VEC and VAR Models

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Vector Error Correction and Vector Autoregressive Models, Part I

- The time-series properties of data and co-integrating relationships between pairs of nonstationary series were assumed that one of the variables was the dependent variable (y_t) and that the other was the independent variable (x_t), and the relationship between y_t and x_t was like a regression model
- A priori, unless we have good reasons not to, we could just as easily have assumed that y_t is the independent variable and x_t is the dependent variable

Vector Error Correction and Vector Autoregressive Models, Part II

- Our models could be:

$$y_t = \beta_{10} + \beta_{11} x_t + e_t^y, \quad e_t^y \sim N(0, \sigma_y^2)$$

$$x_t = \beta_{20} + \beta_{21} y_t + e_t^x, \quad e_t^x \sim N(0, \sigma_x^2)$$

  - For the first equation, we say that we have normalized on *y*, meaning that the coefficient in front of *y* is set to 1
  - For the second equation, we say that we have normalized on *x*, meaning that the coefficient in front of *x* is set to 1

Vector Error Correction and Vector Autoregressive Models, Part III

- We want to explore the causal relationship between pairs of time-series variables
  - We will discuss the **vector error correction (VEC)** and **vector autoregressive (VAR)** models
- We will learn how to estimate a **VEC model** when there is co-integration between I(1) variables and how to estimate a **VAR model** when there is no co-integration

Vector Error Correction and Vector Autoregressive Models, Part IV

- Terminology
  - Univariate analysis examines a single data series
  - Bivariate analysis examines a pair of series
  - The term "vector" indicates that we are considering a number of series: two, three, or more
  - The term ‘‘vector’’ is a generalization of the univariate and bivariate cases

VEC and VAR Models, Part I

- Consider the system of equations:

$$y_t = \beta_{10} + \beta_{11} y_{t-1} + \beta_{12} x_{t-1} + v_t^y$$

$$x_t = \beta_{20} + \beta_{21} y_{t-1} + \beta_{22} x_{t-1} + v_t^x$$

- Together, the equations constitute a system known as a VAR
  - In this example, because the maximum lag is of order 1, we have a VAR(1)

VEC and VAR Models, Part II

- If y and x are stationary I(0) variables, the above system can be estimated using least squares applied to each equation
- If *y* and *x* are nonstationary I(1) and not co-integrated, then we work with the first differences:

$$\Delta y_t = \beta_{11}\Delta y_{t-1} + \beta_{12}\Delta x_{t-1} + v_t^{\Delta y}$$

$$\Delta x_t = \beta_{21}\Delta y_{t-1} + \beta_{22}\Delta x_{t-1} + v_t^{\Delta x}$$

VEC and VAR Models, Part III

- Consider two nonstationary variables y_t and x_t that are integrated of order 1 so that:

$$y_t = \beta_0 + \beta_1 x_t + e_t$$

- The VEC model is:

$$\Delta y_t = \alpha_{10} + \alpha_{11}(y_{t-1} - \beta_0 - \beta_1 x_{t-1}) + v_t^y$$

$$\Delta x_t = \alpha_{20} + \alpha_{21}(y_{t-1} - \beta_0 - \beta_1 x_{t-1}) + v_t^x$$

VEC and VAR Models, Part IV

- We can expand these as:

$$y_t = \alpha_{10} + (\alpha_{11} + 1) y_{t-1} - \alpha_{11}\beta_0 - \alpha_{11}\beta_1 x_{t-1} + v_t^y$$

$$x_t = \alpha_{20} + \alpha_{21} y_{t-1} - \alpha_{21}\beta_0 - (\alpha_{21}\beta_1 - 1) x_{t-1} + v_t^x$$

- The coefficients α_{11}, α_{21} are known as error correction coefficients
  - They show how much Δy_t and Δx_t respond to the co-integrating error

$$y_{t-1} - \beta_0 - \beta_1 x_{t-1} = e_{t-1}$$

VEC and VAR Models, Part V

- Let's consider the role of the intercept terms
- Collect all the intercept terms and rewrite equation 13.5b as:

$$y_t = (\alpha_{10} - \alpha_{11}\beta_0) + (\alpha_{11} + 1)y_{t-1} - \alpha_{11}\beta_1 x_{t-1} + v_t^y$$

$$x_t = (\alpha_{20} - \alpha_{21}\beta_0) + \alpha_{21}y_{t-1} - (\alpha_{21}\beta_1 - 1)x_{t-1} + v_t^x$$

- If we estimate each equation by least squares, we obtain estimates of composite terms:

$$(\alpha_{10} - \alpha_{11}\beta_0)\; and (\alpha_{20} - \alpha_{21}\beta_0)$$

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Estimating a Vector Error Correction Model

