# Linear Algebra Proofs

Below are several proof techniques that you should KNOW how to apply by the end of 3191 …this means that any of these is fair game for the final exam. Each one below comes with several examples.

1. Let $H$ be a subset of a vector space $V$. Prove that $H$ is a subspace of $V$.

   Technique: Let $\boldsymbol{u}, \boldsymbol{v} \in H$ and let $c, d \in \mathbb{R}$. Show that $c\boldsymbol{u} + d\boldsymbol{v} \in H$ by checking that the membership criterial for $H$ are satisfied.
   Examples:

   (a) Let $A$ be an $m \times n$ matrix. Prove that $Nul(A) = \{\boldsymbol{x} : A\boldsymbol{x} = \boldsymbol{0}\}$ is a subspace of $\mathbb{R}^n$

   (b) Let $A$ be an $m \times n$ matrix. Prove that $Col(A) = \{\boldsymbol{b} : A\boldsymbol{x} = \boldsymbol{b} \text{ for some } \boldsymbol{x} \in \mathbb{R}^n\}$ is a subspace of $\mathbb{R}^m$.

   (c) Let $C[a, b]$ be the set of continuous function on the interval $[a, b]$. Show that $S = \{f \in C[a, b] : f(a) = f(b)\}$ is a subspace of $C[a, b]$.

   (d) Show that the set of all matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is a subspace of the $2 \times 2$ matrices

   (e) Let $\lambda$ be an eigenvalue of a square matrix $A$. Prove that the eigenspace, $E_\lambda$, is a subspace of $\mathbb{R}^n$.

   (f) Prove that the orthogonal complement, $W^\perp = \{\boldsymbol{v} \in \mathbb{R}^n : \boldsymbol{v} \cdot \boldsymbol{w} = 0 \,\forall \boldsymbol{w} \in W\}$ is a subspace of $\mathbb{R}^n$.

2. Prove that the transformation $T : V \to W$ is linear.

   Technique: Let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and $c, d \in \mathbb{R}$ and show that $T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2) = c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2)$ using the definition of $T$ (and possibly properties from the spaces $V$ and $W$).
   Examples:

   (a) Prove that if $T(\boldsymbol{x}) = A\boldsymbol{x}$ where $A$ is an $m \times n$ matrix, then $T$ is a linear transformation.

   (b) Let $M_{2\times2}$ be the vector space of all $2 \times 2$ matrices, and define $T : M_{2\times2} \to M_{2\times2}$ by $T(A) = A + A^T$. Prove that $T$ is a linear transformation. (also describe the kernel of $T$)

   (c) Define $T : C[0, 1] \to C[0, 1]$ as follows: For $f \in C[0, 1]$, let $T(f)$ be the antiderivative, $F$, of $f$ such that $F(0) = 0$. Prove that $T$ is a linear transformation. (also describe the kernel of $T$)

3. Prove that the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is linearly independent.

   Technique: Let $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Prove that the equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots c_n\boldsymbol{v}_n = \boldsymbol{0}$ only has the trivial solution.
   Examples:

   (a) Prove that the set $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$ is linearly independent.

   (b) Prove that the set of polynomials $\{1, 1 - t, 1 + t - t^2\}$ is linearly independent.

   (c) Prove that the set of matrices
   $$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \right\}$$
   is linearly independent.

   (d) Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^n$. Prove that $S$ is a linearly independent set.

4. Prove that a set $S = \{\boldsymbol{v}_1, \ldots \boldsymbol{v}_n\}$ is a basis for a vector space $V$.

Technique: Prove that $S$ spans the vector space and prove that $S$ is linearly independent.
Examples:

(a) Let $A \in M_{n\times n}$ such that $A^{-1}$ exists. Prove that the columns of $A$ form a basis for $\mathbb{R}^n$.

(b) Prove that the set of polynomials $\{1, 1-t, 1+t-t^2\}$ is a basis for $\mathbb{P}_2$.

