Bayesian Inference

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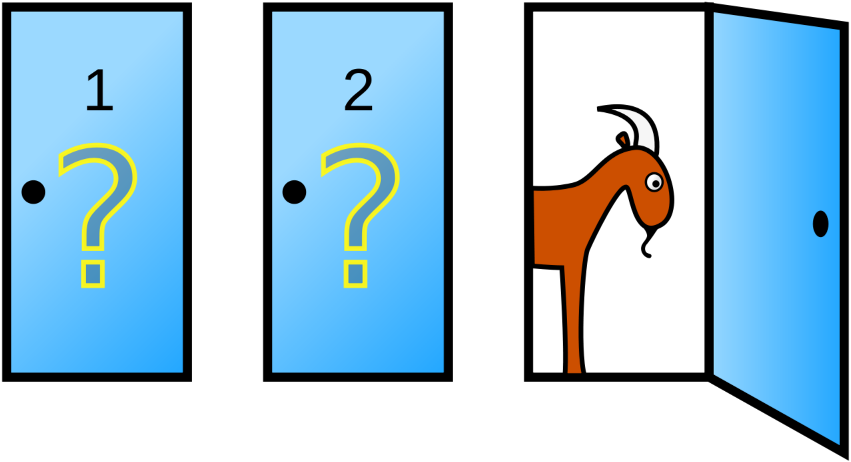
(With thanks to Mark Holder,and Paul Lewis for slides!)

Monte Hall problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?



Monte Hall problem

Exercise:

- Find a partner.
- Take 3 playing cards - two red (goats), and one black (car).
- Play the Monte Hall game. (take turns as the host and the contestant)
- Record if you got the car or the goat, and if you chose to switch or not. Run at least 12 games per tactic.
- Try out other variants! (what if the host doesn't know which door has the car? What if you have 3 goats and a car?)

Bayesian statistics

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- You decide to get tested.
- Test is correct 99 percent of the time. (in other words, if you have the disease, it shows that you do with 99 percent probability, and if you don't have the disease, it shows that you do not with 99 percent probability).
- This disease is actually quite rare, occurring randomly in the general population in only one of every 10,000 people.

If your test results come back positive, what are your chances that you actually have the disease?

Example from Su, Francis E., et al. "Medical Tests and Bayes' Theorem." Math Fun Facts. <http://www.math.hmc.edu/funfacts>.

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- .99
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- .10
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Congratulations, there is less than 1 percent chance that you have the disease!

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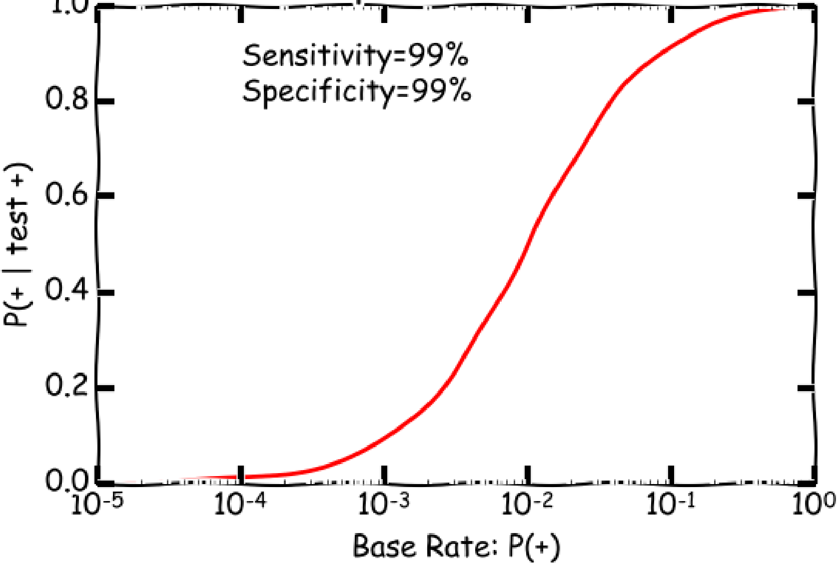
- Event A is the event you have this disease, and event B is the event that you test positive.
- $P(B|notA)$ is the probability of a "false positive": that you test positive even though you don't have the disease.
- In this case: $P(B|A) = .99$
- but $P(A) = .0001$.
- $P(B)$ may be derived by conditioning on whether event A does or does not occur:
- $P(B) = P(B|A)P(A) + P(B|notA)P(notA)$
- or .99*.0001+.01*.9999.

Thus the probability that you have the disease is less than 1 percent.