Estimating a Vector Error Correction Model

- There are many econometric methods to estimate the VEC model
- A two-step least squares procedure is:
  1. Use least squares to estimate the co-integrating relationship and generate the lagged residuals
  2. Use least squares to estimate the equations:

$$\Delta y_t = \alpha_{10} + \alpha_{11}\hat{e}_{t-1} + v_t^y$$

$$\Delta x_t = \alpha_{20} + \alpha_{21}\hat{e}_{t-1} + v_t^x$$

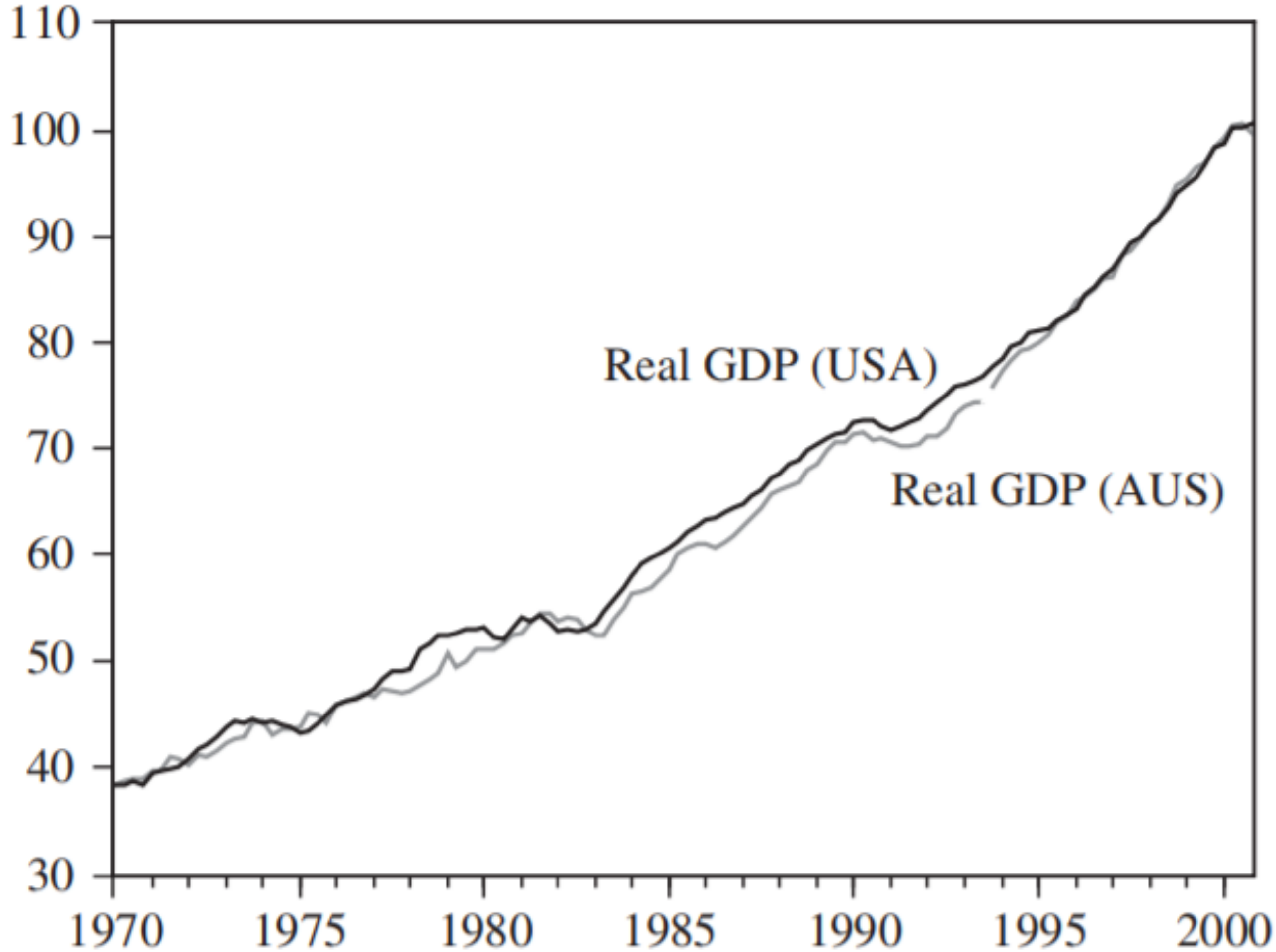

FIGURE 13.1 **Real gross domestic product (GDP = 100 in 2000).**

Example: VEC Model for GDP, Part I

- To check for co-integration, we obtain the fitted equation (the intercept term is omitted because it has no economic meaning):

$$\hat{A}_t = 0.985U_t$$

- A formal unit root test is performed and the estimated unit root test equation is:

