Interval Estimation

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Interval Estimation

- Interval estimation proposes a range of values in which the true parameter β_2 is likely to fall.
- Providing a range of values gives a sense of what the parameter value might be, and the precision with which we have estimated it.
- Such intervals are often called confidence intervals.
- We prefer to call them interval estimates because the term “confidence” is widely misunderstood and misused.

The t-Distribution, Part I

- The normal distribution of b_2, the least squares estimator of β_2, is:

$$b_2 \mid x \sim N\left(\beta_2, \frac{\sigma^2}{\sum\left(x_i - \bar{x}\right)^2}\right)$$

- A standardized normal random variable is obtained from b_2 by subtracting its mean and dividing by its standard deviation:

$$Z = \frac{b2 - \beta_2}{\sqrt{\sigma^2 / \sum\left(x_i - \bar{x}\right)^2}} \sim N(0,1)$$

The t-Distribution, Part II

- We know that:

$$\mathrm{P}\left(-1.96 \le \mathrm{Z} \le 1.96\right) = 0.95$$

- Substituting:

$$P\left(-1.96 \le \frac{b2 - \beta_2}{\sqrt{\sigma^2 / \sum \left(x_i - \bar{x}\right)^2}} \le 1.96\right) = 0.95$$

- Rearranging:

$$P\left(b_2 - 1.96\sqrt{\sigma^2 / \sum \left(x_i - \bar{x}\right)^2} \le \beta_2 \le b_2 + 1.96\sqrt{\sigma^2 / \sum \left(x_i - \bar{x}\right)^2}\right) = 0.95$$

The t-Distribution, Part III

- The two end-points $b_2 \pm 1.96\sqrt{\sigma^2 / \sum (x_i - \bar{x})^2}$ provide an interval estimator
- In repeated sampling, 95% of the intervals constructed this way will contain the true value of the parameter β_2
- This easy derivation of an interval estimator is based on the assumption SR6 *and* that we know the variance of the error term σ^2

The t-Distribution, Part IV

- Replacing σ^2 with creates a random variable *t:*
  - (3.2) $t = \frac{b_2 - \beta_2}{\sqrt{\sigma^2 / \sum (x_i - \bar{x})^2}} = \frac{b_2 - \beta_2}{\sqrt{\widehat{\mathrm{var}}(b_2)}} = \frac{b_2 - \beta_2}{se(b_2)} \sim t_{(N-2)}$
- This substitution changes the probability distribution from standard normal to a t-distribution with N − 2 degrees of freedom
- We denote this as $\mathrm{t} \sim \mathrm{t}_{(\mathrm{N}-2)}$

The t-Distribution, Part V

- The *t*-distribution is a bell-shaped curve centered at zero
- It looks like the standard normal distribution, except it is more spread out, with a larger variance and thicker tails
- The shape of the *t*-distribution is controlled by a single parameter called the **degrees of freedom**, often abbreviated as *df*

Obtaining Interval Estimates, Part I

- We can find a "critical value" from a t-distribution such that:

$$P(t \geq t_c) = P(t \leq -t_c) = \alpha/2$$

where α is a probability often taken to be α = 0.01 or α = 0.05

- The critical value t_c for degrees of freedom m is the percentile value $t_{(1-\alpha/2,\, m)}$

Obtaining Interval Estimates, Part II

- Each shaded "tail" area contains α/2 of the probability so that 1 – α of the probability is contained in the center portion
  - Consequently, we can make the probability statement:
    - (3.4) $$P\left(-t_c \le t \le t_c\right) = 1 - \alpha$$
    - Or $$P\left(-t_c \le \frac{b_k - \beta_k}{se\left(b_k\right)} \le t_c\right) = 1 - \alpha$$
    - (3.5) $$P\left[b_k - t_c se\left(b_k\right) \le \beta_k \le t_c + t_c se\left(b_k\right)\right] = 1 - \alpha$$

Obtaining Interval Estimates, Part III

- When b_k and $se(b_k)$ are estimated values (numbers) based on a given sample of data, then $b_k \pm t_c se(b_k)$ is called a $100(1 - \alpha)\%$ interval estimate of b_k
  - Equivalently, it is called a $100(1 - \alpha)\%$ confidence interval
  - Usually $\alpha = 0.01$ or $\alpha = 0.05$ so that we obtain a 99% confidence interval or a 95% confidence interval