Even a *very good* test will be wrong most of the time when positives are rare



Bayes' theorem (aka Bayes' rule or Bayes law) helps us find the probability of event A given event B, ($P(A|B)$,
in terms of the probability of B given A, $P(B|A)$, and the probabilities of A and B:
$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$


Reverend Bayes

Monte Hall problem

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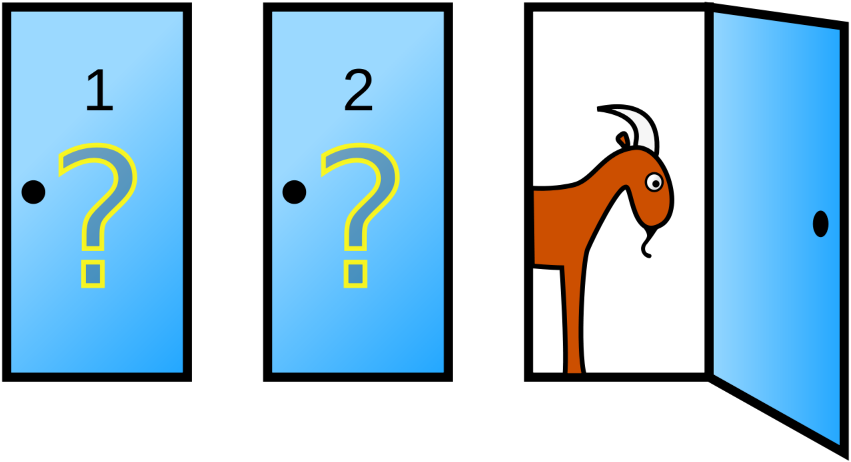

Applying Bayes theorem to the Monty Hall problem:

Assume we pick Door 1 and then Monty shows us a goat behind Door 2.
Now let A be the event that the car is behind Door 1.
B be the event that Monty shows us a goat behind Door 2.

$$\Pr(A \mid B) = \frac{\Pr(B \mid A) \times \Pr(A)}{\Pr(B)}$$

`http://angrystatistician.blogspot.com/2012/06/bayes-solution-to-monty-hall.html`

The tricky calculation is Pr(B).

- We initially chose Door 1.
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- If the car is behind Door 2, Monty never shows us a goat behind Door 2. $1/3 \times 0$
- Finally, if the car is behind Door 3, Monty shows us a goat behind Door 2 every time. $1/3 \times 1$

Thus, $\Pr(B) = 1/3 \times 1/2 + 1/3 \times 0 + 1/3 \times 1 = 1/2$.

Applying Bayes theorem to the Monty Hall problem:

Assume we pick Door 1 and then Monty shows us a goat behind Door 2.
Now let A be the event that the car is behind Door 1.
B be the event that Monty shows us a goat behind Door 2.

$$\begin{aligned}
\Pr(A \mid B) &= \frac{\Pr(B \mid A) \times \Pr(A)}{\Pr(B)} \\
&= \frac{1/2 \times 1/3}{1/3 \times 1/2 + 1/3 \times 0 + 1/3 \times 1} \\
&= 1/3.
\end{aligned}$$

`http://angrystatistician.blogspot.com/2012/06/bayes-solution-to-monty-hall.html`

We saw a goat behind door 2. The car is either behind Door 1 or Door 3, since the probability that it is behind Door 1 is $1/3$ and the sum of the two probabilities must equal 1, the probability the car is behind Door 3 is $1 - 1/3 = 2/3$.

Key aspect of this problem:
The added information is not THAT Monty Hall revealed a goat, it is that he picked a door specifically because it had a goat.

An R script to simulate the Monte Hall problem by Corey Chivers, (`https://bayesianbiologist.com/`)
`https://raw.githubusercontent.com/McTavishLab/GradPhylo/master/docs/scripts/MonteHall.R`

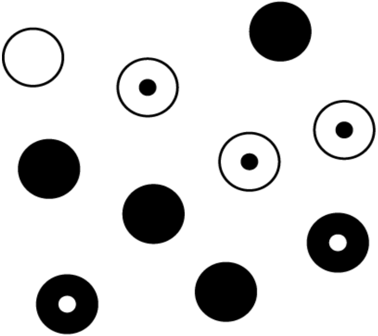
HW: What is the probability you get the car after switching if there are 10 doors? (1 car, 9 goats, everything else the same.)

