HW7

1. a) $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$V^T\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$DV^T\vec{x} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$UDV^T\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$V^T\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$DV^T\vec{x} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$UDV^T\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$M = UDV^T$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\therefore M\vec{x} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$M\vec{x} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

b) $N = UDV^T$

$N^T N = (UDV^T)^T UDV^T$

$= (V^T)^T D^T U^T U D V^T$

$\because U$ is orthagonal

$= V D^T D V^T$ $\quad \therefore U^T U = I$

$= V D^2 V^T$ $\quad \because D$ is diagonal matrix

$\therefore D^T D = D^2$

2. $\tilde{p} = (a, b, c, d)$

When world points are on the plane defined by $\tilde{p}$: $\bar{x}^w \cdot \tilde{p} = 0$

$$\Rightarrow a x^w + b y^w + c z^w + d = 0 \quad \text{— ①}$$

"Show that if all world points are co-planar, then A cannot have rank greater than 9 by finding 3 independant non-zero vector $\vec{m}$ such that $A\vec{m} = \vec{0}$"

consider $\vec{m} = [a\ b\ c\ d\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^T$

$\to 0$ (using eq. ①)

$$\Rightarrow A\vec{m} = \begin{bmatrix} a x_1^w + b y_1^w + c z_1^w + d + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= 0$$

∴

consider $\vec{m} = [0\ 0\ 0\ 0\ a\ b\ c\ d\ 0\ 0\ 0\ 0]^T$

$$A\vec{m} = \begin{bmatrix} 0+0+0+0+0+0+0+0 \quad + 0+0+0+0 \\ \cdots 0 \cdots \quad \underbrace{a x_i^w + b y_i^w + c z_i^w + d}_{0} + \quad 0 \\ \\ \end{bmatrix}$$

$= 0$

consider $\vec{m} = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ a\ b\ c\ d]^T$

$$A\vec{m} = \begin{bmatrix} 0+0 \cdots\cdots \quad + x_i^p(\underbrace{a x_i^w + b y_i^w + c z_i^w + d}_{0}) \\ 0+\cdots\cdots \quad + y_i^p(\underbrace{a x_i^w + b y_i^w + c z_i^w + d}_{0}) \\ \\ \end{bmatrix}$$

$= 0$

Ideally there should be a unique $\vec{m}$ that satisfies $A\vec{m} = 0$, but there are at least 3 independent $\vec{m}$ vectors such that $A\vec{m} = 0$

∴ Nullspace of A has rank at least 3

∴ max rank(A) $= 12 - 3 = 9$

3. $$\vec{X}^I = \frac{|\vec{f}|^2 \vec{X}^c}{\vec{f} \cdot \vec{X}^c}$$

$$V^I \equiv \frac{dX^I}{dt} = \frac{d}{dt}\left(\frac{|\vec{f}|^2 \vec{X}^c}{\vec{f} \cdot \vec{X}^c}\right)$$

using $\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$

$$= \frac{\frac{d}{dt}(|\vec{f}|^2 \vec{X}^c) \cdot (\vec{f} \cdot \vec{X}^c) - \frac{d}{dt}(\vec{f} \cdot \vec{X}^c) \cdot (|\vec{f}|^2 \cdot \vec{X}^c)}{(\vec{f} \cdot \vec{X}^c)^2}$$

$$= \frac{(|\vec{f}|^2 \vec{V}^c)(\vec{f} \cdot \vec{X}^c) - (|\vec{f}|^2 \cdot \vec{X})(\vec{f} \cdot \vec{V}^c)}{(\vec{f} \cdot \vec{X}^c)^2} \quad \text{— (1)}$$

$$X^I_{FOE} = \frac{|\vec{f}|^2 \vec{V}^c}{\vec{f} \cdot \vec{V}^c}$$

$$\Rightarrow |\vec{f}|^2 \vec{V}^c = X^I_{FOE} (\vec{f}\, \vec{V}^c) \quad \text{— (2)}$$

$$X^I = \frac{|\vec{f}|^2 \vec{X}^c}{\vec{f}\, \vec{X}^c}$$

$$\Rightarrow |\vec{f}|^2 \vec{X}^c = X^I (\vec{f} \cdot \vec{X}^c) \quad \text{— (3)}$$

Substituting ②, ③ into ①

$$V^I = \frac{X^I_{FOE}(\vec{f}\cdot\vec{V}^c)(\vec{f}\cdot\vec{X}^c) - X^I(\vec{f}\cdot\vec{X}^c)(\vec{f}\cdot\vec{V}^c)}{(\vec{f}\cdot\vec{X}^c)^2}$$

$$= \frac{(\vec{f}\cdot\vec{V}^c)\cancel{(\vec{f}\cdot\vec{X}^c)}}{(\vec{f}\cdot\vec{X}^c)^{\cancel{2}}}\left(X^I_{FOE} - X^I\right)$$

$$= -\frac{\vec{f}\ \vec{V}^c}{\vec{f}\ \vec{X}^c}\left(X^I - X^I_{FOE}\right)$$

$$\Rightarrow k = \frac{-\vec{f}\cdot\vec{V}^c}{\vec{f}\cdot\vec{X}^c}$$