Double Barrier Options

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FRE-6233 Option Pricing and Stochastic Calculus

Contents

1. Abstract

In this report, we will find the double knock-out call option pricing formula, calculate its analytically Delta. During deriving, we will use Image Method, Fourier Method to get two analytically formulas, and then verified by Monte Carlo Method. Also, we will check the model by other ways. Last, we will use real stock data to do it empirically.

Zheng, Xin and Yan, Qi work on Image Method, Xu, Steven Kenneth works on Fourier Method, Liu, Victor and Liu, Xiao work on Coding part, including model testing and empirical study.

Keywords: double barrier option, image method, Fourier Method, Monte Carlo Method

2. Double barrier option

A double barrier option is knocked out or in if the asset price touches a lower (L) or upper (U) barrier level within the option lifetime. Double barrier option valuation has been described by many researchers. In our report, we consider a double knock-out call option with strike K with several assumptions:

- The asset price follows the Geometric Brownian Motion with constant volatility σ.
$$\frac{d\big(S(t)\big)}{dt} = \mu(t)S(t)dt + \sigma(t)S(t)dW(t)$$
- The interest rate r is constant and positive, no dividends.
- Initial asset price S is in between two barriers, $L < S < U$.
- When strike $K \geq U$, the barrier will be triggered and expire worthless, so we only consider the case where $K < U$.
- Pricing barrier options with flat boundaries, L and U are constant.

3. PDE Approach

The distinction between up and down types is lost for double barrier options; we only have knock-out and knock-in types. We have seen that single barrier options have just one image price associated with the given barrier. Unfortunately, this is not the case for double barrier options.

The deriving PDE part is the same as simple knock-out option and classical Black-Scholes formula:

$$rc = c_t + rSc_s + \frac{1}{2}\sigma^2 S^2 c_{SS}$$

We use the change of variables,

$$S = Le^{x}, t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, c(t,S) = Kv(x,\tau), v = e^{\alpha x + \beta\tau}u(\tau, x)$$

$$\alpha = -\frac{1}{2}(k-1), \beta = -\frac{1}{4}(k+1)^2, k = \frac{r}{\frac{1}{2}\sigma^2}$$

In these new variables the barrier transforms to the point $x = 0$, and the barrier option problem becomes

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

For $0 < x < x_+, x_+ = \log\left(\frac{U}{L}\right), \tau > 0, with$

$$u(x,0) = U(x) = \max\left(e^{\frac{1}{2}(k+1)x} - \left(\frac{K}{L}\right)e^{\frac{1}{2}(k-1)x}, 0\right)$$

And

$$u(0,t) = u(x_+, t) = 0$$

Finally, we define our payoff at maturity $\mathrm{t} = \mathrm{T}$, or $\tau = 0$. Let C be the price of a vanilla European call option. Then:

$$C(S, K, \tau = 0) = (S - K)^{+}, C = Lv(x,\tau)$$

$$v(x,\tau) = \frac{C}{L}$$

$$v(x,\tau) = e^{\alpha x + \beta\tau}u(x,\tau)$$

$$v(x,0) = e^{\alpha x}u(x,0)$$

$$u(x,0) = e^{-\alpha x}v(x,0)$$

As a shorthand, from now on, let C be the payoff of a vanilla European call option at maturity:

$$u(x,0) = e^{-\alpha x}\frac{C}{L}$$

4. Solution

4.1 Image Method

We say $F^*(x) = \left(\frac{x_+}{x}\right)^{-2\alpha} F(\frac{x_+{}^2}{x})$ is the image of $F(x)$ wrt $x = x_+$. And we shall write the image of $F(x)$ wrt $x = x_+$, in operator notation as $F^*(x) = \mathcal{I}_{x_+}\{F(x)\}$, where $\mathcal{I}_{x_+}$ denotes the image operator wrt $x = x_+$. The image function has several important properties,

The image operator $\mathcal{I}_U$ is an involution, ie $\mathcal{I}_{x_+}^2 = I$, where I is the identity operator.
$U^*(x,t)$ satisfies the BS-pde, whenever $U(x,t)$ does. $U^*(x,t) = U(x,t)$ when $x = x_+$.

If $x > x_+$ (resp. $x < x_+$) is the active domain of $U(x,t)$, then $x < x_+$(resp. $x > x_+$) is the active domain of $U^*(x,t)$. If $U(x,t)$ is a derivative price, then we refer to $U^*(x,t)$ as the corresponding image price. Write $U_0(x)$ for the transformed version of the extended initial data, and $U(x,\tau)$ for the corresponding solution of the heat equation.

Now reflect in $x = 0$, considering the initial value $U_0(x) - U_0(-x)$.

The corresponding solution of heat equation is zero at $x = 0$, $U_0(0) - U_0(0) = 0$, but not at $x = x_+$, $U_0(x_+) - U_0(-x_+) = U_0(x_+)$. We can satisfy this second boundary condition by reflecting in $x = x_+$, and it is easy to see that the new initial value created by this reflection is just translating to the right by $2x_+$. This gives a term of the form

$$U_0(x - 2x_+) - U_0(2x_+ - x), x_+ < x < 3x_+$$

Now $U_0(x_+) - U_0(-x_+) + U_0(x_+ - 2x_+) - U_0(2x_+ - x_+) = 0$, but at the first boundary condition, $U_0(0) - U_0(0) + U_0(0 - 2x_+) - U_0(2x_+ - 0) \neq 0$, we fix that up by adding to the initial condition translated to the left by $2x_+$. This gives a term of the form