Advantages of Bayesian inference:

- integrating prior information
- marginalizing inferences over parameter values

Joint probabilities

B = Black   S = Solid
W = White   D = Dotted



Pr(B) = 0.6   Pr(S) = 0.5

Pr(W) = 0.4   Pr(D) = 0.5

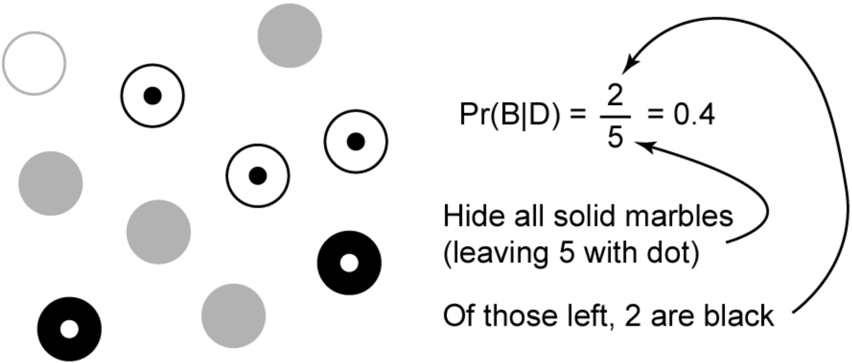
Pr(◉) = Pr(B, D) = 0.2

Pr(●) = Pr(B, S) = 0.4

Pr(⊙) = Pr(W, D) = 0.3

Pr(○) = Pr(W, S) = 0.1

Conditional probabilities

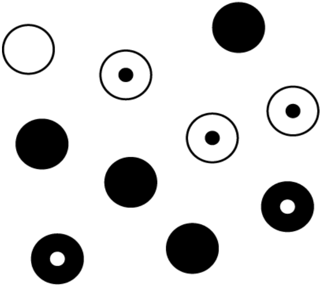


Bayes' rule

$$\Pr(B, D)$$

$$\Pr(D)\Pr(B|D) = \Pr(B)\Pr(D|B)$$

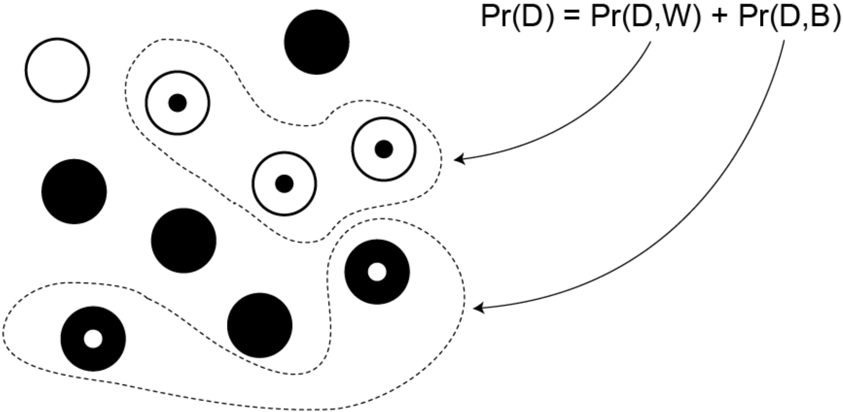
$$\frac{1}{2} \times \frac{2}{5} = \frac{3}{5} \times \frac{1}{3}$$



$$\Pr(B|D) = \frac{\Pr(B)\Pr(D|B)}{\Pr(D)}$$

$$= \frac{\frac{3}{5} \times \frac{1}{3}}{\frac{1}{2}} = \frac{2}{5}$$

Probability of "Dotted"



Bayes' rule (cont.)

$$\Pr(B|D) = \frac{\Pr(B)\Pr(D|B)}{\Pr(D)}$$

$$= \frac{\Pr(D,B)}{\Pr(D,B) + \Pr(D,W)}$$

Pr(*D*) is the **marginal probability** of being dotted
To compute it, we **marginalize over colors**

Bayes' rule (cont.)

It is easy to see that Pr(D) serves as a *normalization constant*, ensuring that Pr(B|D) + Pr(W|D) = 1.0