Obtaining Interval Estimates, Part IV

- The interpretation of confidence intervals requires a great deal of care
  - The properties of the interval estimation procedure are based on the notion of repeated sampling
  - Any one interval estimate, based on one sample of data, may or may not contain the true parameter β_k, and because β_k is unknown, we will never know whether it does or does not
  - When ‘‘confidence intervals’’ are discussed, remember that our confidence is in the procedure used to construct the interval estimate; it is not in any one interval estimate calculated from a sample of data

The Sampling Context

- The household food example variation is due to the fact that in each sample household food expenditures are different
- Sampling variability causes the:
  - Center of each of the interval estimates to change with the values of the least squares estimates
  - The widths of the intervals to change with the standard errors
- Interval estimators are a convenient way to report regression results because they combine point estimation with a measure of sampling variability to provide a range of values in which the unknown parameters might fall

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Hypothesis Tests

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Hypothesis Tests

- Hypothesis testing procedures compare a conjecture we have about a population with the information contained in a sample of data
- In each and every hypothesis test, five ingredients must be present:
  1. A null hypothesis
  2. An alternative hypothesis
  3. A test statistic
  4. A rejection region
  5. A conclusion

The Null Hypothesis

- The **null hypothesis**, which is denoted by (H-naught), specifies a value for a regression parameter
  - Which for generality we denote as , for k = 1 or 2
- The null hypothesis is stated as : = c, where c is a constant
- A null hypothesis is the belief we will maintain until we are convinced by the sample evidence that it is not true, in which case we reject the null hypothesis

The Alternative Hypothesis

- Paired with every null hypothesis is a logical **alternative hypothesis** that we will accept if the null hypothesis is rejected
- For the null hypothesis : = c, the three possible alternative hypotheses are as follows:
    1. $H_1: \beta_k > c$: in this case, leads us to accept the conclusion that > c
    2. $H_1: \beta_k < c$: in this case, leads us to accept the conclusion that < c
    3. $H_1: \beta_k \neq c$: takes a value either larger or smaller than c

The Test Statistic

- The sample information about the null hypothesis is embodied in the sample value of a test statistic
- A test statistic has a special characteristic.
  - Its probability distribution is completely known when the null hypothesis is true
  - It has some other distribution if the null hypothesis is not true
- If the null hypothesis H0:βk = c is true, then we can substitute c for βk, and it follows that:
  - (3.7) $t = \dfrac{b_k - c}{\text{se}\left(b_k\right)} \sim t_{(N-2)}$

The Rejection Region

- The rejection region depends on the form of the alternative. It is the range of values of the test statistic that leads to rejection of the null hypothesis
- It is possible to construct a rejection region only if we have:
    - A test statistic whose distribution is known when the null hypothesis is true
    - An alternative hypothesis
    - A level of significance
- The rejection region consists of values that are unlikely and that have low probability of occurring when the null hypothesis is true

The Rejection Region (cont.)

- The level of significance of the test α is usually chosen to be 0.01, 0.05, or 0.10
- If we reject the null hypothesis when it is true, then we commit what is called a **Type I error**
  - We can specify the amount of Type I error we will tolerate by setting the level of significance α
- If we do not reject a null hypothesis that is false, then we have committed a **Type II error**
  - We cannot control or calculate the probability of this type of error

A Conclusion

- When you have completed testing a hypothesis, you should state your conclusion.
- Do you reject the null hypothesis, or do you not reject the null hypothesis?
- You should avoid saying that you “accept” the null hypothesis, which can be very misleading.
- Say what the conclusion means in the economic context of the problem you are working on and the economic significance of the finding.

