Computational Methods in Macroeconomics
Simple Dynamic Models and Stability Conditions

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Outline

Why Study Dynamic Models?

Economic variables move at different speeds:

▶ Asset prices change rapidly (stocks, bonds, exchange rates)

▶ Consumer prices adjust more slowly

▶ Capital stocks evolve gradually over time

Key questions we'll address:

▶ How do we model variables with different adjustment speeds?

▶ When do economic systems converge to steady states?

▶ Which variables "jump" and which "crawl"?

USA Monetary Growth and Inflation

Inflation and Monetary Growth
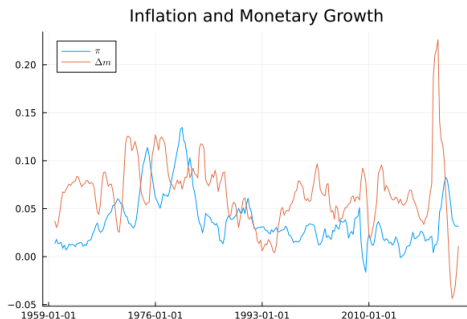
Observation: Money growth and inflation show similar long-run trends, but inflation adjusts more slowly to changes in money growth.

USA Monetary Growth, Inflation and Share Price Growth

Inflation and Monetary Growth
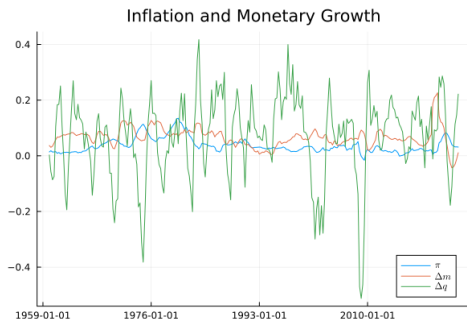
Key insight: Share prices (asset prices) are much more volatile than goods prices. This difference in volatility is a central feature we need to capture in our models.

Different Types of Variables

In dynamic macro models, we distinguish between:

1. "Crawlers" (Predetermined/State Variables):

- ▶ Move slowly and gradually
- ▶ Examples: capital stock, inflation, physical inventories
- ▶ Cannot jump discontinuously

2. "Jumpers" (Forward-Looking/Control Variables):

- ▶ Can adjust instantly to news or policy changes
- ▶ Examples: asset prices, exchange rates, consumption (PIH)
- ▶ Respond to expectations about the future

Understanding this distinction is crucial for analyzing stability!

State Space Representation

General form of a dynamic system:

$$\Delta X_t = F(X_t, Z_t)$$

Where:

- X_t = vector of endogenous variables (determined within the model)
- Z_t = vector of exogenous variables (taken as given)
  - Can be fixed parameters or stochastic shocks
- $F(\cdot)$ = transition function describing the system's dynamics
  - Linear: easy to analyze analytically
  - Nonlinear: may require numerical methods

Understanding the Difference Operator

The symbol Δ denotes change over time:

Forward difference:

$$\Delta X_t = X_{t+1} - X_t$$

"How much will X change from today to tomorrow?"

Backward difference:

$$\Delta X_t = X_t - X_{t-1}$$

"How much did X change from yesterday to today?"

Pro tip: Logarithms are your friend!

▶ If $x_t = \log(X_t)$, then $\Delta x_t \approx$ percentage growth rate of X_t

▶ Makes many calculations simpler and more interpretable

Dynamic System of One Variable

Consider a simple first-order system:

$$\Delta X_t = -\rho X_t + \gamma Z_t$$

In levels (forward difference):

$$X_{t+1} = (1 - \rho)X_t + \gamma Z_t = \tilde{\rho} X_t + \gamma Z_t$$

where $\tilde{\rho} = 1 - \rho$.

This is called an AR(1) process:

▶ "AR" = Autoregressive

▶ "(1)" = Order one (depends only on previous period)

Stability condition: $|\tilde{\rho}| < 1$

▶ The root must lie inside the unit circle

▶ If $|\tilde{\rho}| \geq 1$, the system explodes!

Why Does $|\tilde{\rho}| < 1$ Matter?

Consider $X_{t+1} = \tilde{\rho} X_t$ with no forcing term:

Case 1: $\tilde{\rho} = 0.8 < 1$ (stable)

- $X_1 = 0.8 \times X_0$
- $X_2 = 0.8 \times X_1 = 0.64 \times X_0$
- $X_t = (0.8)^t X_0 \to 0$ as $t \to \infty$ (stable)

Case 2: $\tilde{\rho} = 1.2 > 1$ (unstable)

- $X_1 = 1.2 \times X_0$
- $X_2 = 1.2 \times X_1 = 1.44 \times X_0$
- $X_t = (1.2)^t X_0 \to \infty$ as $t \to \infty$ (unstable)

We need the system to converge, not explode!

