Eg:-

Consider flipping 5 fair coins.
What is the probability distribution of random variable X : number of heads out of 5 flips.

Possible values of X : 0, 1, 2, 3, 4, 5

$$P(A) = \frac{n(A)}{n(\Omega)}$$

$$\underset{2}{\uparrow \_\_} \quad \underset{2}{\uparrow \_\_} \quad \underset{2}{\_\_} \quad \underset{2}{\_\_} \quad \underset{2}{\_\_}$$

There are $2^5 = 32$ outcomes in Ω. (ie $n(\Omega) = 32$).

$$P(X=0) = \frac{\text{\# of ways to 0 heads}}{\text{total \# of outcomes}} = \frac{{}^5C_0}{32} = \frac{1}{32}$$

$$P(X=1) = \frac{{}^5C_1}{32} = \frac{5}{32}$$

$$P(X=2) = \frac{{}^5C_2}{32} = \frac{10}{32}$$

$$P(X=3) = \frac{{}^5C_3}{32} = \frac{10}{32}$$

$$P(X=4) = \frac{{}^5C_4}{32} = \frac{5}{32}$$

$$P(X=5) = \frac{{}^5C_5}{32} = \frac{1}{32}$$

$\_\_ \; \_\_ \; \_\_ \; \_\_ \; \_\_$

$\therefore$ Probability distribution of X:

$$P(X=x) = \begin{cases} 1/32 & : x = 0, 5 \\ 5/32 & : x = 1, 4 \\ 10/32 & : x = 2, 3 \\ 0 & : \text{otherwise.} \end{cases}$$

Note:

1. ${}^nC_0 = {}^nC_n = 1$.

2. ${}^nC_m = {}^nC_{n-m}$.

Proof:

$${}^nC_m = \frac{n!}{(n-m)! \cdot m!} = \frac{n!}{m!\,(n-m)!} = \frac{n!}{(n-(n-m))!\,(m-m)!} = {}^nC_{n-m}.$$

3. $\ ^{n-1}C_{k-1} + \ ^{n-1}C_{k} = \ ^{n}C_{k}$.

Proof – HW.

4. ^nC_r values can be obtained from pascal's triangle

Pascal's triangle:

$$
\begin{array}{ccccccccc}
 & & & & 1\ (^0C_0) & & & & \text{— } n=0 \\
 & & & 1\ (^1C_0) & & 1\ (^1C_1) & & & \text{— } n=1 \\
 & & 1\ (^2C_0) & & 2\ (^2C_1) & & 1\ (^2C_2) & & \text{— } n=2 \\
 & 1\ (^3C_0) & & 3\ (^3C_1) & & 3\ (^3C_2) & & 1\ (^3C_3) & \text{— } n=3 \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Binomial Theorem

For any real numbers x and y and positive integer n,

$$(x+y)^n = \sum_{m=0}^{n} \ ^nC_m \, x^{n-m} \cdot y^m \qquad (n+1 \text{ terms})$$

$$= \ ^nC_0\, x^n + \ ^nC_1\, x^{n-1} y^1 + \ ^nC_2\, x^{n-2} y^2 + \cdots + \ ^nC_{n-1}\, x^1 y^{n-1} + \ ^nC_n\, y^n.$$

Note:

$$(x+y)^2 = \ ^2C_0 x^2 y^0 + \ ^2C_1 xy + \ ^2C_2 x^0 y^2$$

$$= x^2 + 2xy + y^2$$

$$(x+y)^3 = ?$$

Eg:-

From a group of 15 smokers and 21 nonsmokers, a researcher wants to randomly select 7 smokers and 6 nonsmokers for a study. In how many ways can the study group be selected?

multiplication rule.

$$\#\text{ of Study groups} = \left(\begin{array}{c}\#\text{ of ways to Select}\\ 7\text{ Smokers out of }15\end{array}\right) * \left(\begin{array}{c}\#\text{ of ways to Select}\\ 6\text{ non Smokers out of }21\end{array}\right)$$

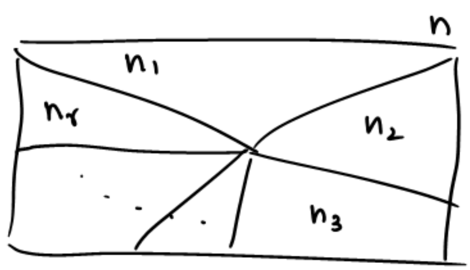
$$= {}^{15}C_{7} * {}^{21}C_{6}$$

$$= \frac{15!}{8!\cdot 7!} * \frac{21!}{15!\cdot 6!}$$

$$= 349{,}188{,}840.$$

Partitions

Suppose a Set of n elements is divided in "r" disjoint subsets such that i^{th} subset has "n_i" elements, $i = 1, 2, \ldots r$, and $n_1 + n_2 + \cdots + n_r = n$.



$$\left.\begin{array}{c}\text{Then the total number of}\\ \text{choices}\end{array}\right\} = {}^{n}C_{n_1} * {}^{n-n_1}C_{n_2} * {}^{n-(n_1+n_2)}C_{n_3} \cdots * {}^{n-(n_1+n_2+\cdots n_{r-1})}C_{n_r}$$

$$= \frac{n!}{n_1!\,\cancel{(n-n_1)!}} * \frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-(n_1+n_2))!}} * \cancel{\cdots} \cdots * \frac{\cancel{(n-n_1-n_2\cdots} - n_{r-1})!}{n_r!\cdot (n-n_1-\cdots-n_{r-1}-n_r)!}$$

$$= \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_r!}$$

$$= \binom{n}{n_1, n_2, n_3 \ldots, n_r}$$