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Rejection Regions for Specific Alternatives

Rejection Regions for Specific Alternatives

- In this section, we hope to be very clear about the nature of the rejection rules for each of the three possible alternatives to the null hypothesis
- To have a rejection region for a null hypothesis:
  1. We need a test statistic
  2. We need a specific alternative, $\beta_k > c$, $\beta_k < c$, $or\ \beta_k \neq c$
  3. We need to specify the level of significance of the test

One-Tail Tests With Alternative "Greater Than" (>)

- When testing the null hypothesis $H_0{:}\beta_k = c$ against the alternative hypothesis, reject the null hypothesis and accept the alternative hypothesis if
- The test is called a "one-tail" test because unlikely values of the t-statistic fall only in one tail of the probability distribution

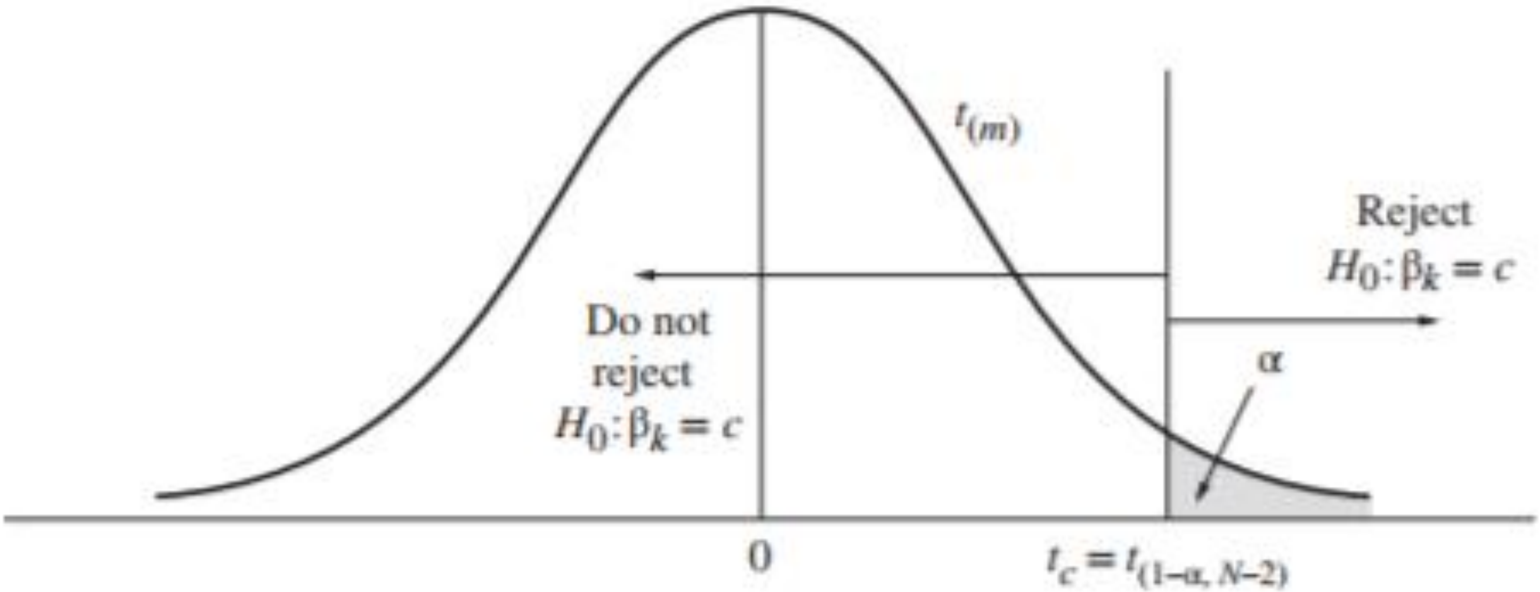

FIGURE 3.2 **Rejection region for a one-tail test of $H_0: \beta_k = c$ against $H_1: \beta_k > c$.**

One-Tail Tests With Alternative "Less Than" (<)

- When testing the null hypothesis $H_0\text{:}\beta_k = c$ against the alternative hypothesis, reject the null hypothesis and accept the alternative hypothesis if $t \leq t_{(1 - \alpha;\, N - 2)}$
- When using Statistical Table 2 to locate critical values, recall that the t-distribution is symmetric about zero so that the α-percentile is the negative of the percentile