(c) Prove that the set of matrices

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \right\}$$

is a basis for $M_{2\times 2}$.

**Solutions**

1. Let $H$ be a subset of a vector space $V$. Prove that $H$ is a subspace of $V$.

   Technique: Let $\boldsymbol{u}, \boldsymbol{v} \in H$ and let $c, d \in \mathbb{R}$. Show that $c\boldsymbol{u} + d\boldsymbol{v} \in H$ by checking that the membership criterial for $H$ are satisfied.
   Examples:

   (a) Let $A$ be an $m \times n$ matrix. Prove that $Nul(A) = \{\boldsymbol{x} : A\boldsymbol{x} = \boldsymbol{0}\}$ is a subspace of $\mathbb{R}^n$

   (Soln) Let $\boldsymbol{u}, \boldsymbol{v} \in Nul(A)$ and let $c, d \in \mathbb{R}$. Observe that $A(c\boldsymbol{u} + d\boldsymbol{v}) = A(c\boldsymbol{u}) + A(d\boldsymbol{v}) = cA\boldsymbol{u} + dA\boldsymbol{v} = c\boldsymbol{0} + d\boldsymbol{0} = \boldsymbol{0} + \boldsymbol{0} = \boldsymbol{0}$. Therefore $c\boldsymbol{u} + d\boldsymbol{v} \in Nul(A)$. Therefore, $Nul(A)$ is a subspace of $\mathbb{R}^n$.

   (b) Let $A$ be an $m \times n$ matrix. Prove that $Col(A) = \{\boldsymbol{b} : A\boldsymbol{x} = \boldsymbol{b} \text{ for some } \boldsymbol{x} \in \mathbb{R}^n\}$ is a subspace of $\mathbb{R}^m$.

   (Soln) Let $\boldsymbol{b}_1, \boldsymbol{b}_2 \in Col(A)$ and let $c, d \in \mathbb{R}$. If $\boldsymbol{b}_j \in Col(A)$ then there exists an $\boldsymbol{x}_j$ such that $A\boldsymbol{x}_j = \boldsymbol{b}_j$. Define $\boldsymbol{x} = (c\boldsymbol{x}_1 + d\boldsymbol{x}_2)$ and observe that $A\boldsymbol{x} = A(c\boldsymbol{x}_1 + d\boldsymbol{x}_2) = cA\boldsymbol{x}_1 + dA\boldsymbol{x}_2 = c\boldsymbol{b}_1 + d\boldsymbol{b}_2$. Therefore there is an $\boldsymbol{x}$ such that $A\boldsymbol{x} = c\boldsymbol{b}_1 + d\boldsymbol{b}_2$, and this means that $c\boldsymbol{b}_1 + d\boldsymbol{b}_2 \in Col(A)$. Therefore, $Col(A)$ is a subspace of $\mathbb{R}^m$.

   (c) Let $C[a, b]$ be the set of continuous function on the interval $[a, b]$. Show that $S = \{f \in C[a, b] : f(a) = f(b)\}$ is a subspace of $C[a, b]$.

   (Soln) Let $f, g \in S$ and let $c, d \in \mathbb{R}$. First observe that since $f$ and $g$ are continouous on $[a, b]$ we must have $cf + dg$ continuous on $[a, b]$ (from calculus). Furthermore, $(cf + dg)(a) = (cf)(a) + (dg)(a) = cf(a) + dg(a) = cf(b) + dg(b) = (cf)(a) + (dg)(b) = (cf + dg)(b)$ where the third equal sign is true since $f, g \in S$. Hence, $cf + dg \in S$. Therefore, $S$ is a subspace of $C[a, b]$.

   (d) Show that the set of all matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is a subspace of the $2 \times 2$ matrices

   (Soln) (we'll use a theorem to do the work for us on this one). Observe that matrices of this form must belong to the set
   $$S = span\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$
   Since all matrices of the desired form can be written as the span of a set then they must form a subspace (theorem 1 pg 194).

