Tax-Rate Rules for Reducing Government Debt: An Application of Computational Methods for Macroeconomic Stabilization

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Abstract

This chapter extends the simple closed-economy New Keynesian model by incorporating the zero lower bound and asymmetric wage adjustment (in which wages are much more rigid in the downward direction). We examine the dynamics of adjustment, given a sharp increase in government debt due to a once-only big increase in spending. We find that selective tax-rate rules, incoporating a degree of tax relief in a period of fiscal consolidation, are effective instruments for rapidly reducing the overhang of a large stock of public debt.

JEL Classificaltion: E17, E61

1 Introduction

The focus of this chapter is on the computational approach to the analysis of macroeconomic adjustment. Specifying, calibrating, solving and simulating a model for evaluating alternative policy rules can appear to be a cumbersome task. There are of course many different types of models to choose from, alternative views about likely parameter values, multiple approximation methods to try, and different options about simulation.

In this chapter we work through an example to demonstrate the steps of specifying, calibrating, solving and simulating a macroeconomic model to evaluate alternative policies for reducing domestic public debt. The particular application is to consider macroeconomic adjustment in a closed economy following a fiscal expansion when government debt expands. Which policy combinations work best to reduce the burden of public debt? We focus on the case when the interest rate is close to the zero lower bound (so that monetary policy cannot be used to inflate away a sizable portion of the real value of government debt) and when there is asymmetry in wage rigidity (with greater flexiblity in upward movements and greater rigidity in the negative direction).

This question is more that just academic. Figure 1 shows the large increases in the debt/GDP ratios of selected OECD countries. Two facts emerge from

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this figure. First, the debt/GDP ratio for Japan far exceeds that in Canada and those in the smaller highly indebted OECD countries in Europe. Secondly, and even more troubling, is that the debt/GDP ratio for Japan appears to be on an upward trajectory, while the debt/GDP ratios for Canada and the European economies appear to be much more stable. Thus, for Japan and for a number of OECD countries, stabilization of, or reduction in, the level of public debt, is a pressing policy goal in its own right, apart from questions of welfare, or other macroeconomic goals set in terms of inflation targets, or output-gaps.

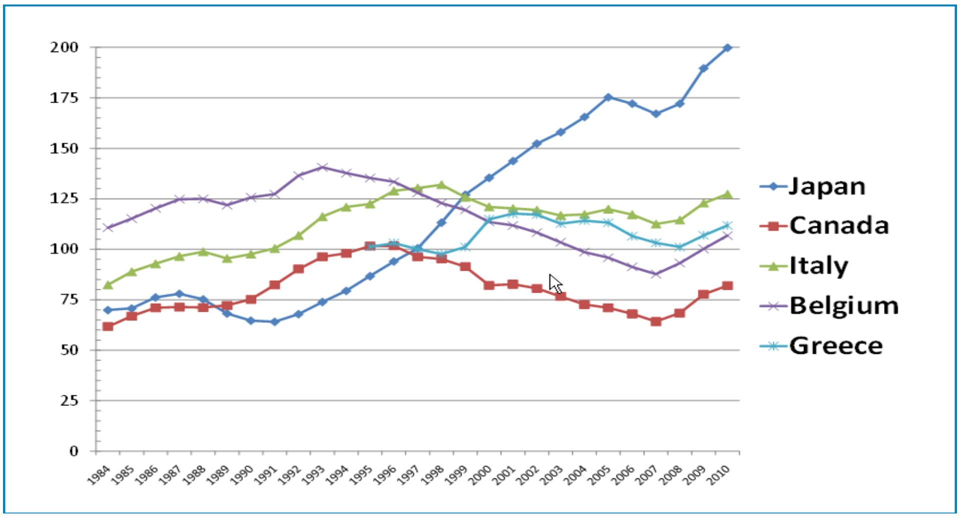

Figure 1: Government Debt/GDP Ratios: Selected Countries

We make use of a simple model, following the widely-used New Keynesian framework, with sticky prices and wages but draw attention to the challenges coming from asymmetries, such as the zero lower bound for nominal interest rates and where wage adjustment is asymmetric, in terms of greater downward nominal rigidity. We show in this chapter that the incorporation of even a little bit of additional complexity (in the form of the zero lower bound and asymmetric wage adjustment) coupled with large shocks, involves a fundamental shift in the way we go about solving and simulating models. Setting up the solution framework is a little harder (since most user-friendly programs are for linear models) and more complex computation algorithms (and more time) are needed. It is easy to understand why nonlinear models are not as popular as linear ones! But as Kenneth Judd (1998) has reminded us, in the spirit of Occam's razor, good models should be simple, but they should not be too simple, so that pressing problems are brushed aside in research.

The chapter is organized as follows. The first section presents a particular specification of a closed economy model. This will be our base model. This is followed by a discussion about calibration and model solutions. The next section discusses solution methods, approximating functions and optimization algorithms for solving the non-linear model. Finally we present a few economic simulations to illustrate applications of the base model.

2 Model Specification

The model we have chosen to work with is designed to examine alternatives to monetary policy as a debt stabilization tool when interest rates are very close to the zero lower bound. However, even when interest rates are not at the zero bound, inflating the debt away, even partially, is often not a serious option. Under central bank independence, the mandate for monetary policy is price stability. The central bank's mission is not to reduce the interest expenses of the fiscal authority by inflating away debt. This means that the reduction in debt needs to come from changes in government spending and/or tax revenues.

The issue is by no means straightforward and depends on when the economy is experiencing the zero lower bound. If the economy is in a boom cycle, debt can be reduced by austerity measures (reduction in government spending) and/or increases in tax rates (to increase current or future tax revenues). However, if interest rates are close to the zero lower bound and the economy is in a recession, questions would be raised about the need to implement austerity measures (including increases in tax rates) solely for the purpose of managing government debt.

The application considered in this chapter is to use the base macroeconomic model to compare a number of scenarios about the effectiveness of government debt-continent rules for tax rates (on labor income and consumption) as a means to reduce the size of the debt. Our experiment considers how to reduce public debt (which came about because of a large initial expansion in government spending) when interest rates are close to the zero lower bound.

2.1 A Simple Closed Economy Model with Bounds and Asymmetric Adjustments

The model considered is in the class of models called New Keynesian Dynamic Stochastic General Equilibrium (DSGE) models. The name conveys the idea that the macroeconomic model contains stickiness, adjustments, expectations and uncertainty, with interactions among all sectors of the economy. The simple model has three sectors: a household sector, a production sector and a government sector with detailed policy rules about the budget deficit.

We state right at the outset that this is a simple closed economy model. There is neither an external sector nor a financial sector. The model also abstracts away from issues associated with habit persistence as well as capital accumulation and investment dynamics with adjustment costs.

2.1.1 Households and Calvo Wage Setting Behavior

A household typically chooses the paths of consumption C, labor L and bonds B to maximize the present value of its utility function $U(C, L_t)$ subject to the budget constraint: The objective function of the household is given by the following expression:

$$
\begin{aligned}
\underset{\{C_t,L_t,B_t\}}{Max}\ \mathcal{L} &= \sum_{\iota=0}^{\infty}\beta^{\iota}\left\{\begin{array}{c} U(C_{t+\iota},L_{t+\iota}) \\ -\Lambda_{t+\iota}\left[\begin{array}{c} P_{t+\iota}C_{t+\iota}(1+\tau^c_{t+\iota})+B_{t+\iota} \\ -(1+R_{t-1+\iota})B_{t-1+\iota} \\ -(1-\tau^w_{t+\iota})W_{t+\iota}L_{t+\iota}-\Pi_{t+i} \end{array}\right] \end{array}\right\} \\
U(C,L_t) &= \frac{(C_t)^{1-\eta}}{1-\eta}-\frac{L_t^{1+\varpi}}{1+\varpi}
\end{aligned}
$$

Overall utility is a positive function of consumption and a negative function of labor[1]; the parameter η is the relative risk aversion coefficient, ϖ is the Frisch labor supply elasticity, while β represents the constant, exogenous discount factor. In addition to buying consumption good, C_t, households hold government bonds B_t which pays return R_t, and receive dividends from the firms Π_t. The household pays taxes on labor income $\tau^w_t W_t L_t$, and on consumption expenditures $\tau^c_t P_t C_t$. The tax rates τ^w_t, τ^c_t are treated as given policy variables.

The Euler equations implied by household optimization of its intertemporal utility with respect to C_t and B_t are:

$$
\begin{aligned}
C_t^{-\eta} &= \Lambda_t P_t(1+\tau^c_t) && (1) \\
\Lambda_t &= \beta\Lambda_{t+1}(1+R_t) && (2)
\end{aligned}
$$

The first equation, 1 tells us that the marginal utility of consumption, divided by the tax-adjusted price level, is equal to the marginal utility of wealth Λ_t. The next equation is the Keynes-Ramsey rule for optimal saving: the marginal utility of wealth today should be equal to the discounted marginal utility tomorrow, multiplied by the gross rate of return on saving.