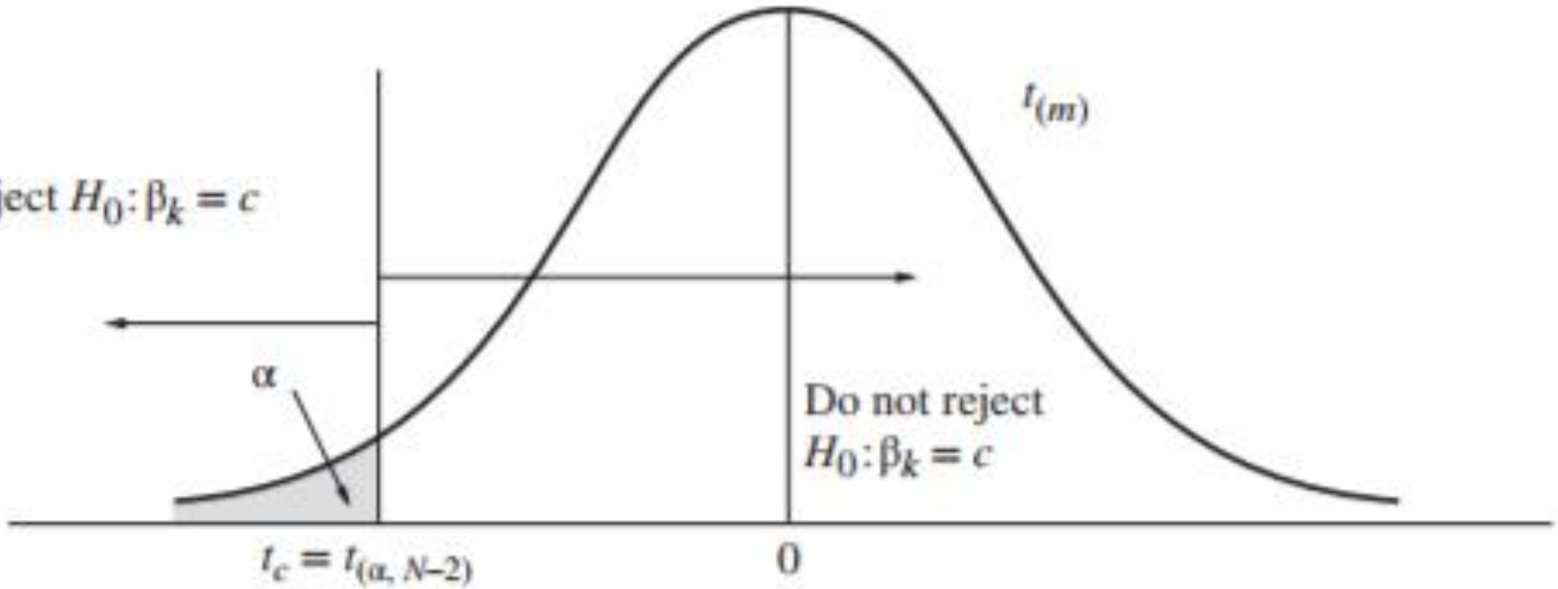

FIGURE 3.3 **The rejection region for a one-tail test of $H_0: \beta_k = c$ against $H_1: \beta_k < c$.**

Two-Tail Tests With Alternative "Not Equal To" (≠)

- When testing the null hypothesis H_0:$\beta_k = c$ against the alternative hypothesis H_1:$\beta_k \neq c$, reject the null hypothesis and accept the alternative hypothesis if

  $t \leq t_{(1-\alpha;\, N-2)}$ or $t \geq t_{(1-\alpha;\, N-2)}$
- Because the rejection region is composed of portions of the t-distribution in the left and right tails, this test is called a **two-tail test**

