The ARCH Model

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Time-Varying Volatility and ARCH Models

- The nonstationary nature of the variables studied earlier implied that they had means that change over time.
- Now we are concerned with stationary series, but with conditional variances that change over time.
  - The model is called the autoregressive conditional heteroskedastic (ARCH) model.
  - Financial time series have characteristics that are well represented by models with dynamic variances.

The ARCH Model, Part I

- Consider a model with an AR(1) error term:
  - (14.1a) $y_t = \phi + e_t$
  - (14.1b) $e_t = \rho e_{t-1} + v_t, \qquad |\rho| < 1$
  - (14.1c) $v_t \sim N(0,\ \sigma_v^2)$

The ARCH Model, Part II

- The **unconditional mean** of the error is:

$$E[e_t] = E[v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \cdots] = 0$$

- The **conditional mean** for the error is:

$$E[e_t | I_{t-1}] = E[\rho e_{t-1} | I_{t-1}] + E[v_t] = \rho e_{t-1}$$

The ARCH Model, Part III

- The **unconditional variance** of the error is:

$$\begin{aligned}E\left[e_t - 0\right]^2 &= E\left[v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \cdots\right]^2 \\ &= E\left[v_t^2 + \rho^2 v_{t-1}^2 + \rho^4 v_{t-2}^2 + \cdots\right] \\ &= \sigma_v^2\left[1 + \rho^2 + \rho^4 + \cdots\right] \\ &= \frac{\sigma_v^2}{1-\rho^2}\end{aligned}$$

- The conditional variance for the error is:

$$E[(e_t - \rho e_{t-1})^2 | I_{t-1}] = E[v_t^2 | I_{t-1}] = \sigma_v^2$$

The ARCH Model, Part IV

- Suppose that, instead of a conditional mean that changes over time, we have a conditional variance that changes over time
  - Consider a variant of the above model:
    - (14.2a) $y_t = \beta_0 + e_t$
    - (14.2b) $e_t | I_{t-1} \sim N(0, h_t)$
    - (14.2c) $h_t = \alpha_0 + \alpha_1 e_{t-1}^2, \qquad \alpha_0 > 0, \qquad 0 \leq \alpha_1 < 1$

The ARCH Model, Part V

- Equations 14.2b and 14.2c describe the ARCH class of models
- Equation 14.2b says that the error term is conditionally normal, where I_{t-1} represents the information available at time t − 1 with mean 0 and time-varying variance, denoted as h_t
- Equation 14.2c models h_t as a function of a constant term and the lagged error squared

The ARCH Model, Part VI

- The name ARCH conveys the fact that we are working with time-varying variances (heteroskedasticity) that depend on (are conditional on) lagged effects (autocorrelation)
  - This particular example is an ARCH(1) model

The ARCH Model, Part VII

- The standardized errors are standard normal:

$$\left(\frac{e_t}{\sqrt{h_t}}|I_{t-1}\right) = z \sim N(0,1)$$

- We can write:

$$E(e_t) = E(z_t)E\left(\sqrt{\alpha_0 + \alpha_1 e_{t-1}^2}\right)$$

- And:

$$E(e_t^2) = E(z_t^2)E(\alpha_0 + \alpha_1 e_{t-1}^2) = \alpha_0 + \alpha_1 E(e_{t-1}^2)$$

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Time-Varying Volatility

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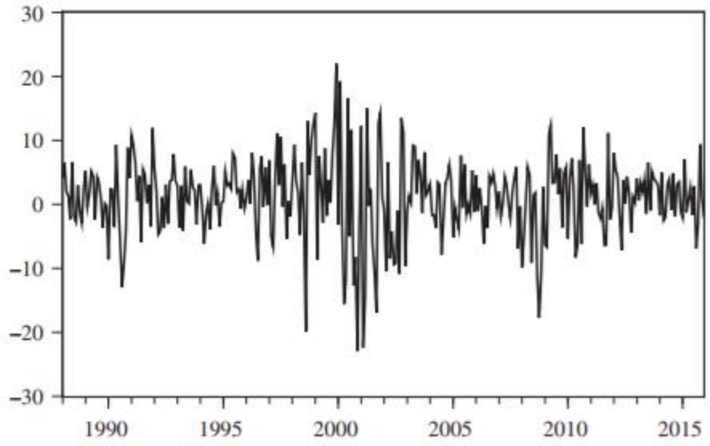
Time-Varying Volatility

The ARCH model has become a popular one because its variance specification can capture commonly observed features of the time series of financial variables.

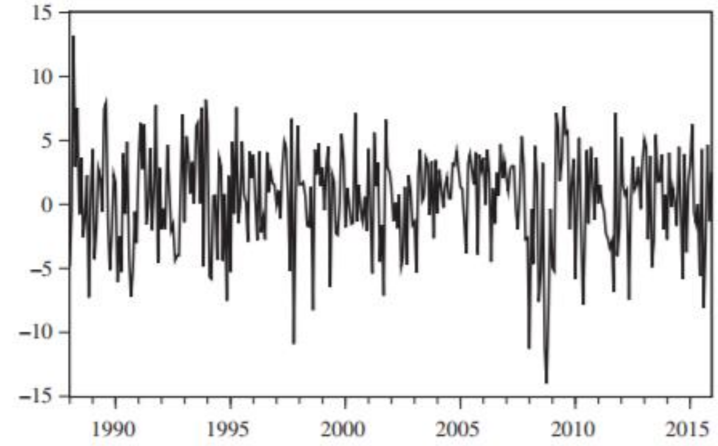
- It is useful for modeling **volatility** and especially changes in volatility over time.