$$\Delta\hat{e}_t = -0.128\hat{e}_{t-1}$$
$$(tau)\ \ (-2.889)$$

Example: VEC Model for GDP, Part II

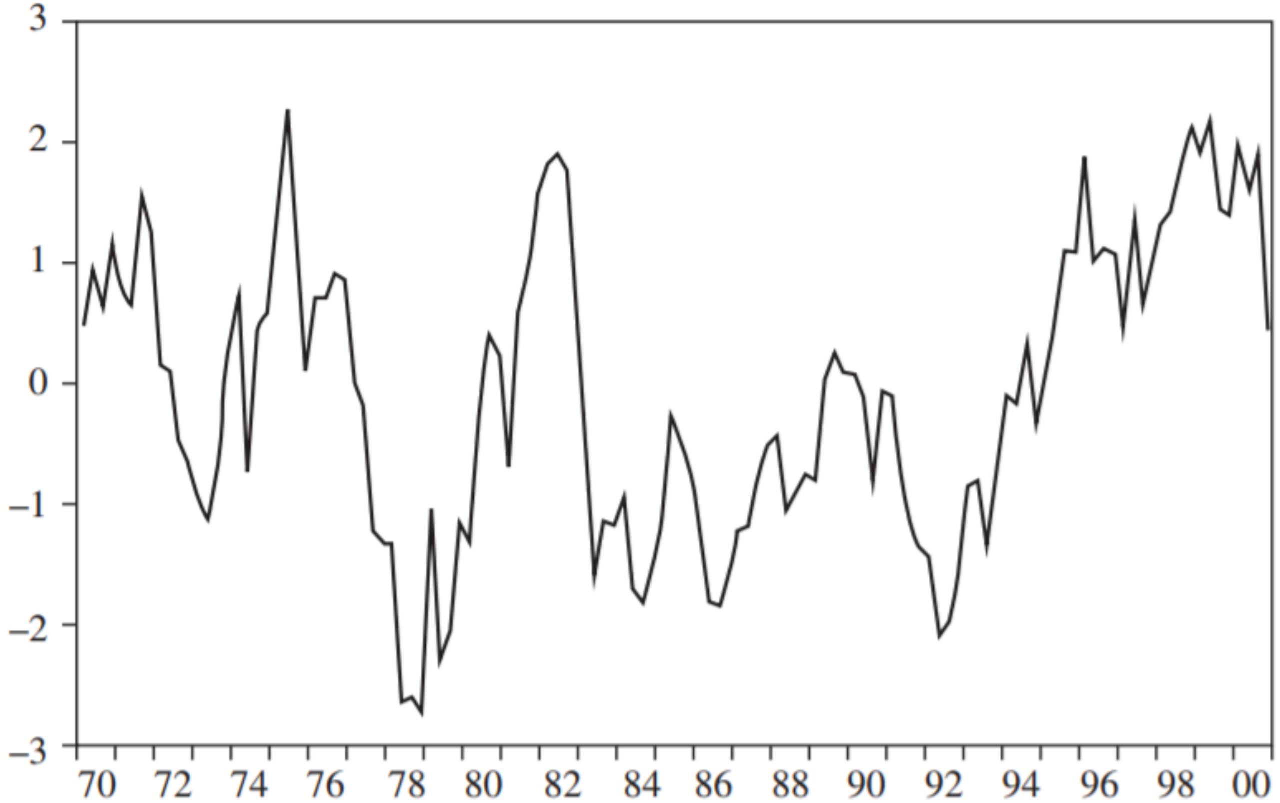

FIGURE 13.2 Residuals derived from the cointegrating relationship.

Example: VEC Model for GDP, Part III

- The estimated VEC model for $\{A_t, U_t\}$ is:

$$\widehat{\Delta A_t} = 0.492 - 0.099\hat{e}_{t-1}$$
$$(t) \qquad\qquad\qquad (\text{-}2.077)$$

$$\widehat{\Delta U_t} = 0.510 + 0.030\hat{e}_{t-1}$$
$$(t) \qquad\qquad\qquad (0.789)$$

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Estimating a VAR Model

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Estimating a VAR Model

- The VEC is a multivariate dynamic model that incorporates a co-integrating equation.
- It is relevant when, for the bivariate case, we have two variables, say y and x, that are both I(1), but are co-integrated.
- We now ask: What should we do if we are interested in the interdependencies between y and x but they are not co-integrated?