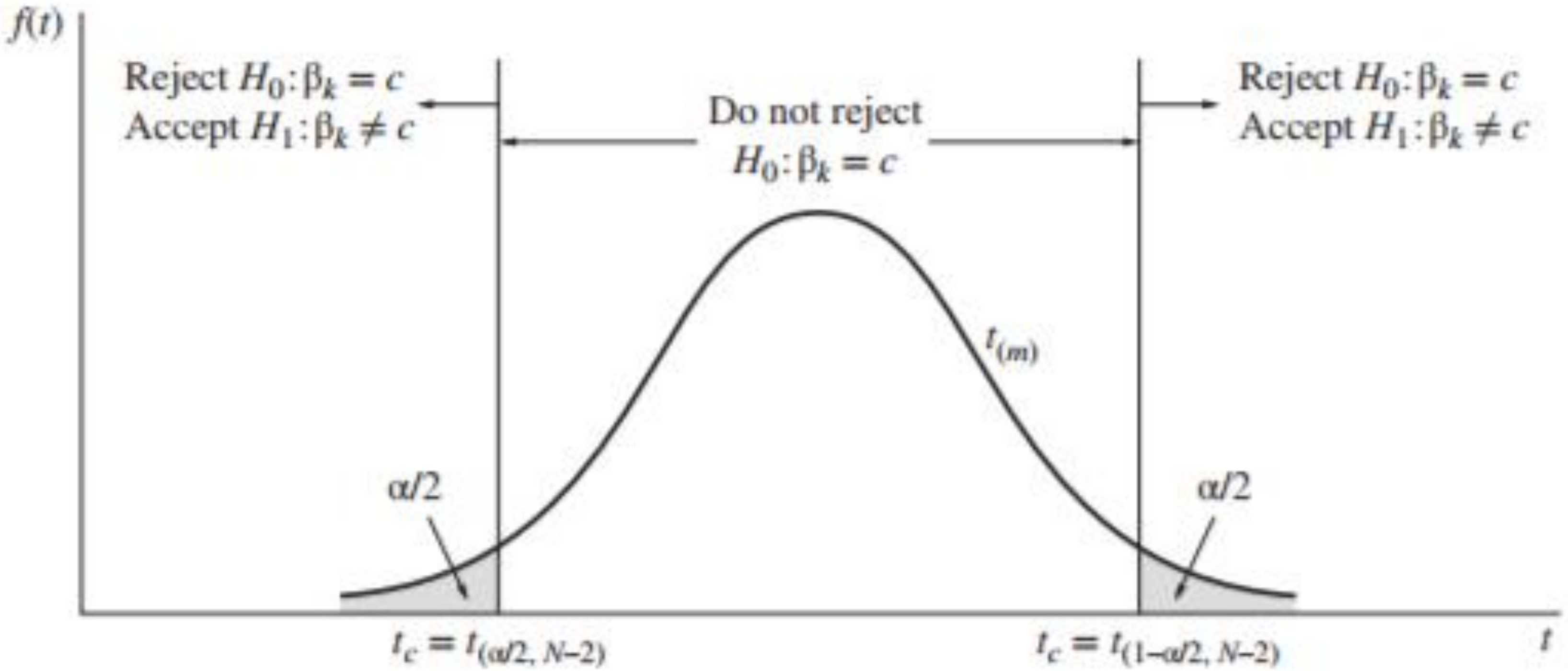

FIGURE 3.4 Rejection region for a test of $H_0: \beta_k = c$ against $H_1: \beta_k \neq c$.

Examples of Hypothesis Tests

Step-by-step procedure for testing hypotheses

1. Determine the null and alternative hypotheses
2. Specify the test statistic and its distribution if the null hypothesis is true
3. Select α and determine the rejection region
4. Calculate the sample value of the test statistic
5. State your conclusion

Example: Right-Tail Test of Significance

- The null hypothesis is $H_0{:}\beta_2 = 0$; the alternative hypothesis is $H_1{:}\beta_2 > 0$
- The test statistic is 3.7; in this case, c = 0, so $P(-1.96 \le Z \le 1.96) = 0.95$ if the null hypothesis is true
- Select α = 0.05
  - The critical value for the right-tail rejection region is the 95th percentile of the *t*-distribution with $N - 2 = 38$ degrees of freedom, $t_{(0.95,38)} = 1.686$
  - Thus, we will reject the null hypothesis if the calculated value of $t \geq 1.686$
  - If $t < 1.686$, we will not reject the null hypothesis

Example: Right-Tail Test of Significance (cont.)