Example 1: Characteristics of Financial Variables

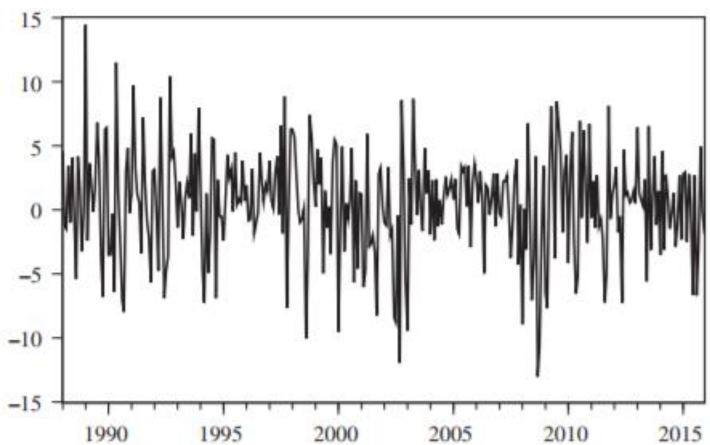
- The values of these series change rapidly from period to period in an apparently unpredictable manner; we say the series are volatile.
- There are periods when large changes are followed by further large changes and periods when small changes are followed by further small changes.
- Distributions where there are more observations around the mean and in the tails are said to be **leptokurtic**.



