CS 3133 Foundations of Computer Science
C term 2019

Solutions for Homework 2

READING: Chapters 3, 4, 5, 18.

1. Exercise 2 on page 97.

   Solution:

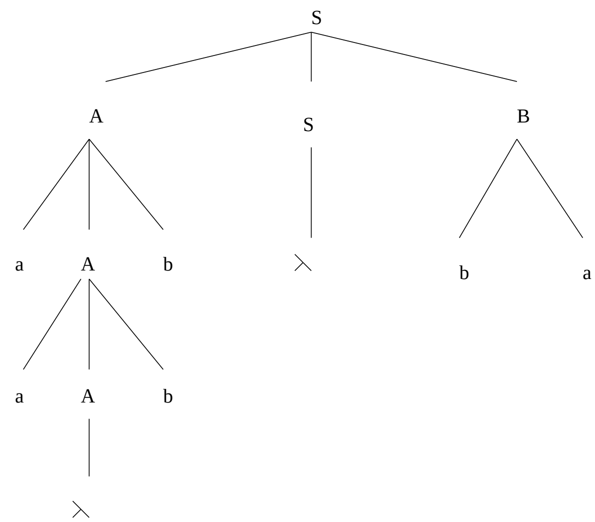
   (a) The following is a leftmost derivation of $aabbba$:

$$
\begin{aligned}
S &\Rightarrow ASB \\
&\Rightarrow aAbSB \\
&\Rightarrow aaAbbSB \\
&\Rightarrow aabbSB \\
&\Rightarrow aabbB \\
&\Rightarrow aabbba
\end{aligned}
$$

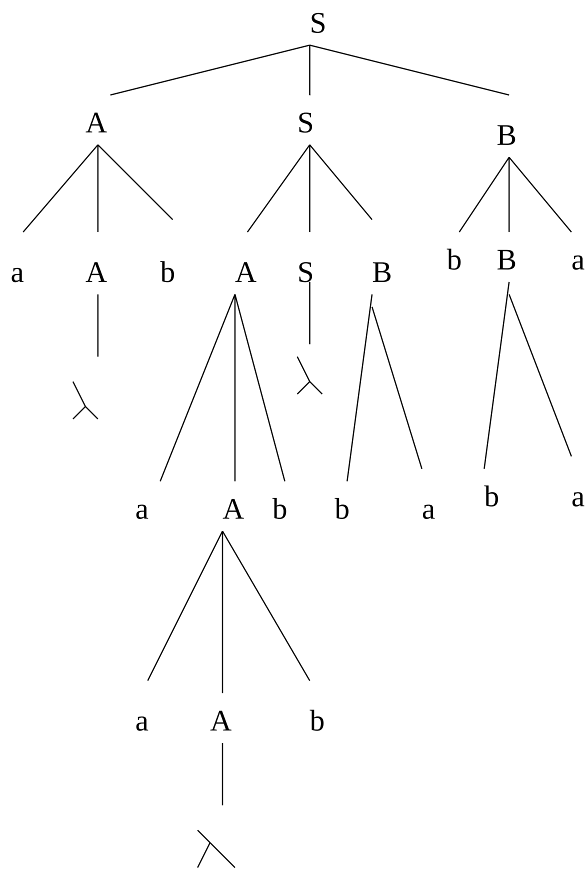
   (b) The following is a rightmost derivation of $abaabbbabbaa$:

$$
\begin{aligned}
S &\Rightarrow ASB \\
&\Rightarrow ASbBa \\
&\Rightarrow ASbbaa \\
&\Rightarrow AASBbbaa \\
&\Rightarrow AASbabbaa \\
&\Rightarrow AAbabbaa \\
&\Rightarrow AaAbbabbaa \\
&\Rightarrow AaaAbbbabbaa \\
&\Rightarrow Aaabbbabbaa \\
&\Rightarrow aAbaabbbabbaa \\
&\Rightarrow abaabbbabbaa
\end{aligned}
$$

(c) Derivation tree for (a):



Derivation tree for (b):



(d)

$L(G) = \{a^{n_1}b^{n_1}\ldots a^{n_k}b^{n_k}b^{m_1}a^{m_1}\ldots b^{m_l}a^{m_l} | n_i, m_j > 0, l \geq 0, k \geq 0, k \leq l\}$

(15 points)

2. Exercise 4 on page 98.

Solution:

(a) The following is a leftmost derivation that generates the given tree

DT:

$$\begin{aligned} S &\Rightarrow AB \\ &\Rightarrow aAB \\ &\Rightarrow aaB \\ &\Rightarrow aaAB \\ &\Rightarrow aaaB \\ &\Rightarrow aaab \end{aligned}$$

(b) The following is a rightmost derivation that generates the given tree DT:

$$\begin{aligned} S &\Rightarrow AB \\ &\Rightarrow AAB \\ &\Rightarrow AAb \\ &\Rightarrow Aab \\ &\Rightarrow aAab \\ &\Rightarrow aaab \end{aligned}$$

(c) There are 20 derivations that generate DT. (20 points)

3. Exercise 7 on page 98.

Solution: The following is a grammar over $\{a, b, c\}$ whose language is exactly $\{a^n b^{2n} c^m \mid n, m > 0\}$:

$$\begin{aligned} S &\rightarrow AC \\ A &\rightarrow aAbb \mid abb \\ C &\rightarrow Cc \mid c \end{aligned}$$

(15 points)

4. Show by induction that for every natural number n, 3 is a divisor of $n^3 + 2n$.

Solution: Basis: It is true for $n = 0$.

