HW - 6 : T (10/10)

Class -26

Final Exam : R (10/12)

Practice Exam : Discuss in the class T (10/10)

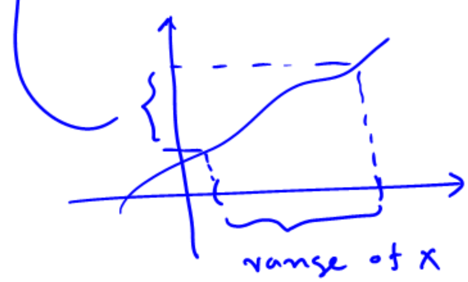
Cover materials after Exam -1
One double sided handwritten sheet is allowed
Bring your calculator

2) Transformation Method

Suppose X is a continuous random variable and g is a function of x such that g is monotonic (strictly increasing or strictly decreasing) over the range of X. Let $Y = g(X)$ and $g^{-1}(x)$ is differentiable over the range of x.

Then the pdf of Y is

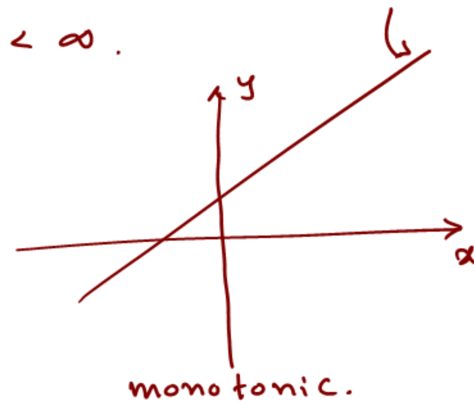
$$f_Y(y) = f_X\left(g^{-1}(y)\right) \cdot \left| \frac{d}{dy}\left(g^{-1}(y)\right) \right| \quad ; \; y \in \underbrace{g(\text{range of } X)}_{\text{may be from a graph.}}$$



Eg:- Let $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$ (a, b – constants)
Find pdf of Y, (ie show that $Y \sim N(a\mu + b, a^2\sigma^2)$.)

Pdf of X :

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad ; \; -\infty < x < \infty .$$



Let $y = ax+b = g(x) \Rightarrow ax = y-b$

$\Rightarrow x = \dfrac{y-b}{a} = g^{-1}(y)$

$$\Rightarrow \frac{d(g^{-1}(y))}{dy} = \frac{d}{dy}\left(\frac{y-b}{a}\right) = \frac{1}{a}$$

$$\Rightarrow \left|\frac{d(g^{-1}(y))}{dy}\right| = \frac{1}{|a|}$$

Pdf of Y:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{d(g^{-1}(y))}{dy}\right|$$

$$= \underbrace{f_X\left(\frac{y-b}{a}\right)} \cdot \frac{1}{|a|}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}} \cdot \frac{1}{|a|}$$

$$= \frac{1}{\sqrt{2\pi}\,\underbrace{\sigma|a|}_{\sigma_N}} \cdot e^{-\frac{(y-\overbrace{(a\mu+b)}^{\mu_N})^2}{2\underbrace{\sigma^2 a^2}_{\sigma_N^2}}}$$

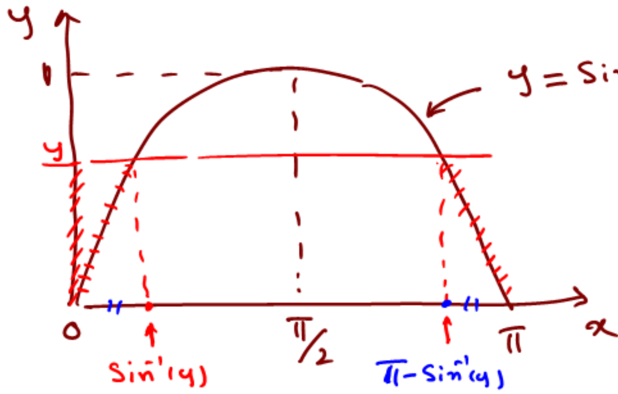
$$\underbrace{f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}_{N(\mu,\sigma^2)}$$

This is the pdf of $N(a\mu+b, a^2\sigma^2)$.

