Quiz -4 : F (9/27) | HW - 5 : W (10/04), in the conference | Class -20

Rest of the term:
Sec 5.1 : Continuous Random Variables
Sec 5.2 : Cumulative distribution function.
Sec 6.4 : Normal distribution
Sec 6.5 : Central Limit theorem
Sec 5.3 : Functions
Sec 5.4 : Joint distributions (continuous)
Sec 5.5 : Marginal distributions (continuous)

Chapter-5 : Distribution of continuous Random variables

When a random variable can take any value in an interval, it is called a continuous random variable. Probability distribution of a continuous random variable is called a probability density function (pdf).

Note: Probability distribution of a discrete random variable is called a probability mass function (pmf).

Defn Pdf

A continuous random variable X is said to have a probability density function "f" if for all $a \le b$,

$$P(a \le X \le b) = \int_a^b f(x)\,dx.$$

Note:

* $P(X=a)=0$ (ie Probability of a Single value is 0).

* $P(a \le X \le b) = P(a \le X < b) = P(a < X \le b) = P(a < X < b)$.

Proof:

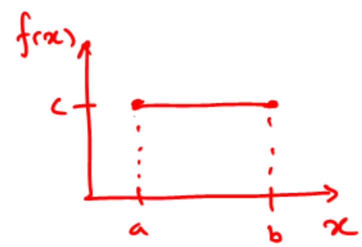
$$P(a \le X \le b) = \underbrace{P(X=a)}_{=0} + P(a < X < b) + \underbrace{P(X=b)}_{=0}.$$

$$= P(a < X < b).$$

* $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1.$

[In discrete case $\sum_{x} p(x) = 1$].

Eg:- 1. Let $f(x) = \begin{cases} c & : a \le x \le b \\ 0 & : \text{otherwise} \end{cases}$ be a pdf.



Find c.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \underbrace{\int_{-\infty}^{a} 0\,dx}_{=0} + \int_{a}^{b} c\,dx + \underbrace{\int_{b}^{\infty} 0\,dx}_{=0} = 1$$

$$c\,x\Big|_a^b = 1$$

$$c[b-a] = 1$$

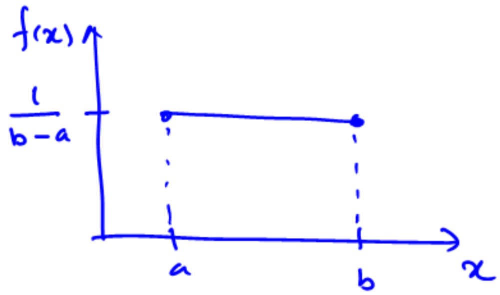
$$\Rightarrow c = \frac{1}{b-a}.$$

$$\boxed{\int_a^b k\,dx = kx\Big|_a^b = k[b-a]}$$

Note:

If $f(x) = \frac{1}{b-a}$: $a \le x \le b$, then X is called uniform random variable. (ie $X \sim \text{uniform}(a, b)$)

parameters.



Eg-2) Let $f(x) = \begin{cases} a/\sqrt{x} & : 0 < x < 1 \\ 0 & : \text{otherwise}, \end{cases}$ be a pdf.

Find a.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 a/\sqrt{x}\,dx = 1$$

$$\Rightarrow a \int_0^1 x^{-1/2}\,dx = 1$$

$$a \left. \frac{x^{1/2}}{1/2} \right|_0^1 = 1$$

$$2a = 1 \quad \Rightarrow \quad a = 1/2$$

* $\int x^k\,dx = \frac{x^{k+1}}{k+1}$, $k \neq -1$

* when $k = -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x|$

$E(X) = \sum_x x\, p(x)$

Expectation

* Expected value of a cts random variable X is

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

* If g is a function of X, then,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

Note:

$$E(X^k) = \int_{-\infty}^{\infty} x^k f(x)\, dx \quad - \quad k^{th} \text{ moment.}$$

* Variance

In discrete case,

$$\text{Var}(X) = E\left[(X - E(X))^2\right].$$
$$= \sum (x - E(x))^2 p(x)$$

$$\text{var}(X) = E\left[(X - E(X))^2\right] = \int_{-\infty}^{\infty} (x - E(X))^2 f(x)\, dx.$$

Note:

* $\text{Var}(X) = E(X^2) - (E(X))^2.$
* $E(aX + b) = aE(X) + b.$
* $\text{Var}(aX + b) = a^2 \text{Var}(X).$

Eg:- Let $X \sim \text{uniform}(a, b)$, Find $E(X)$ and $\text{var}(X)$.

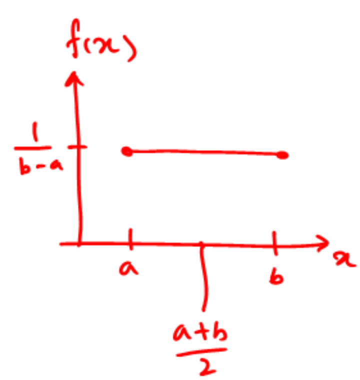
Pdf: $f(x) = \dfrac{1}{b-a}$: $a \le x \le b$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_a^b x \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b x\, dx$$

$$= \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b = \frac{1}{b-a} \left[\frac{b^2 - a^2}{2} \right]$$

$$= \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \boxed{\frac{a+b}{2}}$$



$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(a^2+ab+b^2)}{3\cancel{(b-a)}}$$

$$= \frac{a^2+ab+b^2}{3}$$

$$Var(X) = E(x^2) - (E(x))^2$$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{(b-a)^2}{12} \quad \frac{b^2-2ab+a^2}{12}$$

* If $X \sim uniform(a,b)$,

$E(X) = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$.

Exponential Distribution

Exponential distribution is used to model time between two successive events.

Eg:- 1. time between two calls.

2. time between two busses.

Defn

Random variable X follows an exponential distribution with parameter λ (ie $X \sim Exp(\lambda)$), if

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & : x \geq 0 \\ 0 & : \text{otherwise.} \end{cases}$$

Here X – time between two events

$\frac{1}{\lambda}$ – average time between two events.

Note:

① It is legitimate.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx$$

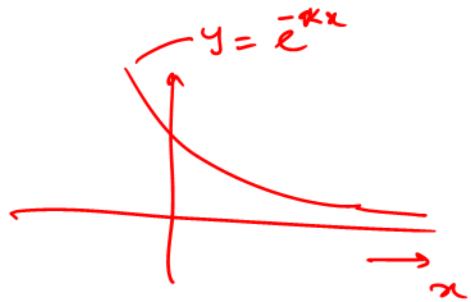
$$= \lambda \int_0^{\infty} e^{-\lambda x}\,dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{\infty}$$

$$= -\Big[\underbrace{\lim_{a\to\infty} e^{-\lambda a}}_{=0} - e^{0} \Big]$$

$$= 1 \quad \square$$

$$\boxed{* \int_a^b e^{kx}\,dx = \frac{e^{kx}}{k}\Big|_a^b}$$



② $E(X) = \frac{1}{\lambda}$