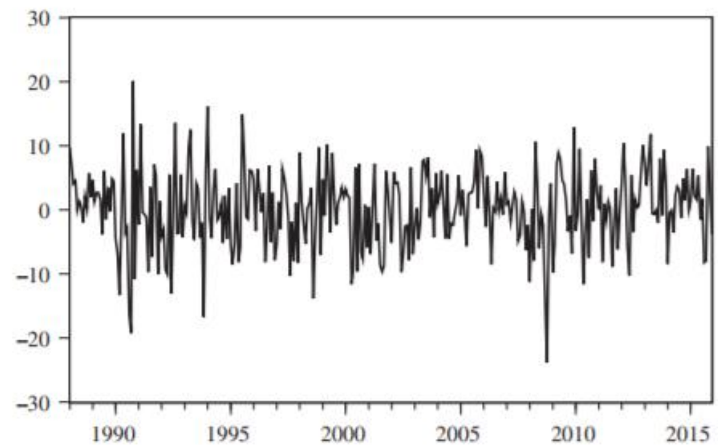
(a) United States: Nasdaq



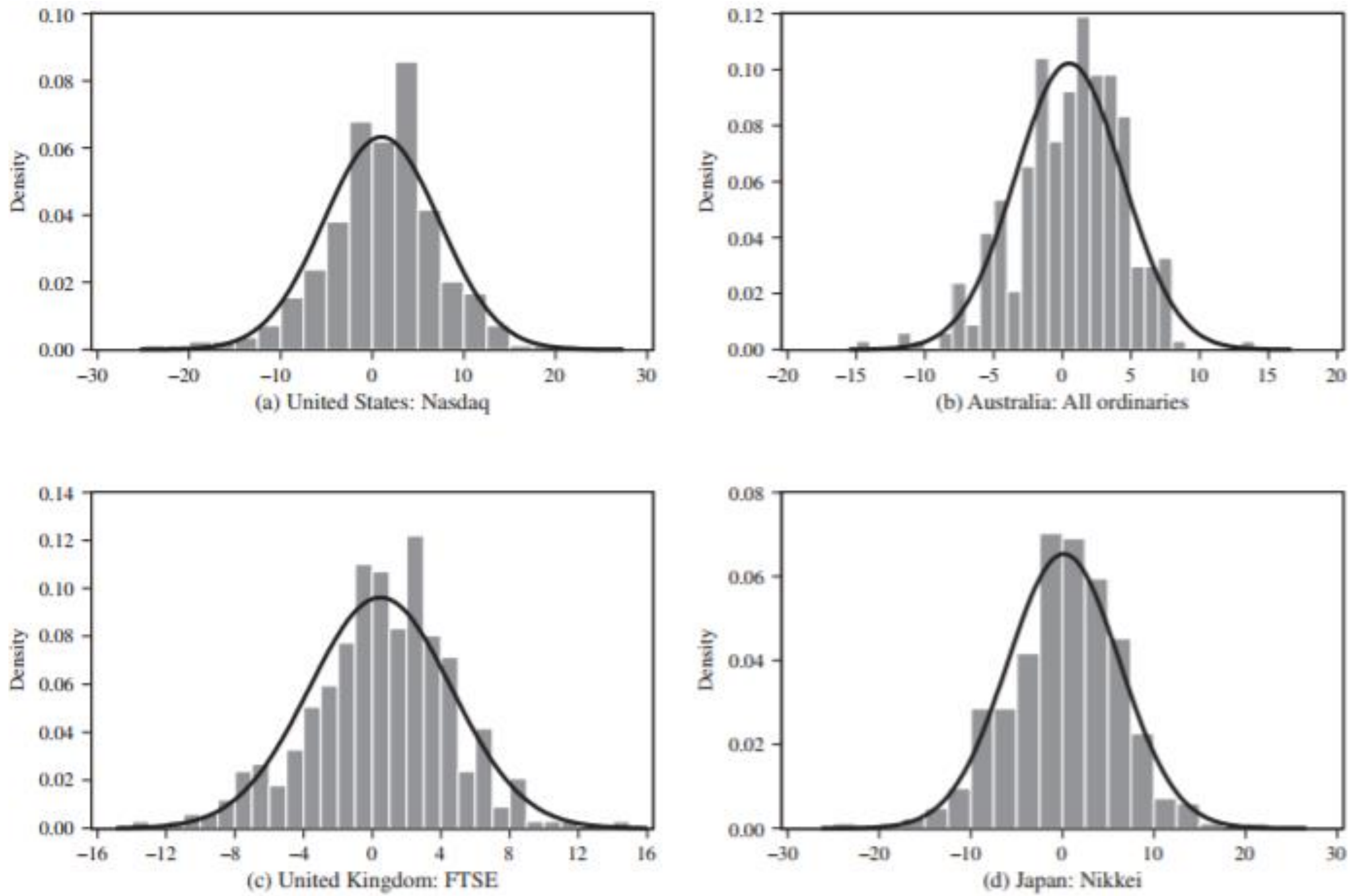
(b) Australia: All ordinaries



(c) United Kingdom: FTSE



(d) Japan: Nikkei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(a) United States: Nasdaq
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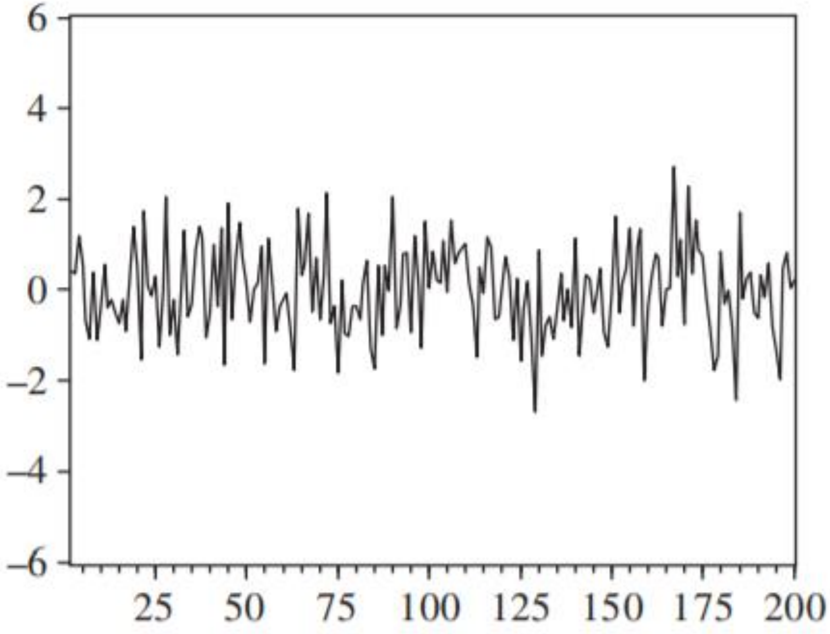
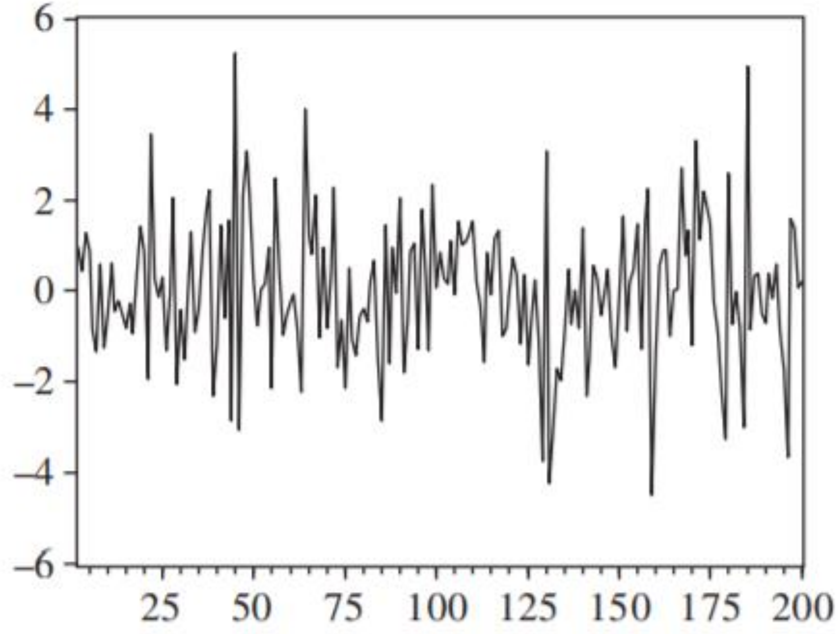
(d) Japan: Nikkei