The Euler equation with respect to L_t is: $L_t^{\varpi} = \Lambda_t(1-\tau^w_t)W_t$ and it relates the marginal dis-utility of Labor, adjusted by the after-tax wage, to the foregone marginal utility of wealth. However, since labor markets rarely clear, we shall replace this Euler condition (which determines labor, given the wage rate) with an alternative specification - the assumption that wages are set as staggered contracts. A fraction $(1-\xi^w)$ of households renegotiate their contracts each period. Each household chooses the optimal wage W^o_t by maximizing the expected discounted utility subject to the demand for its labor $L^h_t : L^h_t = \left(\frac{W^o_t}{W_t}\right)^{-\zeta^w} L_t$. Taking derivative with respect to W^o_t yields the first order condition:

$$
E_t\sum_{\iota=0}^{\infty}\left\{\begin{array}{c} (\xi^w\beta)^{\iota}\left(-L^{\varpi}_{t+\iota}\right)(W_{t+\iota})^{\zeta^w}L_{t+\iota}\left[-\zeta^w\left(W^o_{t+\iota}\right)^{-\zeta^w-1}\right] \\ +\Lambda_{t+\iota}(1-\tau^w_t)(W_{t+\iota})^{\zeta^w}L_{t+\iota}\left[(-\zeta^w+1)\left(W^o_{t+\iota}\right)^{-\zeta^w}\right] \end{array}\right\}=0
$$

which in turn can be rearranged as (assuming the usual assumption of a subsidy to eliminate the mark-up effects:

$$
(W^o_t)^{1+\zeta^w\varpi} = \frac{\sum_{t=0}^{\infty}(\xi^w\beta)^t(W_t)^{\zeta^w+\zeta^w\varpi}\left(L_t^{1+\varpi}\right)}{\sum_{t=0}^{\infty}(\xi^w\beta)\Lambda_t(1-\tau^w_t)(W_t)^{\zeta^w}L_t}
$$

[1]The coefficient of the disutility of labor is set at unity.

Note that, in the steady-state (or when $\xi^w = 0$), this collapses to the same condition as the competitive case:

$$\begin{aligned} (W)^{1+\xi^w \varpi} &= \frac{(W)^{\zeta^w \varpi}(L^{\varpi})}{\Lambda(1-\tau^w)} \\ W &= \frac{(L^{\varpi})}{\Lambda(1-\tau^w)} \end{aligned}$$

The wage equation can be rewritten using auxiliary equations N_t^w and D_t^w:

$$N_t^w = (W_t)^{\zeta^w + \zeta^w \varpi}\left(L_t^{1+\varpi}\right) + \xi^w \beta . N_{t+1}^w \tag{3}$$

$$D_t^w = \Lambda_t(1-\tau_t^w)(W_t)^{\zeta^w} L_t + \xi^w \beta . D_{t+1}^w \tag{4}$$

$$(W_t^o)^{1+\xi^w \varpi} = \frac{N_t^w}{D_t^w} \tag{5}$$

$$W_t = \left[\xi^w (W_{t-1})^{1-\zeta^w} + (1-\xi^w)(W_t^o)^{1-\zeta^w}\right]^{\frac{1}{1-\zeta^w}} \tag{6}$$

$$\xi^w = \begin{cases} \xi_{down}^w & if \quad W_t^o \leq W_{t-1} \\ \xi_{up}^w & if \quad W_t^o > W_{t-1} \end{cases}$$

Since changes to wages also tend to be more sticky downwards and less sticky upwards, we have allowed the stickiness factor ξ^w to be different. More specifically, $\xi_{down}^w > \xi_{up}^w$.

2.1.2 Production and Calvo Price Setting Behavior

Output is a function of labor only (that is we abstract from issues associated with capital formation).

$$Y_t = Z L_t \tag{7}$$

where the productivity term Z is assumed to be fixed (for convenience at unity). Total output is for both household and government consumption:

$$Y_t = C_t + G_t \tag{8}$$

$$G_t = \rho^g G_{t-1} + (1-\rho^g)\overline{G} + \epsilon_t^g; \qquad \epsilon_t^g \sim N(0, \sigma_g^2) \tag{9}$$

where government spending G_t is assumed to follow a simple exogenous autoregressive process, with autoregressive coefficient ρ^g,steady state $\overline{G}$, and a stochastic shock ϵ_t^g normally distributed with mean zero and variance σ_g^2.

The profits of the firms are given by the following relation, and distributed to the households:

$$\Pi_t = P_t Y_t - W_t L_t$$

We assume sticky monopolistically competitive firms. In the Calvo price setting world, there are forward-looking domestic-goods price setters and backward looking setters. Assuming at time t that ξ^p is the probability of persistence, with demand for the product from firm j given by $Y_t\left(P_t^j/P_t^c\right)^{-\zeta^p}$, the optimal domestic-goods price, P_t^o can be written in forward recursive formulation as:ϱ

$$\begin{aligned}
A &= W_t/Z_t && (10)\\
P_t^o &= \frac{N_t^p}{D_t^p} && (11)\\
N_t^p &= Y_t\left(P_t\right)^{\zeta^p} A_t + \beta\xi^p N_{t+1}^p && (12)\\
D_t^p &= Y_t\left(P_t\right)^{\zeta^p} + \beta\xi^p D_{t+1}^p && (13)\\
P_t &= \left[\xi^p\left(P_{t-1}\right)^{1-\zeta^p} + (1-\xi^p)\left(P_t^o\right)^{1-\zeta^p}\right]^{\frac{1}{1-\zeta^p}} && (14)
\end{aligned}$$

where A_t is the marginal cost at time t, while the domestic price level P_t is a CES aggregator of forward and backward-looking prices.

2.1.3 Monetary and Fiscal Policy

The Central Bank is responsible for monetary policy and it is assumed to adopt a Taylor rule, with smoothing. We model the Taylor rule subject to a zero lower bound on the official interest rate as:

$$\begin{aligned}
R_t &= \max[0, \;\; \rho^r(R_{t-1}) + (1-\rho^r)(\overline{R} + \phi^p(\pi_t - \pi^*) + \phi^y(\theta_t - \theta^*))] && (15)\\
\pi_t &= \frac{P_t}{P_{t-1}} - 1 && (16)\\
\theta_t &= \frac{Y_t}{Y_{t-1}} - 1 && (17)
\end{aligned}$$

where the variable π_t is the inflation rate at time t, π^* is the target inflation rate, θ_t is the growth rate at time t, θ^* is the target growth rate. ρ^r is the smoothing coefficient, with $0 < \rho^r < 1$. The parameter $\phi^p > 1$ is the Taylor rule inflation coefficient, ϕ^y is the Taylor rule growth coefficient, while $\overline{R}$ is the steady state interest rate.

The Treasury is responsible for fiscal policy and the fiscal borrowing requirement is given as follows:

$$B_t = (1 + R_{t-1})B_{t-1} + P_t G_t - \tau_t^w W_t L_t - \tau_t^c P_t C_t \tag{18}$$

2.1.4 Summary

In summary, the 18 equations derived above described a simple model of a closed economy where the nominal rate is subjected to a lower bound at zero, and where wage adjustments are asymmetric (more sticky downwards). The eighteen variables are: $G, R, C, Y, L, \pi, \theta, \Lambda, A, P, W, N^w, D^w, W^o, N^p, D^p, P^o, B$. In this simple model, there is only one shock (ϵ_t^g) with one unknown standard error (σ_g^2). There are seven behavioral parameters ($\beta, \eta, \varpi, \zeta^p, \zeta^w, \xi^p, \xi^w$), six policy parameters ($\tau^w, \tau^c, \rho^g, \rho^r, \phi^p, \phi^y$). We set $\pi^* = \theta^* = 0$.

To solve the model, we need some estimates of the parameters as well as a way to solve a model with forward-looking variables ($\Lambda_{t+1}, N_{t+1}^p, D_{t+1}^p, N_{t+1}^w, D_{t+1}^w$). These variables, unlike the backward looking (lagged) variables ($W_{t-1}, G_{t-1}, P_{t-1}, R_{t-1}, Y_{t-1}$), are of course unknown at time t.

2.2 Calibration and steady-state values

The model is calibrated rather than estimated - the recent development of estimation techniques for DSGE models deserves a more detailed treatment. However, the parameters are based on estimates which are widely accepted. The calibrated base model we use is a widely-shared, if not consensus, model of a closed economy that may be used for policy evaluation in the short run (fixed capital).

Table 1 gives the values of the parameters. The discount parameter β corresponds to an annualized risk free rate of return equal to 2%. In other words, we start our analysis at the point when interest rates are low, but it has not yet hit the zero lower bound. Values for the Frisch elasticity of labor supply, ϖ, usually range from 0.5 to 1.5. We have set $\varpi = 1$. The coefficient of relative risk aversion (equal to the reciprocal of the elasticity of intertemporal substitution), η, is usually greater than 1, since empirical estimates tend to vary between 1 to 4. We have set this to 2.5. The demand elasticity for labor ζ^w and goods ζ^p have been set at the usual value of 6 (corresponding to a mark-up factor of 20 per cent) while the two Calvo stickiness parameters ξ^w, ξ^p have been set at 0.8 to capture inertia in wage and price adjustments. The tax parameters τ^w and τ^c are set at 0.3 and 0.1 respectively. The coefficients governing the Taylor rule allow for some autoregressive behavior, while satisfying the Taylor principle.