The Richardson Arms Race Model

Historical motivation (Lewis Richardson, 1930s):

- ▶ How do countries decide on military spending?
- ▶ Each country responds to the other's armament level
- ▶ Can lead to escalation or de-escalation

Why study this in a macro class?

- ▶ Classic example of a *bivariate dynamic system*
- ▶ Shows interaction between two variables
- ▶ Mathematical structure appears in many macro models:
  - ▶ Inflation and output (Phillips curve dynamics)
  - ▶ Capital and consumption (Ramsey model)
  - ▶ Debt and GDP (fiscal dynamics)

The techniques we learn here apply broadly!

Richardson Model: First Differences

Matrix format:

$$\Delta x_t = A x_t + B z_t$$

where

$$\Delta x_t = \begin{bmatrix} \Delta x_{1,t} \\ \Delta x_{2,t} \end{bmatrix}, \quad A = \begin{bmatrix} -\alpha & \beta \\ \gamma & -\delta \end{bmatrix}, \quad B = \begin{bmatrix} \theta & 0 \\ 0 & \eta \end{bmatrix}$$

Equation format (easier to interpret):

$$\Delta x_{1,t} = -\alpha x_{1,t} + \beta x_{2,t} + \theta z_{1,t}$$
$$\Delta x_{2,t} = \gamma x_{1,t} - \delta x_{2,t} + \eta z_{2,t}$$

Interpretation:

- ▶ $-\alpha, -\delta$: own-variable damping (negative feedback)
- ▶ β, γ: cross-variable effects (strategic interaction)
- ▶ $z_{1,t}, z_{2,t}$: external factors (grievances, threats)

Richardson Model: Level Form

Matrix format:

$$x_{t+1} = \tilde{A}x_t + Bz_t$$

where

$$\tilde{A} = \begin{bmatrix} 1-\alpha & \beta \\ \gamma & 1-\delta \end{bmatrix}$$

Equation format:

$$x_{1,t+1} = (1-\alpha)x_{1,t} + \beta x_{2,t} + \theta z_{1,t}$$
$$x_{2,t+1} = \gamma x_{1,t} + (1-\delta)x_{2,t} + \eta z_{2,t}$$

This is a VAR(1) system:

▶ VAR = Vector Autoregression

▶ Each variable depends on lagged values of both variables

▶ Widely used in empirical macroeconomics

Finding the Steady State

Steady state: All variables stop changing ($\Delta x_t = 0$)

Setting $\Delta x_t = 0$ in matrix form:

$$Ax_t + Bz_t = 0 \implies x_t = -A^{-1}Bz_t$$

The inverse of A is:

$$A^{-1} = \frac{1}{\alpha\delta - \gamma\beta} \begin{bmatrix} -\delta & -\beta \\ -\gamma & -\alpha \end{bmatrix}$$

Steady state solution:

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \frac{1}{\alpha\delta - \gamma\beta} \begin{bmatrix} \delta\theta & \beta\eta \\ \gamma\theta & \alpha\eta \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}$$

Note: For A^{-1} to exist, we need $\alpha\delta - \gamma\beta \neq 0$

Tutorial: Matrix Multiplication

Remember the rule: Row $\times$ Column

Given matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

The product $C = AB$ is computed as:

- $c_{11} = a_{11}b_{11} + a_{12}b_{21}$ (row 1 of A $\times$ column 1 of B)
- $c_{12} = a_{11}b_{12} + a_{12}b_{22}$ (row 1 of A $\times$ column 2 of B)
- $c_{21} = a_{21}b_{11} + a_{22}b_{21}$ (row 2 of A $\times$ column 1 of B)
- $c_{22} = a_{21}b_{12} + a_{22}b_{22}$ (row 2 of A $\times$ column 2 of B)

$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

What is Stability?

Intuitive definition:

- ▶ A system is *stable* if, after a shock, it eventually returns to its steady state
- ▶ It's *unstable* if deviations grow without bound

Why do we care?

- ▶ Real economies don't explode to infinity
- ▶ Understanding stability helps us:
    - ▶ Determine valid parameter ranges
    - ▶ Predict long-run behavior
    - ▶ Design stabilizing policies

Key tool: Eigenvalues (characteristic roots)

- ▶ These determine whether solutions converge or diverge
- ▶ We need them to lie inside the unit circle

The Characteristic Equation

To find eigenvalues, solve:

$$\det[A - \lambda I] = 0$$

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix.