$$\Pr(B|D) = \frac{\Pr(D,B)}{\Pr(D,B) + \Pr(D,W)} \quad \longleftarrow \Pr(D)$$

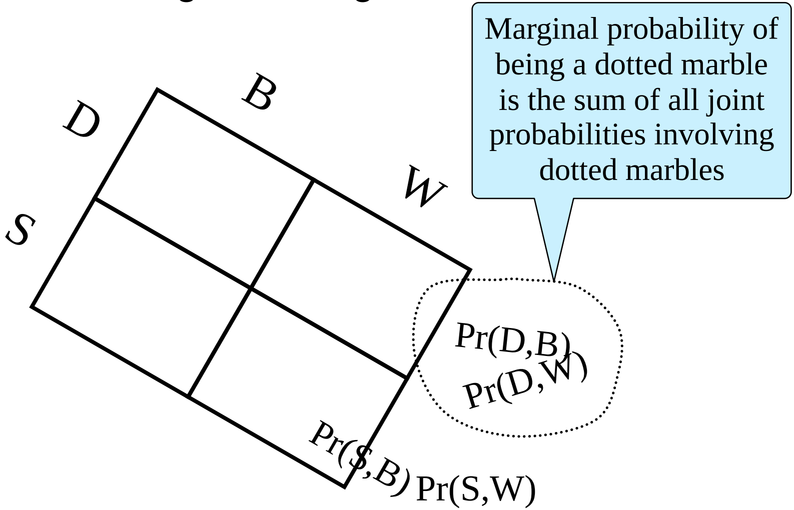
$$\Pr(W|D) = \frac{\Pr(D,W)}{\Pr(D,B) + \Pr(D,W)} \quad \longleftarrow \Pr(D)$$

$$\Pr(B|D) + \Pr(W|D) = \cancel{\frac{\Pr(D,B) + \Pr(D,W)}{\Pr(D,B) + \Pr(D,W)}} = 1$$

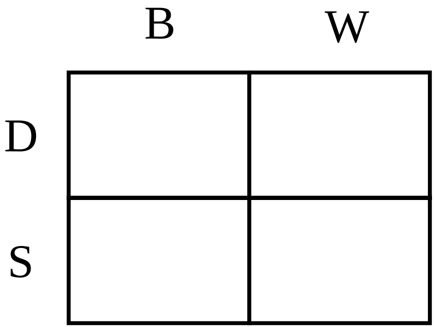
Joint probabilities

	B	W
D	Pr(D,B)	Pr(D,W)
S	Pr(S,B)	Pr(S,W)

Marginalizing over colors



Marginal probabilities



Pr(D) = marginal probability
of being dotted

Pr(D,B) + Pr(D,W)

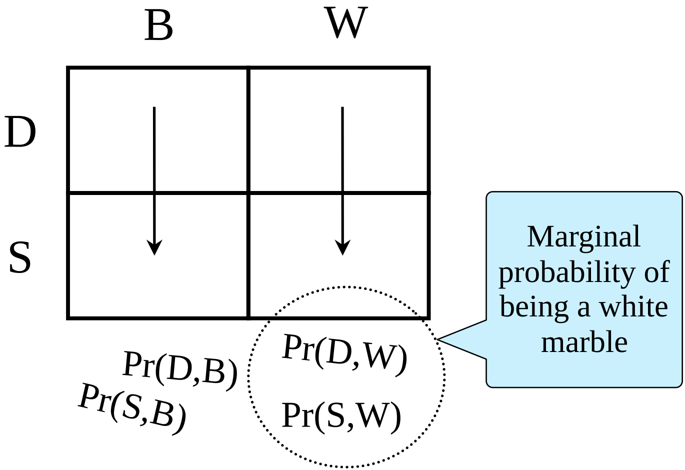
Pr(S,B) + Pr(S,W)

Pr(S) = marginal probability
of being solid

Joint probabilities

	B	W
D	Pr(D,B)	Pr(D,W)
S	Pr(S,B)	Pr(S,W)

Marginalizing over "dottedness"



Bayes' rule (cont.)

$$\begin{aligned}\Pr(B|D) &= \frac{\Pr(B)\Pr(D|B)}{\Pr(D,B)+\Pr(D,W)} \\ &= \frac{\Pr(B)\Pr(D|B)}{\Pr(B)\Pr(D|B)+\Pr(W)\Pr(D|W)} \\ &= \frac{\Pr(B)\Pr(D|B)}{\sum_{\theta\in\{B,W\}}\Pr(\theta)\Pr(D|\theta)}\end{aligned}$$

Bayes' rule in Statistics

$$\Pr(\theta|D) = \frac{\Pr(D|\theta)\Pr(\theta)}{\sum_\theta \Pr(D|\theta)\Pr(\theta)}$$

D refers to the "observables" (i.e. the **Data**)

θ refers to one or more "unobservables" (i.e. **parameters** of a model, or the **model itself**):

- *tree model* (i.e. tree topology)
- *substitution model* (e.g. JC, F84, GTR, etc.)
- *parameter* of a substitution model (e.g. a branch length, a base frequency, transition/transversion rate ratio, etc.)
- *hypothesis* (i.e. a special case of a model)
- a *latent variable* (e.g. ancestral state)

