HW - 6 : Today,T (10/10)

Class -27

Final Exam : R (10/12)

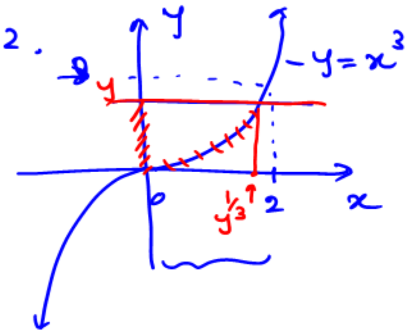
Practice Exam : Discuss in the class T (10/10

Cover materials after Exam -1
One double sided handwritten sheet is allowed
Bring your calculator
No electronics Except a calculator

6. a) Suppose X is uniform over [0, 2] and $Y = X^3$. Find the probability density function of Y.

* cdf method:

Pdf of X: $f_X(x) = \frac{1}{2}$: $0 \le x \le 2$.



monotonic over [0,2]

(both methods work.)

1) $F_Y(y) = P(Y \le y) = P(X \le y^{1/3})$

$= F_X(y^{1/3})$

2) d. w.r.t. y,

$$f_Y(y) = f_X(y^{1/3}) \cdot \frac{d}{dy}(y^{1/3})$$

$$= \frac{1}{2} \cdot \frac{1}{3} \cdot y^{-2/3}$$

$$= \frac{1}{6\, y^{2/3}} \; : \; 0 < y \le 8$$

strict

* Transformation Method

$$y = g(x) = x^3 \implies x = y^{1/3} = g^{-1}(y)$$

$$\implies \frac{d}{dy}(g^{-1}(y)) = \frac{d}{dy}(y^{1/3}) = \frac{1}{3} \cdot y^{-2/3}$$

$$\left| \frac{d(g^{-1}(y))}{dy} \right| = \frac{1}{3} \cdot y^{-2/3}$$

Pdf of Y:

$f_X(x) = \frac{1}{2}$: $0 \le x \le 2$

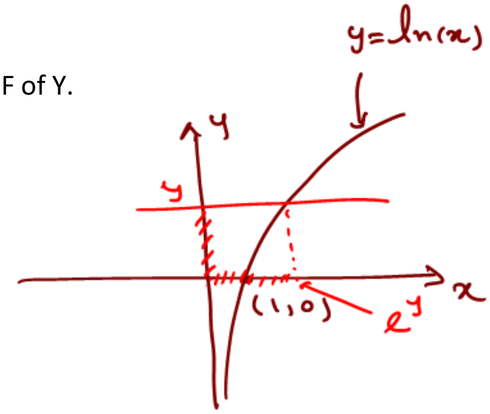
$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d(g^{-1}(y))}{dy} \right|$$

$$= \underbrace{f_X(y^{1/3})}_{1/2} \cdot \frac{1}{3} \cdot y^{-2/3}$$

$$= \frac{1}{6\, y^{2/3}} \; : \; 0 < y \le 8$$

from the graph.

b) Let X is Exponential with parameter λ and $Y = \ln(X)$. Find PDF of Y.



monotonic

Both methods work.

Pdf of X: $f_X(x) = \lambda e^{-\lambda x} \; : x > 0$

$$: x \geq 0$$
$$x > 0$$

* cdf method

1) $F_Y(y) = P(Y \le y)$
$= P(X \le e^y)$
$= F_X(e^y)$

2) d.w.r.t. y,

$$f_Y(y) = \underbrace{f_X(e^y)}\cdot \frac{d}{dy}(e^y)$$
$$= \lambda e^{-\lambda e^y} \cdot e^y$$
$$= \lambda e^y e^{-\lambda e^y} \quad : -\infty < y < \infty.$$

* Transformation method:

$$y = \ln(x) = g(x) \Rightarrow x = e^y = g^{-1}(y)$$

$$\frac{d}{dy}(g^{-1}(y)) = \frac{d}{dy}(e^y) = e^y \Rightarrow \left|\frac{d}{dy}(g^{-1}(y))\right| = e^y$$

