HW-5 (Solution Key)

Q1

a) Marginal distribution of X:

Y \ X	1	2	3	4	
4	0	1/20	1/20	1/2	3/20
3	1/20	2/20	3/20	1/20	7/20
2	1/20	2/20	3/20	1/20	7/20
1	1/20	1/20	1/20	0	3/20
	3/20	6/20	8/20	3/20	

$$P_X(x) = \begin{cases} 3/20 & ; \; x = 1, 4 \\ 6/20 & ; \; x = 2 \\ 8/20 & ; \; x = 3 \end{cases}$$

Marginal Distribution of Y:

$$P_Y(y) = \begin{cases} 3/20 & ; \; y = 1, 4 \\ 7/20 & ; \; y = 2, 3 \end{cases}$$

b) $Z = X + 2Y$

Possible values of Z: 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

- 3 ← (1,1)
- 4 ← (2,1)
- 5 ← {(1,2), (3,1)}
- 6 ← {(2,2), (4,1)}
- 7 ← {(3,2), (1,3)}
- 8 ← {(2,3), (4,2)}
- 9 ← {(1,4), (3,3)}
- 10 ← {(2,4), (4,3)}
- 11 ← (3,4)
- 12 ← (4,4)

$P(Z=3) = P(1,1) = 1/20$

$P(Z=4) = P(2,1) = 1/20$

$P(Z=5) = P(1,2) + P(3,1) = \frac{1}{20} + \frac{1}{20} = \frac{2}{20}$

$P(Z=6) = P(2,2) + P(4,1) = \frac{2}{20} + 0 = 2/20$

⋮

∴ Probability distribution of Z:

$$P(Z=z) = \begin{cases} 1/20 & ; \; z = 3, 4, 11, 12 \\ 2/20 & ; \; z = 5, 6, 10 \\ 3/20 & ; \; z = 8, 9 \\ 4/20 & ; \; z = 7 \end{cases}$$

OR

z	3	4	5	6	7	8	9	10	11	12
$P(Z=z)$	1/20	1/20	2/20	2/20	4/20	3/20	3/20	2/20	1/20	1/20

c) $E(Z) = 3\cdot\frac{1}{20} + 4\cdot\frac{1}{20} + 5\cdot\frac{2}{20} + 6\cdot\frac{2}{20} + 7\cdot\frac{4}{20} + 8\cdot\frac{3}{20} + 9\cdot\frac{3}{20} + 10\cdot\frac{2}{10} + 11\cdot\frac{1}{20} + 12\cdot\frac{1}{20}$

$= 7.55$

d) $E(X) = 1\cdot\frac{3}{20} + 2\cdot\frac{6}{20} + 3\cdot\frac{8}{20} + 4\cdot\frac{3}{20} = 2.55$

$E(Y) = 1\cdot\frac{3}{20} + 2\cdot\frac{7}{20} + 3\cdot\frac{7}{20} + 4\cdot\frac{3}{20} = 2.5$

$E(Z) = E(X+2Y) = E(X) + 2E(Y) = 2.55 + 2(2.5) = 7.55$

e) Since,

$P(X=1, Y=4) = 0 \neq P(X=1)\cdot P(Y=4) = \frac{3}{20}\cdot\frac{3}{20} = \frac{9}{400},$

X and Y are not independent.

Q2

$Y \backslash X$	0	1	2	
0	1/3	0	1/3	2/3
1	0	1/3	0	1/3
	1/3	1/3	1/3	

Marginal distributions:

$$P(X=x) = \begin{cases} 1/3 & ; x = 0,1,2 \\ 0 & ; \text{otherwise} \end{cases}$$

$$P(Y=y) = \begin{cases} 2/3 & ; y = 0 \\ 1/3 & ; y = 1 \\ 0 & ; \text{otherwise} \end{cases}$$

Since $P(X=1, Y=0) = 0 \neq P(X=1)\cdot P(Y=0) = 1/3 \cdot 2/3 = 2/9$,

X and Y are not independent.

Q3) Let X_i – # of eggs John eats in i^{th} day, $i = 1, 2, 3, \ldots, 10$.

Then Probability distribution of X_i for each $i = 1, \ldots 10$, is

x	1	2	3	4	5
$P(X_i = x)$	$1/5$	$1/5$	$1/5$	$1/5$	$1/5$

$$\therefore E(X_i) = 1 \cdot 1/5 + 2 \cdot 1/5 + 3 \cdot 1/5 + 4 \cdot 1/5 + 5 \cdot 1/5 = 3$$

$$E(X_i^2) = 1^2 \cdot 1/5 + 2^2 \cdot 1/5 + 3^2 \cdot 1/5 + 4^2 \cdot 1/5 + 5^2 \cdot 1/5 = 11$$

$$\therefore Var(X_i) = E(X_i^2) - (E(X_i))^2$$

$$= 11 - 3^2 = 2 \quad , \; i = 1, 2, \ldots, 10,$$

Let X – # of eggs he eats in 10 days, then

$$X = X_1 + X_2 + \cdots + X_{10}$$

$$\Rightarrow E(X) = E(X_1 + X_2 + \cdots + X_{10})$$

$$= E(X_1) + E(X_2) + \cdots + E(X_{10})$$

$$= 3 + 3 + \cdots + 3$$

$$= 30$$

$$Var(X) = Var(X_1 + X_2 + \cdots + X_{10})$$

$$= Var(X_1) + Var(X_2) + \cdots + Var(X_{10}) \quad [\because X_i\text{s are independent}]$$

$$= 2 + 2 + \cdots + 2$$

$$= 20$$

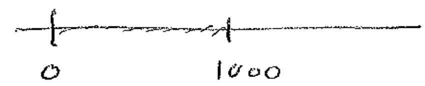
Q4) Let X – time untill a component fails.