Bayes' rule in statistics

Likelihood of hypothesis θ

Prior probability of hypothesis θ

$$\Pr(\theta|D) = \frac{\Pr(D|\theta)\Pr(\theta)}{\sum_\theta \Pr(D|\theta)\Pr(\theta)}$$

Posterior probability of hypothesis θ

Marginal probability of the data (marginalizing over hypotheses)

Bayes' rule: continuous case

Likelihood

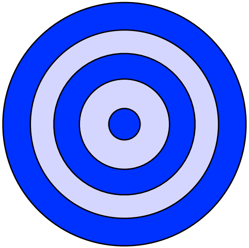
Prior probability ***density***

$$\boxed{f(\theta|D)} = \frac{\boxed{f(D|\theta)}\,\boxed{f(\theta)}}{\boxed{\int f(D|\theta)f(\theta)d\theta}}$$

Posterior probability ***density***

Marginal probability of the data

If you had to guess…



1 meter

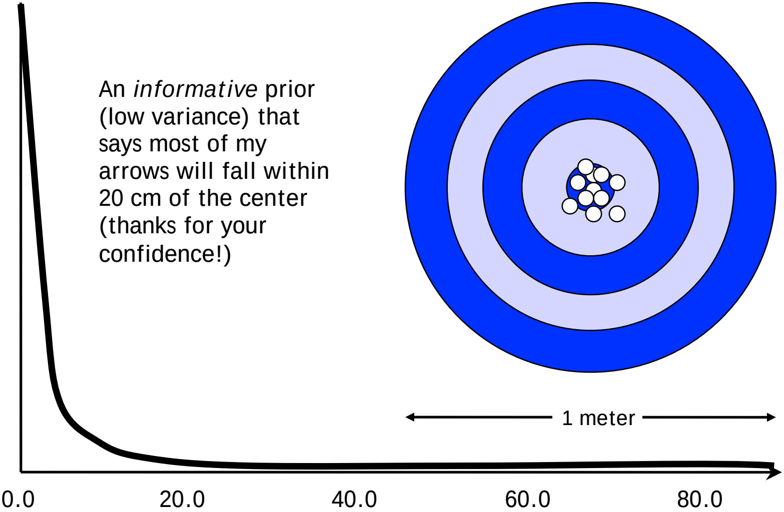
Photo by Tracy Heath

Not knowing anything about my archery abilities, draw a curve representing your view of the chances of my arrow landing a distance d centimeters from the center of the target (if it helps, I'm standing 50 meters away from the target)

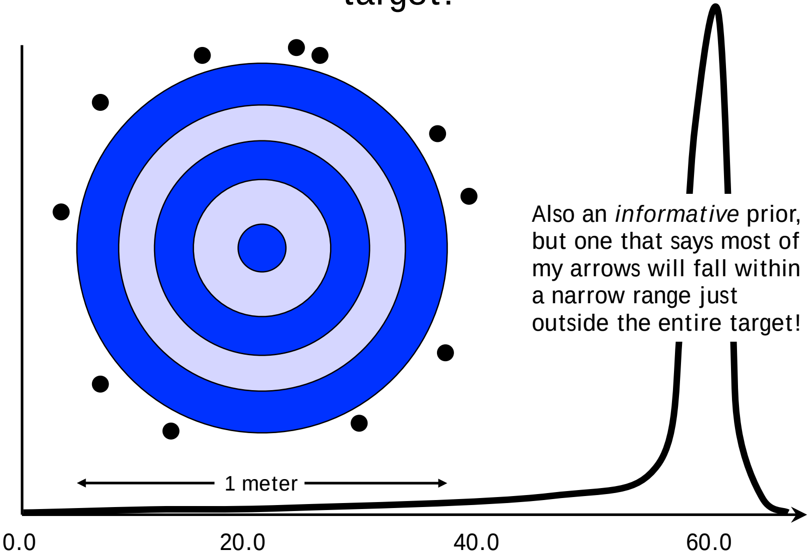
0.0 d ∞

Case 1: assume I have talent

An *informative* prior (low variance) that says most of my arrows will fall within 20 cm of the center (thanks for your confidence!)