- Using the food expenditure data, we found that b_2 = 10.21 with standard error se(b_2) = 2.09
  - The value of the test statistic is $t = \frac{b_2}{\mathrm{se}(b_2)} = \frac{10.21}{2.09} = 4.88$
- Because t = 4.88 > 1.686, we reject the null hypothesis that $\beta_2 = 0$ and accept the alternative that $\beta_2 > 0$
- That is, we reject the hypothesis that there is no relationship between income and food expenditure, and conclude that there is a *statistically significant* positive relationship between household income and food expenditure

Example: Right-Tail Test of an Economic Hypothesis

- The null hypothesis is $H_0 : \beta_2 \leq 5.5$; the alternative hypothesis is $H_1 : \beta_2 > 5.5$
- The test statistic is $t = (b_2 - 5.5)/\text{se}(b_2) \sim t_{(N-2)}$ if the null hypothesis is true
- Select $\alpha = 0.01$
  - The critical value for the right-tail rejection region is the 99th percentile of the t-distribution with $N - 2 = 38$ degrees of freedom, $t_{(0.99,38)} = 2.429$
  - Thus, we will reject the null hypothesis if the calculated value of $t \geq 2.429$
  - If $t < 2.429$, we will not reject the null hypothesis

Example: Right-Tail Test of an Economic Hypothesis (cont.)

- Using the food expenditure data, we found that b_2 = 10.21 with standard error se(b_2) = 2.09
- The value of the test statistic is $t = \frac{b_2 - 5.5}{\text{se}(b_2)} = \frac{10.21 - 5.5}{2.09} = 2.25$
- Because t = 2.25 < 2.429, we do not reject the null hypothesis that $\beta_2 \leq 5.5$
- We are *not* able to conclude that the new supermarket will be profitable and will not begin construction

Example: Left-Tail Test of an Economic Hypothesis

- The null hypothesis is $H_0 : \beta_2 \geq 15$; the alternative hypothesis is $H_1 : \beta_2 < 15$
- The test statistic is $t = (b_2 - 15)/\text{se}(b_2) \sim t_{(N-2)}$ if the null hypothesis is true
- Select $\alpha = 0.05$
  - The critical value for the left-tail rejection region is the 5th percentile of the t-distribution with $N - 2 = 38$ degrees of freedom, $t_{(0.05,38)} = -1.686$
  - Thus, we will reject the null hypothesis if the calculated value of $t \leq -1.686$
  - If $t > -1.686$, we will not reject the null hypothesis

Example: Left-Tail Test of an Economic Hypothesis (cont.)

- Using the food expenditure data, we found that b_2 = 10.21 with standard error se(b_2) = 2.09
- The value of the test statistic is $t = \dfrac{b_2 - 5.5}{\mathrm{se}\left(b_2\right)} = \dfrac{10.21 - 15}{2.09} = -2.29$
- Because t = –2.29 < –1.686, we reject the null hypothesis that $\beta_2 \geq 15$ and accept the alternative that $\beta_2 < 15$
- We conclude that households spend less than $15 from each additional $100 income on food

Example: Two-Tail Test of an Economic Hypothesis

- The null hypothesis is $H_0{:}\beta_2 = 7.5$; the alternative hypothesis is $H1{:}\beta_2 \neq 7.5$
- The test statistic is $t = (b_2 - 7.5)/\mathrm{se}(b_2) \sim t_{(N-2)}$ if the null hypothesis is true
- Select $\alpha = 0.05$
  - The critical value for the two-tail rejection region is the 2.5th percentile of the t-distribution with $N - 2 = 38$ degrees of freedom, $t(0.025,38) = -2.024$ and the 97.5th percentile $t(0.975,38) = 2.024$
  - Thus, we will reject the null hypothesis if the calculated value of $t \geq 2.024$ or if $t \leq -2.024$

Example: Two-Tail Test of an Economic Hypothesis (cont.)

- Using the food expenditure data, we found that b_2 = 10.21 with standard error se(b_2) = 2.09
- The value of the test statistic is $t = \frac{b_2 - 5.5}{\text{se}(b_2)} = \frac{10.21 - 7.5}{2.09} = 1.29$
- Because −2.024 < t = 1.29 < 2.024, we do not reject the null hypothesis that β_2 = 7.5
- The sample data are consistent with the conjecture households will spend an additional $7.50 per additional $100 income on food

Example: Two-Tail Test of a Significance

- The null hypothesis is $H_0{:}\beta_2 = 0$; the alternative hypothesis is $H_1{:}\beta_2 \neq 0$
- The test statistic is $t = (b_2\ )/\text{se}(b_2) \sim t_{(N-2)}$ if the null hypothesis is true
- Select α = 0.05
  - The critical value for the two-tail rejection region is the 2.5th percentile of the t-distribution with N − 2 = 38 degrees of freedom, t(0.025,38) = −2.024 and the 97.5th percentile t(0.975,38) = 2.024
  - Thus, we will reject the null hypothesis if the calculated value of t ≥ 2.024 or if t ≤ −2.024

Example: Two-Tail Test of a Significance (cont.)