Pdf of Y:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy}(g^{-1}(y))\right|$$
$$= \underbrace{f_X(e^y)} \cdot e^y$$
$$= \lambda e^{-\lambda e^y} \cdot e^y$$
$$= \lambda e^y \cdot e^{-\lambda e^y} \quad : -\infty < y < \infty.$$

8. a) Suppose $X_1, X_2, X_3, \cdots, X_{100}$ be a random sample and X_i is exponential with $\lambda = 1/1000$ for $i = 1,2,3,\cdots,100$. Find the probability that the sum $S_n = X_1 + X_2 + X_3, \cdots + X_{100}$ is in between 95000 and 105000.

$$X_i \sim \text{Exp}\left(\lambda = \frac{1}{1000}\right), \quad i = 1, 2, \ldots, 100.$$

$$\mu = E(X_i) = \frac{1}{\lambda} = 1000 \quad , \quad \sigma^2 = \text{Var}(X_i) = \frac{1}{\lambda^2} = (1000)^2, \quad i = 1, 2, \ldots 100.$$

$$E(S_n) = n\mu = 100 \cdot 1000 = 10^5 \checkmark$$

$$\text{Var}(S_n) = n\sigma^2 = 100 \cdot 1000^2 = 10^8$$

Standard deviation of $S_n = \sqrt{10^8} = 10^4 \checkmark$

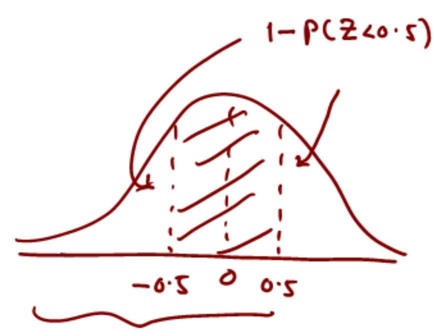
$$P(95000 \le S_n \le 105000) = P\left(\frac{95000 - 10^5}{10^4} \le \underbrace{\frac{S_n - 10^5}{10^4}} \le \frac{105000 - 10^5}{10^4}\right)$$

$$= P(-0.5 \le Z \le 0.5) \quad [\text{By C.L.T}]$$

$$= 1 - 2[1 - \underbrace{P(Z < 0.5)}]$$

$$= 1 - 0.617$$

$$= \boxed{0.383}$$



b) The time it takes students in a cooking school to learn to prepare seafood gumbo is a random variable with a normal distribution where the average is 3.2 hours with a standard deviation 1.8 hours. Find the probability that the average time it will take a class of 36 students to learn to prepare seafood gumbo is less than 3.4 hours.

$\mu = 3.2, \quad \sigma^2 = 1.8^2$

Let X_i - time it takes for the i^{th} student, $i = 1, 2, \dots 36$

$$X_1, X_2, \dots, X_{36} \sim N(3.2, 1.8^2)$$

Let M_n - average time $= \dfrac{X_1 + \dots + X_{36}}{36}$ then,

$$E(M_n) = \mu = 3.2$$

$$\mathrm{var}(M_n) = \sigma^2/n = \frac{1.8^2}{36}$$

Standard deviation $= \dfrac{1.8}{6} = 0.3$.

$$P(M_n \le 3.4) = P\left(\frac{M_n - 3.2}{0.3} \le \frac{3.4 - 3.2}{0.3}\right)$$

$$= P(Z \le 0.67)$$

$$= \boxed{0.7486}$$

7. An engineering company advertises a job in three papers, A, B and C. It is known that these papers attract undergraduate engineering readerships in the proportions 2:3:1. The probabilities that an engineering undergraduate sees and replies to the job advertisement in these papers are 0.002, 0.001 and 0.005 respectively. Assume that the undergraduate sees only one job advertisement.

a) What is the probability a randomly selected engineering undergraduate sees and replies to the job advertisement?

Let A_1 – advertising the job in paper A

A_2 " " " " " B

A_3 " " " " " C

D – undergraduate sees the job advertisment.