Example 2: Simulating Time-Varying Volatility

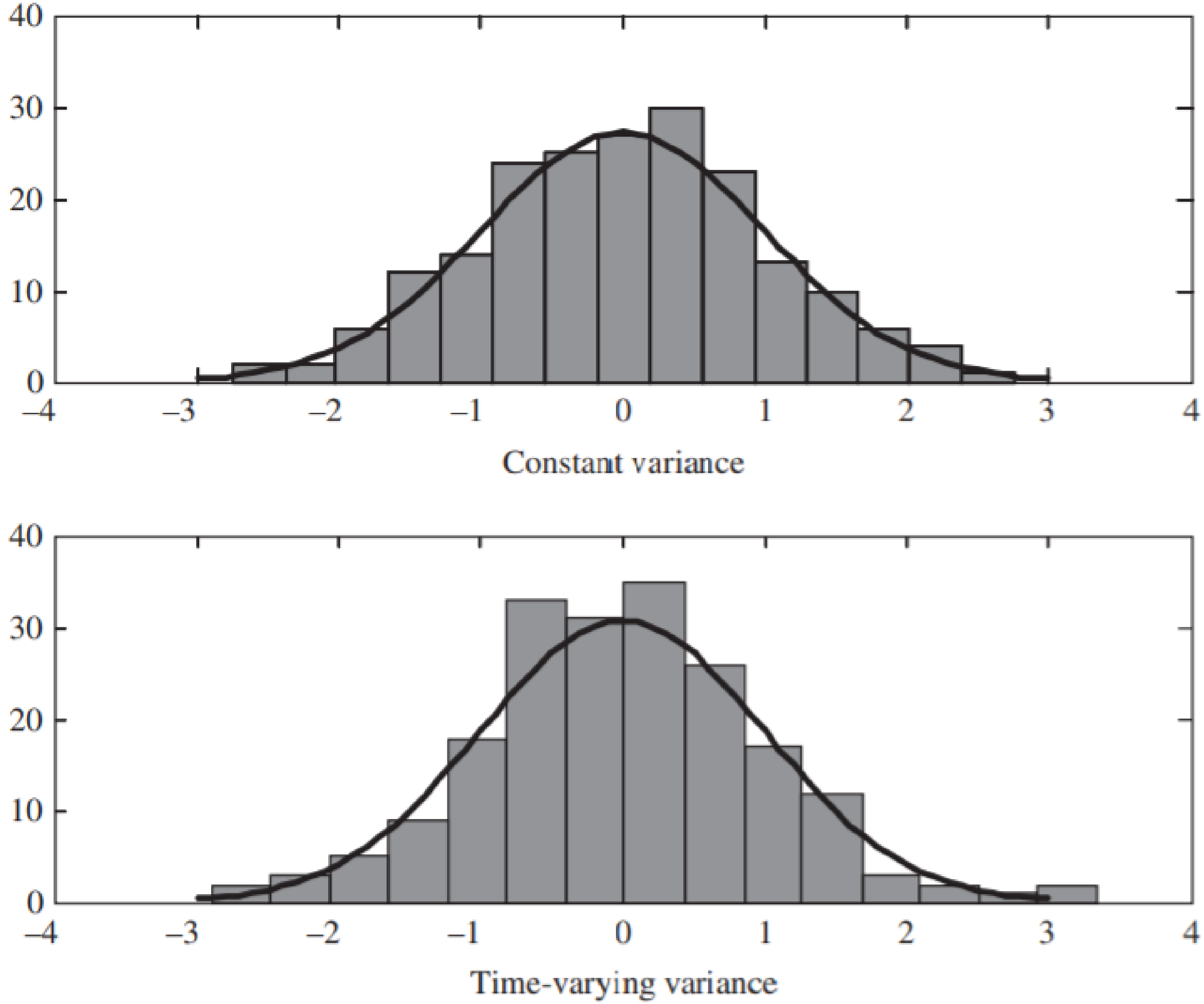
- For Figure 14.3
  - Note that, relative to the series in the top panel, volatility in the bottom panel is not constant
  - It changes over time and it clusters—there are periods of small changes and periods of big changes
- For Figure 14.4
  - The second distribution has higher frequencies around the mean (zero) and higher frequencies in the tails (outside ± 3)



Constant variance: $h_t = 1$



Time-varying variance: $h_t = 1 + 0.8e^2_{t-1}$


40
30
20
10
0
–4
–3
–2
–1
0
1
2
3
4
Constant variance
Time-varying variance

Time-Varying Volatility

The ARCH model is intuitively appealing because it seems sensible to explain volatility as a function of the errors e_t.

