{i+1,j} + I_{i-1,j} + I_{i,j+1} + I_{i,j-1} - 4\, I_{i,j}$$

Discrete scheme

We recall the heat equation : $\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} = \Delta u(x,y,t) \text{ on } \Omega\times]0,T[\\ u(x,y,0) = I^{noise}(x,y) \text{ on } \Omega \\ \dfrac{\partial u(x,y,t)}{\partial \vec{n}} = 0 \text{ on } \partial\Omega\times]0,T[. \end{cases}$ To solve this PDE's, we need a discretization of $\dfrac{\partial u(x,y,t)}{\partial t}$, we begin with the definition :

$$\frac{\partial u(x,y,t)}{\partial t} = \lim_{\delta t \to 0} \frac{u(x,y,t+\delta t) - u(x,y,t)}{\delta t}$$

where $t \in]0,T[$, in other words :

$$\frac{\partial u(x,y,t)}{\partial t} \simeq \frac{u(x,y,t+\delta t) - u(x,y,t)}{\delta t}$$

Discrete scheme

For u, if we use the discrete form of the second order derivatives of u and we use the following time discretization :

$$t_0 = 0, t_1 = \delta t, t_2 = 2\delta t, \ldots, t_n = n\delta t = T,$$

with $u(x,y,t_k) = u(x,y,k\delta t) = u^k(x,y)$. Moreover if $(x,y) \in \Omega$, Ω the domain of the image I, then we have $(x,y) = (i,j)$ and $u^k(x,y) = u^k(i,j) = u^k_{i,j}$.
Now, it is obvious to obtain :

$$\frac{\partial u^k_{i,j}}{\partial t} \simeq \frac{u^{k+1}_{i,j} - u^k_{i,j}}{\delta t}$$

By using the discrete form of the laplacian :

$$\Delta u^k_{i,j} = u^k_{i+1,j} + u^k_{i-1,j} + u^k_{i,j+1} + u^k_{i,j-1} - 4\,u^k_{i,j}$$

Image denoising - Heat equation - A discrete scheme

Discrete scheme

We can rewrite the heat equation in its discrete form :

$$\frac{u_{i,j}^{k+1} - u_{i,j}^{k}}{\delta t} = \Delta u_{i,j}^{k},$$

or in this more convenient form :

$$u_{i,j}^{k+1} = u_{i,j}^{k} + \delta t \left(u_{i+1,j}^{k} + u_{i-1,j}^{k} + u_{i,j+1}^{k} + u_{i,j-1}^{k} - 4\, u_{i,j}^{k} \right).$$

with $u_{i,j}^{0} = I_{i,j}^{noise}$, $1 \leq i \leq N$, $1 \leq j \leq M$, $0 \leq k \leq n$ and the Neumann conditions at the boundary $\partial\Omega$ of Ω (the domaine of the image I^{noise}).

Exercice

1. Write the octave program of the heat equation using a noisy version of the image “lena” (lena+noise) then with the image “test”.
2. What can you conclude about this restoration ?

Image denoising - The divergence operator div

Reminder : the divergence operator div

$\mathbf{u} : \Omega \subset \mathbb{R}^{2} \to \mathbb{R}^{2}$ is a $\mathscr{C}^1(\Omega)$ function, where $\mathbf{u} = \left(u^1, u^2\right)$:

$$\mathrm{div}\left(\mathbf{u}(x,y)\right) = \frac{\partial u^1(x,y)}{\partial x} + \frac{\partial u^2(x,y)}{\partial y}$$

In mathematics, it is well-known that, we have a strong relation between the divergence div and gradient ∇ operators, namely :

$$\langle \varphi, \mathrm{div}\,\mathbf{u} \rangle = -\langle \nabla\varphi, \mathbf{u} \rangle$$

$\forall \varphi$, a $\mathscr{C}^1(\Omega)$ function with compact support and $\langle ., . \rangle$ is the dot product and its discrete version for 2 images I and J :

$$\langle I, J \rangle_d = \sum_{i,j} I_{i,j} J_{i,j}.$$

This relation is very important (but its explanation is beyond the scope of this course) that is why we would like to preserve it in the discrete case !

Image denoising - The discrete divergence operator div in octave

Reminder : the discrete divergence operator div & octave

```
function M = div(px,py)
% compute the divergence of a vector (p1,p2)
% px and py have the same size
% (satistiying div = -(grad)*)
% Syntaxe : div(px,py)
px_x = (px(:,[2:nx nx],:) - px(:,[1 1:nx-1],:))/2;
py_y = (py([2:ny ny],:,:) - py([1 1:ny-1],:,:))/2;
   M = px_x+py_y;
```

Remark : $\mathrm{div}\,(\nabla u) = \Delta u$, compare this result with the previous one (on the laplacian operator).