   (e) Let $\lambda$ be an eigenvalue of a square matrix $A$. Prove that the eigenspace, $E_\lambda$, is a subspace of $\mathbb{R}^n$.

   (Soln) Let $\boldsymbol{u}, \boldsymbol{v} \in E_\lambda$ and let $c, d \in \mathbb{R}$. Observe that $A(c\boldsymbol{u} + d\boldsymbol{v}) = cA\boldsymbol{u} + dA\boldsymbol{v} = c(\lambda\boldsymbol{u}) + d(\lambda\boldsymbol{v}) = \lambda(c\boldsymbol{u} + d\boldsymbol{v})$. Therefore, $c\boldsymbol{u} + d\boldsymbol{v} \in E_\lambda$. Therefore, $E_\lambda$ is a subspace of $\mathbb{R}^n$

   (f) Prove that the orthogonal complement, $W^\perp = \{\boldsymbol{v} \in \mathbb{R}^n : \boldsymbol{v} \cdot \boldsymbol{w} = 0\, \forall \boldsymbol{w} \in W\}$ is a subspace of $\mathbb{R}^n$.

   (Soln) Let $\boldsymbol{u}, \boldsymbol{v} \in W^\perp$ and let $c, d \in \mathbb{R}$. Since $\boldsymbol{u}, \boldsymbol{v} \in W^\perp$ we know that $\boldsymbol{u} \cdot \boldsymbol{w} = 0 = \boldsymbol{v} \cdot \boldsymbol{w}$ for all $\boldsymbol{w} \in W$. Observe that $(c\boldsymbol{u} + d\boldsymbol{v}) \cdot \boldsymbol{w} = c\boldsymbol{u} \cdot \boldsymbol{w} + d\boldsymbol{v} \cdot \boldsymbol{w} = c(0) + d(0) = 0 + 0 = 0$. Therefore, $c\boldsymbol{u} + d\boldsymbol{v} \in W^\perp$. Therefore, $W^\perp$ is a subspace of $\mathbb{R}^n$

2. Prove that the transformation $T : V \to W$ is linear.

   Technique: Let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and $c, d \in \mathbb{R}$ and show that $T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2) = c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2)$ using the definition of $T$ (and possibly properties from the spaces $V$ and $W$).
   Examples:

   (a) Prove that if $T(\boldsymbol{x}) = A\boldsymbol{x}$ where $A$ is an $m \times n$ matrix, then $T$ is a linear transformation.

   (Soln) Let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and let $c, d \in \mathbb{R}$. Observe that $T(c\boldsymbol{v}_1 + d\boldsymbol{v}_2) = A(c\boldsymbol{v}_1 + d\boldsymbol{v}_2) = cA\boldsymbol{v}_1 + d\boldsymbol{v}_2 = cT(\boldsymbol{v}_1) + dT(\boldsymbol{v}_2)$. Therefore $T$ is linear.

(b) Let $M_{2\times2}$ be the vector space of all $2\times 2$ matrices, and define $T: M_{2\times2} \to M_{2\times2}$ by $T(A) = A + A^T$. Prove that $T$ is a linear transformation. (also describe the kernel of $T$)

(Soln) Let $A, B \in M_{2\times2}$ and let $c, d \in \mathbb{R}$. Observe that $T(cA+dB) = (cA+dB)+(cA+dB)^T = cA+dB+cA^T+dA^T = c(A+A^T)+d(B+B^T) = cT(A)+dT(B)$. Therefore $T$ is linear.
The kernel described as

$$ker(T) = \left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} : a \in \mathbb{R} \right\}.$$

(c) Define $T: C[0,1] \to C[0,1]$ as follows: For $f \in C[0,1]$, let $T(f)$ be the antiderivative, $F$, of $f$ such that $F(0) = 0$. Prove that $T$ is a linear transformation. (also describe the kernel of $T$)

(Soln) Let $f, g \in C[0,1]$ and let $c, d \in \mathbb{R}$. Observe that $T(cf+dg) = \displaystyle\int (cf+dg)dx = c\int f dx + d\int g dx = cF + dG = cT(f) + dT(g)$. Therefore $T$ is linear.
The kernel contains only the zero function.