For our Richardson model:

$$\det \begin{bmatrix} -\alpha - \lambda & \beta \\ \gamma & -\delta - \lambda \end{bmatrix} = 0$$

Expanding the determinant:

$$(-\alpha - \lambda)(-\delta - \lambda) - \beta\gamma = 0$$

$$\lambda^2 + (\alpha + \delta)\lambda + (\alpha\delta - \gamma\beta) = 0$$

This is a quadratic equation in λ with two solutions (the eigenvalues)

Understanding the Characteristic Polynomial

Standard form:

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$

where:

- $\text{tr}(A) = a_{11} + a_{22}$ is the *trace* (sum of diagonal elements)
- $\det(A)$ is the *determinant*

For our model:

- $\text{tr}(A) = -\alpha - \delta$ (sum of own-variable effects)
- $\det(A) = \alpha\delta - \gamma\beta$ (combines all parameters)

Properties:

- $\lambda_1 + \lambda_2 = \text{tr}(A)$
- $\lambda_1 \times \lambda_2 = \det(A)$

These relationships are useful for checking stability conditions!

Stability Conditions

The characteristic polynomial yields two eigenvalues: λ_1, λ_2

For full stability, both roots must be inside the unit circle:

Case 1: Real eigenvalues

$$|\lambda_i| < 1 \quad \text{for } i = 1, 2$$

Case 2: Complex eigenvalues $(\lambda = a \pm bi)$

$$\text{Modulus: } \sqrt{a^2 + b^2} < 1$$

Special case: Saddle-path stability

▶ One root inside, one root outside the unit circle:
$|\lambda_1| < 1 < |\lambda_2|$

▶ System has one stable direction and one unstable direction

▶ *This is the most common case in forward-looking macro models!*

Why Saddle-Path Stability is Important

In many macro models:
- ▶ One eigenvalue > 1 (unstable root)
- ▶ One eigenvalue < 1 (stable root)

Economic interpretation:
- ▶ The system would explode if left on its own
- ▶ But agents have *forward-looking expectations*
- ▶ They choose the "jump variable" to put the system on the stable path
- ▶ This is called the *saddle path*

Examples in macro:
- ▶ Consumption (jumper) vs. capital (crawler) in Ramsey model
- ▶ Exchange rate (jumper) vs. price level (crawler) in open economy
- ▶ Asset prices (jumper) vs. fundamentals (crawler)

This explains why asset prices are so volatile!

The Stable Path in a Saddle-Point System

Setup: Two roots λ_1, λ_2 with $|\lambda_1| < 1$ and $|\lambda_2| > 1$

General stable solution:

$$\Delta X_t = \lambda_1(X_t - \bar{X})$$

where $\bar{X}$ is the steady state value.

For variable $x_{1,t}$ (the jumper):

$$\Delta x_{1,t} = \lambda_1(x_{1,t} - \bar{x}_1)$$

From the original system:

$$\Delta x_{1,t} = -\alpha x_{1,t} + \beta x_{2,t} + \theta z_{1,t}$$

Equating these two expressions and solving for $x_{1,t}$:

$$x_{1,t} = \frac{\beta}{\alpha + \lambda_1} x_{2,t} + \frac{\theta}{\alpha + \lambda_1} z_{1,t} + \frac{\lambda_1}{\alpha + \lambda_1} \bar{x}_1$$

Interpreting the Stable Solution

Recall the solution for the jump variable:

$$x_{1,t} = \frac{\beta}{\alpha + \lambda_1} x_{2,t} + \frac{\theta}{\alpha + \lambda_1} z_{1,t} + \frac{\lambda_1}{\alpha + \lambda_1} \bar{x}_1$$

Key insights:

- $x_{1,t}$ (the jumper) immediately adjusts to current values of:
  - The predetermined variable $x_{2,t}$
  - The exogenous shock $z_{1,t}$
  - The steady state $\bar{x}_1$
- This instantaneous adjustment ensures the system stays on the stable path
- Meanwhile, $x_{2,t}$ (the crawler) evolves according to:

$$\Delta x_{2,t} = \gamma x_{1,t} - \delta x_{2,t} + \eta z_{2,t}$$

The jumper "looks ahead" and adjusts to keep the system stable!

Summary of the Two-Equation System

Jumper (control variable):

$$x_{1,t} = \frac{\beta}{\alpha + \lambda_1} x_{2,t} + \frac{\theta}{\alpha + \lambda_1} z_{1,t} + \frac{\lambda_1}{\alpha + \lambda_1} \bar{x}_1$$

- ▶ Responds instantly to keep system on stable path
- ▶ Forward-looking behavior

Crawler (state variable):

$$\Delta x_{2,t} = \gamma x_{1,t} - \delta x_{2,t} + \eta z_{2,t}$$

- ▶ Evolves gradually over time
- ▶ Predetermined from past decisions

Together, these form a stable dynamic system where one variable jumps and the other crawls toward equilibrium.