The Perona-Malik equation

To enhance the result of the heat equation, Perona and Malik introduced their model by modifiying the heat equation. The main idea is to introduce a edge detection process. Then with this new model, we compute the solution (the denoised image) where the initial condition is the observed image I^{noise} (the noisy image !) :

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} = \mathrm{div}\,\left(c(|\nabla u|)\nabla u(x,y,t)\right) \text{ on } \Omega\times]0,T[\\ u(x,y,0) = I^{noise}(x,y) \text{ on } \Omega \\ \dfrac{\partial u(x,y,t)}{\partial \vec{n}} = 0 \text{ on } \partial\Omega\times]0,T[. \end{cases}$$

where u is function defined on $\Omega \times [0,T]$, $\vec{n}$ the normal vector, Ω is the domain of the image I^{noise}, T is the maximum time and the function c an edge detector.

Image denoising - Perona-Malik equation - A discrete scheme

Discrete scheme

Exercice

1. As the heat equation, write the octave program of the Perona-Malik equation using a noisy version of the image “lena” (lena+noise) then with the image “test” and with the function $c(t) = \dfrac{1}{\sqrt{1+\left(\frac{t}{\alpha}\right)^2}}$ then with $c(t) = \dfrac{1}{1+\left(\frac{t}{\alpha}\right)^2}$.
2. What can you conclude about this restoration ?

Image denoising - An enhancement of the Perona-Malik equation

The alternative equation

To enhance the result of the Perona-Malik equation, we can put a smoothing on the gradient of the image in the function c. This smoothing will regularize the edge detector avoiding to consider the noise as an edge. Then with this new model, we compute the solution (the denoised image) where the initial condition is the observed image I^{noise} (the noisy image !) :

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} = \mathrm{div}\left(c(|G_\sigma * \nabla u|)\nabla u(x,y,t)\right) \text{ on } \Omega\times]0,T[\\ u(x,y,0) = I^{noise}(x,y) \text{ on } \Omega \\ \dfrac{\partial u(x,y,t)}{\partial \vec{n}} = 0 \text{ on } \partial\Omega\times]0,T[. \end{cases}$$

where u is function defined on $\Omega \times [0,T]$, $\vec{n}$ the normal vector, Ω is the domain of the image I^{noise}, T is the maximum time, c an edge detector and G_σ a gaussian filter.

Exercice (discrete scheme)

1. As for the previous equations (heat and Perona-Malik), write the octave program using a noisy version of the image "lena" (lena+noise) then with the image "test" and with the function $c(t) = \dfrac{1}{\sqrt{1+\left(\frac{t}{\alpha}\right)^2}}$ then with

$$c(t) = \frac{1}{1+\left(\frac{t}{\alpha}\right)^2}.$$

```
% Don't forget to load the octave package image processing
% for fspecial and imfilter!
pkg load image
% try for sigma =1, 3, 10, 15
sigma = 3
k=fspecial("gaussian",sigma);
% Convolution k*I where I is an image
Iconv = imfilter(I,k,'conv','symmetric')
```

2. What can you conclude about this restoration?

Image restoration-The mathematical model

- $\Omega \subset \mathbb{R}^2$: the domain of the image
- $u : \Omega \longrightarrow \mathbb{R}$: the original image.
- $u_0 : \Omega \longrightarrow \mathbb{R}$: the observed image (a degradation of u).

$$u_0 = u + n,$$

- n : is the noise (a random variable).

An approximation of u is given by solving the minimization problem

$$\min_u \frac{1}{2}\int_\Omega |u-u_0|^2 dx + \lambda \int_\Omega \varphi(|\nabla u|)dx.$$

Formally, u, the minimum, satisfies the equation

$$u - \lambda \mathrm{div}\left(\frac{\varphi'(|\nabla u|)}{|\nabla u|}\nabla u\right) = u_0.$$

Image restoration-Regularization function

Thus, the function φ could be chosen such that

- Encourage *smoothing* (isotropic diffusion) in regions of *weak variations* of u ($\nabla u \approx 0$).
- Direct the diffusion along the edges and not across them (to preserve edges).

$$\lim_{t \longrightarrow +\infty} \frac{t\varphi''(t)}{\varphi'(t)} = 0.$$

Example 1 : $\varphi(t) = \sqrt{1+t^2}$
Example 2 : $\varphi(t) = t$ (Total variation model).
(the usual function $\varphi(t) = t^2$ does not work : edges are not preserved).

Image restoration-A descent algorithm with a constant stepsize

For each u Set $w(u) = (u - u_0) - \lambda \mathrm{div}\left(\frac{\varphi'(|\nabla u|)}{|\nabla u|}\nabla u\right)$.

- *Initial data :* $n = 0$, $u^{(0)} = u_0$, α.
- *Step 1 :* if $w(u^{(n)}) = 0$, then stop.
- *Step 2 :* $d^{(n)} = w(u^{(n)})$.
- *Step 4 :* $u^{(n+1)} = u^{(n)} - \alpha d^{(n)}$.
  $n \leftarrow n + 1$. Go to Step 1.

Image restoration-A descent algorithm with a constant stepsize

Exercice (discrete scheme)

1. Write the octave program using a noisy version of the image “lena” (lena+noise) then with the image “test” and with $\varphi(t) = t^2$ then with the function $\varphi(t) = \sqrt{1+t^2}$.
2. What can you conclude about this restoration ?

Thank you for your attention!