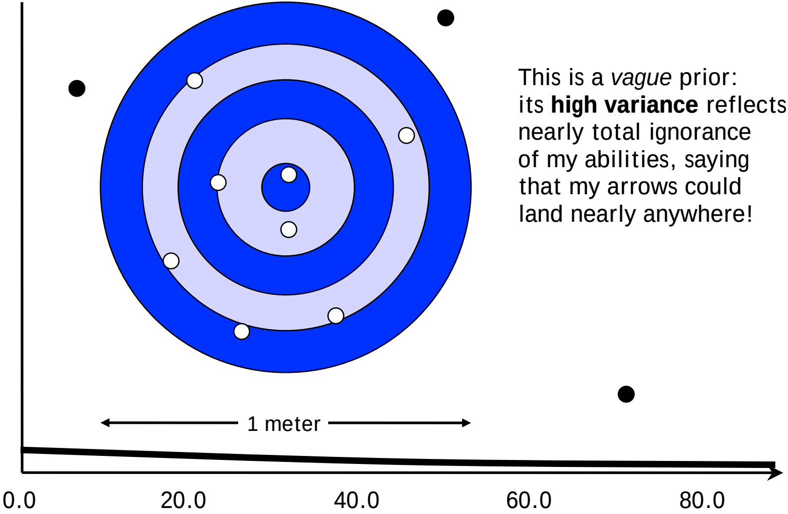


Case 2: assume I have a talent for missing the target!



Also an *informative* prior, but one that says most of my arrows will fall within a narrow range just outside the entire target!

Case 3: assume I have no talent



This is a *vague* prior: its **high variance** reflects nearly total ignorance of my abilities, saying that my arrows could land nearly anywhere!

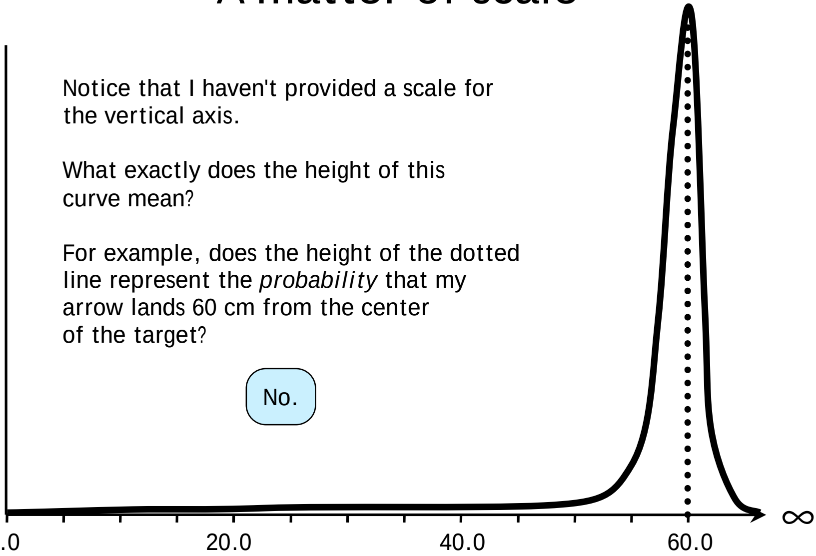
A matter of scale

Notice that I haven't provided a scale for the vertical axis.

What exactly does the height of this curve mean?

For example, does the height of the dotted line represent the *probability* that my arrow lands 60 cm from the center of the target?

No.