- These errors are often called "shocks" or "news" by financial analysts.
- According to the ARCH model, the larger the shock, the greater the volatility in the series.
- This model captures volatility clustering, as big changes in e_t are fed into further big changes in h_t via the lagged effect e_{t-1}.

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Testing, Estimating, and Forecasting

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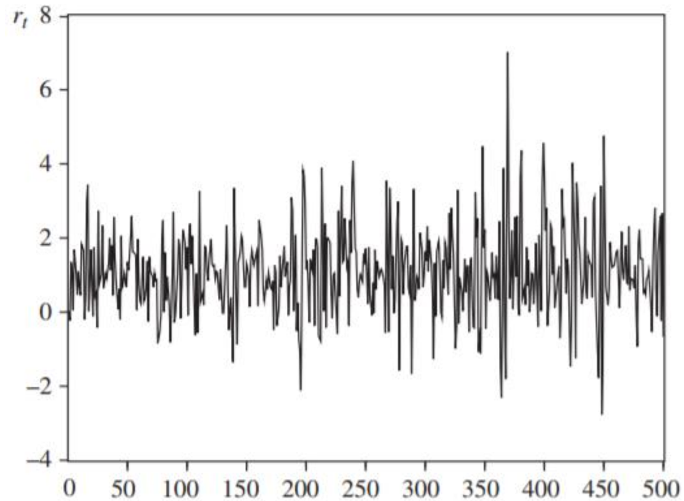
Testing, Estimating, and Forecasting

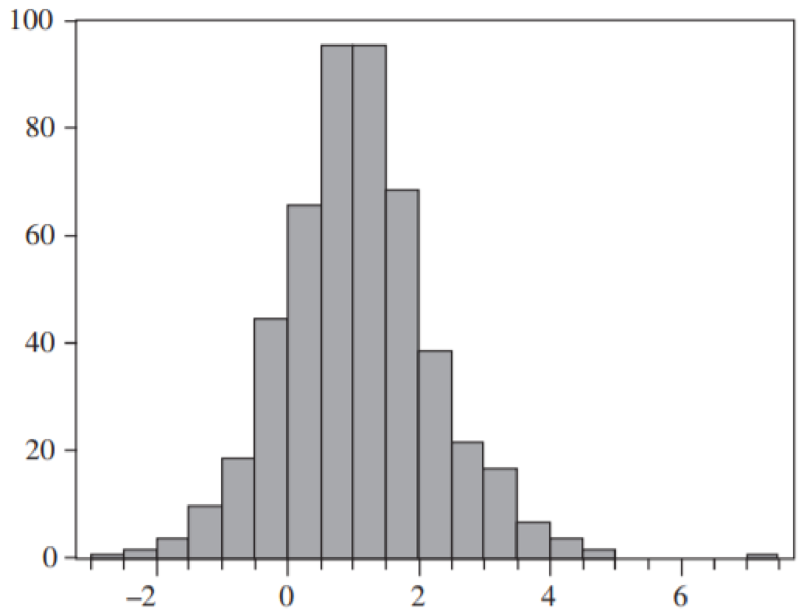
- A Lagrange multiplier (LM) test is often used to test for the presence of ARCH effects
- To perform this test, first estimate the mean equation:
  - (14.3) $\hat{e}_t^2 = \gamma_0 + \gamma_1 \hat{e}_{t-1}^2 + v_t$
- The null and alternative hypotheses are:

$$H_0 : \gamma_1 = 0 \qquad H_1 : \gamma_1 \neq 0$$

Example 3: Testing for ARCH in BYD Lighting

- Consider the returns from buying shares in the hypothetical company Brighten Your Day (BYD) Lighting.
- The time series shows evidence of time-varying volatility and clustering, and the unconditional distribution is non-normal.





Series: Returns
Sample 1 500
Observations 500

Mean	1.078294
Median	1.029292
Maximum	7.008874
Minimum	–2.768566
Std. Dev.	1.185025
Skewness	0.401169
Kurtosis	4.470080
Jarque Bera	58.43500
Probability	0.000000

Example 3: Testing for ARCH in BYD Lighting (cont.)

- The results for an ARCH test are:

$$\hat{e}_t^2 = 0.908 + 0.353\hat{e}_{t-1}^2 \qquad R^2 = 0.124$$
$$(t) \qquad\qquad (8.409)$$

  - The *t*-statistic suggests a significant first-order coefficient
  - The sample size is 500, giving an LM test value of $(T - q)R^2 = 61.876$
  - Comparing the computed test value to the 5% critical value of a $\chi^2_{(1)}$ distribution ($\chi^2_{(0.95,\ 1)} = 3.841$) leads to the rejection of the null hypothesis
    - The residuals show the presence of ARCH(1) effects

