Chapter-2: Combinatorial Probability

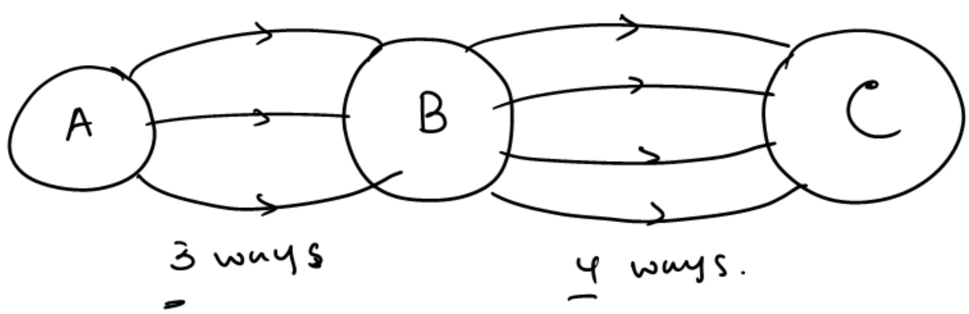
If Ω has finite equally likely outcomes, then for any event A,

$$P(A) = \frac{n(A)}{n(\Omega)}.$$

The multiplication Rule

If a task consists of a sequence of choices in which there are n_1 selections for the 1^{st} choice, n_2 selections for the second choice, and so on, then the task of making the selections can be done in

$n_1 \cdot n_2 \cdot n_3 \cdots\cdots$ different ways.



$\#$ of ways $= 3 * 4 = 12$.

Eg:-

The United States postal service currently uses 5-digits zip codes in most of the areas.

a) How many zip codes are possible if there are no restrictions?

b) How many would be possible if the first number could not be zero?

c) How many would be possible if the numbers should be different?

a)

$$\underset{\underset{10}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{10}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{10}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{10}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{10}{\uparrow}}{\underline{\quad}}$$

\# of codes $= 10 * 10 * 10 \cdots * 10$
$= 10^5.$

b)

$$\underset{\underset{9}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{10}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{10}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{10}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{10}{\uparrow}}{\underline{\quad}}$$

\# of codes $= 9 \cdot 10^4.$

c)

$$\underset{\underset{10}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{9}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{8}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{7}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{6}{\uparrow}}{\underline{\quad}}$$

\# of codes $= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 =$ ________

Eg.-2) How many ways can 5 people stand on a line?

$$\underset{\underset{5}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{4}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{3}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{2}{\uparrow}}{\underline{\quad}}\quad \underset{\underset{1}{\uparrow}}{\underline{\quad}}$$

\# of ways $= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

<u>General formula:</u>

Total number of ways to arrange "n" different objects on a line is

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 = n!$$

Defn <u>factorial n</u> $\mathbb{Z}$

Let $n \in \mathbb{Z}_0^+$ (non negative integers). Then the factorial n is

$$n! = \begin{cases} 1 & ; n = 0 \\ n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1 & ; n > 0 \end{cases}$$

Note:

* $0! = 1$
* $1! = 1$
* $2! = 2\cdot 1 = 2$
* $3! = 3\cdot 2\cdot 1 = 6$

$\vdots$

* $n! = n\cdot \underbrace{(n-1)\cdot \overbrace{(n-2)\cdots\cdots 2\cdot 1}^{(n-2)!}}_{(n-1)!}$