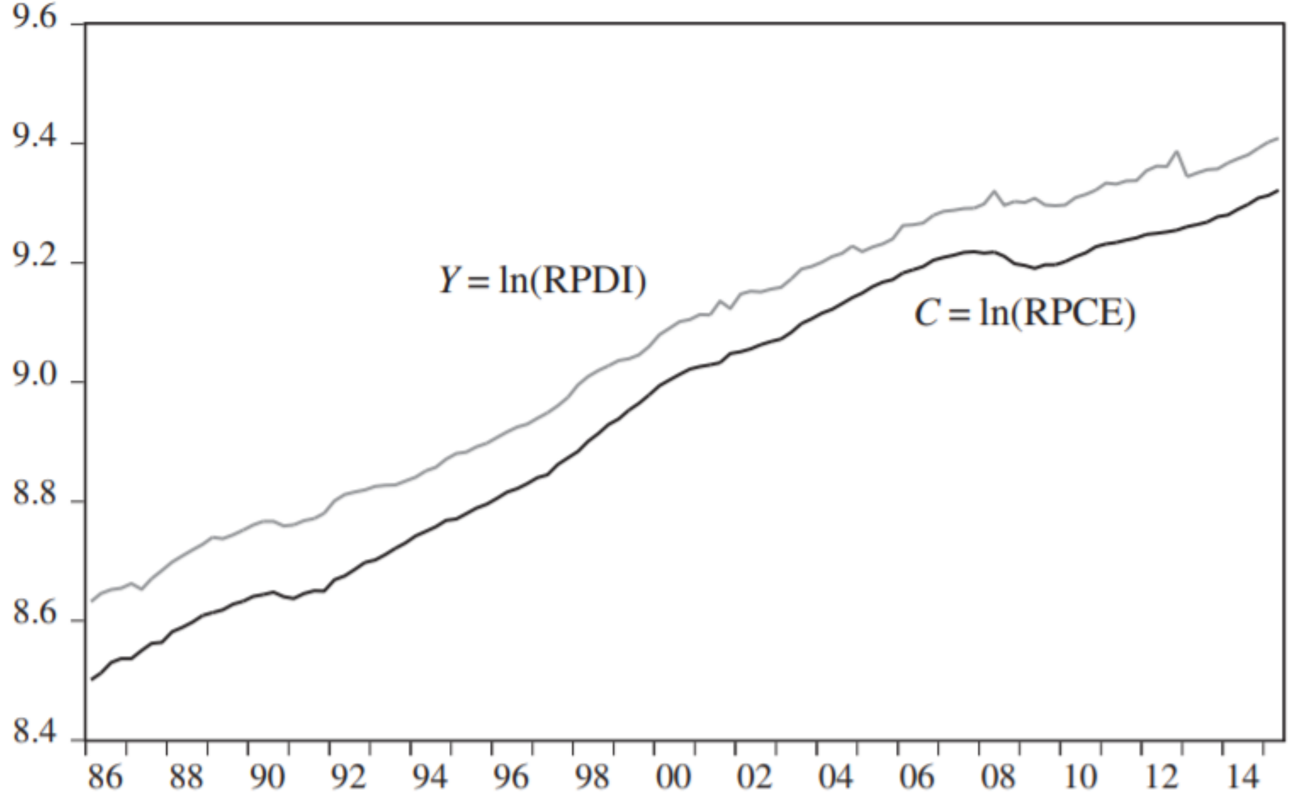

FIGURE 13.3 **The logarithms of real personal disposable income (RPDI) and real personal consumption expenditure (RPCE).**

Example: VAR Model for Consumption and Income, Part I

- Testing for co-integration yields the following results:

$$\hat{e}_t = C_t + 0.543 - 1.049Y_t$$

$$\widehat{\Delta \hat{e}_t} = -0.203\hat{e}_{t-1} - 0.290\Delta\hat{e}_{t-1}$$
$$(\tau) \quad (-3.046)$$

- An intercept term has been included to capture the component of (log) consumption that is independent of disposable income

Example VAR Model for Consumption and Income, Part II

- For illustrative purposes, the order of lag in this example has been restricted to one
  - In general, we should test for the significance of lag terms greater than one
  - The results are:

$$\Delta\hat{C}_t = 0.00367 + 0.348\Delta C_{t-1} + 0.131\Delta Y_{t-1}$$
$$(t) \quad (4.87) \qquad (4.02) \qquad (2.52)$$

$$\Delta\hat{Y}_t = 0.006 + 0.475\Delta C_{t-1} - 0.217\Delta Y_{t-1}$$
$$(t) \quad (3.38) \quad (3.96) \qquad (-3.25)$$

Example VAR Model for Consumption and Income, Part III

- The first equation (13.11a) shows that the quarterly growth in consumption (ΔC_t) is significantly related to its own past value ($\Delta\, C_{t-1}$) and also significantly related to the quarterly growth in last period's income ($\Delta\, Y_{t-1}$)
- The second equation (13.11b) shows that $\Delta\, Y_t$ is significantly negatively related to its own past value but significantly positively related to last period's change in consumption

Example VAR Model for Consumption and Income, Part II

- For illustrative purposes, the order of lag in this example has been restricted to one
  - In general, we should test for the significance of lag terms greater than one
  - The results are:

$$\Delta\hat{C}_t = 0.00367 + 0.348\Delta C_{t-1} + 0.131\Delta Y_{t-1}$$
$$(t) \quad (4.87) \qquad (4.02) \qquad (2.52)$$

$$\Delta\hat{Y}_t = 0.006 + 0.475\Delta C_{t-1} - 0.217\Delta Y_{t-1}$$
$$(t) \quad (3.38) \quad (3.96) \qquad (-3.25)$$

