Bayesian Inference

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(With thanks to Mark Holder,and Paul Lewis for slides!)

Bayesian statistics

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- You decide to get tested.
- Test is correct 99 percent of the time. (in other words, if you have the disease, it shows that you do with 99 percent probability, and if you don't have the disease, it shows that you do not with 99 percent probability).
- This disease is actually quite rare, occurring randomly in the general population in only one of every 10,000 people.

If your test results come back positive, what are your chances that you actually have the disease?

Example from Su, Francis E., et al. "Medical Tests and Bayes' Theorem." Math Fun Facts. <http://www.math.hmc.edu/funfacts>.

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Congratulations, there is less than 1 percent chance that you have the disease!

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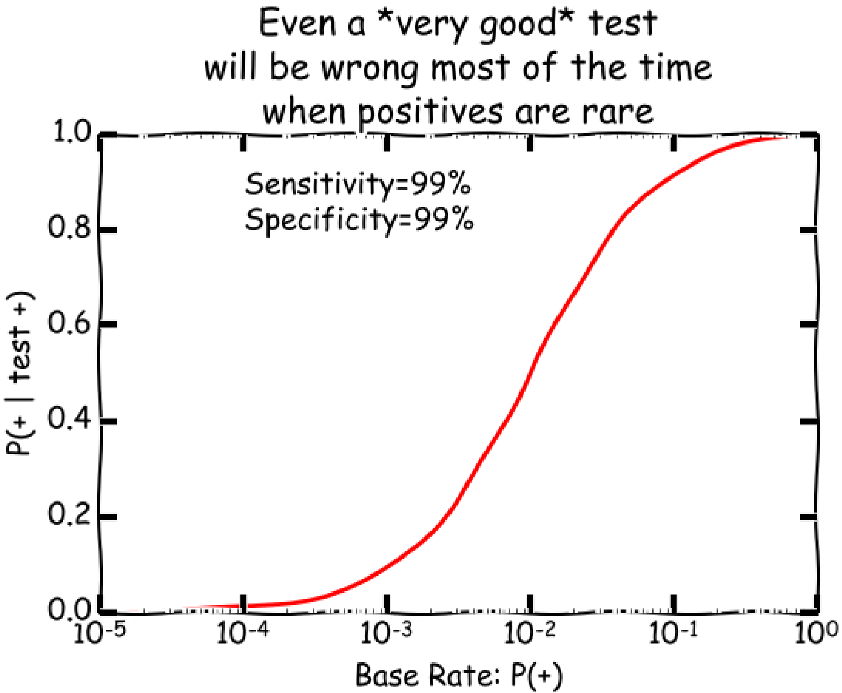
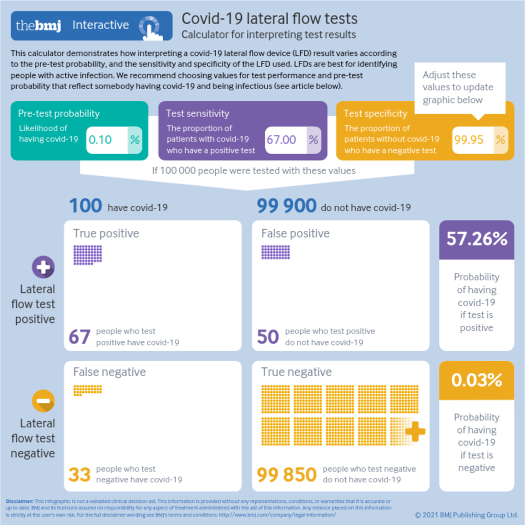
- Event A is the event you have this disease, and event B is the event that you test positive.
- $P(B|notA)$ is the probability of a "false positive": that you test positive even though you don't have the disease.
- In this case: $P(B|A) = .99$
- but $P(A) = .0001$.
- $P(B)$ may be derived by conditioning on whether event A does or does not occur:
- $P(B) = P(B|A)P(A) + P(B|notA)P(notA)$
- or .99*.0001+.01*.9999.

Thus the probability that you have the disease is less than 1 percent.


