CS/RBE 549 Computer Vision Name:________________________________
Fall 2019 Due: Monday 28 Oct 2019

HW #7

Note the following class changes:
Prof. Gennert will be away the week of Nov 4. To make up for lost class time, we will meet as follows:

Mon 28 Oct.	4:30-7:20pm with breaks. HW #7 due.
Wed 30 Oct.	4:30-5:50pm as usual.
Mon 4 Nov.	No class. HW #8 due.
Wed 6 Nov.	No class.
Mon 11 Nov.	4:30-7:20pm with breaks. HW #9 due.
Wed 13 Nov.	Resume usual schedule.

1. **Singular Value Decomposition (4 pts):** This is an important tool that allows us to solve systems of equations $A\vec{x} \approx \vec{0}, |\vec{x}| = 1$. This is useful in camera calibration, where we get such a system of equations and want a non-zero parameter vector $\vec{x}$. See Szeliski for an introduction to SVD.

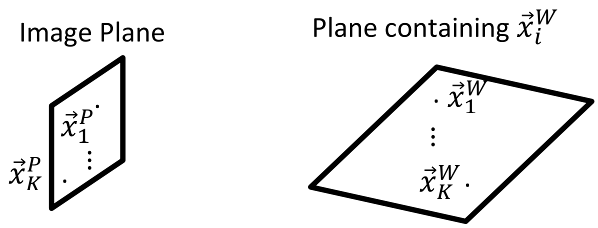
   You can think of the SVD as follows:
   1. It takes the component of input $\vec{x}$ in the u_1 direction, scales it by σ_1 and outputs it in the v_1 direction.
   2. Repeat for all u_i, σ_i, and v_i.
   3. The output vector is the sum of all the contributions

   a. Suppose that 2×2 matrix *M* has singular value decomposition
   $$U = [u_1|u_2] = \frac{1}{\sqrt{2}}\left[\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array}\right], D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, V^T = \begin{bmatrix} v_1 \\ \hline v_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ \hline 1 & -1 \end{bmatrix}$$
   Without computing M, predict the value of $M\vec{x}$ for $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ based on the SVD properties. Hint: Note that in both cases $V^T\vec{x}$ has a simple form, so does $DV^T\vec{x}$, and finally, so does $UDV^T\vec{x}$. Now compute *M* and use it to verify $M\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $M\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   b. Let matrix *N* have singular value decomposition $N = UDV^T$. Show that matrix N^TN has SVD $N^TN = VD^2V^T$.

2. **Camera Calibration (10 pts):** One restriction in camera calibration is that the world points chosen must not lie in a single plane, that is, they cannot be co-planar, otherwise calibration will fail. To see this, suppose that there are *K* world points $\vec{x}_i^W, 1 \le i \le K$. We know that we

need at least 6 points for calibration, $K \geq 6$. Consider what happens if all world points lie in a single plane represented by $\tilde{p} = (a, b, c, d)$, defined by $\tilde{x}^W \cdot \tilde{p} = 0$ (*Szeliski*, eqn. 2.7).



The calibration equation is given by $A\vec{m} = \vec{0}$, written out as

$$\begin{bmatrix} x_1^W & y_1^W & z_1^W & 1 & 0 & 0 & 0 & 0 & x_1^P x_1^W & x_1^P y_1^W & x_1^P z_1^W & x_1^P \\ 0 & 0 & 0 & 0 & x_1^W & y_1^W & z_1^W & 1 & y_1^P x_1^W & y_1^P y_1^W & y_1^P z_1^W & y_1^P \\ & \vdots & & & & & & & & \vdots & & \\ 0 & 0 & 0 & & & \cdots & z_K^W & 1 & y_K^P x_K^W & y_K^P y_K^W & y_K^P z_K^W & y_K^P \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{12} \end{bmatrix}$$

Ideally, there should be a single vector $\vec{m}$ such that $A\vec{m} = \vec{0}$ (or $\approx \vec{0}$). This is the $\vec{m}$ that we hope to find through the singular value decomposition of A. The key concept here is the *rank* of a matrix, which is the number of independent rows, also the number of independent columns. For this $\vec{m}$ to exist and be unique, A must have rank = 11 (or 12 with a very small value for the smallest singular value σ_{12}. Show that if all world points are co-planar, then A cannot have rank greater than 9 by finding 3 independent non-zero vectors $\vec{m}$ such that $A\vec{m} = \vec{0}$. (These independent vectors establish that the nullspace of A has rank at least 3; hence A's rank cannot exceed 12-3=9.) In this case, it is not possible to find a unique $\vec{m}$ such that $A\vec{m} = \vec{0}$ and the calibration procedure fails.

Hint: The non-zero vectors $\vec{m}$ are mostly 0s. Use the fact that if all world points are co-planar, then $\tilde{x}_i^W \cdot \tilde{p} = 0$.

3. **Focus of Expansion (8 pts):** Suppose that the viewer (camera) is moving. We can model this in the imaging equations as

$$\vec{X}^C = R\vec{X}^W + \vec{T} \text{ and } \vec{X}^I = \frac{\left|\vec{f}\right|^2 \vec{X}^C}{\vec{f} \cdot \vec{X}^C}$$

by letting R and $\vec{T}$ depend on time, $R = R(t), \vec{T} = \vec{T}(t)$. Assume that the camera is translating with velocity $\vec{V}^C \equiv \frac{d}{dt}\vec{T}(t)$ and that there is no rotation, $\frac{d}{dt}R(t) = 0$. As we know from experience, points in the image will seem to move. As we saw in class, image points will appear to move with velocity

$$\vec{V}^I \equiv \frac{d}{dt}\vec{X}^I(t) = \frac{1}{\vec{f} \cdot \vec{X}^C}\left(\left(\vec{X}^I \times \vec{V}^C\right) \times \vec{f}\right).$$

The Focus of Expansion (FOE) is the point in the image toward which the camera appears to be moving, given by the projection of $\vec{V}^C$ into the image. That is,

$$\vec{X}^I{}_{\text{FOE}} = \frac{\left|\vec{f}\right|^2 \vec{V}^C}{\vec{f} \cdot \vec{V}^C}.$$

Show that for any world point $\vec{X}^W$, its image point $\vec{X}^I$ will appear to move with velocity

$$\vec{V}^I = k\left(\vec{X}^I - \vec{X}^I{}_{\text{FOE}}\right),$$

proving that image points appear to move toward or away from the FOE. What is k?