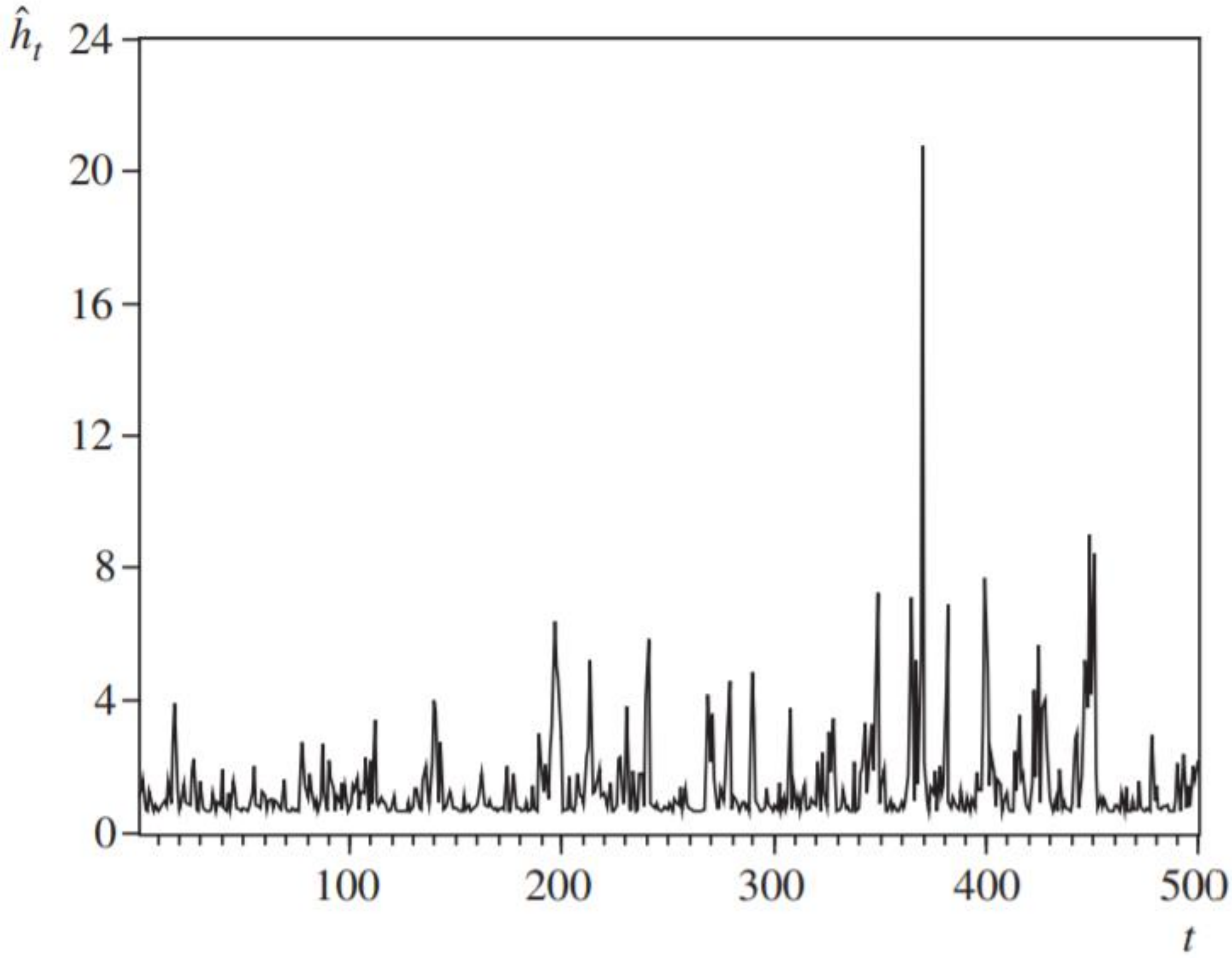
Example 4: ARCH Model Estimates for (BYD) Lighting

- ARCH models are estimated by the maximum likelihood method
- Equation 14.4 shows the results from estimating an ARCH(1) model applied to the monthly returns from buying shares in BYD Lighting
- (14.4a) $\hat{r}_t = \hat{\beta}_0 = 1.063$
- (14.4b) $\hat{h}_t = \hat{\alpha}_0 + \hat{\alpha}_1 \hat{e}_{t-1}^2 = 0.642 + 0.569\hat{e}_{t-1}^2$

$(t) \qquad\qquad\qquad\qquad (5.536)$

Example 5: ARCH Model Estimates for (BYD) Lighting

- The forecast return and volatility are:
  - (14.5a) $\hat{r}_{t+1} = \hat{\beta}_0 = 1.063$
  - (14.5b) $\hat{h}_{t+1} = \hat{\alpha}_0 + \hat{\alpha}_1\left(r_t - \hat{\beta}_0\right)^2$
    $= 0.642 + 0.569(r_t - 1.063)^2$
- Equation 14.5a gives the estimated return that is both the conditional and unconditional mean return


$\hat{h}_t$
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Extensions

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Extensions

- The ARCH(1) model can be extended in a number of ways
  - One obvious extension is to allow for more lags
  - An ARCH(*q*) model would be:
    - (14.6) $h_t = \alpha_0 + \alpha_1 e_{t-1}^2 + \alpha_2 e_{t-2}^2 \ldots + \alpha_q e_{t-q}^2$
- Testing, estimating, and forecasting are natural extensions of the case with one lag