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Impulse Responses and Variance Decompositions

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Impulse Responses and Variance Decompositions

Impulse response functions and variance decompositions are techniques that are used by macroeconometricians to analyze problems such as:

- The effect of an oil price shock on inflation and GDP growth
- The effect of a change in monetary policy on the economy

Impulse Response Functions

- Impulse response functions show the effects of shocks on the adjustment path of the variables
- We consider:
  - The univariate case
  - The bivariate case

Impulse Response Functions: The Univariate Case

- Consider a univariate series: $y_t = \rho y_{t-1} + v_t$
  - The series is subject to a shock of size *v* in period 1
  - At time *t* = 1 following the shock, the value of *y* in period 1 and subsequent periods will be:

$$t = 1,\ \ y_1 = \rho y_0 + v_1 = v$$

$$t = 2,\ y_2 = \rho y_1 = \rho v$$

$$t = 3,\ \ y_3 = \rho y_2 = \rho(\rho y_1) = \rho^2 v$$

$$\ldots$$

$$\text{the shock is } v,\ \rho v,\ \rho^2 v, \ldots$$

Impulse Response Functions: The Univariate Case (cont.)

- The values of the coefficients $\{1, \rho, \rho^2, ...\}$ are known as multipliers, and the time path of *y* following the shock is known as the impulse response function
- To illustrate, assume that $\rho = 0.9$ and let the shock be unity: $v = 1$; according to the analysis, *y* will be {1, 0.9, 0.81, …}, approaching zero over time

