HW-2 Solutions

#1 $p(x) = 1/10, \quad x = 0, 1, 2, \dots 10.$

a) $Y \equiv \bmod(4)$

Possible values for y: 0, 1, 2, 3

$0 \leftarrow (x = 0, 4, 8)$

$1 \leftarrow (x = 1, 5, 9)$

$2 \leftarrow (x = 2, 6)$

$3 \leftarrow (x = 3, 7)$

$$p(y=0) = p(X=0) + p(X=4) + p(X=8) = 3/10$$

$$p(y=1) = p(X=1) + p(X=5) + p(X=9) = 3/10$$

$$p(y=2) = p(x=2) + p(x=6) = 2/10$$

$$p(y=3) = p(x=3) + p(x=7) = 2/10$$

$\therefore$ Probability distribution of Y,

$$P_y(y) = \begin{cases} 3/10 & ; \ y = 0, 1 \\ 2/10 & ; \ y = 2, 3. \end{cases}$$

b) $Y = 6 \bmod (X+1)$

$X = 0 \longrightarrow Y = 0$

$X = 1 \longrightarrow Y = 0$

$X = 2 \longrightarrow Y = 0$

$X = 3 \longrightarrow Y = 2$

$X = 4 \longrightarrow Y = 1$

$X = 5 \longrightarrow Y = 0$

$\left.\begin{matrix} X = 6 \\ X = 7 \\ X = 8 \\ X = 9 \end{matrix}\right\} \longrightarrow Y = 6$

Possible values for y: 0, 1, 2, 6

$0 \leftarrow (X = 0, 1, 2, 5)$

$1 \leftarrow (X = 4)$

$2 \leftarrow (X = 3)$

$6 \leftarrow (X = 6, 7, 8, 9)$

$\therefore$ Probability distribution of Y:

$$P(y) = \begin{cases} 4/10 & ; \ y = 0 \\ 1/10 & ; \ y = 1 \\ 1/10 & ; \ y = 2 \\ 4/10 & ; \ y = 6. \end{cases}$$

#2 $$p(x) = \begin{cases} x^2/a & ; \quad x = -2, -1, 0, 1, 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

a) $$P(-2) + P(-1) + P(0) + P(1) + P(2) = 1$$

$$\frac{(-2)^2}{a} + \frac{(-1)^2}{a} + \frac{0^2}{a} + \frac{1^2}{a} + \frac{2^2}{a} = 1$$

$$\frac{10}{a} = 1 \implies a = 10$$

$$E(X) = (-2)\cdot\frac{(-2)^2}{10} + (-1)\frac{(-1)^2}{10} + 0\cdot\frac{0^2}{10} + 1\cdot\frac{1^2}{10} + 2\cdot\frac{2^2}{10}$$

$$= -\frac{8}{10} - \frac{1}{10} + 0 + \frac{1}{10} + \frac{8}{10}$$

$$= 0$$

b) $$Z = (X - E(X))^2 = (X-0)^2 = X^2$$

possible values for Z: 4, 1, 0

$x = -2, 2$ $\quad$ $x = -1, 1$ $\quad$ $x = 0$

$$P(Z=0) = P(X=0) = 0$$

$$P(Z=1) = P(X=-1) + P(X=1) = 1/10 + 1/10 = 2/10$$

$$P(Z=4) = P(X=-2) + P(X=2) = \frac{4}{10} + 4/10 = 8/10$$

$\therefore$ Probability distribution of Z,

$$P(z) = \begin{cases} 2/10 & ; \quad z = 1 \\ 8/10 & ; \quad z = 4 \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

c)

$$\mathrm{Var}(X) = E\Big(\underbrace{(X - E(X))^2}_{Z}\Big) = E(Z)$$

$$= 1 \cdot \frac{2}{10} + 4 \cdot \frac{8}{10} = \frac{34}{10} = 3.4$$

d) $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

but $E(X^2) = (-2)^2 \cdot \frac{(-2)^2}{10} + (-1)^2 \frac{(-1)^2}{10} + 0^2 \cdot \frac{0^2}{10} + 1^2 \cdot \frac{1^2}{10} + 2^2 \cdot \frac{2^2}{10}$

$$= \frac{16}{10} + \frac{1}{10} + \frac{0}{10} + \frac{1}{10} + \frac{16}{10}$$

$$= \frac{34}{10}$$

$$\mathrm{Var}(X) = \cancel{34}\, E(X^2) - (E(X))^2 = \frac{34}{10} - 0 = 3.4$$

#3 Let X - winning amount, then

$$E(X) = (-1)(0.3) + 0(0.4) + 3(0.2) + 10(0.1)$$

$$= 1.3$$

$$E(X^2) = (-1)^2(0.3) + 0^2(0.4) + 3^2(0.2) + 10^2(0.1)$$

$$= 12.1$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 12.1 - (1.3)^2 = \boxed{10.41}$$

b) Let $Y = X - 1$, then

$E(Y) = E(X-1) = 1.3 - 1 = 0.3$

$Var(Y) = Var(X-1) = Var(X) = 10.41$

We use:

$E(aX+b) = aE(X) + b$ and

$Var(aX+b) = a^2 Var(X)$.

c) Let $Z = 2Y$, then

$E(Z) = E(2\cancel{X}) = 2E(\cancel{X}) = 2(0.3) = 0.6$.