Table 1: Calibrated Values

Symbol	Definition	Value
β	Discount factor	0.995
ϖ	Elasticty of labor supply	1
η	Relative risk aversion	2.5
ζ^w	Demand elasticity for ;abor	6
ζ^p	Demand elasticity for goods	6
ξ^w	Calvo wage coefficient	0.8
ξ^p	Calvo price coefficient	0.8
τ^w	Labor income tax rate	0.3
τ^c	Consumption tax rate	0.1
ρ^g	Government spending coefficient	0
ρ^r	Taylor mmoothing coefficient	0.5
ϕ^p	Taylor inflation coefficient	1.5
ϕ^y	Taylor growth coefficient	0.5
σ^g	Standard deviation-spending shocks	0.01

Given the parameter configuration, we can solve for the steady state. Note first that $\pi = \theta = 0$, $A = P^o = P$ and $W^o = W$. The values of the 8 key endogenous variables (G, R, C, Y, L, P, W, B) comes from solving the following system of equations:

$$
\begin{aligned}
PG &= \tau^w WL + \tau^c PC \\
Y &= L \\
Y &= C + G \\
L^{\varpi}(1+\tau^c)P &= C^{-\eta}(1-\tau^w)W \\
W &= P \\
(1+R) &= 1/\beta
\end{aligned}
$$

which is predicated on the assumption that there is no outstanding public debt in the steady-state, $B = 0$ and where the steady state price level, P, is normalized at unity. For the government sector, a balanced budget means that the tax revenue just covers the government expenditure. In the steady state, goods market equilibrium for the closed economy requires that production of goods be equal to the demand for consumption and government goods. For the labor market, the marginal disutility of labor should be equal to the productivity of labor, net of taxes, times the marginal utility of consumption. In this simple model, without capital, $W = P$ (because the productivity factor Z is fixed at unity). Finally, the steady state gross interest rate,$(1+R)$, is equal to the inverse of the social discount rate β. Solving the system of nonlinear equations gives:

Table 2: Steady-state values

Symbol	Definition	Value
B_0	Bonds	0
C_0	Consumption	0.7724
G_0	Government spending	0.4414
L_0	Labor	1.2137
P_0	Prive level	1
W_0	Wage level	1
Y_0	Output	1.2137
R_0	Interest rate (quarterly)	0.005

For incompleteness, the values for the remaining five variables are given by the following equations:

$$
\begin{aligned}
\Lambda_0 &= C_0^{-\eta}/(1+\tau_0^c) \\
N_0^p &= Y_0/(1-\beta\xi^p) \\
D_0^p &= Y_0(1-\beta\xi^p) \\
N_0^w &= L_0^{1+\varpi}/(1-\xi^w\beta) \\
D_0^w &= \Lambda_0(1-\tau_0^w)L_0/(1-\xi^w\beta)
\end{aligned}
$$

We also note that given the relationship between steady-state tax rates, consumption, government spending, real labor income shares, and bond/GDP ratio:

$$
(1-R)\frac{B}{PY} = \frac{G}{Y} - \tau^c\frac{C}{Y} - \tau^w\frac{WL}{PY}
$$

Given our assumption that $B = 0$, and $\frac{WL}{PY} = 1$ (because labor is the only factor of production), the implied steady-state share of consumption to GDP is given by the following ratio::

$$\frac{C_0}{Y_0} = \frac{(1 - \tau_0^w)}{(1 + \tau_0^c)}$$

In the absence of investment in this model, the government spending ratio is the remaining share.

3 Solution Methods

DSGE models, no matter how simple, do not have closed form solutions except under very restrictive circumstances (such as logarithmic utility functions and full depreciation of capital). We have to use computational methods if we are going to find out how the models behave for a given set of initial conditions and parameter values. However, the results may differ, depending on the solution method. Moreover, there is no benchmark exact solution for this model, against which we can compare the accuracy of alternative numerical methods.

There are of course a variety of solution methods. Every practicing computational economist has a favorite solution method (or two). Even with a given solution method there are many different options, such as the functional form to use in any type of approximating function, or the way in which we measure the errors for finding accurate decision rules for the model's control variables. The selection of one method or another is as much a matter of taste as well as convenience, based on speed of convergence and the amount of time it takes to set up a computer program.

Briefly, there are two broad classes of solution methods - perturbation and projection methods. Both are widely used and have advantages and drawbacks. We can illustrate these differences with reference to the well-known example of an agent choosing a stream of consumption (c_t) which maximizes her utility function (U) and which then defines the capital (k) accumulation, given the production function f and productivity process z_t:

$$\begin{aligned} & \max_{c_t} \sum_{t=1}^{\infty} \beta^t U(c_t) \\ k_{t+1} &= f(z_t, k_t) - c_t \\ z_t &= \rho z_{t-1} + \varepsilon_t,\ \varepsilon_t \tilde{} N(0, \sigma^2) \end{aligned}$$

The first order condition for the problem is:

$$U'(c_t) = \beta U'(c_{t+1}) f'(k_{t+1})$$

The system has one forward looking variable (also known as "jumper" or "control") for the evolution of c_t, and one state variable k_t which depends on the values of the forward-looking variables, c_t, and the previous-period values, k_{t-1}. The key to solving the model is to find ways to represent functional forms ("decision rules")[2] for these controls, which depend on the lagged values of the state

[2] The computational literature refers to these decision rules for variables which depend on their own and other expected future variables as "policy functions". The word "policy" in this case is not to be confused with the interest rate policy function given by the Taylor rule. The terms "policy function" or "decision rule" refer to functional equations (functions of functions) which we use for the forward-looking control variables.

variables. Once we do this, the system becomes fully recursive and the dynamic process is generated (given an initial value for k).[3]

3.1 Perturbation Method

The first method - perturbation method - involves a local approximation based on a Taylor expansion. For example, let $h(x_t)$ represent the decision rule (or policy function) for c_t based on the vector of state variables $x_t = [z_t, k_t]$ around the steady-state x_0 :

$$h(x_t) = h(x_0) + h'(x_o)(x_t - x_0) + \frac{1}{2}h''(x_0)(x_t - x_0)^2 + ...$$

Perturbation methods have been extensively analyzed by Schmidt-Grohé and Uribe (2004). The first-order perturbation approach (a first-order Taylor expansion around the steady state) is identical to the most widely used solution method for dynamic general equilibrium models, namely linearization or log linearization of the Euler equations around a steady state (see Uribe (2003) for examples). The linear model is then solved using the methods for forward-looking rational expectations such as those put forward by Blanchard and Kahn (1980) and later discussed by Sims (2001).

Part of the appeal of this approach lies with the fact that the solution algorithm is fast. The linearized system is quickly and efficiently solved by exploiting the fact that it can be expressed as a state-space system. Vaughan's method, popularized by Blanchard and Khan (1980), established the conditions for the existence and uniqueness of a rational expectations solution as well as providing the solution. Canova (2007) summarizes this method as essentially an eigenvalue-eigenvector decomposition on the matrix governing the dynamics of the system by dividing the roots into explosive and stable ones. For instance, the Blanchard-Kahn condition states that the number of roots outside the unit circle must be equal to the number of forward-looking variables for there to be a unique stable trajectory.

This first-order approach can be extended to higher-order Taylor expansions. Moving from a first to a second or third-order approximation simply involves adding second order terms linearly in the specification of the decision rules. Since the Taylor expansion has both forward-looking and backward-looking state variables, these methods also use the same Blanchard-Kahn method as the first-order approach. Collard and Julliard (2001a, 2001b) offer first, second, and third order perturbation methods in the latest version (4.3) of their DYNARE system.

Log-linearization, is an example of the "change of variable" method for a first-order perturbation method. Fernández-Villaverde and Rubio-Ramírez (2006) take this idea one step further within the context of the perturbation method. The essence of the Fernández-Villaverde and Rubio-Ramírez approach is to use a first or second-order perturbation method but with transformation of the variables in the decision rule from levels to power-functions. Just as a

[3] Taylor and Uhlig (1990) edited a speical issue of the Journal of Business and Economic Statistics centered on the solution of the stochastic nonlinear growth model. Authors were asked to solve the model with different methods, for a given set of parameters governing the model and stochastic shocks. Not surprisingly, when the shocks became progressively large, the results of the different methods started to diverge.

log-linear transformation is easily applied to the linear or first order perturbation representation, these power transformations may be done in the same way. The process simply involves iterating on a set of parameters for the power functions, in transforming the state variables, for minimizing the Euler equation errors. The final step is to back out the level of the series from the power transformations, once the best set of parameters is found. They argue that this method preserves the fast linear method for efficient solution while capturing model nonlinearities that would otherwise not be captured by the first-order perturbation method.

We note that the second and higher order methods remain, like the first-order method, a local method. As Fernandez-Villaverde [2006: p. 39] observes, it approximates the solution around the deterministic steady state and it is only valid within a specific radius of convergence. Overall, the perturbation method is especially useful when the dynamics of the model consists of small deviations from the steady-state values of the variables. It assumes that there are no asymmetries, no threshold effects, no types of precautionary behavior and no big transitional changes in the economy. The perturbation methods are local approximations, in the sense that they assume that the shocks represent small deviations from the steady state.