Inductive Hypothesis: Assume that it is true for all values $k = 0, 1, \ldots, n$, i.e.

$$3 | k^3 + 2k.$$

Inductive Step: We need to show that it is true for $n + 1$, i.e.

$$3 | (n+1)^3 + 2(n+1).$$

Indeed,

$$(n+1)^3 + 2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 2 =$$

$$= (n^3 + 2n) + 3n^2 + 3n + 3 = (n^3 + 2n) + 3(n^2 + n + 1),$$

which is divisible by three. (The first part because of the Inductive Hypothesis and the second part trivially.) (15 points)

5. Let G be the grammar

$$S \to ASB|\lambda$$
$$A \to a$$
$$B \to b.$$

   (a) What is $L(G)$?

   (b) Prove formally (so using induction on the length of the derivations) that $L(G)$ is the set given in (a).

Solution:

(a) $L(G) = \{a^n b^n | n \geq 0\}$.

(b) First we show that $L(G) \subseteq \{a^n b^n | n \geq 0\}$. For this purpose we will show by induction on the length of the derivations that $n(a) + n(A) = n(b) + n(B)$ and that in all the strings in the derivation the a-s and A-s form a prefix of the string and the b-s and B-s form a suffix of the string.

Basis: Derivations of length 0, so S. True. Remark: Sometimes you have to start with derivations of length 1 as basis. Here the statement is true for $n = 0$ so we can start with $n = 0$ as basis, but sometimes this is not case.

Inductive Hypothesis: We assume that this statement is true for all strings w that can be obtained by n rule applications, so $S \overset{n}{\Rightarrow} w$.

Inductive Step: We have to show that the statement is true for all strings w that can be obtained by $n+1$ rule applications, so $S \overset{n+1}{\Rightarrow} w$. Once again the key step is to reformulate the derivation to apply the inductive hypothesis. The derivation of w can be written $S \overset{n}{\Rightarrow} w' \Rightarrow w$. By the Inductive Hypothesis we know that the statement is true for w', so $n_{w'}(a) + n_{w'}(A) = n_{w'}(b) + n_{w'}(B)$ (say $= k$) and the a-s and A-s form a prefix of w' and the b-s and B-s form a suffix of w'. But these obviously remain true when we apply one more rule:

Rule	$n_w(a) + n_w(A)$	$n_w(b) + n_w(B)$
$S \to ASB$	$k+1$	$k+1$
$S \to \lambda$	k	k
$A \to a$	k	k
$B \to b$	k	k

Next we show that $\{a^n b^n | n \geq 0\} \subseteq L(G)$. Indeed,

$$\begin{aligned} S &\stackrel{n}{\Rightarrow} \underbrace{A\ldots A}_{n} S \underbrace{B\ldots B}_{n} \\ &\Rightarrow \underbrace{A\ldots A}_{n} \underbrace{B\ldots B}_{n} \\ &\stackrel{n}{\Rightarrow} \underbrace{a\ldots a}_{n} \underbrace{B\ldots B}_{n} \\ &\stackrel{n}{\Rightarrow} \underbrace{a\ldots a}_{n} \underbrace{b\ldots b}_{n} \end{aligned}$$

(20 points)

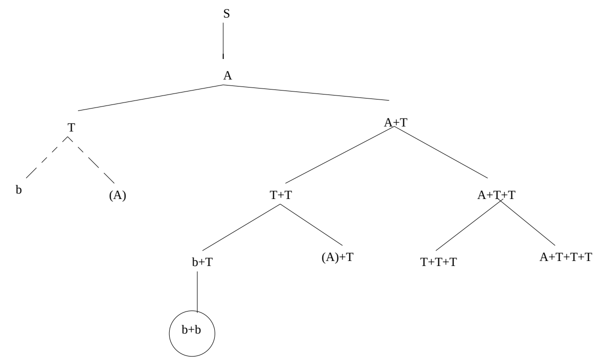
6. In this problem we consider the grammar of arithmetic expressions AE, so

$$\begin{array}{lll} AE: & V = & \{S, A, T\} \\ & \Sigma = & \{b, +, (,)\} \\ & P: & 1. S \to A \\ & & 2. A \to T \\ & & 3. A \to A + T \\ & & 4. T \to b \\ & & 5. T \to (A) \end{array}$$

Build the search tree constructed by the breadth-first top-down parsing algorithm for the string $b + b$.

Solution:

The breadth-first top-down search tree:



(15 points)