Even a *very good* test
will be wrong most of the time
when positives are rare
Sensitivity=99%
Specificity=99%
1.0
0.8
0.6
0.4
0.2
0.0
P(+ | test +)
10-5
10-4
10-3
10-2
10-1
100
Base Rate: P(+)



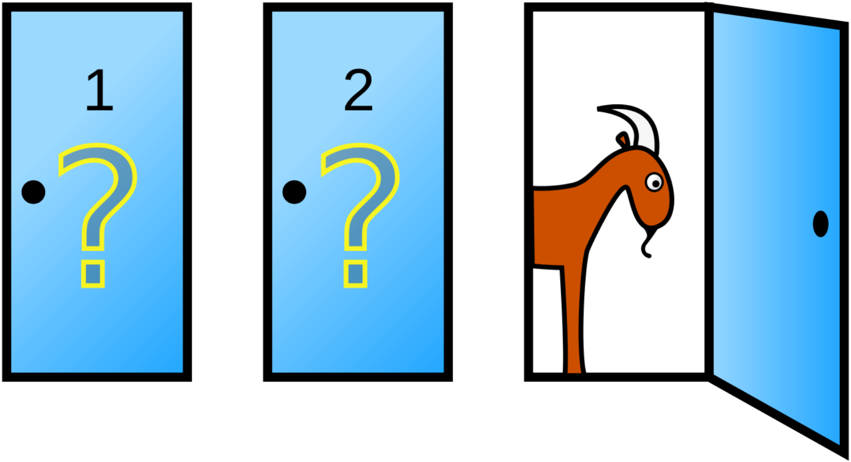
https://www.bmj.com/content/373/bmj.n1411

Bayes' theorem (aka Bayes' rule or Bayes law) helps us find the probability of event A given event B, $(P(A|B)$,
in terms of the probability of B given A, $P(B|A)$, and the probabilities of A and B:
$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$

Reverend Bayes

Monte Hall problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 2, which has a goat. He then says to you, "Do you want to pick door No. 3?" Is it to your advantage to switch your choice?



Monte Hall problem

Exercise:

- Find a partner.
- Take 3 playing cards - two red (goats), and one black (car).
- Play the Monte Hall game. (take turns as the host and the contestant)
- Record if you got the car or the goat, and if you chose to switch or not. Run at least 10 games per tactic, and add your scores to the spreadsheet.
- Try out other variants! (what if the host doesn't know which door has the car? What if you have 3 goats and a car?)

Applying Bayes theorem to the Monty Hall problem:

Assume we pick Door 1 and then Monty shows us a goat behind Door 2.
Now let A be the event that the car is behind Door 1.
B be the event that Monty shows us a goat behind Door 2.

$$\Pr(A \mid B) = \frac{\Pr(B \mid A) \times \Pr(A)}{\Pr(B)}$$

http://angrystatistician.blogspot.com/2012/06/bayes-solution-to-monty-hall.html
https://brilliant.org/wiki/monty-hall-problem/

The tricky calculation is Pr(B).

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- If the car is behind Door 2, Monty never shows us a goat behind Door 2. $1/3 \times 0$
- Finally, if the car is behind Door 3, Monty shows us a goat behind Door 2 every time. $1/3 \times 1$

Thus, $\Pr(B) = 1/3 \times 1/2 + 1/3 \times 0 + 1/3 \times 1 = 1/2$.

Applying Bayes theorem to the Monty Hall problem:

Assume we pick Door 1 and then Monty shows us a goat behind Door 2.
Now let A be the event that the car is behind Door 1.
B be the event that Monty shows us a goat behind Door 2.

$$\begin{aligned}\Pr(A \mid B) &= \frac{\Pr(B \mid A) \times \Pr(A)}{\Pr(B)} \\ &= \frac{1/2 \times 1/3}{1/3 \times 1/2 + 1/3 \times 0 + 1/3 \times 1} \\ &= 1/3.\end{aligned}$$

`http://angrystatistician.blogspot.com/2012/06/bayes-solution-to-monty-hall.html`

We saw a goat behind door 2. The car is either behind Door 1 or Door 3, since the probability that it is behind Door 1 is $1/3$ and the sum of the two probabilities must equal 1, the probability the car is behind Door 3 is $1 - 1/3 = 2/3$.

Key aspect of this problem:
The added information is not THAT Monty Hall revealed a goat, it is that he picked a door specifically because it had a goat.

An R script to simulate the Monte Hall problem by Corey Chivers, (`https://bayesianbiologist.com/`)
`https://raw.githubusercontent.com/McTavishLab/GradPhylo/master/docs/scripts/MonteHall.R`

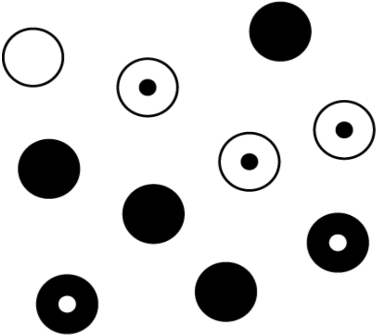
HW: What is the probability you get the car after switching if there are 10 doors? (1 car, 9 goats, everything else the same.)