3.2 Projection Methods

The projection solution method, put forward by den Haan and Marcet (1990, 1994), (so called Parameterized Expectations Algorithm or PEA), seeks decision rules for c_t that are "rational" in that they satisfy the Euler equation in a sufficiently robust way. It may be viewed intuitively as a computer analogue of the method of undetermined coefficients. The steps in the algorithm are:

- specify decision rules for the forward looking variables; for example, $\widehat{c}_t = \psi(\Omega, x_t)$ where Ω are parameters, x_t contains variables known at time t (e.g z_t, k_{t-1}) and ψ is the approximating function;

- estimate Ω using various optimizing algorithm so that the Euler equation residuals ($\epsilon_t = U'(\widehat{c}_t) - \beta U'(\widehat{c}_{t+1})f'(k_{t+1})$) or the difference between the left and right hand sides of the Euler equation, is close to zero.

3.2.1 Approximating Functions

The function ψ may be any approximating functions and the decision variables x_t are typically observations on the shocks and other state variables. In fact, approximating functions are just flexible functional forms, which are parameterized to minimize Euler equation errors well-defined by *a priori* theoretical restrictions based on the optimizing behavior of the agents in the underlying the model.

Neural-network (typically logistic) or the Chebychev orthogonal polynomial specifications are the two most common approximating functions used. The question facing the researcher here is one of robustness. First, given a relatively simple model, should one use a low-order Chebychev polynomial approximation or are there gains to using slightly higher-order expansion for obtaining the decision rules for the forward-looking variable? Will the results change very much if we use a more complex Chebyshev polynomial, or a neural network alternative?

Are there advantages to using a more complex approximating function, even if a less complex approximation does rather well? In other words, is the functional form of the decision rule robust with respect to the complexity of the model?

The question of using slightly more complex approximating functions, even when they may not be needed for simple models, illustrates a trade-off noted by Olaf Wolkenhauer, that more complex approximations are often not specific or precise enough for a particular problem while simple approximations may not be general enough for more complex models [Wolkenhauer (2001): p.xx]. In general though, the "discipline" of Occam's razor still applies: relatively simple and more transparent approximating functions should be preferred over more complex and less transparent functions. Canova (2007) recommends starting with simple approximating functions such as a first or second-order polynomial, and later checking the robustness of the solution with more complex functions.

Logistic Neural Networks Sirakaya, Turnovsky, and Alemdar (2006) cite several reasons for using neural networks as approximating functions. First, as noted by Hornik, Stinchcombe, and White (1989), a sufficiently complex feedforward network can approximate any member of a class of function to any degree of accuracy. Secondly, they use fewer parameters to achieve the same degree of accuracy than orthogonal polynomials, which require an exponential increase in parameters. While the curse of dimensionality is still there, it's "sting", to borrow an expression from St. Paul, and expanded by Kenneth Judd,[4] is reduced. Thirdly, such networks, with logsigmoid functions, easily deliver control bounds on endogenous variables. Finally, such networks can be easily applied to models which admit bang-bang solutions [Sirakaya, Turnovsky, and Alemdar (2006): p.3]. For all these reasons, neural networks can serve as a useful and readily available alternative or robustness check to the more commonly used Chebyshev approximating functions.

Like orthogonal polynomial approximation methods, a logistic neural network relates a set of input variables to a set of one or more output variables but the difference is that the neural network makes use of one or more hidden layers, in which the input variables are squashed or transformed by a special function, known as a logistic or logsigmoid transformation. The following equations describe this form of approximation:

$$n_{j,t} = \omega_{j,0} + \sum_{i=1}^{i*} \omega_{j,i} x^*_{i,t} \tag{19}$$

$$N_{j,t} = \frac{1}{1 + e^{-n_{j,t}}} \tag{20}$$

$$y^*_t = \gamma_0 + \sum_{j=1}^{j^*} \gamma_j N_{j,t} \tag{21}$$

Equation (19) describes a variable $n_{j,t}$ as a linear combination of a constant term, $\omega_{j,0}$ and input variables observed at time t, $\{x_{i,t}\}$, $i = 1, ..., i^*$, with coefficient vector or set of "input weights" $\omega_{j,i}, i = 1, ..., i^*$. Equation (21)

[4] At the 2006 Meetings of the Society of Computational Economics and Finance in Cyprus, the title of Kenneth Judd's plenary session was "O Curse of Dimensionality, Where Is Thy Sting".

shows how this variable is squashed by the logistic function, and becomes a neuron $N_{j,t}$ at time or observation t. The set of j^* neurons are then combined in a linear way with the coefficient vector $\{\gamma_j\}, j = 1, ...j^*$, and taken with a constant term γ_0, to form the forecast $\widehat{y}_t^*$ at time t.

This system is known as a feedforward network and when coupled with the logsigmoid activation functions it is also known as the "multi-layer perception" or MLP network. An important difference between neural network and orthogonal polynomial approximation is that the neural network approximation is not linear in parameters.

3.2.2 Optimizing Algorithm

The parameters Ω are obtained by minimizing the squared residuals ϵ. A variety of optimization methods can be used to obtain the global optimum. We use an algorithm similar to the parameterized expectations approach developed by Marcet (1988), and further developed in Den Haan and Marcet (1990, 1994) and in Marcet and Lorenzoni (1999). We solve for the parameters as a fixed-point problem. We make an initial guess of the parameter vector $[\Omega]$, draw a large sequence of shocks (ε_t), and then generate time series for the endogenous variables of the model (c_t, k_t). We then iterate on the parameter set $[\Omega]$ to minimize a loss function $\mathcal{L}$ based on the Euler equation errors ϵ for a sufficiently large T.[5] We continue this process till convergence. Judd (1996) classifies this approach as a "projection" or a "weighted residual" method for solving functional equations, and notes that the approach was originally developed by Williams and Wright (1984, 1991).

There are, however, drawbacks of this approach. One is that for more complex models, the iterations may take quite a bit of time for convergence. Then there is also the ever present curse of dimensionality. The larger the number of state variables, the greater the number of parameters needed to solve for the decision rules. Also, the method relies on the sufficiency of the Euler equation errors. If the utility function is not strictly concave, for example, then the method may not give appropriate solutions. As Canova (2007) suggested, minimization of Euler equations may fail, when there are large number of parameters or when there is a high degree of complexity or nonlinearity.

Heer and Maußner (2005) note another type of drawback of the approach. They point out that the Monte Carlo simulation will more likely generate data points near the steady state values than far away from the steady state in the repeated simulations for estimating the parameter set $[\Omega]$. [Heer and Maußner (2005): p. 163]. We have used normally distributed errors here but we note that fat tails and volatility clustering are pervasive features of observed macroeconomic data so there is no reason not to use wider classes of distributions for solving and simulating dynamic stochastic models. As Fernandez-Villaverde (2005) and Justiniano and Primiceri (2006) emphasize, there is no reason for a stochastic dynamic general equilibrium model not to have a richer structure than normal innovations. However, for the first-order perturbation approach, small normally-distributed innovations are necessary. That is not the case for projection methods.

With this method, as noted by Canova (2007), the approximation is globally

[5] Den Haan and Marcet (1994) recommend a sample size of $T = 30,000$.

valid as opposed to being valid only around a particular steady state point as is the case for perturbation methods. While the projection method is computationally more time-consuming than the perturbation method, the advantage of using this method is that the researcher or policy analyst can undertake experiments which are far away from the steady state, or involve more dramatic regime changes in the policy rule. It is also more suitable for models with thresholds and/or inequality constraints.[6]

Another point that is worth mentioning is that an algorithmic can be specified to impose, for example, non-negativity constraints for all of the variables. The usual no-Ponzi game can only be applied to the evolution of government debt:

$$\lim_{t\to\infty} B_t \exp^{-it} = 0$$

3.3 Linking Perturbation and Projection Methods

The perturbation methods, as mentioned above, involve Taylor expansions around a steady state. These methods assume that the shocks involve small deviations from a steady state, and do not allow asymmetries (in the form of zero lower bound on interest rates or greater downward rigidity in nominal wages). The advantage of these methods is that they are fast in terms of computing time. Many software programs, such as DYNARE, are user friendly. So whenever one wants to start analyzing the dynamic properties of a model, a good first step is still to implement a perturbation solution method.

Another advantage for starting with perturbation methods, is that we may use it to obtain starting values for the projection method. First run the model with a perturbation method, and generate time series of the variables (e.g z_t, k_{t-1}). Next estimate the function ψ using nonlinear methods to obtain the coefficients of the approximating equation $[\Omega]$. Good starting values go a long way towards speeding up an iterative solution method.

3.4 Accuracy Test: Judd-Gaspar Statistic

While the outcomes using different approximating functions, will not be identical, we would like the results to be sufficiently robust, in terms of basic dynamic properties. Since the model does not have any exact closed-form solution against which we can benchmark numerical approximations, we have to use indirect measures of accuracy. Too often, these accuracy checks are ignored when researchers present simulation results based on stochastic dynamic models. This is unfortunate, since the credibility of the results, even apart from matching key characteristics of observable data, rests on acceptable measures of computational accuracy as well as theoretical foundations.

