HW - 4 : W (9/27), in the conference

Quiz -4 : F (9/27)

Rest of the term:
Sec 5.1 : Continuous Random Variables
Sec 5.2 : Cumulative distribution function.
Sec 6.4 : Normal distribution
Sec 6.5 : Central Limit theorem
Sec 5.3 : Functions
Sec 5.4 : Joint distributions (continuous)
Sec 5.5 : Marginal distributions (continuous)

Marginal distribution of X :

$X=x$	0	1	2
$P(X=x)$	$\frac{36}{105}$	$\frac{54}{105}$	$\frac{15}{105}$

OR

$$P(X=x) = \begin{cases} 36/105 & : x=0 \\ 54/105 & : x=1 \\ 15/105 & : x=2 \end{cases}$$

Marginal distribution of Y :

$$P(Y=y) = \begin{cases} 45/105 & : y=0 \\ 50/105 & : y=1 \\ 10/105 & : y=2. \end{cases}$$

Functions of Multiple Random Variables

Let X and Y are two random variables and g is a function of X and Y.

Then $Z = g(X,Y)$ is also a random variable and Probability distribution is given by

$$P(Z=z) = \sum_{\{(x,y) \mid g(x,y)=z\}} p(x,y).$$

Note:

* $E(Z) = E(\underline{g(X,Y)}) = \sum_x \sum_y g(x,y)\, P(x,y)$

* $E(\underline{aX+bY+c}) = aE(X) + bE(Y) + c$, a, b, c – constants.

* More Generally,

For any sequence of random variables $X_1, X_2, \dots, X_n$ and constants $a_1, a_2, \dots a_n$,

$$E(a_1X_1 + a_2X_2 + \dots + a_nX_n) = a_1E(X_1) + a_2E(X_2) + \dots + a_nE(X_n).$$

Eg: consider the joint distribution function of X and Y.

X \ Y	1	2	
1	1/5	1/5	2/5
2	2/5	1/5	3/5
	3/5	2/5	

marginals:

$$P(X=x) = \begin{cases} 2/5 & : x=1 \\ 3/5 & : x=2. \end{cases}$$

$$P(Y=y) = \begin{cases} 3/5 & : y=1 \\ 2/5 & : y=2. \end{cases}$$

a) Let $Z = \overbrace{X+2Y}^{g(X,Y)}$, find the distribution of Z.

Possible values of Z: $\underset{(1,1)}{3}, \underset{(2,1)}{4}, \underset{(1,2)}{5}, \underset{(2,2)}{6}$

$P(Z=3) = P(1,1) = \frac{1}{5}$

$P(Z=4) = P(2,1) = 2/5$

$P(Z=5) = P(1,2) = 1/5$

$P(Z=6) = P(2,2) = 1/5$

Probability distribution of Z:

$$P(Z=z) = \begin{cases} 1/5 & : z = 3, 5, 6 \\ 2/5 & : z = 4. \end{cases}$$

b) Find $E(Z)$.

1st method (using the ~~prob~~ distribution Z)

$$E(Z) = 3\cdot\frac{1}{5} + 4\cdot 2/5 + 5\cdot 1/5 + 6\cdot 1/5 = \boxed{\frac{22}{5}}$$

2nd method (using $E(X)$ and $E(Y)$).

$$E(X) = 1\cdot 2/5 + 2\cdot 3/5 = 8/5$$

$$E(Y) = 1\cdot 3/5 + 2\cdot 2/5 = 7/5$$

$$E(Z) = E(X+2Y) = E(X) + 2E(Y)$$
$$= 8/5 + 2\cdot 7/5 = \boxed{\frac{22}{5}}.$$

Independence of Random Variables

Two random variables X and Y are independent iff

$$p(X=x, Y=y) = p(X=x)\cdot p(Y=y)$$

for all $\underline{x}$ and $\underline{y}$.

events.

* ~~If~~ A and B are independent iff
$P(A\cap B) = P(A)\cdot P(B)$.

Note:

1. If X and Y are independent

$$E(\overbrace{XY}^{g(X,Y)}) = E(X)\cdot E(Y).$$

Proof:

$$E(XY) = \sum_x \sum_y xy \underbrace{p(x,y)}$$

$$= \sum_x \sum_y xy\, p(x)\cdot p(y) \quad (\because X \text{ and } Y \text{ are independent}).$$

$$= \underbrace{\left(\sum_x x\,p(x)\right)}\left(\sum_y y\,p(y)\right)$$

$$= E(X)\cdot E(Y)_{\text{a}}.$$

2. If X and Y independent

$$Var(aX+bY+c) = a^2 var(X) + b^2 var(Y).$$

$E(aX+bY+c) = aE(X)+bE(Y)+c$

This is true for any rvs X and Y

3. More Generally, if $X_1, X_2, \ldots, X_n$ are independent,

$$Var(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1^2 var(X_1) + a_2^2 var(X_2) + \cdots + a_n^2 var(X_n),$$

$a_1, a_2, \ldots, a_n$ - constants.

Eg:- Let $X \sim$ Binomial (n, p). Show that $E(X) = np$ and $Var(X) = np(1-p)$

We consider n- independent and identical trials and

$P(\text{Success}) = p.$

$$\text{Let} \quad X_i = \begin{cases} 1 : & \text{the } i^{th} \text{ trial is a Success} \\ 0 : & \text{otherwise} \end{cases}, \quad i = 1, 2, \ldots n.$$

Probability distribution of X_i,

$$P(X_i = x_i) = \begin{cases} p : & x_i = 1 \\ 1-p : & x_i = 0 \end{cases}$$

$$E(X_i) = 1 \cdot p + 0 \cdot (1-p) = p, \quad i = 1, 2, \ldots n.$$

$$E(X_i^2) = 1^2 \cdot p + 0^2 \cdot (1-p) = p$$

$$Var(X_i) = E(X_i^2) - (E(X_i))^2$$
$$= p - (p)^2 = p(1-p), \quad i = 1, 2, \ldots n.$$

But

$$X = X_1 + X_2 + X_3 + \cdots + X_n$$

$$E(X) = E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$$
$$= p + p + \cdots + p \text{ (n-times)}$$
$$= np.$$

$$\begin{aligned} Var(X) &= var(X_1 + X_2 + \cdots + X_n) \\ &= var(X_1) + var(X_2) + \cdots + var(X_n) \quad [\because X_i\text{s are independent}]. \\ &= p(1-p) + p(1-p) + \cdots + p(1-p) \quad (n\text{-times}) \\ &= np(1-p) \quad \square. \end{aligned}$$

Chapter-5 : Distribution of Continuous Random variables

When a random variable can take any value in an interval, it is called a continuous random variable. Probability distribution of a continuous random variable is called a probability density function (pdf).