- Using the food expenditure data, we found that b_2 = 10.21 with standard error se(b_2) = 2.09
- The value of the test statistic is $t = \frac{b_2}{\operatorname{se}(b_2)} = \frac{10.21}{2.09} = 4.88$
- Because 4.88 > 2.024, we reject the null hypothesis that $\beta_2 = 0$
- We conclude that there is a statistically significant relationship between income and food expenditure

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The p-Value

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The p-Value

- When reporting the outcome of statistical hypothesis tests, it has become standard practice to report the *p*-value (an abbreviation for probability value) of the test
  - If we have the *p*-value of a test, *p*, we can determine the outcome of the test by comparing the *p*-value to the chosen level of significance, α, without looking up or calculating the critical values
  - This is much more convenient

The p-Value (cont.)

- If t is the calculated value of the t-statistic, then:
  - If H_1:$\beta_K > c$
    - p = probability to the right of t
  - If H_1:$\beta_K < c$
    - p = probability to the left of t
  - If H_1:$\beta_K \neq c$
    - p = **sum** of probabilities to the right of $|t|$ **and** to the left of $-|t|$

The p-Value Rule

- Reject the null hypothesis when the *p*-value is less than, or equal to, the level of significance α. That is, if $p \le \alpha$, then reject H_0. If $p > \alpha$, then do not reject H_0.

Example, Part I

- The null hypothesis is $H_0\!:\beta_2 \le 5.5$
- The alternative hypothesis is $H_1\!:\beta_2 > 5.5$

$$t = \frac{b_2 - 5.5}{\mathrm{se}\left(b_2\right)} = \frac{10.21 - 5.5}{2.09} = 2.25$$

- The *p*-value is:

$$\mathrm{p} = \mathrm{P}\left[\mathrm{t}_{(38)} \ge 2.25\right] = 1 - \mathrm{P}\left[\mathrm{t}_{(38)} \le 2.25\right] = 1 - 0.9848 = 0.0152$$

Example, Part II

- The null hypothesis is $H_0\!:\beta_2 \geq 15$
- The alternative hypothesis is $H_1\!:\beta_2 < 15$

$$t = \frac{b_2 - 15}{\mathrm{se}\left(b_2\right)} = \frac{10.21 - 15}{2.09} = -2.29$$

- The *p*-value is:

$$p = P\left[t_{(38)} \leq -2.29\right] = 0.0139$$

Example, Part III

- The null hypothesis is $H_0{:}\beta_2 = 7.5$
- The alternative hypothesis is $H_1{:}\beta_2 \neq 7.5$

$$t = \frac{b_2 - 7.5}{\mathrm{se}\left(b_2\right)} = \frac{10.21 - 7.5}{2.09} = 1.29$$

- The p-value is:

$$p = P\left[t_{(38)} \geq 1.29\right] + P\left[t_{(38)} \leq -1.29\right] = 0.2033$$

Example, Part IV

- The null hypothesis is H_0:$\beta_2 = 0$
- The alternative hypothesis is H_1:$\beta_2 \neq 0$

$$t = \frac{b_2 - 7.5}{\text{se}\left(b_2\right)} = \frac{10.21 - 7.5}{2.09} = 1.29$$

- The *p*-value is:

$$p = P\left[t_{(38)} \geq 4.88\right] + P\left[t_{(38)} \leq -4.88\right] = 0.0000$$

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Linear Combinations of Parameters

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Linear Combinations of Parameters, Part I

- We may wish to estimate and test hypotheses about a linear combination of parameters $\lambda = c_1\beta_1 + c_2\beta_2$, where c_1 and c_2 are constants that we specify
- Under assumptions SR1–SR5, the least squares estimators b_1 and b_2 are the best linear unbiased estimators of β_1 and β_2
- It is also true that $= c_1b_1 + c_2b_2$ is the best linear unbiased estimator of $\lambda = c_1\beta_1 + c_2\beta_2$