A natural way to check for accuracy is to see if the Euler equations are satisfied, in the sense that the Euler equation errors are close to zero. Judd and Gaspar (1997) suggest transforming the Euler equation errors as follows:

$$JG_t = \frac{|\epsilon_t|}{c_t} \tag{22}$$

[6] Duffy and McNelis (2001) applied the Parameterized Expectatios method with neural network specification for the solution of the stochastic growth model. Lim and McNelis (2008) apply these methods to a series of progressively more complex models of a small open economy.

that is they suggest checking the accuracy of the approximations by examining the absolute Euler equation errors relative to their respective forward looking variable. If the mean absolute values of the Euler equation errors, deflated by the forward looking variable c_t, is 10^{-2}, Judd and Gaspar note that the Euler equation is accurate to within a penny per unit of consumption or per unit spent on foreign currency.

Since consumption is an important variable in these type of models, the last point we note here is that it is conventional to substitute out the marginal utility of wealth Λ_t and work instead with the Euler equation below to generate the Judd and Gaspar statistics.

$$\frac{C_t^{-\eta}}{P_t(1+\tau_t^c)} = \beta \frac{C_{t+1}^{-\eta}}{P_{t+1}(1+\tau_{t+1}^c)}(1+R_t)$$

3.5 Application

Consider now the model described in Section 2. To solve the model, we start by parameterizing decision rules for $C_t, N_t^p, D_t^p, N_t^w, D_t^w$:

$$\begin{aligned}
C_t &= \psi^C(\mathbf{x}_t; \Omega_C) \\
N_t^p &= \psi^{Np}(\mathbf{x}_t; \Omega_{Np}) \\
D_t^p &= \psi^{Dp}(\mathbf{x}_{t-1}; \Omega_{Dp}) \\
N_t^w &= \psi^{Nw}(\mathbf{x}_t; \Omega_{Nw}) \\
D_t^w &= \psi^{Dw}(\mathbf{x}_t; \Omega_{Dw})
\end{aligned}$$

where the symbols Ω_C, $\Omega_{Np}, \Omega_{Dp}, \Omega_{Nw}$, and Ω_{Dw} represent the parameters, and $\psi^C, \psi^{Np}, \psi^{Dp}, \psi^{Nw}$and ψ^{Dw} represent the expectation approximation functions. The symbol $\mathbf{x}_t$ contains a vector of observable state variables known at time t:

$$\mathbf{x}_t = [G_t, R_{t-1}, Y_{t-1}, P_{t-1}, W_{t-1}]$$

We use a neural network specification with one neuron for each of the decision variables. There are 25 parameters to estimate for our model with five Euler equations and five state variables. The starting values for Ω, are obtained from estimating ψ using artificial data generated by the perturbation method in DYNARE. An optimising algorithm was then applied to minimise the Euler errors.

The accuracy of the simulations was checked using the Judd-Gaspar statistic defined for consumption C, the Calvo price P, and the Calvo wage W :

$$\begin{aligned}
JG_t^C &= \frac{1}{C_t}\left|\left(\frac{C_t^{-\eta}}{P_t(1+\tau_t^c)} - \beta \frac{C_{t+1}^{-\eta}}{P_{t+1}(1+\tau_{t+1}^c)}(1+R_t)\right)\right| \\
JG_t^P &= \frac{1}{P_t^o}\left|\left(\frac{N_t^p}{D_t^p} - \frac{Y_t\,(P_t)^{\zeta^p} A_t + \beta\xi^p N_{t+1}^p}{Y_t\,(P_t)^{\zeta^p} + \beta\xi^p D_{t+1}^p}\right)\right| \\
JG_t^W &= \frac{1}{(W_t^o)^{1+\zeta\varpi}}\left(\frac{N_t^w}{D_t^w} - \frac{(W_t)^{\zeta^w+\zeta^w\varpi}\left(L_t^{1+\varpi}\right) + \xi^w\beta.N_{t+1}^w}{\frac{C_t^{-\eta}}{P_t(1+\tau_t^c)}(1-\tau_t^w)\,(W_t)^{\zeta^w} L_t + \xi^w\beta.D_{t+1}^w}\right)
\end{aligned}$$

The Judd-Gaspar statistic was specified with reference to the price and the wage variable as they were easier to interpret than their respective auxilliary components N_t^p, D_t^p, N_t^w, D_t^w. We solved and simulated the model for $T = 100$, for 1000 realizations of the stochastic process governing G_t.Figure 2 gives the distribution of the mean and maximum values of the Judd-Gaspar statistics. We see that the decision rules have a high degree of accuracy. Both the mean and the maxima of the absolute Euler equation errors are quite small, well less than one per cent.

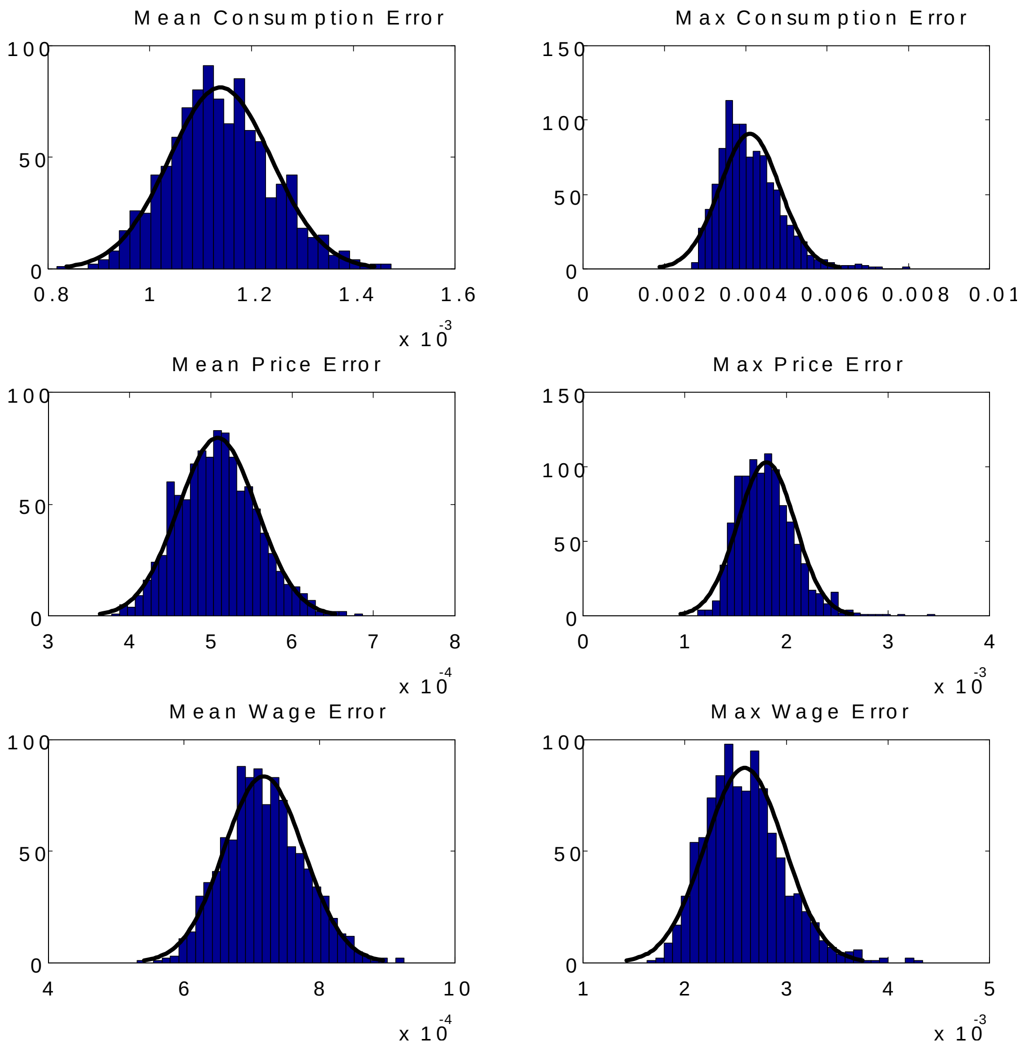

Figure 2: Distribution of Mean and Maximum Judd-Gaspar Errors

4 Fiscal Shock

The impulse response functions for a one-off (recall $\rho^g = 0$) shock to government spending G appear in Figures 2(a-e). By construction, the shock takes place in period 5.

4.1 Simulations with fixed tax rates

Two scenarios are presented; the base case when the nominal interest rate cannot be negative and wages are more sticky downwards (solid line) and the alternative linear case when the zero lower bound and asymmetric wage stickiness is not in operation (dashed line). We have dubbed the alternative case as "linear" because a DSGE model without bounds and asymmetric adjustments can be re-written equivalently in log-linearised form. Both income and consumption tax rates are pre-determined and fixed in these impulses.

What is reassuring about the results in Figure 3 is that the adjustment paths of the linear and the PEA cases are not markedly different. This is because, the non-linearities under study are not huge departures from the linear case. Be that as it may, the figure illustrates the point that the use of a linear model is often a good starting point before proceeding to the implementation of complex solution methods.