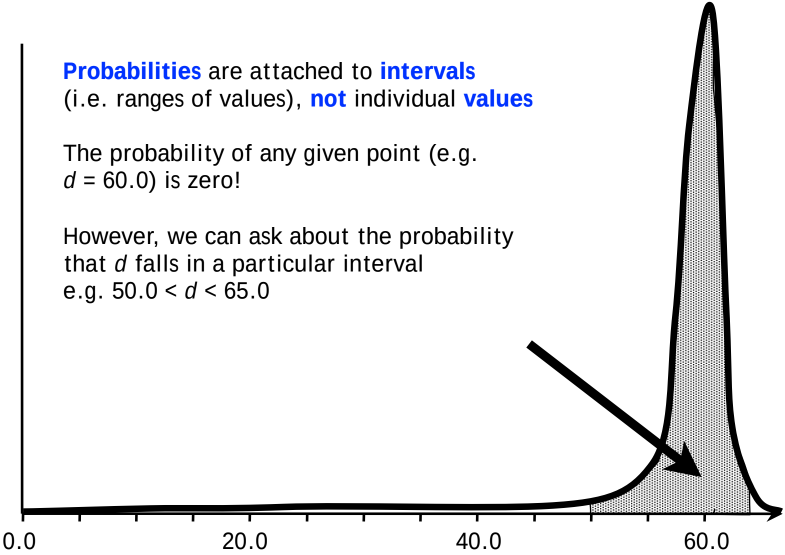


Probabilities are associated with intervals

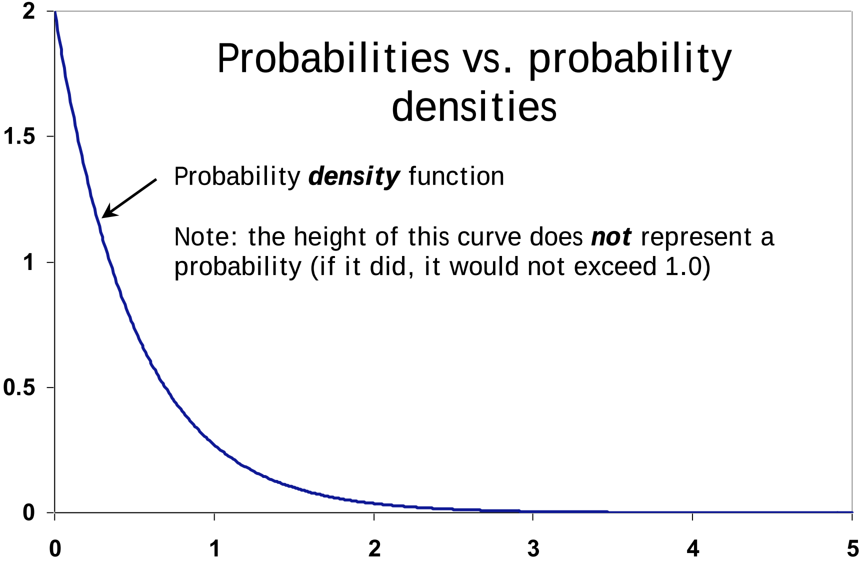
Probabilities are attached to **intervals** (i.e. ranges of values), **not** individual **values**

The probability of any given point (e.g. $d = 60.0$) is zero!

However, we can ask about the probability that d falls in a particular interval e.g. $50.0 < d < 65.0$



Probabilities vs. probability densities

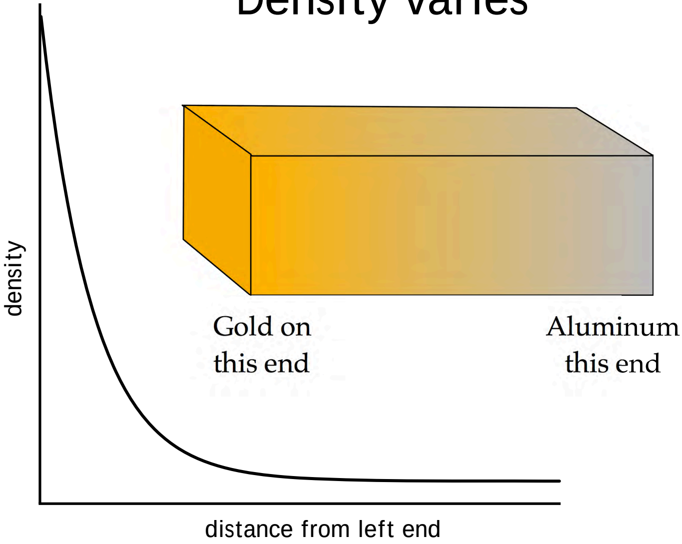


Densities of various substances

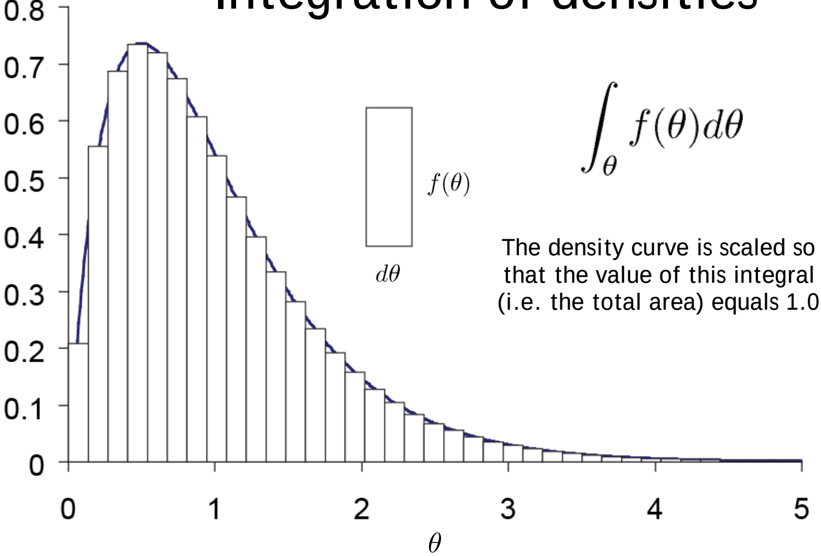
Substance	Density (g/cm^3)
Cork	0.24
Aluminum	2.7
Gold	19.3