So $Y \sim N(a\mu+b, a^2\sigma^2)$.

Eg:-2) Let X be a random variable with pdf $\overbrace{f_X(x) = \dfrac{2x}{\pi^2}}^{\text{Cauchy distribution,}}$ $: 0 \le x \le \pi$.

Let $Y = \sin(X)$, find the pdf of Y.



$y = \sin(x)$

$\Rightarrow x = \sin^{-1}(y)$

not monotonic. (So transformation method can not be used).

* Using cdf method

$F_X(x) = P(X \le x)$

1) cdf of Y:

$$F_Y(y) = P(Y \le y)$$
$$= P(0 \le X \le \sin^{-1}(y)) + P(\pi - \sin^{-1}(y) \le X \le \pi)$$
$$= P(X \le \sin^{-1}(y)) - P(X \le 0) + P(X \le \pi) - P(X \le \pi - \sin^{-1}(y))$$
$$= F_X(\sin^{-1}(y)) - \underbrace{F_X(0)}_{\text{constant}} + \underbrace{F_X(\pi)}_{\text{constant}} - F_X(\pi - \sin^{-1}(y))$$

$f_X(x) = \frac{2x}{\pi^2}$

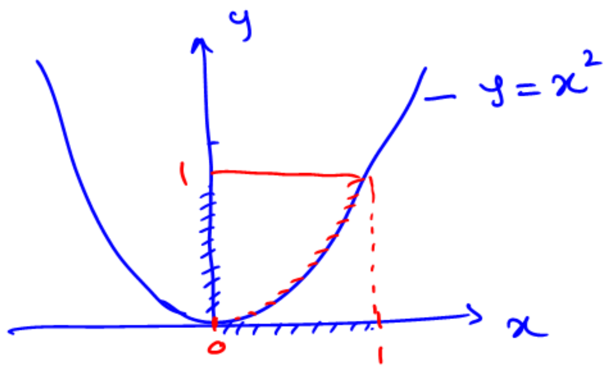
2) d.w.r.t. y,

$$f_Y(y) = f_X(\sin^{-1}(y)) \cdot \frac{d}{dy}(\sin^{-1}(y)) - 0 + 0 - f_X(\pi - \sin^{-1}(y)) \cdot (-1) \frac{d}{dy}(\sin^{-1}(y))$$
$$= \frac{2(\sin^{-1}(y))}{\pi^2} \cdot \frac{1}{\sqrt{1-y^2}} + \frac{2(\pi - \sin^{-1}(y))}{\pi^2} \cdot \frac{1}{\sqrt{1-y^2}}$$
$$= \frac{2\pi}{\pi^2} \cdot \frac{1}{\sqrt{1-y^2}}$$
$$= \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}} \; ; \; 0 \le y < 1$$

$0 \le x \le \pi$

Eg-3) Let $X \sim$ uniform $[0,1]$ and $Y = X^2$. Find the pdf of Y.

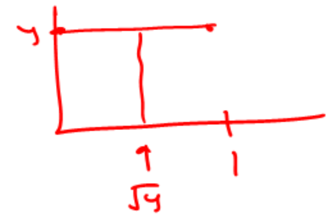
Graph:



monotonic over $[0,1]$.

pdf of X:

$f(x) = 1 \quad : \quad 0 \le x \le 1$



* Transformation method:

Let $y = x^2 = g(x) \Rightarrow x = \sqrt{y} = g^{-1}(y)$

$$\Rightarrow \frac{d}{dy}(g^{-1}(y)) = \frac{d}{dy}(\sqrt{y}) = \frac{1}{2\sqrt{y}}$$

$$\left| \frac{d}{dy}(g^{-1}(y)) \right| = \frac{1}{2\sqrt{y}}$$

pdf of Y:

$$f_Y(y) = \underbrace{f_X(g^{-1}(y))} \cdot \left| \frac{d}{dy}(g^{-1}(y)) \right|$$

$$= \underbrace{f_X(\sqrt{y})} \cdot \frac{1}{2\sqrt{y}}$$

$$= 1 \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{2\sqrt{y}} \quad : \quad 0 < y \le 1$$