With respect to the results, we can see, the fiscal shock caused an increase in G which then stimulates an increase in aggregate demand. Since the expenditure is bond-financed, B increase and pressure is put on the nominal interest rate to rise. Price rise (but the increase in inflation is trivial), and the aggregate supply increases as output expands. The increase in the demand for labor is met by an increase in wages and consumption improves in this scenario. Note too that since wages are rising, the stickiness factor falls giving less weight to the lagged wage rate. The result is a higher real wage when compared to the case with a fixed stickiness parameter.

When G drops off in the following period, the reverse process occurs. Output, consumption, prices and wages fall. With asymmetric wage adjustment, the persistence parameter falls during the adjustment process giving more weight to past wage rate so that the aggregate wage becomes more rigid downwards.

We also see in this figure that the interest rate in the "linear" model falls below zero. In the nonlinear base case, the zero-lower bound is binding, so the fall in consumption is greater. Allowing the nominal rate to turned negative kept consumption at a higher level than than in case when the nominal interest rate is subjected to a lower zero bound.

The final point we want to make here is to draw attention to the trajectory of government debt. Since the evolution of bonds is determined by an accumulation equation, the stock of bonds rises initially with the increase in government spending. In these scenarios B_t remains at the higher level because the debt service $(1 + R_{t-1})B_{t-1}$ has not been offset by tax receipts $(\tau_t^w W_t L_t + \tau_t^c P_t C_t)$. Although debt service has dropped with the fall in the nominal interest rate, the economy is on a downward trajectory following the drop in G to its steady-state level. This is an example where fiscal austerity is not the policy option if the aim is to reduce the size of the government debt. Should measures be taken to keep the economy growing?

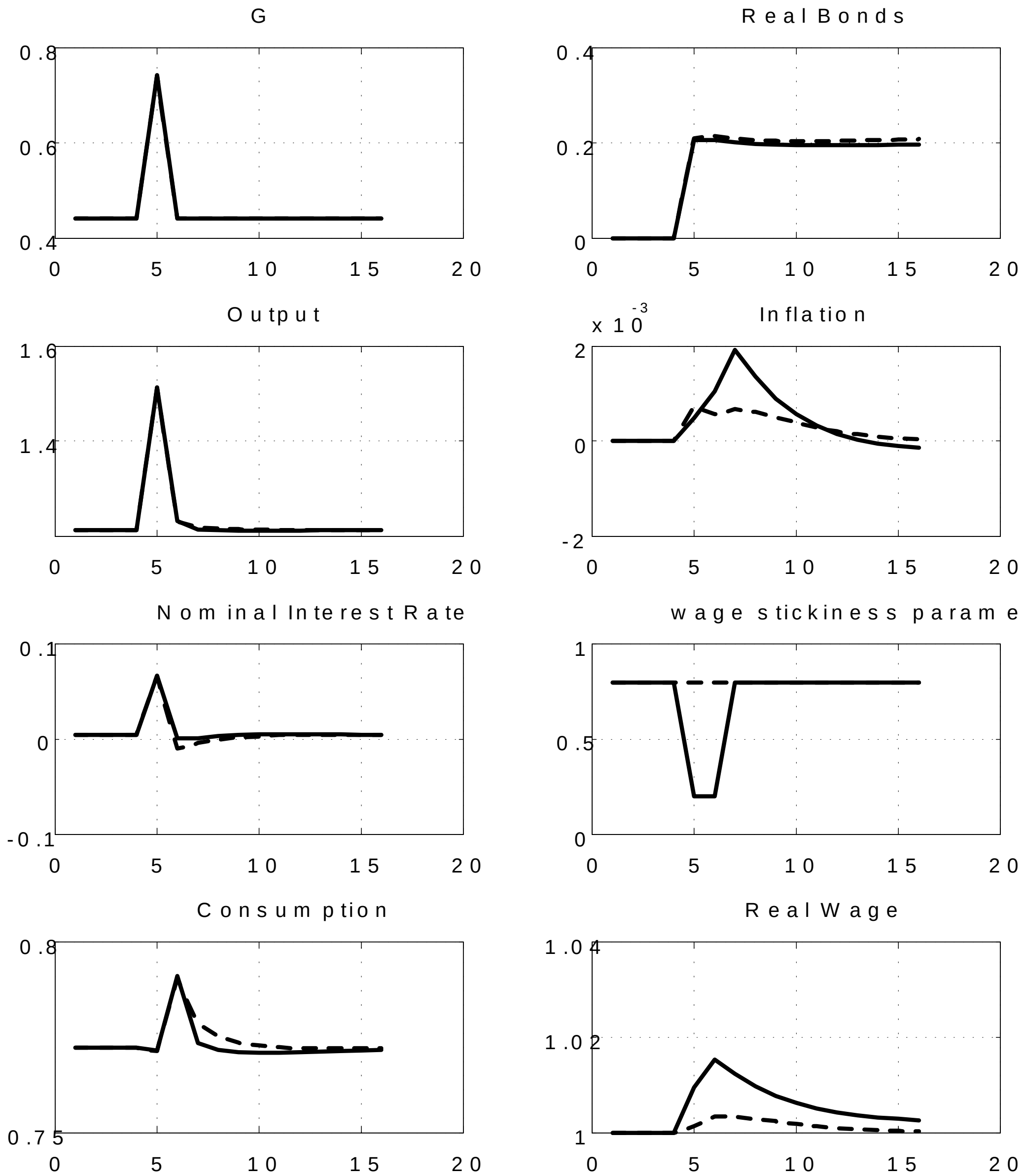

Figure 3a: Impulse Responses with Constant Tax Rates

4.2 Simulations with debt-contingent taxes

The question we pose is: what type of taxation should be in place to ensure that debt is gradually reduced. We suggest a number of scenarios based on tax-contingent rules as follows:

$$\begin{aligned} \tau_t^w &= \rho^w \tau_{t-1}^w + (1-\rho^w)(\tau_0^w + \phi^w(B_{t-1} - B^*)) \\ \tau_t^c &= \rho^c \tau_{t-1}^c + (1-\rho^c)(\tau_0^c + \phi^c(B_{t-1} - B^*)) \end{aligned}$$

In this scenario, B^* is the target debt level and we assume that it is reduced via changes in the tax rates on labor income and consumption. The steady-state non-contingent tax rates are τ_0^w and τ_0^c. The tax rates have persistence coefficients ρ^w and ρ^c which allows for some inertia in the adjustment of the tax rates to changes in debt.

We consider 4 scenarios with different values for the reaction coefficients ϕ_w and ϕ_c. The values of these reaction coefficients were chosen to keep tax changes within reasonable bounds. For example, in the absence of lagged effects ($\rho^w = \rho^c = 0$), when B jumped to 0.2 (see Figure 2a), a positive reaction coefficient of 0.2 would bring about a change in τ^w from 0.3 to 0.34 and a change in τ^c from 0.1 to 0.14, while a negative reaction coefficient of 0.1 would bring about a change in τ^w from 0.3 to 0.28 and a change in τ^c from 0.1 to 0.08.

Table 3: Scenarios

	Base case (a)	case (b)	case (c)	case (d)	case (e)
ρ^w, ρ^c	0.0	0.5	0.5	0.5	0.5
ϕ_w	0.0	0.2	-0.1	0.2	-0.1
ϕ_c	0.0	0.2	-0.1	-0.1	0.2

To obtain some idea about the likely effects of tax changes, we generated impulse responses for the scenarios and compared them to the base case (where debt increased following the fiscal stimulus). The impulses are shown in Figures 3 (b)-(e). In all cases similar dynamic patterns for output, inflation interest rate and the real wage prevailed. However, the key result here is that there are combinations of tax rates that can be used to stimulate the economy and bring down debt.

Recall, in the base case (a) with fixed tax rates, that B increased even though government spending was back at its steady-state level. In case (b) both tax rates were increased and we see that they were effective instruments for reducing government debt. Of course, additional austerity measures had been imposed to reduce the debt.

Since tax cuts can stimulate activity and thereby increase tax revenue, in case (c) we allowed for tax cuts in the state-contingent manner on both labor income and consumption. However in this case, debt became destabilized because the stimulus generated by the fall in both tax rates was insufficient to generate enough tax revenue to bring down debt.

In case (d) and (e) we checked out combinations of tax cuts and tax hikes. In case (d) we let the income tax increase, but let the consumption tax fall. This is a program of some tax relief in a period of fiscal consolidation. We find that this combination of a tax cut on consumption along with a tax increase on income, increased consumption and tax revenue by enough to reduce government debt.

Similarly in case (e) when we allowed the income tax rate to fall, but the consumption tax rate to rise, the policy combination also reduced the debt. In this combination the increase in consumption tax revenue compensated for the loss of tax revenue from the labor income tax cut. At the same time, while the increased tax rate on consumption reduced demand in the adjustment process, the fall was more than compensated for by the stronger increase in consumption from the income tax cuts.

The reduction in case (e) is however not as fast as that in case (d). The main reason for this is associated with the relative size of tax revenue from income and from consumption. For most economies (and in our model), the revenue from consumption taxes is smaller than the tax revenue from income. Hence the policy combination in case (d), of an income tax rise accompanied by a consumption tax cut, stabilized debt quickly because the reduction in tax revenue was more than compensated for by the increase in tax revenue from income tax.