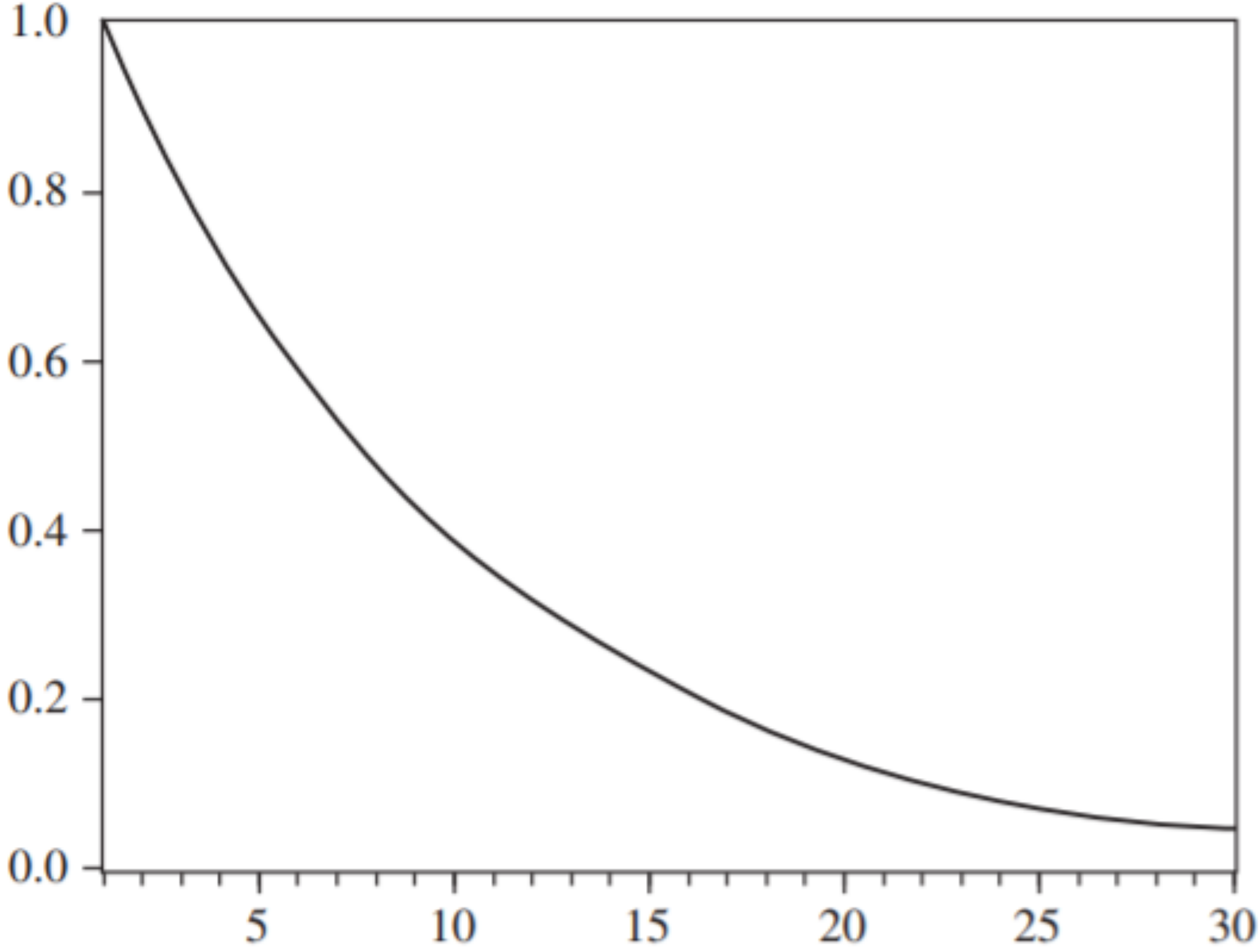


FIGURE 13.4 Impulse responses for an AR(1) model $y_t = 0.9y_{t-1} + e_t$ following a unit shock.

Impulse Response Functions: The Bivariate Case, Part I

- Consider an impulse response function analysis with two time series based on a bivariate VAR system of stationary variables:

$$y_t = \delta_{10} + \delta_{11} y_{t-1} + \delta_{12} x_{t-1} + v_t^y$$

$$x_t = \delta_{20} + \delta_{21} y_{t-1} + \delta_{22} x_{t-1} + v_t^x$$

Impulse Response Functions: The Bivariate Case, Part II

- The mechanics of generating impulse responses in a system is complicated by the fact that:
  1. One has to allow for interdependent dynamics (the multivariate analog of generating the multipliers)
  2. One has to identify the correct shock from unobservable data
- Together, these two complications lead to what is known as the **identification problem**

Impulse Response Functions: The Bivariate Case, Part III

- Consider the case when there is a one–standard deviation shock (alternatively called an **innovation**) to *y*
  1. When t = 1, the effect of a shock of size σ_y on y is $y_1 = v_1^y = \sigma_y$, and the effect on $x_1 = v_1^x = 0$
  2. When t = 2, the effect of the shock on y is $y_2 = \delta_{11}y_1 + \delta_{12}x_1 = \delta_{11}\sigma_y + \delta_{12}0 = \delta_{11}\sigma_y$
     - And the effect on x is $x_2 = \delta_{21}y_1 + \delta_{22}x_1 = \delta_{21}\sigma_y + \delta_{22}0 = \delta_{21}\sigma_y$

Impulse Response Functions: The Bivariate Case, Part IV

3. When t = 3, the effect of the shock on y is $y_3 = \delta_{11}y_2 + \delta_{12}x_2 = \delta_{11}\delta_{11}\sigma_y + \delta_{12}\delta_{21}\sigma_y$
   - And the effect on x is $x_3 = \delta_{21}y_2 + \delta_{22}x_2 = \delta_{21}\delta_{11}\sigma_y + \delta_{22}\delta_{21}\delta_{21}\sigma_y$

- Impulse response to y on y: $\sigma_y\{1,\ \delta_{11},\ (\delta_{11}\delta_{11} + \delta_{12}\delta_{21}), \ldots\}$
- Impulse response to y on x: $\sigma_y\{0,\ \delta_{21},\ (\delta_{21}\delta_{11} + \delta_{22}\delta_{21}), \ldots\}$

Impulse Response Functions: The Bivariate Case, Part V

Now let $v_1^x = \sigma_x$, $v_t^x = 0$ for $t > 1$, $v_t^y = 0$ for all t:

