Note:

1) $P(\Phi) = 0$

Proof:-

For any event A,

$$A \cap \Phi = \Phi$$

$\Rightarrow$ A and Φ are disjoint.

But $A \cup \Phi = A$

$$\Rightarrow \underbrace{P(A \cup \Phi)} = P(A)$$

$$\cancel{P(A)} + P(\Phi) = \cancel{P(A)} \quad (\because \text{Axiom-2})$$

$$\Rightarrow P(\Phi) = 0.$$

2) If $A_1, A_2, \ldots, A_n$ are disjoint then,

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$

3) If the Sample Space consists "n" equally likely outcomes (not true in general).

then $$P(A) = \frac{n(A)}{n(\Omega)} = \frac{n(A)}{n}$$

$n(A)$: # of outcomes in A

1. Consider an experiment of rolling a pair of 4 sided fair dice. What is the probability of

a) A- the sum of the rolls is even.

b) B- the sum of the rolls is odd.

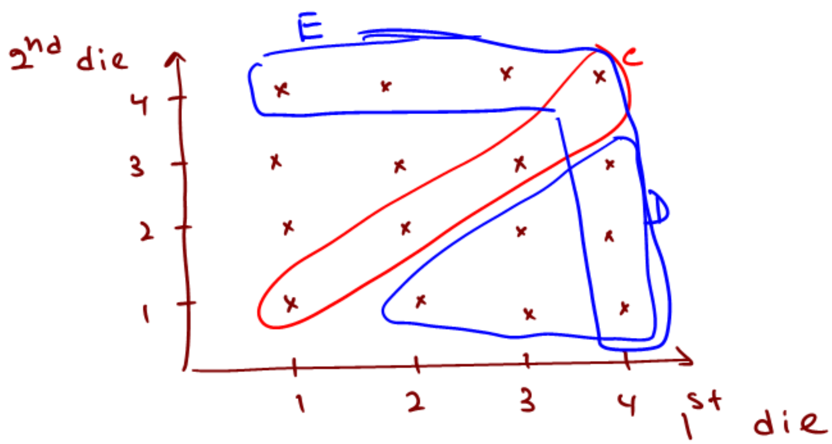
c) C- the first roll is equal to the second roll.

d) D- the first roll is larger than the second roll.

e) E- at least one roll is equal to 4.

Sample Space $= \Omega = \{ (1,1), (1,2), \ldots\ldots, (4,4) \}$

(16- outcomes)



a) $A = \{ (1,1), (1,3), (2,2), (2,4), (3,1), (3,3), (4,2), (4,4) \}$.

$$P(A) = \frac{n(A)}{n(\Omega)} = 8/16 = 1/2$$

b) $A \cup B = \Omega$

$P(A \cup B) = P(\Omega)$

$\underbrace{P(A \cup B)} = 1$ (Axiom - I)

$P(A) + P(B) = 1$ (A and B are disjoint)

$\frac{1}{2} + P(B) = 1$

$\Rightarrow P(B) = 1/2$.

c) $P(C) = \dfrac{n(C)}{n(\Omega)} = \dfrac{4}{16} = 0.25$

$(3,4)$ $(4,3)$

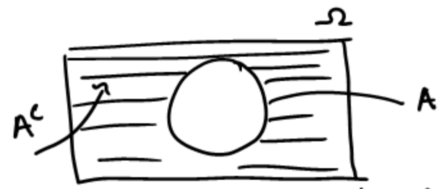
$(4,4)$

d) $P(D) = \dfrac{n(D)}{n(\Omega)} = 6/16 = 3/8$

e) $P(E) = \dfrac{n(E)}{n(\Omega)} = 7/16$

Properties of Probability

Let A, B, C be events,



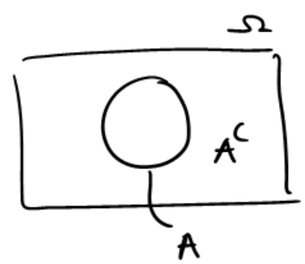
1. Let A^c (or $\bar{A}$) be the complement of event A, then,

$$A^c = \{x \in \Omega \mid x \notin A\}$$
$$= \{\text{all the outcomes in } \Omega \text{ but not in } A\}.$$