From Saddle Paths to Blanchard–Khan

So far we've learned:

- ▶ Systems can have multiple eigenvalues
- ▶ Saddle-path stability occurs when some roots are stable, others unstable
- ▶ Jump variables adjust to put us on the stable path

But important questions remain:

- ▶ When does a unique stable solution exist?
- ▶ What if we have more jump variables than unstable roots?
- ▶ What if we have fewer jump variables than unstable roots?

Answer: The Blanchard–Khan conditions

- ▶ Provide a precise counting rule
- ▶ Tell us when solutions exist and are unique
- ▶ Named after Olivier Blanchard and Charles Khan (1980)

The Blanchard–Khan Conditions

Consider a linear rational expectations model with:

- ▶ n predetermined (state) variables
- ▶ m jump (forward-looking) variables
- ▶ Total system size: $n + m$ variables

The Blanchard–Khan theorem states:

1. **Existence:** A solution exists if the number of *stable* roots ($|\lambda| < 1$) is at least n
2. **Uniqueness:** The solution is unique if the number of stable roots is *exactly* n
3. **Problems:**
   - ▶ Stable roots $< n$: No solution exists (system explodes)
   - ▶ Stable roots $> n$: Multiple equilibria (indeterminacy)

The golden rule:

$$\# \text{ of stable roots} = \# \text{ of predetermined variables}$$

Intuition Behind Blanchard–Khan

Why does the counting rule work?

Think of it this way:

- ▶ Each predetermined variable needs a stable root to converge
- ▶ Each jump variable can "absorb" an unstable root
- ▶ By jumping appropriately, jump variables eliminate explosive paths

Example with $n = 2$ predetermined, $m = 1$ jump variable:

- ▶ System has 3 eigenvalues total
- ▶ For unique stability: need exactly 2 stable roots, 1 unstable root
- ▶ The single jump variable adjusts to eliminate the unstable path

This matching of unstable roots to jump variables is the essence of rational expectations!

Interpretation and Economic Meaning

The Blanchard–Khan conditions tell us:

1. **Why some variables must jump:**
   - ▶ Without jump variables, systems with unstable roots would explode
   - ▶ Forward-looking agents choose jumpers to stabilize the system

2. **Why some variables crawl:**
   - ▶ Predetermined variables (capital, debt, etc.) can't jump
   - ▶ They need stable roots to gradually return to equilibrium

3. **Role of expectations:**
   - ▶ Rational expectations enforce the counting rule
   - ▶ Agents choose actions today that rule out explosive future paths
   - ▶ This explains seemingly "excessive" volatility in asset markets

Saddle-path stability is not a bug—it's a feature of forward-looking behavior!

Practical Application: Checking Stability

Step-by-step procedure for any model:

Step 1: Write the system in matrix form

$$x_{t+1} = Ax_t + Bz_t$$

Step 2: Count predetermined vs. jump variables

- n = number of state variables (crawlers)
- m = number of control variables (jumpers)

Step 3: Find eigenvalues by solving $\det[A - \lambda I] = 0$

Step 4: Count how many eigenvalues satisfy $|\lambda| < 1$

Step 5: Apply Blanchard–Khan:

- If # stable roots $= n$: unique stable solution (YES)
- If # stable roots $< n$: no solution (system explodes) (NO)
- If # stable roots $> n$: multiple equilibria (indeterminacy) (WARNING)

Key Takeaways

What we've learned:

1. Dynamic systems have two types of variables:
- ▶ Crawlers (predetermined): adjust slowly
- ▶ Jumpers (forward-looking): adjust instantly

2. Stability depends on eigenvalues:
- ▶ All roots inside unit circle $\rightarrow$ full stability
- ▶ Mixed roots $\rightarrow$ saddle-path stability

3. Blanchard–Khan conditions ensure uniqueness:
- ▶ # of stable roots must equal # of predetermined variables
- ▶ This "counting rule" is fundamental to modern macro

4. This framework applies to many models:
- ▶ Asset pricing, optimal growth, monetary policy, fiscal dynamics

Looking Ahead

Next steps in your study:

Computational methods:

- ▶ Solving nonlinear systems numerically
- ▶ Log-linearization around steady states
- ▶ Simulating impulse responses

Applications to real models:

- ▶ New Keynesian DSGE models
- ▶ Asset pricing with adjustment costs
- ▶ Fiscal and monetary policy interactions

Extensions:

- ▶ Stochastic systems (adding randomness)
- ▶ Higher-dimensional systems
- ▶ Occasionally binding constraints

Any questions?

Thank you for your attention!

Office hours and contact information on syllabus