Advantages of Bayesian inference:

- integrating prior information
- marginalizing inferences over parameter values

Joint probabilities

B = Black S = Solid
W = White D = Dotted



Pr(B) = 0.6 Pr(S) = 0.5

Pr(W) = 0.4 Pr(D) = 0.5

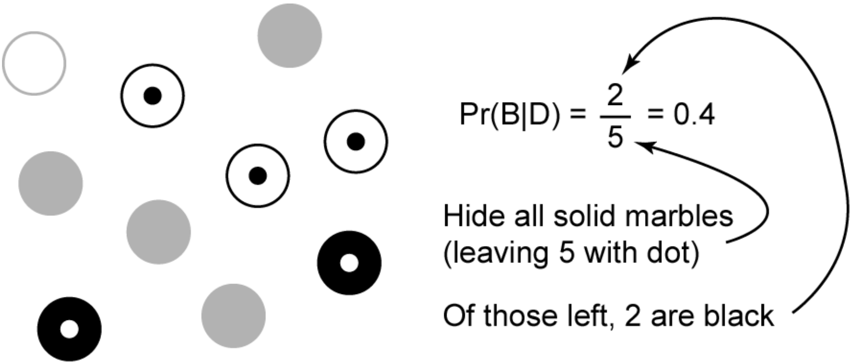
Pr(◉) = Pr(B, D) = 0.2

Pr(●) = Pr(B, S) = 0.4

Pr(⊙) = Pr(W, D) = 0.3

Pr(○) = Pr(W, S) = 0.1

Conditional probabilities

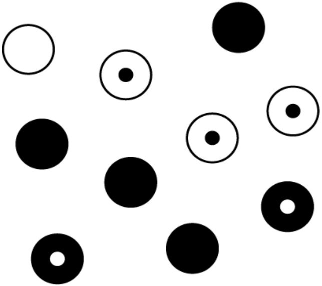


Bayes' rule

$$\Pr(B, D)$$

$$\Pr(D)\Pr(B|D) = \Pr(B)\Pr(D|B)$$

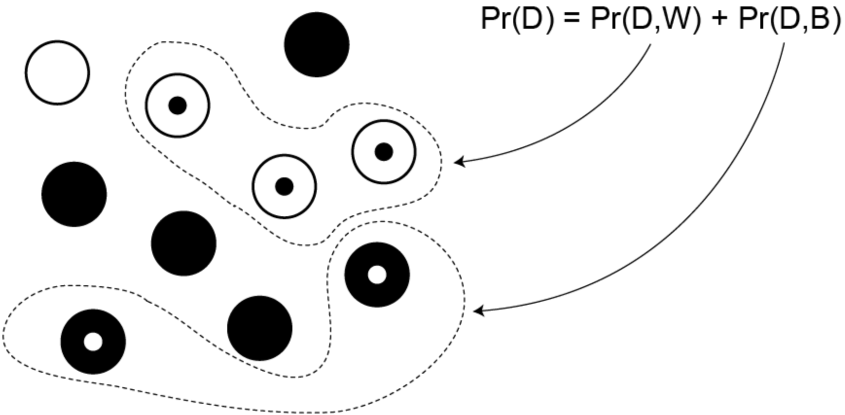
$$\frac{1}{2} \times \frac{2}{5} = \frac{3}{5} \times \frac{1}{3}$$



$$\Pr(B|D) = \frac{\Pr(B)\Pr(D|B)}{\Pr(D)}$$

$$= \frac{\frac{3}{5} \times \frac{1}{3}}{\frac{1}{2}} = \frac{2}{5}$$

Probability of "Dotted"



Bayes' rule (cont.)

$$\Pr(B|D) = \frac{\Pr(B)\Pr(D|B)}{\Pr(D)}$$
$$= \frac{\Pr(D,B)}{\Pr(D,B) + \Pr(D,W)}$$

Pr(*D*) is the **marginal probability** of being dotted
To compute it, we **marginalize over colors**

Bayes' rule (cont.)

It is easy to see that Pr(D) serves as a *normalization constant*, ensuring that Pr(B|D) + Pr(W|D) = 1.0