$Var(Z) = Var(2Y) = 4 Var(Y) = 4(10.41) = 41.64$.

#3) (Did not grade)

Let X = time between mosquito bites OR

= time until the next mosquito bite OR

= # of Seconds until the next mosquito bite.

Let A – a mosquito lands on your neck.

B – it will bite you, then

$P(A) = 0.5$, $P(B|A) = 0.2$

$\therefore \underbrace{P(\text{Success on each trial})}_{p} = P(\text{mosquito bites at each Second})$

$= P(A \cap B) = P(A) \cdot P(B|A) = (0.5)(0.2) = 0.1$

Then X has a Geometric ~~dis~~distribution with $p = 0.1$.

$E(X) = 1/p = \frac{1}{0.1} = 10$, $Var(X) = \frac{1-p}{p} = \frac{0.9}{(0.1)^2} = 90$

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Let X – Sum of three sides.

x	combination	# of outcomes	
3	1 + 1 + 1	1	
4	1 + 1 + 2	3	
5	1 + 1 + 3	3	6
	1 + 2 + 2	3	
6	1 + 1 + 4	3	10
	1 + 2 + 3	6	
	2 + 2 + 2	1	
7	1 + 2 + 4	6	12
	1 + 3 + 3	3	
	2 + 2 + 3	3	
8	1 + 3 + 4	6	12
	2 + 2 + 4	3	
	2 + 3 + 3	3	
9	1 + 4 + 4	3	10
	2 + 3 + 4	6	
	3 + 3 + 3	1	
10	2 + 4 + 4	3	6
	3 + 3 + 4	3	
11	3 + 4 + 4	3	
12	4 + 4 + 4	1	

∴ Probability distribution of X :

$$p(x) = \begin{cases} 1/64 & ; \quad x = 3, 12 \\ 3/64 & ; \quad x = 4, 11 \\ 6/64 & ; \quad x = 5, 10 \\ 10/64 & ; \quad x = 6, 9 \\ 12/64 & ; \quad x = 7, 8 \end{cases}$$

#6) Let X – ~~price~~ winning amount, then,

x	100	25	10	0
$P(X=x)$	$1/100$	$2/100$	$5/100$	$92/100$

$$E(X) = 100 \cdot \frac{1}{100} + 25 \cdot \frac{2}{100} + 10 \cdot \frac{5}{100} + 0 \cdot \frac{92}{100}$$

$$= 1 + 0.5 + 0.5 = \cancel{6} \; 2$$

$\therefore$ So I am willing to pay \$ ~~1~~ 2 for a ticket.

[redacted]

#7)

G	D
3	2

$P(D) = P(\text{defective}) = 2/5$

N has Geometric distribution with $p = 2/5$.

$$\therefore E(N) = \frac{1}{p} = 5/2 \simeq (3 \text{ trials})$$

$$Var(N) = \frac{1-p}{p^2} = \frac{3/5}{(2/5)^2} = \frac{3}{5} \cdot \frac{5^2}{2^2} = \frac{15}{4} = 3.75.$$

#8)

$$P(1) + P(2) + P(3) = 1 \longrightarrow (1)$$

$$1 \cdot P(1) + 2 \cdot P(2) + 3 \cdot P(3) = 2.5 \longrightarrow (2) \quad (\because E(X) = 2.5)$$

(2) – (1) :

$$P(2) + 2P(3) = 1.5$$

$$\Rightarrow P(2) = 1.5 - 2P(3) \longrightarrow (3)$$

from 1 and 2 :

$$p(1) + (1.5 - 2p(3)) + p(3) = 1$$

$$p(1) + 1.5 - p(3) = 1$$

$$p(1) = p(3) - 0.5 \longrightarrow (4)$$

$(3) \Rightarrow \quad p(2) \geq 0 \quad \Rightarrow \quad 1.5 - 2p(3) \geq 0$

$$p(3) \leq 0.75$$

$(4) \Rightarrow \quad p(1) \geq 0 \quad \Rightarrow \quad p(3) - 0.5 \geq 0$

$$\Rightarrow \quad p(3) \geq 0.5$$

$$\therefore \quad 0.5 \leq p(3) \leq 0.75$$

$$\begin{aligned}
Var(X) &= 1^2 \cdot p(1) + 2^2 \cdot p(2) + 3^2 p(3) - (2.5)^2 \\
&= 1(p(3) - 0.5) + 4(1.5 - 2p(3)) + 9p(3) - 6.25 \\
&= p(3) - 0.5 + 6 - 8p(3) + 9p(3) - 6.25 \\
&= 2 \cdot p(3) - 0.75
\end{aligned}$$

$$\therefore \quad 2(0.5) - 0.75 \leq Var(X) \leq 2(0.75) - 0.75$$

$$0.25 \leq Var(X) \leq 0.75$$