Let $P(A_3) = x$ then $P(A_1) = 2x$, $P(A_2) = 3x$

But $P(A_1) + P(A_2) + P(A_3) = 1$

$2x + 3x + x = 1 \Rightarrow 6x = 1 \Rightarrow x = 1/6$

$P(A_1) = 2/6$, $P(A_2) = 3/6$, $P(A_3) = 1/6$

$P(D/A_1) = 0.002$, $P(D/A_2) = 0.001$, $P(D/A_3) = 0.005$

$P(D) = ?$

By total probability theorem,

$$P(D) = P(D|A_1)\cdot P(A_1) + P(D|A_2)\cdot P(A_2) + P(D|A_3)\cdot P(A_3)$$
$$= (0.002)\cdot 2/6 + (0.001)\cdot 3/6 + (0.005)\cdot 1/6 = \boxed{0.002}$$

b) Probability that the applicant has seen the job advertised in paper A.

By Bayes' theorem,

$$P(A_1|D) = \frac{P(D|A_1)\cdot P(A_1)}{P(D)}$$
$$= \frac{(0.002)(2/6)}{0.002} = \boxed{2/6}.$$

Practice Exam (for Exam-2)

MA 2621 – A 15

1. Joint probability distribution of random variables X and Y is given as,

3	1/10	2/10	1/10	$\to 4/10$
2	0	1/10	1/10	$\to 2/10$
1	1/10	1/10	2/10	$\to 4/10$
Y / X	1	2	3	
	$2/10$	$4/10$	$4/10$	

a) Find the marginal distributions of X and Y.

b) Are X and Y independent?

c) Find $P(X > Y)$

a) marginal distribution of X:

$$p_X(x) = \begin{cases} 2/10 & ; \ x = 1 \\ 4/10 & ; \ x = 2, 3 \\ 0 & ; \ \text{otherwise} \end{cases}$$

Marginal distribution of Y:

$$p_Y(y) = \begin{cases} 2/10 & ; \ y = 2 \\ 4/10 & ; \ y = 1, 3 \\ 0 & ; \ \text{otherwise} \end{cases}$$

b) If X and Y are independent $p(x,y) = p_X(x) \cdot p_Y(y)$, for all x, y.

$p(X=1, Y=2) = p(1,2) = 0$ but $p(X=1) \cdot p(Y=2) = p_X(1) \cdot p_Y(2) = \frac{2}{10} \cdot \frac{2}{10}$

$= \frac{4}{100}$

ie $p(1,2) \neq p_X(1) \cdot p_Y(2)$

$\therefore$ X and Y are not independent.

c) $p(X > Y) = p(2,1) + p(3,1) + p(3,1)$

$= 1/10 + 2/10 + 1/10 = 4/10 = \boxed{0.4}$

2. Consider the Probability density function of random variable Y,

$f_Y(y) = ay^2 \; : 0 \le y \le 4$.

a) Find a,
b) Find E[Y],
c) Find Var[Y],
d) Find P(Y>3).

a) $\int_{-\infty}^{\infty} f_Y(y)\,dy = 1 \Rightarrow \int_0^4 ay^2\,dy = 1 \Rightarrow a \cdot \left[\frac{y^3}{3}\right]_0^4 = 1 \Rightarrow \frac{64a}{3} = 1$

$\Rightarrow \boxed{a = 3/64}$

b) $E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy = \int_0^4 y \cdot \frac{3}{64} y^2\,dy = \frac{3}{64} \int_0^4 y^3\,dy = \frac{3}{64}\left[\frac{y^4}{4}\right]_0^4$

$= \frac{3}{64} \cdot \frac{4^4}{4} = 3$

c) $E(Y^2) = \int_0^4 y^2 \cdot \frac{3}{64} y^2\,dy = \frac{3}{64}\int_0^4 y^4\,dy = \frac{3}{64}\left[\frac{y^5}{5}\right]_0^4 = \frac{3}{64} \cdot \frac{4^5}{5} = 9.6$

$Var(Y) = E(Y^2) - (E(Y))^2 = 9.6 - 3^2 = \boxed{0.6}$

d) $P(Y>3) = \int_3^4 \frac{3}{64} y^2\,dy = \frac{3}{64}\left[\frac{y^3}{3}\right]_3^4 = \frac{1}{64}[64 - 27] = \boxed{\frac{37}{64}}$

$$f_Y(y) = \frac{3}{64} y^2 \quad ; \quad 0 \le y \le 4 .$$

3. a) Find the CDF of the PDF given in Q #2. $F_Y(y) = P(Y \le y) = \int_{-\infty}^{y} f_Y(t)\, dt$