Linear Combinations of Parameters, Part II

- As an example of a linear combination, if we let $c_1 = 1$ and $c_2 = x_0$, then we have:

$$E(\hat{\lambda} \mid x) = E(c_1 b_1 + c_2 b_2 \mid x) = c_1 E(b_1 \mid x) + c_2 E(b_2 \mid x)$$

$$= c_1 \beta_1 +_2 \beta_2 = \lambda$$

- The estimator $\hat{\lambda}$ is unbiased because:

$$E(\hat{\lambda} \mid x) = E(c_1 b_1 + c_2 b_2 \mid x) = c_1 E(b_1 \mid x) + c_2 E(b_2 \mid x)$$

$$= c_1 \beta_1 +_2 \beta_2 = \lambda$$

- The variance of $\hat{\lambda}$ is equation 3.8:

$$\operatorname{var}\left(\hat{\lambda} \mid x\right) = \operatorname{var}\left(c_1 b_1 + c_2 b_2\right) = c_1^2 \operatorname{var}\left(b_1\right) + c_2^2 \operatorname{var}\left(b_2\right) + 2 c_1 c_2 \operatorname{cov}\left(b_1, b_2\right)$$

Linear Combinations of Parameters, Part III

- We estimate λˆ by replacing the unknown variances and covariances with their estimated variances and covariances in (2.20)–(2.22)
- (3.9) $\widehat{var}\left(\hat{\lambda}\middle|x\right) = \widehat{var}(c_1 b_1 + c_2 b_2|x) = c_1^2\ \widehat{var}(b_1|x) + c_2^2\ \widehat{var}(b_2|x) + c_1^2 c_2^2 \widehat{var}(b_1, b_2|x)$
- The standard error of $\hat{\lambda}$ is the square root of the estimated variance
- (3.10) $se(\hat{\lambda}) = \mathrm{se}(c_1 b_1 + c_2 b_2|x) = \sqrt{\widehat{var}(c_1 b_1 + \ c_2 b_2|x)}$

Linear Combinations of Parameters, Part IV

- If in addition SR6 holds, or if the sample is large, the least squares estimators b_1 and b_2 have normal distributions
- It is also true that linear combinations of normally distributed variables are normally distributed, so that:

$$\hat{\lambda} \mid x = c_1 b_1 + c_2 b_2 \sim N\left[\lambda, \operatorname{var}\left(\hat{\lambda} \mid x\right)\right]$$

Testing a Linear Combination of Parameters

- A general linear hypothesis involves both parameters, β_1 and β_2, and may be stated as:

(3.12a) $$H_0: \left(c_1\beta_1 + c_2\beta_2\right) = c_0$$

  - Or, equivalently:

(3.12b) $$H_0: \left(c_1\beta_1 + c_2\beta_2\right) - c_0 = 0$$

Testing a Linear Combination of Parameters (cont.)

- The alternative hypothesis might be any one of the following:

$$(i)\quad H_1: c_1\beta_1 + c_2\beta_2 \neq c_0 \quad \text{two-tail test}$$

$$(ii)\quad H_1: c_1\beta_1 + c_2\beta_2 > c_0 \quad \text{right-tail test}$$

$$(iii)\quad H_1: c_1\beta_1 + c_2\beta_2 < c_0 \quad \text{left-tail test}$$

- The t-statistic is 3.13: $t = \dfrac{\left(c_1\beta_1 + c_2\beta_2\right) - c_0}{\text{se}\left(c_1\beta_1 + c_2\beta_2\right)} \sim t_{(N-2)}$
- The rejection regions for the one- and two-tail alternatives (i)–(iii) are the same as those described in Section 3.3, and conclusions are interpreted the same way as well

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Key Words

- Alternative hypothesis
- Confidence intervals
- Critical value
- Degrees of freedom
- Hypotheses
- Hypothesis testing
- Inference
- Interval estimation
- Level of significance
- Linear combination of parameters
- Linear hypothesis
- Null hypothesis
- One-tail tests
- Pivotal statistic
- Point estimates
- Probability value
- p-value
- Rejection region
- Test of significance
- Test statistic
- Two-tail tests
- Type I error
- Type II error

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