3. Prove that the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is linearly independent.

Technique: Let $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Prove that the equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots c_n\boldsymbol{v}_n = \mathbf{0}$ only has the trivial solution.
Examples:

(a) Prove that the set $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$ is linearly independent.

(b) write the three vectors as the columns of a matrix, $A$, and row reduce ... you will see three pivots, and this implies that the only solution to the equation $A\boldsymbol{x} = \mathbf{0}$. Therefore the vectors are linearly independnet.

(c) Prove that the set of polynomials $\{1, 1-t, 1+t-t^2\}$ is linearly independent.

(d) Let $c_1, c_2, c_3 \in \mathbb{R}$. Observe that $c_1(1) + c_2(1-t) + c_3(1+t-t^2) = (c_1+c_2+c_3) + (c_3-c_2)t - c_3t^3$. If this polynomial is zero then it is clear that $c_3 = 0$, $c_3 - c_2 = 0$, and $c_1 + c_2 + c_3 = 0$. The obvious solution to this system is $c_1 = c_2 = c_3 = 0$. Therefore the polynomials are linearly independent.

(e) Prove that the set of matrices

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \right\}$$

is linearly independent.

(Soln) First write the coordinate vectors for each matrix with respect to the standard basis for $M_{2\times2}$:

$$[\boldsymbol{v}_1] = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, [\boldsymbol{v}_2] = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, [\boldsymbol{v}_3] = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$$

Now write these vectors as the columns of a $4\times 3$ matrix and row reduce. You will find that there are three pivots, and therefore the columns are linearly independent. Since the coordinate vectors are independent we know that the matrices must be independent.

(f) Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^n$. Prove that $S$ is a linearly independent set.

(Soln) Let $c_1, \ldots c_n \in \mathbb{R}$ such that $c_1\boldsymbol{v}_1 + \cdots c_n\boldsymbol{v}_n = \mathbf{0}$. Take the dot product with $\boldsymbol{v}_1$ and we get $0 = \boldsymbol{v}_1 \cdot \displaystyle\sum_{j=1}^{n}(c_j\boldsymbol{v}_j) = c_1||\boldsymbol{v}_1||^2$. Therefore, $c_1 = 0$. Similarly we find that $c_2, c_3, \ldots c_n = 0$. Therefore $S$ is linearly independent.

4. Prove that a set $S = \{\boldsymbol{v}_1, \dots \boldsymbol{v}_n\}$ is a basis for a vector space $V$.

Technique: Prove that $S$ spans the vector space and prove that $S$ is linearly independent.
Examples:

(a) Let $A \in M_{n\times n}$ such that $A^{-1}$ exists. Prove that the columns of $A$ form a basis for $\mathbb{R}^n$.

(b) From the invertible matrix theorem, the columns of an invertible matrix are independent and span $\mathbb{R}^n$. Therefore the columns form a basis for $\mathbb{R}^n$.

(c) Prove that the set of polynomials $\{1, 1-t, 1+t-t^2\}$ is a basis for $\mathbb{P}_2$.

(d) From the proof above we see that this set is linearly independent. It is also clear that in order to form a quadratic polynomial we must have all three vectors from the set. Therefore the set spans $\mathbb{P}_2$.

(e) Prove that the set of matrices

$$\left\{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}\right\}$$

is a basis for $M_{2\times 2}$.

(f) Using a similar method as the proof above we can see that the linearly independent (use coordinate vectors). Also, the set of $2 \times 2$ matrices is four dimensional so we know that any linearly independent set of four vectors must form a basis.