1、 From the optical flow constraint equation:

$$I_x u + I_y v + I_t = 0$$

we can get the line equation:

$$V = \left(\frac{-I_x}{I_y}\right)u - \frac{I_t}{I_y} \quad — ①$$

Because $\vec{V}_{min}$ should perpendicular to the line

we can know the slop should be $\frac{I_y}{I_x}$

And $\vec{V}_{min}$ pass through $(0,0)$

$\Rightarrow$ the line equation is $V = \left(\frac{I_y}{I_x}\right)u \quad - ②$

The intercetion point is

$$\begin{cases} V = \left(\frac{-I_x}{I_y}\right)u - \frac{I_t}{I_y} \\ V = \left(\frac{I_y}{I_x}\right)u \end{cases}$$

$$\Rightarrow \quad u = \frac{-I_x I_t}{I_x^2 + I_y^2}, \quad V = \frac{-I_y I_t}{I_x^2 + I_y^2}$$

$$\Rightarrow \vec{V}_{min} = \left(\frac{-I_x I_t}{I_x^2 + I_y^2}, \frac{-I_y I_t}{I_x^2 + I_y^2}\right)$$

$$|\vec{v}| = \frac{-\frac{I_t}{I_y} \times -\frac{I_t}{I_x}}{\sqrt{\left(0 - \frac{I_t}{I_y}\right)^2 + \left(\frac{-I_t}{I_x} - 0\right)^2}}$$

$$= \frac{I_t^2}{I_x I_y} \Big/ \sqrt{\frac{I_t^2}{I_x^2} + \frac{I_t^2}{I_y^2}}$$

$$= \frac{I_t^2}{I_x I_y} \Big/ \sqrt{\frac{I_x^2 I_t^2 + I_y^2 I_t^2}{I_x^2 \; I_y^2}}$$

$$= \frac{I_t^2}{I_x I_y} \Big/ \sqrt{\frac{I_t^2}{I_x^2 I_y^2}\left(I_x^2 + I_y^2\right)}$$

$$= \frac{I_t^{\cancel{2}}}{\cancel{I_x I_y}} \Big/ \frac{\cancel{I_t}}{\cancel{I_x I_y}} \sqrt{I_x^2 + I_y^2}$$

$$= \frac{I_t}{\sqrt{I_x^2 + I_y^2}}$$

2. a)

$$I(x,y,t) = e^{-\frac{1}{2\sigma^2}\left(t^2 - 2\left(\frac{x}{k_1}+\frac{y}{k_2}\right)t + \left(\frac{x}{k_1}+\frac{y}{k_2}\right)^2\right)}$$

$$= e^{-\frac{1}{2\sigma^2}\left(t - \left(\frac{x}{k_1}+\frac{y}{k_2}\right)\right)^2}$$

$$I_x = I(x,y,t)\left[\frac{-1}{2\sigma^2}\left(2\left(t-\left(\frac{x}{k_1}+\frac{y}{k_2}\right)\right)\right)\left(\frac{-1}{k_1}\right)\right]$$

$$= I(x,y,t)\left[\frac{1}{k_1\sigma^2}\left(t-\left(\frac{x}{k_1}+\frac{y}{k_2}\right)\right)\right]$$

similarly

$$I_y = I(x,y,t)\left[\frac{1}{k_2\sigma^2}\left(t-\left(\frac{x}{k_1}+\frac{y}{k_2}\right)\right)\right]$$

$$I_t = \frac{dI}{dt} = I(x,y,t)\left[\frac{-1}{\sigma^2}\left(t-\left(\frac{x}{k_1}+\frac{y}{k_2}\right)\right)\right]$$

b) $I_x u + I_y V + I_t = 0$

$$\Rightarrow I(x,y,t)\frac{1}{\sigma^2}\left(t-\left(\frac{x}{k_1}+\frac{y}{k_2}\right)\right)\left(\frac{u}{k_1}+\frac{V}{k_2}-1\right) = 0$$

$$\therefore \frac{u}{k_1}+\frac{V}{k_2}-1 = 0$$

$$\frac{u}{k_1}+\frac{V}{k_2} = 1$$

3. a) $I(x,y,t) = I_0 + \frac{1}{2}\left[(x-c_1 t)^2 + (y-c_2 t)^2\right]$

$$I_x = \frac{dI}{dx} = \frac{1}{2}\, 2 \cdot (x - c_1 t)$$

$$= x - c_1 t$$

$$I_y = \frac{dI}{dy} = \frac{1}{2}\, 2(y - c_2 t)$$

$$= y - c_2 t$$

$$I_t = \frac{dI}{dt} = \frac{1}{2}\left[2\cdot(-c_1(x - c_1 t)) + 2 \cdot -c_2\,(y - c_2 t)\right]$$

$$= -c_1 x + c_1^2 t - c_2 y + c_2^2 t$$

b) $I_x u + I_y v + I_t = 0$

$$\Rightarrow (x - c_1 t)u + (y - c_2 t)v - c_1 x + c_1^2 t - c_2 y + c_2^2 t = 0$$

$$\Rightarrow (u - c_1)x + (v - c_2)y + c_1(c_1 - u)t + c_2(c_2 - v) = 0$$

c) $\Rightarrow$ $u = c_1$

$v = c_2$

4. a)