$$t = 1 \quad y_1 = v_1^y = 0$$

$$x_1 = v_t^x = \sigma_x$$

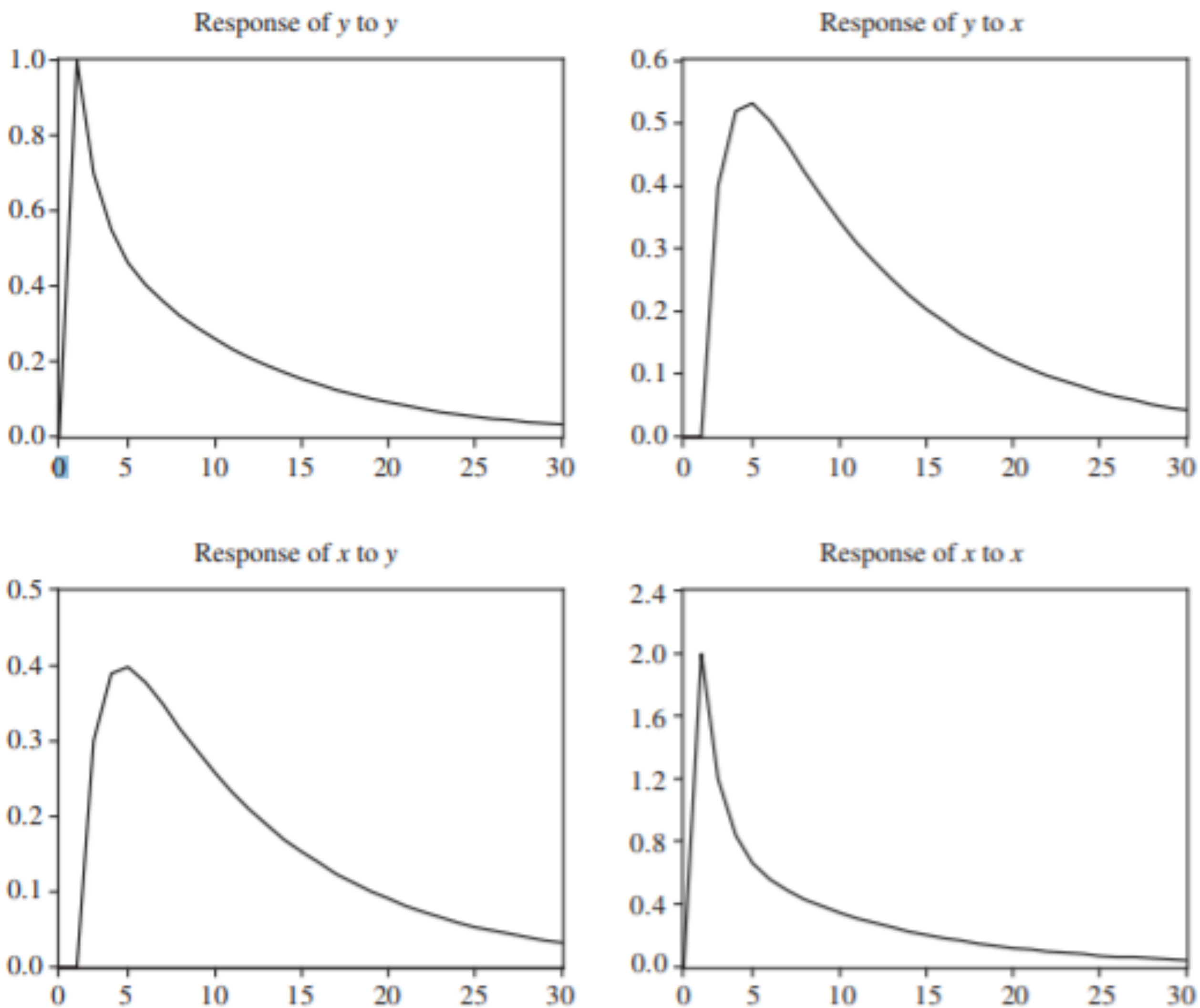
$$t = 2 \quad y_2 = \delta_{11} y_1 + \delta_{12} x_1 = \delta_{11} 0 + \delta_{12} \sigma_x = \delta_{12} \sigma_x$$

$$x_2 = \delta_{21} y_1 + \delta_{22} x_1 = \delta_{21} 0 + \delta_{22} \sigma_x = \delta_{22} \sigma_x$$

...

impulse response to x on y: $\sigma_x \{0,\ \delta_{12},\ \left(\delta_{11}\delta_{12} + \delta_{12}\delta_{22}\right), \ldots\}$

impulse response to x on x: $\sigma_x \{1,\ \delta_{22},\ \left(\delta_{21}\delta_{12} + \delta_{22}\delta_{22}\right), \ldots\}$


Response of y to y
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20
25
30
Response of y to x
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
5
10
15
20
25
30
Response of x to y
0.5
0.4
0.3
0.2
0.1
0.0
0
5
10
15
20
25
30
Response of x to x
2.4
2.0
1.6
1.2
0.8
0.4
0.0
0
5
10
15
20
25
30

Forecast Error Variance Decompositions

Another way to disentangle the effects of various shocks is to consider the contribution of each type of shock to the forecast error variance.

Forecast Error Variance Decompositions: Univariate Analysis

- Consider a univariate series: $y_t = \rho y_{t-1} + v_t$
- The best one-step-ahead forecast (alternatively the forecast one period ahead) is:

$$y_t = \rho y_{t-1} + v_t$$

$$y_{t+1}^F = E_t[\rho y_t + v_{t+1}]$$

$$y_{t+1} - E_t[y_{t+1}] = y_{t+1} - \rho y_t = v_{t+1}$$

$$y_{t+2}^F = E_t[\rho y_{t+1} + v_{t+2}] = E_t[\rho(\rho y_t + v_{t+1}) + v_{t+2}] = \rho^2 y_t$$

$$y_{t+2} - E_t[y_{t+2}] = y_{t+2} - \rho^2 y_t = \rho v_{t+1} + v_{t+2}$$

Forecast Error Variance Decompositions: Univariate Analysis (cont.)