Further

$$\boxed{P(A^c) = 1 - P(A)}$$

Proof:



$$A \cup A^c = \Omega$$

$$\Rightarrow \underbrace{P(A \cup A^c)} = P(\Omega)$$

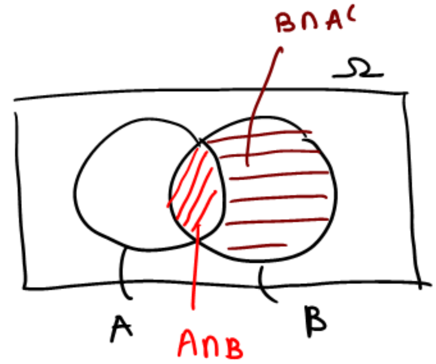
$$P(A) + P(A^c) = \underbrace{P(\Omega)}_{=1}$$ [$\because$ A and A^c are disjoint]

$$P(A) + P(A^c) = 1$$

$$P(A^c) = 1 - P(A) \quad \square.$$

2) * $P(A) = P(A \cap B) + P(A \cap B^c)$

* $P(B) = P(B \cap A) + P(B \cap A^c)$



Proof:

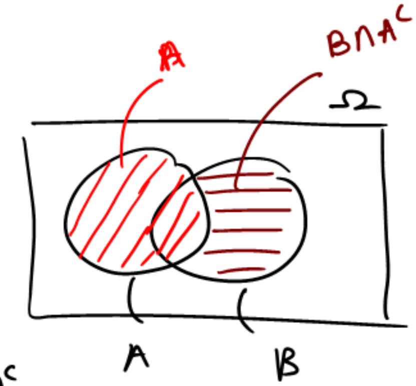
$$B = (B \cap A) \cup (B \cap A^c)$$

$$P(B) = P[(\underline{B \cap A}) \cup (\underline{B \cap A^c})]$$

$$P(B) = P(B \cap A) + P(B \cap A^c) \quad [\because B \cap A \text{ and } B \cap A^c \text{ are disjoint}].$$

$\square$.

3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Proof:

$$A \cup B = A \cup (B \cap A^c)$$

$$P(A \cup B) = P[A \cup (B \cap A^c)]$$

$$= P(A) + \underbrace{P(B \cap A^c)}_{} \quad [A \text{ and } B \cap A^c \text{ are disjoint}]$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad [\text{from } \# 2].$$

$\square$.

4) $P(A \cup B) \leq P(A) + P(B)$

Proof:

From # 3, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow \underbrace{P(A \cap B)}_{\geq 0} = P(A) + P(B) - P(A \cup B)$$

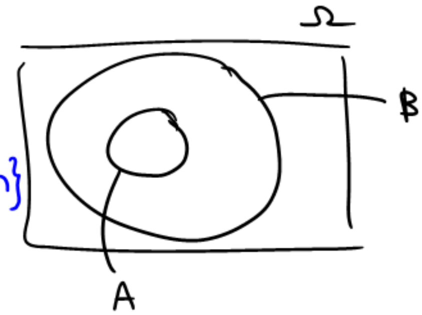
But $P(A \cap B) \geq 0$ (Axiom-I)

$$\Rightarrow P(A) + P(B) - P(A \cup B) \geq 0$$

$$\Rightarrow P(A) + P(B) \geq P(A \cup B)$$

$\square$.

5) If $A \subseteq B$ then $P(A) \leq P(B)$.



* $\left.\begin{array}{l} P(A \cap B) \leq P(A) \\ P(A \cap B) \leq P(B) \end{array}\right\} \Rightarrow P(A \cap B) \leq \min\{P(A), P(B)\}$

* $\left.\begin{array}{l} P(A) \leq P(A \cup B) \\ P(B) \leq P(A \cup B) \end{array}\right\} \Rightarrow \max\{P(A), P(B)\} \leq P(A \cup B)$.