$$\Pr(B|D) = \frac{\Pr(D,B)}{\Pr(D,B) + \Pr(D,W)} \longleftarrow \Pr(D)$$

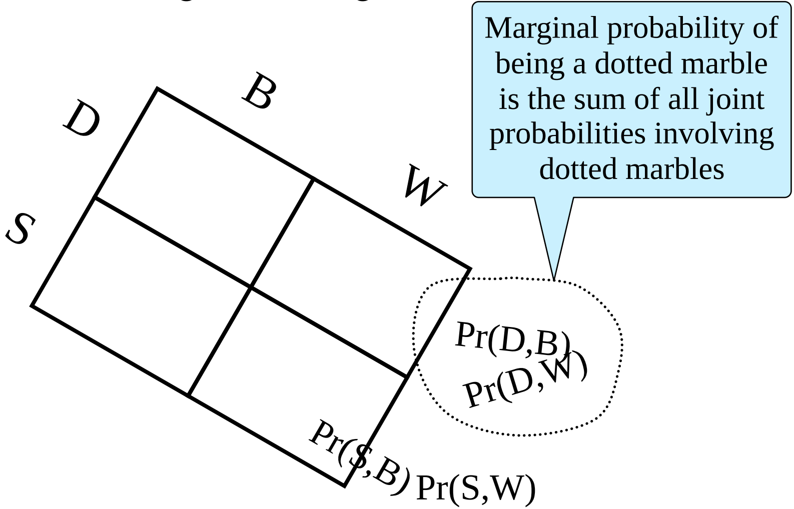
$$\Pr(W|D) = \frac{\Pr(D,W)}{\Pr(D,B) + \Pr(D,W)} \longleftarrow \Pr(D)$$

$$\Pr(B|D) + \Pr(W|D) = \frac{\cancel{\Pr(D,B) + \Pr(D,W)}}{\cancel{\Pr(D,B) + \Pr(D,W)}} = 1$$

Joint probabilities

	B	W
D	Pr(D,B)	Pr(D,W)
S	Pr(S,B)	Pr(S,W)

Marginalizing over colors



Marginal probabilities

	B	W
D		
S		

Pr(D) = marginal probability of being dotted

Pr(D,B) + Pr(D,W)

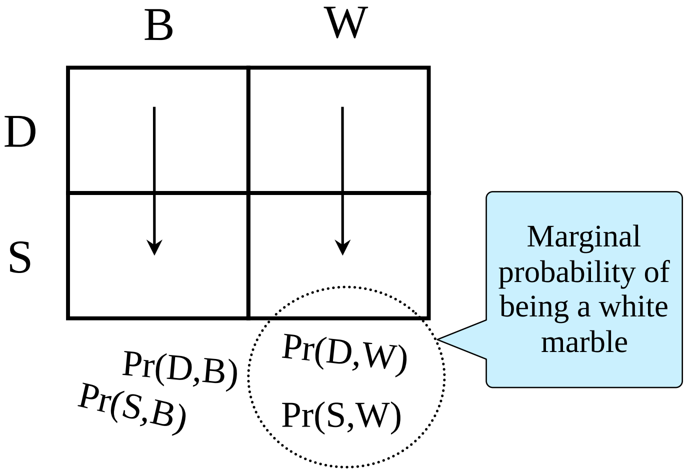
Pr(S,B) + Pr(S,W)

Pr(S) = marginal probability of being solid

Joint probabilities

	B	W
D	Pr(D,B)	Pr(D,W)
S	Pr(S,B)	Pr(S,W)

Marginalizing over "dottedness"



Bayes' rule (cont.)

$$\Pr(B|D) = \frac{\Pr(B)\Pr(D|B)}{\Pr(D,B)+\Pr(D,W)}$$

$$= \frac{\Pr(B)\Pr(D|B)}{\Pr(B)\Pr(D|B)+\Pr(W)\Pr(D|W)}$$

$$= \frac{\Pr(B)\Pr(D|B)}{\sum_{\theta\in\{B,W\}}\Pr(\theta)\Pr(D|\theta)}$$

Bayes' rule in Statistics

$$\Pr(\theta|D) = \frac{\Pr(D|\theta)\Pr(\theta)}{\sum_\theta \Pr(D|\theta)\Pr(\theta)}$$

D refers to the "observables" (i.e. the **Data**)

θ refers to one or more "unobservables" (i.e. **parameters** of a model, or the **model itself**):

- *tree model* (i.e. tree topology)
- *substitution model* (e.g. JC, F84, GTR, etc.)
- *parameter* of a substitution model (e.g. a branch length, a base frequency, transition/transversion rate ratio, etc.)
- *hypothesis* (i.e. a special case of a model)
- a *latent variable* (e.g. ancestral state)

Bayes' rule in statistics

Likelihood of hypothesis θ

Prior probability of hypothesis θ

$$\Pr(\theta|D) = \frac{\Pr(D|\theta)\,\Pr(\theta)}{\sum_\theta \Pr(D|\theta)\,\Pr(\theta)}$$

Posterior probability
of hypothesis θ

Marginal probability
of the data (marginalizing
over hypotheses)

Next class we will discuss how to estimate posterior probabilities, and apply Bayesian statistics to phylogenetic data.