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \lambda I_x^2+4 & \lambda I_x I_y \\ \lambda I_x I_y & \lambda I_y^2+4 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{n\in neighbors} u^{old}(n) - \lambda I_x I_y \\ \sum_{n\in neighbors} v^{old}(n) - \lambda I_x I_y \end{bmatrix}$$

Let $A = \begin{bmatrix} \lambda I_x^2+4 & \lambda I_x I_y \\ \lambda I_x I_y & \lambda I_y^2+4 \end{bmatrix}$

$$\Rightarrow A^{-1} = \begin{bmatrix} \lambda I_x^2+4 & \lambda I_x I_y \\ \lambda I_x I_y & \lambda I_y^2+4 \end{bmatrix}^{-1}$$

$$\because \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} \lambda I_y^2+4 & -\lambda I_x I_y \\ -\lambda I_x I_y & \lambda I_x^2+4 \end{bmatrix}$$

$$= \frac{1}{\lambda^2 I_x I_y^2 + 4\lambda I_x^2 + 4\lambda I_y^2 + 16 - \lambda^2 I_x^2 I_y^2} \begin{bmatrix} \lambda I_y^2+4 & -\lambda I_x I_y \\ -\lambda I_x I_y & \lambda I_x^2+4 \end{bmatrix}$$

$$= \frac{1}{4\lambda I_x^2 + 4\lambda I_y + 16} \begin{bmatrix} \lambda I_y^2+4 & -\lambda I_x I_y \\ -\lambda I_x I_y & \lambda I_x^2+4 \end{bmatrix}$$

b) $\because \dfrac{\sum u^{old}}{4} = \bar{u}^{old}$ , $\dfrac{\sum v^{old}}{4} = \bar{v}^{old}$

from a) we know

$$u^{new} = \frac{1}{4\lambda I_x^2 + 4\lambda I_y^2 + 16}\Big[(\lambda I_y^2 + 4)(\sum u^{old} - \lambda I_x I_t) + (-\lambda I_x I_y)(\sum v^{old} - \lambda I_y I_t)\Big]$$

$$= \frac{1}{4\lambda I_x^2 + 4\lambda I_y^2 + 16}\Big(\lambda I_y^2 \sum u^{old} + 4\sum u^{old} - \cancel{\lambda^2 I_x I_y I_t} - 4\lambda I_x I_t - \lambda I_x I_y \sum v^{old} + \cancel{\lambda^2 I_x I_y I_t}\Big)$$

$$= \frac{1}{I_x^2 + I_y^2 + \frac{4}{\lambda}} \cdot \frac{1}{4\lambda}\Big(\lambda I_y^2 \sum u^{old} + 4\sum u^{old} - 4\lambda I_x I_t - \lambda I_x I_y \sum v^{old}\Big)$$

$$= \frac{1}{I_x^2 + I_y^2 + \frac{4}{\lambda}}\Big(I_y^2 \frac{\sum u^{old}}{4} + \frac{4}{\lambda}\frac{\sum u^{old}}{4} - I_x I_t - I_x I_y \frac{\sum v^{old}}{4}\Big)$$

$$= \frac{1}{I_x^2 + I_y^2 + \frac{4}{\lambda}}\Big(I_y^2 \bar{u}^{old} + \frac{4}{\lambda}\bar{u}^{old} - I_x I_t - I_x I_y \bar{v}^{old}\Big)$$

$$= \frac{1}{I_x^2 + I_y^2 + \frac{4}{\lambda}}\Big(I_x^2 \bar{u}^{old} - I_x^2 \bar{u}^{old} + I_y^2 \bar{u}^{old} + \frac{4}{\lambda}\bar{u}^{old} - I_x I_t - I_x I_y \bar{v}^{old}\Big)$$

$$= \frac{1}{I_x^2 + I_y^2 + \frac{4}{\lambda}}\Big\{(I_x^2 + I_y^2 + \frac{\lambda}{4})\bar{u}^{old} - I_x(I_x \bar{u}^{old} + I_y \bar{v}^{old} + I_t)\Big]$$

$$= \bar{u}^{old} - \frac{I_x}{I_x^2 + I_y + \frac{4}{\lambda}} \left( I_x \bar{u}^{old} + I_y \bar{v}^{old} + I_t \right)$$

with same process, $V^{new}$ can proof

c) $$u^{new} = \bar{u}^{old} - \frac{I_x}{I_x^2 + I_y^2 + \frac{4}{\lambda}} \left( I_x \bar{u}^{old} + I_y \bar{v}^{old} + I_t \right)$$

$$= \bar{u}^{old} - \frac{\lambda I_x}{\lambda I_x^2 + \lambda I_y^2 + 4} \left( I_x \bar{u}^{old} + I_y \bar{v}^{old} + I_t \right)$$

when $\lambda = 0$

$$\Rightarrow u^{new} = \bar{u}^{old} - \frac{0}{4} \left( I_x \bar{u}^{old} + I_y \bar{v}^{old} + I_t \right)$$

$$= \bar{u}^{old}$$

so as $v^{new}$

$$\Rightarrow \begin{bmatrix} u \\ v \end{bmatrix}^{new} = \begin{bmatrix} \bar{u}^{old} \\ \bar{v}^{old} \end{bmatrix}$$