Consider chess tournament problem:
(Ai- had a type i opponent (i=1,2,3), B- won the game)
P(A1)=0.5, p(A2)=.25, P(A3)=0.25, P(B|A1)=0.3, P(B|A2)=0.4, P(B|A3)=0.5.
P(B)= 0.375 (previous section).
Suppose you won the game, what is the probability that you had a type 1 opponent? $P(A_1|B)=?$

By Bayes' theorem.

$$P(A_1|B) = \frac{P(B|A_1)\cdot P(A_1)}{P(B)}$$

$$= \frac{(0.3)(0.5)}{0.375}$$

$$= \_\_\_\_$$

Eg:- 2)

Consider four sided die problem:
(Ai- The first roll is i (i=1,2,3,4), B- the sum is at least four)
P(Ai)=1/4 (i=1,2,3,4),
P(B|A1)=1/2, P(B|A2)=1/3, P(B|A3)=0, P(B|A4)=1.
P(B)= 9/16 (previous section).

a) Suppose the sum is at least four, what is the probability that the first roll is 2? $P(A_2|B)=?$

b) Suppose the sum is less than four, what is the probability that the first roll is 1?

a) $$P(A_2|B) = \frac{P(B|A_2)\cdot P(A_2)}{P(B)}$$

$$= \frac{(1/3)(1/4)}{9/16} = \_\_\_\_$$

b) $$P(A_1|B^c) = \frac{P(B^c|A_1)\cdot P(A_1)}{P(B^c)}$$

$$= \frac{[1 - P(B|A_1)]\cdot P(A_1)}{1 - P(B)}$$

$$= \frac{[1-1/2]\cdot 1/4}{1 - 9/16} = 2/7$$

Eg:-

Three factories make 20, 30, and 50% of the computer chips for a company. The probability of a defective chip is 0.04, 0.03, and 0.02 for three factories. We have a defective chip. What is the probability that it came from factory 1?

Let A_i – chip came from factory i, $i = 1, 2, 3$.

B – the chip is defective.

$P(A_1) = 0.2$, $P(A_2) = 0.3$, $P(A_3) = 0.5$.

$P(B|A_1) = 0.04$, $P(B|A_2) = 0.03$, $P(B|A_3) = 0.02$.

$P(A_1|B) = ?$

By Bayes' theorem

$$P(A_1|B) = \frac{P(B|A_1)\cdot P(A_1)}{P(B|A_1)\cdot P(A_1) + P(B|A_2)\cdot P(A_2) + P(B|A_3)\cdot P(A_3)}$$

$$= \frac{(0.04)(0.2)}{(0.04)(0.2) + (0.03)(0.3) + (0.02)(0.5)}$$

$$= 8/32 = 1/4 = 0.25$$

Sec 3.4 : Discrete Joint Distributions

Let X and Y are two random variables associated with same random experiment. Then the probability distribution of X and Y is given by (i.e the joint distribution of X and Y)

$$P(X=x, Y=y) = P(x,y) = P\Big(\underbrace{\{X=x\} \cap \{Y=y\}}_{\text{event}}\Big)$$

Note:

$$\sum_{x,y} P(x,y) = 1.$$

Eg:- Roll two four Sided fair dice.

Let X – the maximum of the two numbers

Y – the Sum

a) Find the joint distribution of X and Y

b) Find $P(X \le 2, Y \le 3)$

Possible values of X: 1, 2, 3, 4

possible values of Y: 2, 3, 4, 5, 6, 7, 8

Joint distribution of X and Y.

X \ Y	2	3	4	5	6	7	8	
1	$\frac{1}{16}$	0	0	0	0	0	0	← (1,1)
2	0	$2/16$	$1/16$	0	0	0	0	← (1,2),(2,1), (2,2)
3	0	0	$2/16$	$2/16$	$1/6$	0	0	← $\underbrace{(1,3),(3,1)}$, $\underbrace{(2,3),(3,2)}$, (3,3)
4	0	0	0	$2/16$	$2/16$	$2/16$	$1/16$	

$$P(X=1, Y=2) = P(\{(1,1)\}) = \frac{1}{16}$$

b) $P(X \le 2, Y \le 3) = P(1,2) + P(1,3) + P(2,2) + P(2,3)$

$= 1/16 + 0 + 0 + 2/16 = 3/16$

Marginal Distributions

Suppose $p(x,y)$ is the joint distribution of X and Y.

Then the marginal distribution of X:

$$P(X=x) = \sum_y P(x,y).$$

The marginal distribution of Y:

$$P(Y=y) = \sum_x p(x,y)$$

Eg:- Suppose we draw 2 balls out of an urn with 6-red, 5-blue and 4 green balls.

Let X – # of red balls, Y – # of blue balls.

a) Find the joint distribution function of X and Y

b) Find the marginal distributions of X and Y?

6-R
5-B
4-G
$\to$ 2

Possible values of X: 0, 1, 2

Possible values of Y: 0, 1, 2

Joint distribution:

X \ Y	0	1	2	
0	$6/105$	$20/105$	$10/105$	$P(X=0) = \frac{6+20+10}{105} = 36/105$
1	$24/105$	$30/105$	0	$P(X=1) = \frac{54}{105}$
2	$15/105$	0	0	$P(X=2) = 15/105$
	$P(Y=0) = \frac{45}{105}$	$P(Y=2) = \frac{50}{105}$	$P(Y=3) = 10/105$	

$$P(X=0, Y=0) = P(2\text{-green}) = \frac{{}^4C_2}{{}^{15}C_2} = \frac{6}{105}$$

$$P(X=0, Y=1) = P(1\text{-green}, 1\text{-blue}) = \frac{{}^4C_1 \cdot {}^5C_1}{{}^{15}C_2} = \frac{20}{105}$$

$$P(0,2) = P(2\text{-blue}) = \frac{{}^5C_2}{{}^{15}C_2} = 10/105$$

$\vdots$

Marginal distribution of X :

$X=x$	0	1	2
$P(X=x)$	$\frac{36}{105}$	$\frac{54}{105}$	$\frac{15}{105}$

OR

$$P(X=x) = \begin{cases} 36/105 & : x=0 \\ 54/105 & : x=1 \\ 15/105 & : x=2 \end{cases}$$