The GARCH Model: Generalized ARCH, Part I

One of the shortcomings of an ARCH(q) model is that there are $q + 1$ parameters to estimate

- If q is a large number, we may lose accuracy in the estimation
- The generalized ARCH model, or GARCH, is an alternative way to capture long lagged effects with fewer parameters

The GARCH Model: Generalized ARCH, Part II

Consider equation 14.6, but write it as:

$$h_t = \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1 \alpha_1 e_{t-2}^2 + \beta_1^2 \alpha_1 e_{t-3}^2 + \cdots$$

- Add and subtract $\beta_1 \alpha_0$ and rearrange terms as follows:

$$h_t = (\alpha_0 - \beta_1 \alpha_0) + \alpha_1 e_{t-1}^2 + \beta_1 (\alpha_0 + \alpha_1 e_{t-2}^2 + \beta_1 \alpha_1 e_{t-3}^2 + \cdots)$$

- Because, $h_{t-1} = \alpha_0 + \alpha_1 e_{t-2}^2 + \beta_1 \alpha_1 e_{t-3}^2 + \beta_1^2 \alpha_1 e_{t-4}^2 + \cdots$, we may simplify to equation 14.7: $h_t = \delta + \alpha_1 e_{t-1}^2 + \beta_1 h_{t-1}$

The GARCH Model: Generalized ARCH, Part III

This generalized ARCH model is denoted as GARCH(1,1)

- The model is a very popular specification because it fits many data series well
- It tells us that the volatility changes with lagged shocks (e^2_{t-1}), but there is also momentum in the system working via h_{t-1}
- One reason why this model is so popular is that it can capture long lags in the shocks with only a few parameters

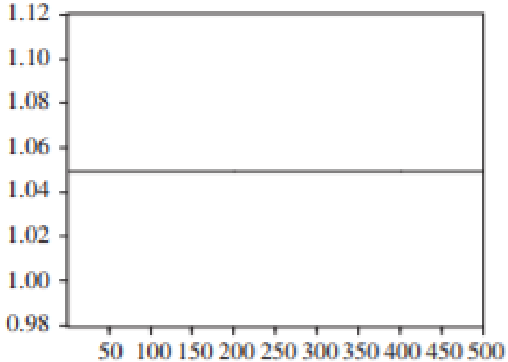
Example 6: A GARCH Model for Brighten Your DAY

Consider again the returns to our shares in BYD Lighting, which we re-estimate (by maximum likelihood) under the new model:

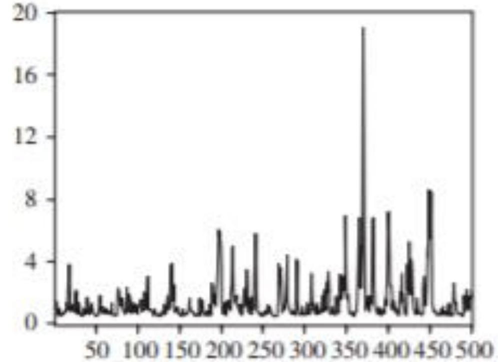
$$\hat{r}_t = 1.049$$

$$\hat{h}_t = 0.401 + 0.492\hat{e}_{t-1}^2 + 0.238\hat{h}_{t-1}$$

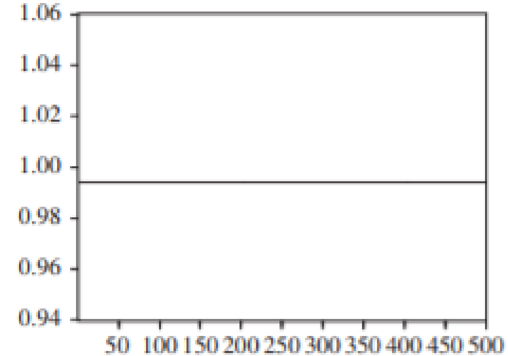
$$(t) \qquad\qquad (4.834) \qquad (2.136)$$



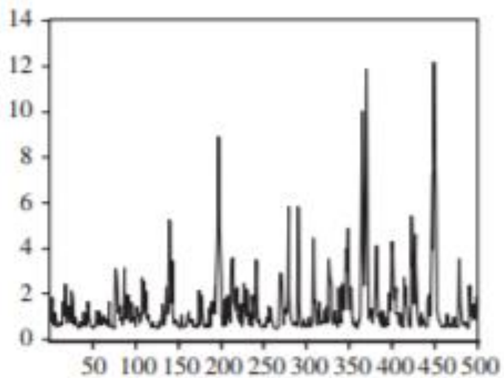
GARCH(1, 1): $E(r_t) = 1.049$



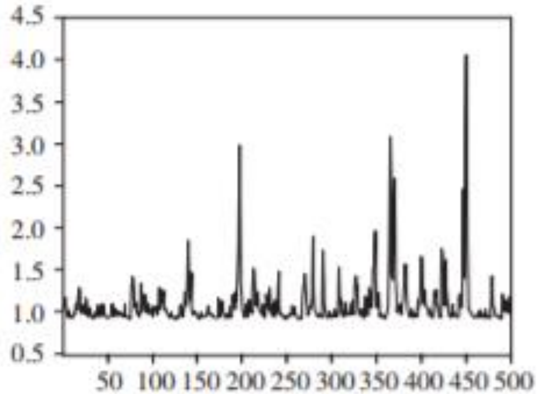
GARCH(1, 1): $h_t = 0.401 + 0.492\epsilon^2_{t-1} + 0.238h_{t-1}$