- In this univariate example, there is only one shock that leads to a forecast error
  - The forecast error variance is 100% due to its own shock
  - The exercise of attributing the source of the variation in the forecast error is known as variance decomposition

Forecast Error Variance Decompositions: Bivariate Analysis, Part I

- We can perform a variance decomposition for our special bivariate example where there is no identification problem
  - Ignoring the intercepts (because they are constants), the one-step-ahead forecasts are:
  - $y^F_{t+1} = E_t[\delta_{11} y_t + \delta_{12} x_t + v^y_{t+1}] = \delta_{11} y_t + \delta_{12} x_t$
  - $x^F_{t+1} = E_t[\delta_{21} y_t + \delta_{22} x_t + v^x_{t+1}] = \delta_{21} y_t + \delta_{22} x_t$

Forecast Error Variance Decompositions: Bivariate Analysis, Part II

- The corresponding one-step-ahead forecast errors and variances are:
  - $FE_1^y = y_{t+1} - E_t[y_{t+1}] = v_{t+1}^y$;   var$(FE_1^y) = \sigma_y^2$
  - $FE_1^x = x_{t+1} - E_t[x_{t+1}] = v_{t+1}^x$;   var$(FE_1^x) = \sigma_x^2$
- The two-steps-ahead forecast for *y* is $y_{t+2}^F = E_t[\delta_{11} y_{t+1} + \delta_{12} x_{t+1} + v_{t+2}^y]$
  $= E_t[\delta_{11}\left(\delta_{11} y_t + \delta_{12} x_t + v_{t+1}^y\right) + \delta_{12}(\delta_{21} y_t + \delta_{22} x_t + v_{t+1}^x) + v_{t+2}^y = \delta_{11}(\delta_{11} y_t + \delta_{12} x_t) + \delta_{12}(\delta_{21} y_t + \delta_{22} x_t)$

Forecast Error Variance Decompositions: Bivariate Analysis, Part III

- The two-steps-ahead forecast for *x* is:
  - $x_{t+2}^{F} = E_t[\delta_{21} y_{t+1} + \delta_{22} x_{t+1} + v_{t+2}^{x}]$
  - $= E_t[\delta_{21}(\delta_{11} y_t + \delta_{12} x_t + v_{t+1}^{y}) + \delta_{22}(\delta_{21} y_t + \delta_{22} x_t + v_{t+1}^{x}) + v_{t+2}^{x}]$
  - $= \delta_{21}(\delta_{11} y_t + \delta_{12} x_t) + \delta_{22}(\delta_{21} y_t + \delta_{22} x_t)$

Forecast Error Variance Decompositions: Bivariate Analysis, Part IV

- The corresponding two-steps-ahead forecast errors and variances are:
  - $FE_2^y = y_{t+2} - E_t[y_{t+2}] = [\delta_{11}v_{t+1}^y + \delta_{12}v_{t+1}^x + v_{t+2}^y]$
  - $var(FE_2^y) = \delta_{11}^2\sigma_y^2 + \delta_{12}^2\sigma_x^2 + \sigma_y^2$
  - $FE_2^x = x_{t+2} - E_t[x_{t+2}] = [\delta_{21}v_{t+1}^y + \delta_{22}v_{t+1}^x + v_{t+2}^x]$
  - $var(FE_2^x) = \delta_{21}^2\sigma_y^2 + \delta_{22}^2\sigma_x^2 + \sigma_x^2$

Forecast Error Variance Decompositions: Bivariate Analysis, Part V

- This decomposition is often expressed in proportional terms
  - The proportion of the two-steps-ahead forecast error variance of *y* explained by its own shock is $\left(\delta_{11}^2\sigma_y^2 + \sigma_y^2\right)/\left(\delta_{11}^2\sigma_y^2 + \delta_{12}^2\sigma_x^2 + \sigma_y^2\right)$
  - The proportion of the two-steps-ahead forecast error variance of *y* explained by the other shock is $(\delta_{12}^2\sigma_x^2)/\left(\delta_{11}^2\sigma_y^2 + \delta_{12}^2\sigma_x^2 + \sigma_y^2\right)$

Forecast Error Variance Decompositions: Bivariate Analysis, Part VI

- Similarly, the proportion of the two-steps-ahead forecast error variance of *x* explained by its own shock is $(\delta_{22}^2\sigma_x^2 + \sigma_x^2)/\left(\delta_{21}^2\sigma_y^2 + \delta_{22}^2\sigma_x^2 + \sigma_x^2\right)$
- The proportion of the forecast error of *x* explained by the other shock is $\left(\delta_{21}^2\sigma_y^2\right)/\left(\delta_{21}^2\sigma_y^2 + \delta_{22}^2\sigma_x^2 + \sigma_x^2\right)$

Forecast Error Variance Decompositions: The General Case

Contemporaneous interactions and correlated errors complicate the identification of the nature of shocks and hence the interpretation of the impulses and decomposition of the causes of the forecast error variance.

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Key Words

- Dynamic relationships
- Error correction
- Forecast error variance decomposition
- Identification problem
- Impulse response functions
- VAR model
- VEC model

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