Density does not equal mass
mass = density × volume

Density varies



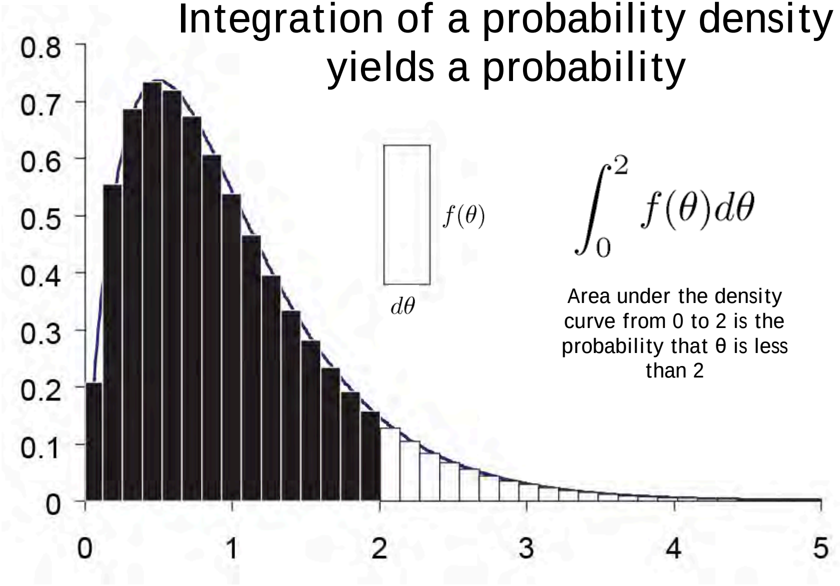
Integration of densities



$$\int_\theta f(\theta)d\theta$$

The density curve is scaled so that the value of this integral (i.e. the total area) equals 1.0

Integration of a probability density yields a probability

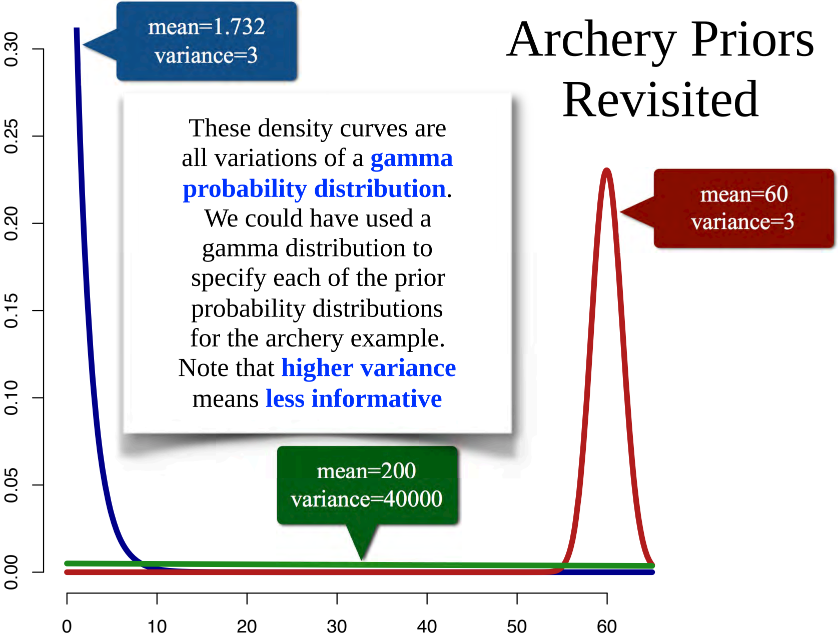


$$\int_0^2 f(\theta)d\theta$$

Area under the density curve from 0 to 2 is the probability that θ is less than 2

Archery Priors Revisited

These density curves are all variations of a **gamma probability distribution**. We could have used a gamma distribution to specify each of the prior probability distributions for the archery example. Note that **higher variance** means **less informative**



Usually there are many parameters...

A 2-parameter example

Prior probability density

Likelihood

$$f(\theta, \phi | D) = \frac{f(D|\theta, \phi)\; f(\theta) f(\phi)}{\int_\theta \int_\phi f(D|\theta, \phi)\; f(\theta) f(\phi)\; d\theta d\phi}$$

Marginal probability of data

Posterior probability density

An analysis of **100 sequences** under the simplest model (JC69) requires 197 branch length parameters. The denominator is a **197-fold integral** in this case! Now consider summing over **all possible tree topologies**! It would thus be nice to avoid having to calculate the marginal probability of the data...

Next class we will discuss how to estimate posterior proabilities, and apply Bayesian statistics to phylogenetic data.