However, the gain in tax revenue is at the expense of forgone consumption. While increases in either tax rates would lead to a negative consumption response, the negative response of consumption to a rise in the income tax is likely to be bigger. We can see this as follows. Given our normalizing assumptions, it works out that we can compute steady state consumption as a function of the tax rates (see below). The derived elasticities with respect to the tax rates show that the effect on consumption of a consumption tax will be smaller than the effect of an income tax.

$$\begin{aligned}
C_0 &= \left(\frac{(1-\tau_0^w)}{(1+\tau_0^c)}\right)^{(1+\varpi)/(\eta+\varpi)} \\
\frac{\partial C_0/C_0}{\partial \tau_o^w/\tau_o^w} &= \frac{(1+\varpi)}{(\eta+\varpi)}\left(\frac{\tau_0^w}{(1-\tau_0^w)}\right)(-1) \\
\frac{\partial C_0/C_0}{\partial \tau_0^c/\tau_0^c} &= \frac{(1+\varpi)}{(\eta+\varpi)}\left(\frac{\tau_0^c}{(1+\tau_0^c)}\right)(-1)
\end{aligned}$$

Thus although tax-rate relief in a period of fiscal consolidation, is more effective if it falls on consumption, it is also important to remember that there were negative effects on consumption from the hike in income tax. Our scenarios (d) and (e) illustrate the tradeoffs - debt reduction versus fall in consumption.

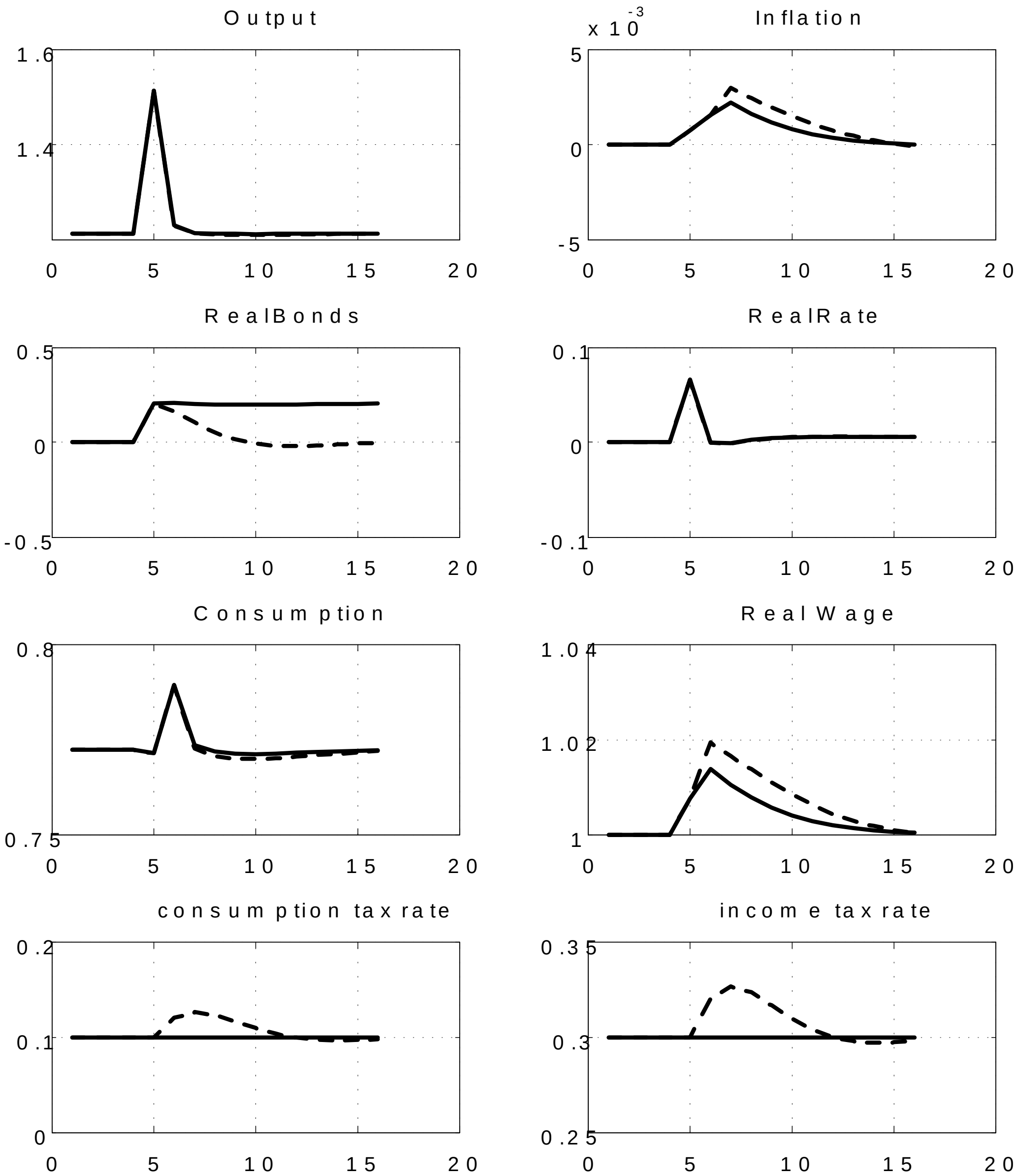

Figure 3b: Impulse Responses: Base case (solid line) and Case with debt-contingent tax rates resulting in an increase in both income and consumption tax rates (dashed line)

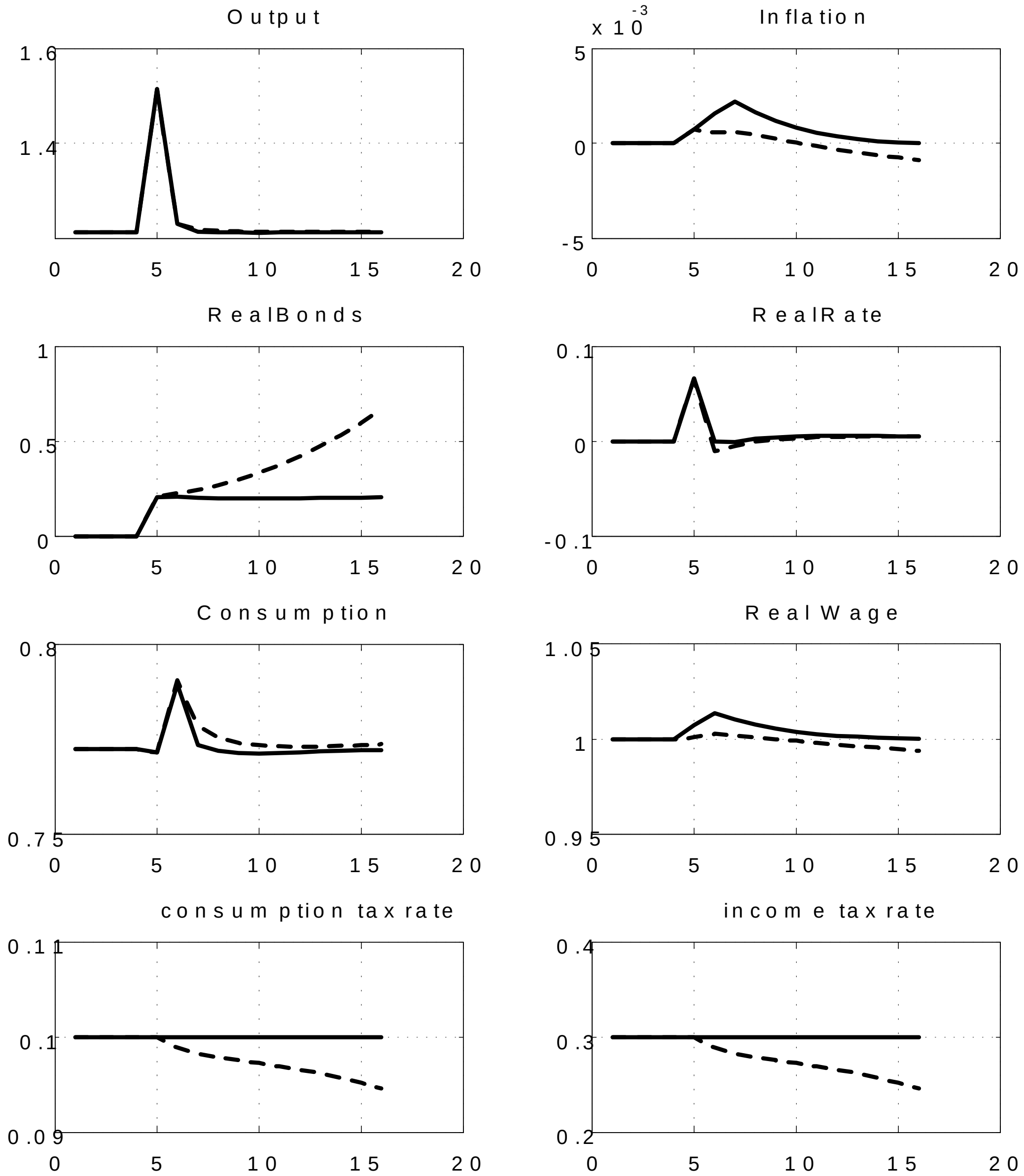

Figure 3c: Impulse Responses: Base case (solid line) and Case with a decrease in both income and consumption tax rates (dashed line)