Assume the parameter is λ.

$$X \sim Exp(\lambda).$$

Since 10% components failed by 1000 hours,

$$P(X \leq 1000) = 0.1$$



$$1 - P(X > 1000) = 0.1$$

$$1 - e^{-\lambda(1000)} = 0.1 \qquad \left(\because P(X>a) = e^{-\lambda a}\right)$$

$$e^{-1000\lambda} = 0.9$$

$$\Rightarrow \quad \lambda = -\frac{\ln(0.9)}{1000} = 1.05 \times 10^{-4}$$

a) $P(X > 5000) = e^{-\lambda(5000)}$

$$= e^{-(1.05 \times 10^{-4}) \times 5000}$$

$$= 0.5915.$$

b) $E(X) = \frac{1}{\lambda} = \frac{1}{1.05 \times 10^{-4}} = 9523.81$ hours

$Var(X) = \frac{1}{\lambda^2} \Rightarrow$ standard deviation $= \frac{1}{\lambda} = 9523.81$ hours.

Q5] $f_X(x) = \begin{cases} ax^2 & ; 0 \leq x < 2 \\ 0 & ; \text{otherwise} \end{cases}$

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^2 ax^2\,dx = 1$

$$a \cdot \left[\frac{x^3}{3}\right]_0^2 = 1$$

$$a\left[\frac{8}{3} - 0\right] = 1 \Rightarrow a = 3/8$$

b) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^2 x \cdot \frac{3}{8}x^2\,dx = \frac{3}{8}\left[\frac{x^4}{4}\right]_0^2$

$$= \frac{3}{8}\left[\frac{\cancel{16}^4}{\cancel{4}} - 0\right] = \boxed{\frac{3}{2}}.$$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^3 x^2 \cdot 3/8\, x^2\,dx = \frac{3}{8}\left[\frac{x^5}{5}\right]_0^2$

$$= \frac{3}{8}\left[\frac{32}{5}\right] = \frac{12}{5}$$

$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{12}{5} - \left(\frac{3}{2}\right)^2 = \frac{3}{20}$

c) when $0 < x < 2$,

$F(x) = \int_0^x f(t)\,dt = \int_0^x 3/8\, t^2\,dt = 3/8\left[\frac{t^3}{3}\right]_0^x = 1/8\, x^3$

$\therefore$ cdf of X,

$$F(x) = \begin{cases} 0 & ; x \leq 0 \\ 1/8\, x^3 & ; 0 < x < 2 \\ 1 & ; x \geq 2. \end{cases}$$

Q6 $f_X(x) = k e^{-2x}$; $x > 0$

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^{\infty} k e^{-2x}\,dx = 1$

$\Rightarrow k \cdot \left[\frac{e^{-2x}}{-2}\right]_0^{\infty} = 1$

$\Rightarrow \frac{k}{-2}[0-1] = 1$

$\Rightarrow k = 2$

2nd Method

Here X follows an exponential distribution with $\lambda = 2$

$\therefore \boxed{k = 2}$

b) since $X \sim Exp(2)$,

$E(X) = \frac{1}{\lambda} = \boxed{\frac{1}{2}}$

The proof ~~of~~ is in the class note

c) $Var(X) = \frac{1}{\lambda^2} = \boxed{\frac{1}{4}}$

d) $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$= \int_0^{x} f(x)\,dx$ (when $x > 0$)

$= P(X \le x)$

$= 1 - P(X > x)$

$= 1 - e^{-2x}$

$\therefore$ cdf of X :

$$F(x) = \begin{cases} 0 & ; x < 0 \\ 1 - e^{-2x} & ; x \geq 0 \end{cases}$$

Q7 $p(x) = c\,(1/3)^x \; ; \; x = 0, 1, 2, \ldots$

a) $$\sum_{x=0}^{\infty} p(x) = 1 \Rightarrow \sum_{x=0}^{\infty} c\,(1/3)^x = 1$$

$$\Rightarrow c + c(1/3) + c(1/3)^2 + \cdots = 1$$

$$\Rightarrow \frac{c}{1 - 1/3} = 1 \Rightarrow c = 2/3$$

b) $$F(x) = \sum_{k=0}^{x} p(k) = \sum_{k=0}^{x} 2/3\,(1/3)^k$$

$$= \frac{2}{3} + \frac{2}{3}(1/3) + 2/3\,(1/3)^2 + \cdots + 2/3\,(1/3)^x$$

Here note that there are $x+1$ terms

$$= \frac{2/3\left[1 - (1/3)^{x+1}\right]}{(1 - 1/3)}$$

$$= 1 - (1/3)^{x+1} \quad , \quad x = 0, 1, 2, \ldots$$

Q8 $f_X(x) = 4x^3 \quad , \quad 0 < x < 1$

when $0 < x < 1$,

$$F(x) = \int_0^x 4t^3\,dt = 4 \cdot \left[\frac{t^4}{4}\right]_0^x = x^4$$

$\therefore$ cdf of X;

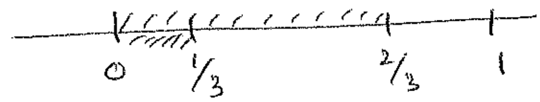
$$F(x) = \begin{cases} 0 & ; \; x \le 0 \\ x^4 & ; \; 0 < x < 1 \\ 1 & ; \; x \ge 1 \end{cases}$$

Q9

b) $P(X \leq 1/2) = F(1/2)$

$= (1/2)^4 = \frac{1}{16}.$

c) $P(1/3 < X < 2/3)$



$= P(X < 2/3) - P(X < 1/3)$

$= F(2/3) - F(1/3)$

$= (2/3)^4 - (1/3)^4$

$= \frac{16-1}{81} = 15/81$