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)\, dt = \int_{0}^{y} \frac{3}{64} t^2\, dt = \frac{3}{64}\left[\frac{t^3}{3}\right]_0^y =$$

$$= \frac{3}{64}\left[y^3 - 0\right] = \frac{3y^3}{64}$$

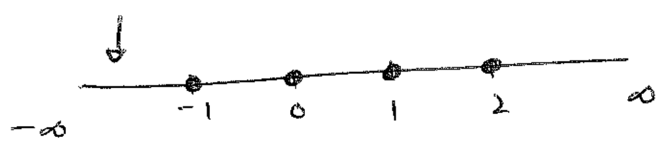
$\therefore$ CDF of Y:

$$F_Y(y) = \begin{cases} 0 & ; \quad y < 0 \\ \dfrac{3y^3}{64} & ; \quad 0 \le y \le 4 \\ 1 & ; \quad y > 1 \end{cases}$$

b) Consider the Probability distribution function of random variable X,

$$P_X(x) = \begin{cases} 1/3 & : x = -1 \\ 1/4 & : x = 0 \\ 1/4 & : x = 1 \\ 1/6 & : x = 2 \end{cases},$$

Find CDF of X.



$$F_X(x) = \begin{cases} 0 & ; \quad x < -1 \\ 1/3 & ; \quad -1 \le x < 0 \\ \frac{1}{3} + 1/4 = \frac{7}{12} & ; \quad 0 \le x < 1 \\ \frac{7}{12} + 1/4 = 5/6 & ; \quad 1 \le x < 2 \\ 1 & ; \quad x \ge 2 \end{cases}$$

3 jobs — 1 hour

1 jobs — $\frac{1}{3}$ hours

4. Jobs are sent to a printer at an average of 3 jobs per hour.
   a) What is the expected time between jobs?

Let X – time between jobs,

$\boxed{E(X) = \frac{1}{3}}$ (in hours)

But $E(X) = \frac{1}{\lambda} \Rightarrow \lambda = 3$, $f_X(x) = 3e^{-3x}$: $x > 0$

b) What is the probability that the next job is sent within 5 minutes? $\left(5 \text{ mins} = \frac{5}{60} \text{ hours}\right)$

$$\boxed{\begin{array}{l} X \sim \text{Exponential}(\lambda) \\ P(X > a) = e^{-\lambda a} \end{array}}$$

$$P\left(X \le \frac{5}{60}\right) = P(X < \tfrac{1}{12})$$

$$= 1 - P(X > \tfrac{1}{12})$$

$$= 1 - e^{-3(1/12)}$$

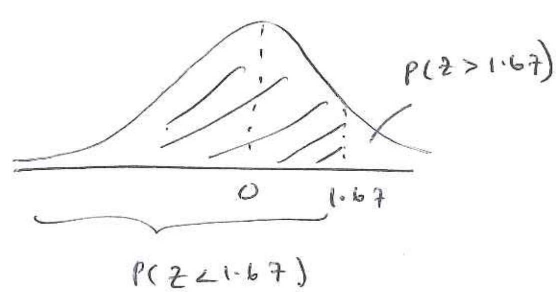
$$= 1 - e^{-1/4} = \boxed{0.221}$$

5. The lifetime of a battery is normally distributed with a mean life of 40 hours and a standard deviation of 1.2 hours. Find the probability that a randomly selected battery lasts longer than 42 hours.

Let X – life time of the battery. $X \sim N(40, 1.2^2)$

$$P(X > 42) = P\left(\frac{X-40}{1.2} > \frac{42-40}{1.2}\right) = P\left(Z > \frac{2}{1.2}\right)$$

$$= P(Z > 1.67)$$



$$= 1 - P(Z < 1.67)$$

$$= 1 - 0.9525$$

$$= \boxed{0.0475}$$