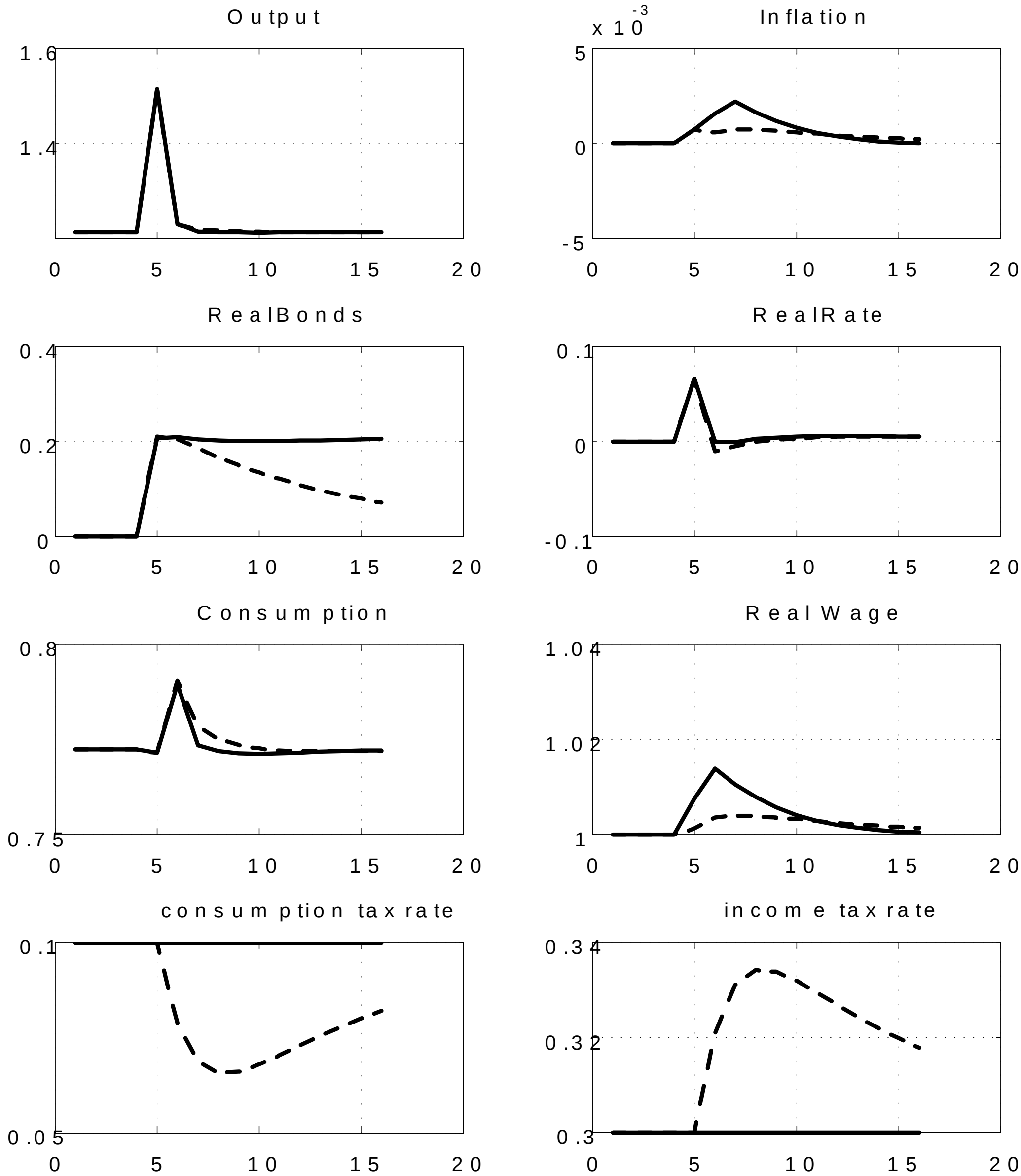

Figure 3d: Impulse Responses: Base case (solid line) and Case with an increase in income tax rate and a decrease in the consumption tax rate (dashed line)

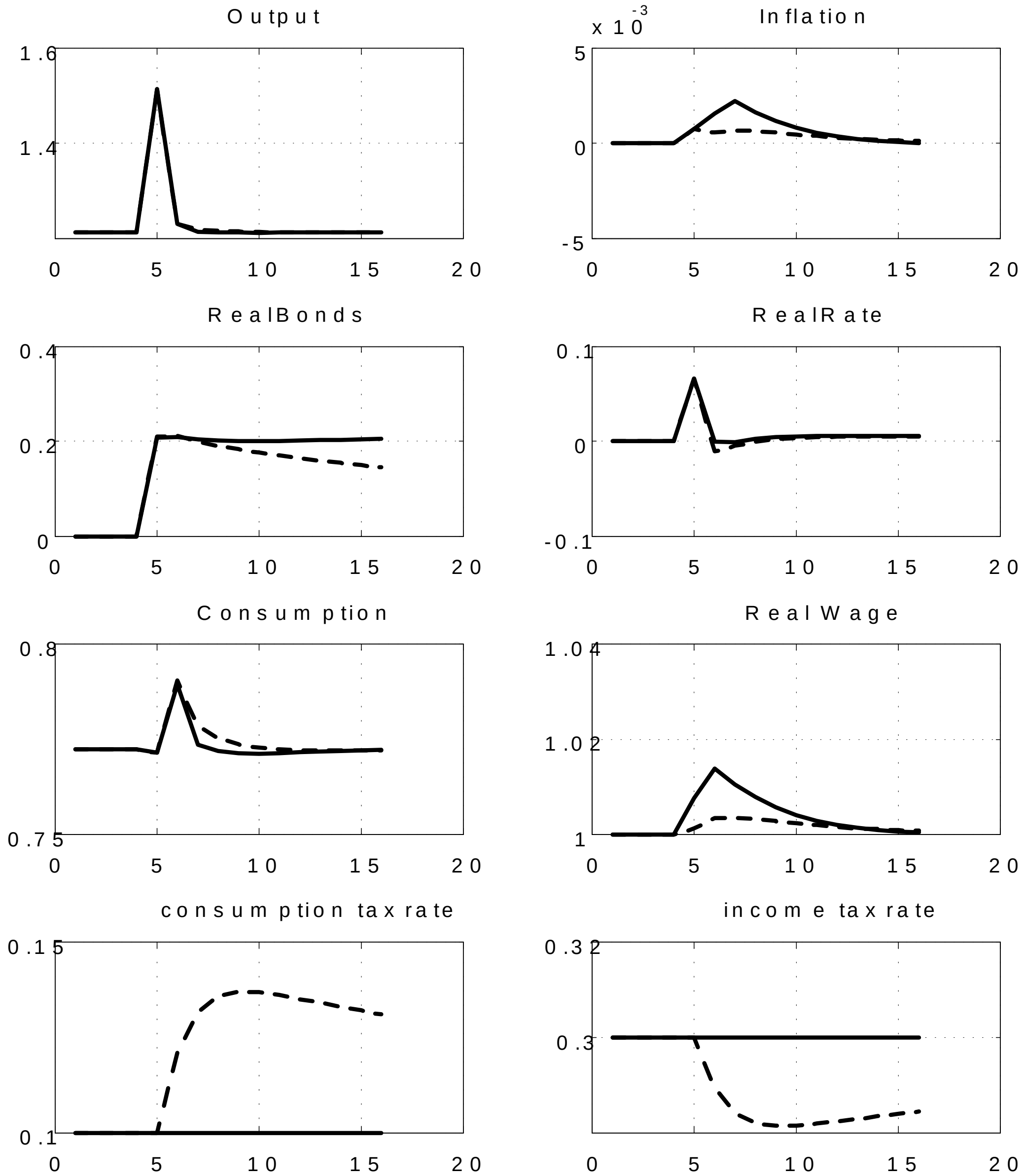

Figure 3e: Impulse Responses: Base case (solid line) and Case with an decrease in income tax rate and an increase in the consumption tax'rate (dashed line)

5 Concluding Remarks

This chapter has promoted the use of simulation methods to solve non-linear models. The specific example is to consider a model with a zero lower bound and with asymmetric wage adjustments. The scenarios in the paper show that the use of debt-contingent tax cuts on labor income, or consumption, can be effective ways to reduce debt, by stimulating labor income, consumption demand and tax revenue. These instruments are powerful and particularly useful when the interest rate is near or at the lower bound. Although the example was simple, the result is profound - there is a case for considered, careful tax relief, during a period of debt stabilization. But, there are trade-offs and the value of computational simulation analysis is that they are the right tools to evaluate the alternatives.

We also note the advantage of using the relatively faster perturbation method to generate the starting values for the decision rules for the forward-looking variables in the projection methods. Furthermore, while the perturbation methods no not allow the imposition of the zero lower bound and asymmetric wage response, for this simple example, the adjustment paths generated from the perturbation method were not very far off from those generated by the projection method. In other words, both perturbation and projection methods have their place in underpinning computational methods for economic analysis.

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