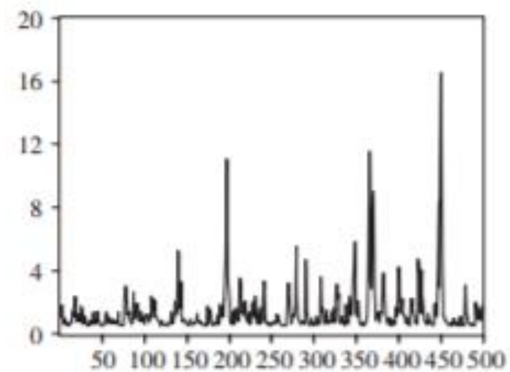
T-GARCH(1, 1): $E(r_t) = 0.994$



T-GARCH(1, 1):
$h_t = 0.356 + (0.263 + 0.492d_{t-1})\epsilon^2_{t-1} + 0.287h_{t-1}$



GARCH-in-mean: $E(r_t) = 0.818 + 0.196h_t$



GARCH-in-mean:
$h_t = 0.370 + (0.295 + 0.321d_{t-1})\epsilon^2_{t-1} + 0.278h_{t-1}$

Allowing for an Asymmetric Effect

The threshold ARCH model, or T-ARCH, is one example where positive and negative news are treated asymmetrically

- In the T-GARCH version of the model, the specification of the conditional variance is:

$$h_t = \delta + \alpha_1 e_{t-1}^2 + \gamma d_{t-1} e_{t-1}^2 + \beta_1 h_{t-1}$$

- (14.8)

$$d_t = \begin{cases} 1 & e_t < 0 \text{ (bad news)} \\ 0 & e_t \geq 0 \text{ (good news)} \end{cases}$$

Example 7: A T-GARCH Model for BYD

- The returns to our shares in BYD Lighting were re-estimated with a T-GARCH(1,1) specification:

$$\hat{r}_t = 0.994$$

$$\begin{array}{llll} \hat{h}_t = 0.356 + 0.263\hat{e}_{t-1}^2 + 0.492 d_{t-1}\hat{e}_{t-1}^2 + 0.287\hat{h}_{t-1} \\ (t) \qquad\qquad (3.267) \qquad (2.405) \qquad\qquad (2.488) \end{array}$$

- Overall, negative shocks create greater volatility in financial markets

GARCH-in-Mean and Time-Varying Risk Premium

- Another popular extension of the GARCH model is the "GARCH-in-mean" model
- (14.9a) $y_t = \beta_0 + \theta h_t + e_t$
- (14.9b) $e_t | I_{t-1} \sim N(0, h_t)$
- (14.9c) $h_t = \delta + \alpha_1 e_{t-1}^2 + \beta_1 h_{t-1},$
  $\delta > 0,\ 0 \leq \alpha_1 < 1,\ 0 \leq \beta_1 < 1$

Example 8:
GARCH-in-Mean Model for BYD

- The returns to shares in BYD Lighting were re-estimated as a GARCH-in-mean model:

$$\hat{r}_t = 0.818 + 0.196 h_t$$
$$(t) \qquad (2.915)$$

$$\hat{h}_t = 0.370 + 0.295 \hat{e}_{t-1}^2 + 0.321 d_{t-1} \hat{e}_{t-1}^2 + 0.278 \hat{h}_{t-1}$$
$$(t) \qquad (3.426) \qquad (1.979) \qquad (2.678)$$

Example 8: GARCH-in-Mean Model for BYD (cont.)

- The results show that, as volatility increases, the returns correspondingly increase by a factor of 0.196
- In other words, this result supports the usual view in financial markets—high risk, high return

Other Developments

- The GARCH, T-GARCH, and GARCH-in-mean models are three important extensions of the original ARCH concept
- The EGARCH model is:

$$\ln(h_t) = \delta + \beta_1 \ln(h_{t-1}) + \alpha \left| \frac{e_{t-1}}{\sqrt{h_{t-1}}} \right| + \gamma \left(\frac{e_{t-1}}{\sqrt{h_{t-1}}} \right)$$

where $\left(\frac{e_{t-1}}{\sqrt{h_{t-1}}} \right)$ are the standardized residuals

Other Developments (cont.)

- The leverage effect refers to the generally observed negative correlation between an asset return and its volatility changes
- Another significant development is to allow the conditional distribution of the error term to be non-normal
- Because empirical distributions of financial returns generally exhibit fat tails and clustering around zero, the t-distribution has become a popular alternative to the assumption of normality

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Key Words

- ARCH
- ARCH-in-mean
- Conditionally normal
- GARCH
- GARCH-in-mean
- T-ARCH and T-GARCH
- Time-varying variance

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