* $n! = n\cdot(n-1)!$

$= n\cdot(n-1)\cdot(n-2)!$

$= n\cdot(n-1)\cdots\cdots(n-k+1)\cdot(n-k)!$

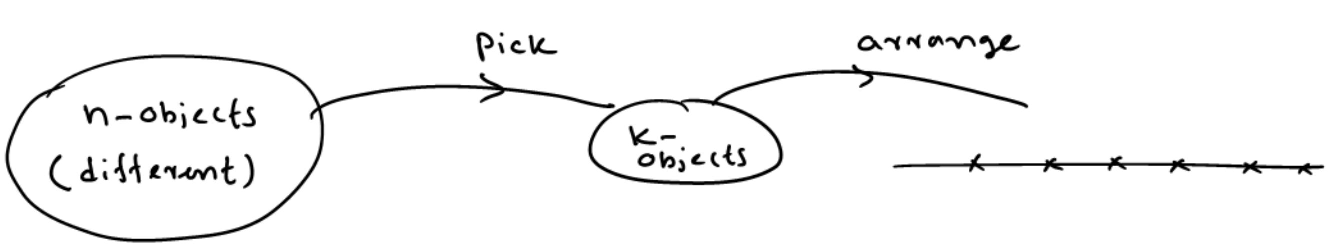
$k \to k-1$

$-(k-1) = -k+1$

Defn <u>Permutations</u> <u>(order matters)</u>

Number of different ways to pick "k" objects out of "n" different objects and arrange them on a line is denoted by "^np_k" and

$$^np_k = \frac{n!}{(n-k)!} \quad ; k \leq n.$$



Note:

* when $k=0$, $\quad {}^{n}P_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$

* when $k=n$, $\quad {}^{n}P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$

* $${}^{n}P_k = \frac{n\cdot(n-1)\cdot(n-2)\cdots\cdot(n-k+1)\cdot\cancel{(n-k)!}}{\cancel{(n-k)!}} = n\cdot(n-1)\cdot\cdots\cdot(n-k+1).$$

Eg:-

A baseball team has 15 players. How many 9 player batting orders are possible?

$n=15$, $k=9$

$$\#\text{ of orders} = \underbrace{{}^{15}P_9} = \frac{15!}{(15-9)!} = \frac{15!}{6!}$$

$$= 15\cdot 14\cdot 13\cdots 7$$

$$= 1\ 816\ 214\ 400.$$

Eg-2)

A student activity club at a college has 32 members. In how many different ways can the club select a president, a vice president, a treasurer, and a secretary?

$n=32$, $k=4$

$$\#\text{ of ways} = {}^{32}P_4 = \frac{32!}{(32-4)!} = \frac{32\cdot 31\cdot 30\cdot 29\cdot \cancel{28!}}{\cancel{28!}}$$

$$= 863040.$$

Defn

Combinations (order doesn't matter).

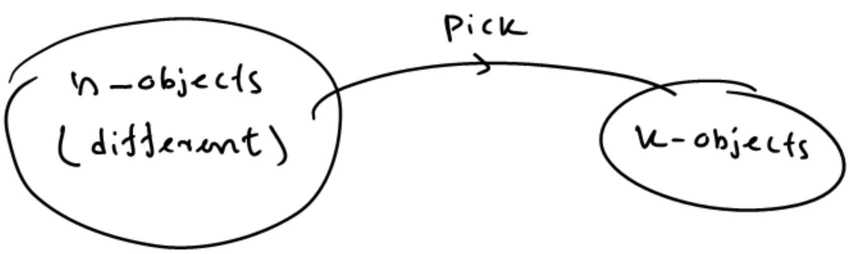
Number of different ways to pick "k" out of "n" objects (do not arrange) is denoted by "${}^{n}C_k$" or "$\binom{n}{k}$" and

$${}^{n}C_k = \binom{n}{k} = \frac{n!}{(n-k)!\cdot k!} \quad : \quad k \leq n.$$

# of arrangements $= {}^nP_k$

# of ways to arrange k objects $\} = k!$

$\therefore$ # groups (combinations) $= \dfrac{{}^nP_k}{k!} = \dfrac{n!}{(n-k)!\cdot k!} = {}^nC_k$.



A group of 4 students is to be selected from a group of 10 students to take part in a class in cell Biology. In how many ways can this be done?

$n = 10, \quad k = 4$

$$\# \text{ of ways} = {}^{10}C_4 = \frac{10!}{(10-4)!\cdot 4!} = \frac{10!}{4!\cdot 6!}$$

$$= 210$$

Eg-2) Consider fliping 5 fair coins. Find the probability distri-bution of

X — # of heads.