$$U_0(x + 2x_+) - U_0(-2x_+ - x), -3x_+ < x < -x_+$$

Now $U_0(0) - U_0(0) + U_0(0 - 2x_+) - U_0(2x_+ - 0) + U_0(0 + 2x_+) - U_0(-2x_+ - 0) = 0$, but for the second boundary condition, $U_0(x_+) - U_0(-x_+) + U_0(x_+ - 2x_+) - U_0(2x_+ - x_+) + U_0(x_+ + 2x_+) - U_0(-2x_+ - x_+) \neq 0$; and so on. In the end, we have an infinite series of copies. The initial data for the heat equation is therefore

$$u(x,0) = \sum_{-\infty}^{\infty} \big(U_0(x - 2nx_+) - U_0(2nx_+ - x)\big)$$

As we show before, to simplify notation, let the operator $\mathcal{I}_{x_+,0}$ denote the double image sequence: $\mathcal{I}_{x_+}\mathcal{I}_0$. The image wrt 0 is performed first and is then followed by the image wrt x_+.

Define the doubly-infinite sequence of image operators, by

$$\mathcal{K}_0^{x_+} = I - \mathcal{I}_0 - \mathcal{I}_{x_+} + \mathcal{I}_{x_+,0} + \mathcal{I}_{0,x_+} - \mathcal{I}_{x_+,0,x_+} - \mathcal{I}_{0,x_+,0} + \cdots$$

where I is the identity operator, and $\mathcal{I}$'s are image operators. Observe the symmetry: $\mathcal{K}_0^{x_+} = \mathcal{K}_{x_+}^0$. Then, the double barrier price satisfying the boundary conditions is given by the expression

$$V(S,t) = \mathcal{K}_L^U\{U_{L,U}(S,t)\} = \sum_{n=-\infty}^{\infty} \left(\frac{U}{L}\right)^{-2n\alpha} \left[V_{LU}\left(\left(\frac{U}{L}\right)^{2n} S, t\right) - V_{LU}^*\left(\left(\frac{U}{L}\right)^{2n} S, t\right)\right]$$

where $V_{LU}^*\left(\left(\frac{U}{L}\right)^{2n} S, t\right) = \left(\frac{L}{S}\right)^{-2\alpha} V_{LU}\left(\frac{L^{2n+2}}{U^{2n}S}, t\right)$ or (resp.)

$V_{LU}^*\left(\left(\frac{U}{L}\right)^{2n} S, t\right) = \left(\frac{U}{S}\right)^{-2\alpha} V_{LU}\left(\frac{L^{2n}}{U^{2n-2}S}, t\right)$ is the image function wrt $S = L$ or (resp.) $S = U$.

Now, we need apply this method with $V(S) = (S-K)^+$, and the strike price satisfies $K < U$. Let $\ell = \max(L, K)$. Then, we calculate,

$$V_{LU}(S,t) = (S-K)^+[\mathbb{I}\{S > L\} - \mathbb{I}\{S > U\}] = (S-K)[\mathbb{I}\{S > \ell\} - \mathbb{I}\{S > U\}]$$

Hence, $V_{LU}(S,t)$ will be given, in terms of gap option Q, by

$$V_{LU}(S,t) = Q_\ell^+(S,t,K) - Q_U^+(S,t,K)$$

for all $t < T$.

The gap call price is

$$C_\xi(S,t,K) = SN(d_1) - Ke^{-rt}N(d_2)$$

where $d_{1,2} = \frac{\log\left(\frac{S}{\xi}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$

By the image method, we get the following formula for the price of a knock-out, double barrier call option:

$$V_c^{ko}(S,t) = \sum_{n=-\infty}^{\infty} \left(\frac{U}{L}\right)^{-2n\alpha} \left[Q_\ell^+\left(\left(\frac{U}{L}\right)^{2n} S, t, K\right) - Q_U^+\left(\left(\frac{U}{L}\right)^{2n} S, t, K\right)\right]$$
$$- \sum_{n=-\infty}^{\infty} \left(\frac{U}{L}\right)^{-2n\alpha} \left[{Q^*}_\ell^+\left(\left(\frac{U}{L}\right)^{2n} S, t, K\right) - {Q^*}_U^+\left(\left(\frac{U}{L}\right)^{2n} S, t, K\right)\right]$$

We can take the gap option pricing formula and do the simplification:

$$c = S\sum_{n=-\infty}^{\infty}\left\{\left(\frac{U^n}{L^n}\right)^{k+1}[\phi(d_1)-\phi(d_2)]-\left(\frac{L^{n+1}}{U^nS}\right)^{k+1}[\phi(d_3)-\phi(d_4)]\right\}$$
$$-Ke^{-rt}\sum_{n=-\infty}^{\infty}\left\{\left(\frac{U^n}{L^n}\right)^{k-1}\left[\phi\left(d_1-\sigma\sqrt{t}\right)-\phi\left(d_2-\sigma\sqrt{t}\right)\right]\right.$$
$$\left.-\left(\frac{L^{n+1}}{U^nS}\right)^{k-1}\left[\phi\left(d_3-\sigma\sqrt{t}\right)-\phi\left(d_4-\sigma\sqrt{t}\right)\right]\right\}$$

where $d_1 = \frac{\log\left(\frac{SU^{2n}}{KL^{2n}}\right)+\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$, $d_2 = \frac{\log\left(\frac{SU^{2n-1}}{L^{2n}}\right)+\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$,

$d_3 = \frac{\log\left(\frac{L^{2n+2}}{KSU^{2n}}\right)+\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$, $d_4 = \frac{\log\left(\frac{L^{2n+2}}{KU^{2n+1}}\right)+\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$, $k = \frac{2r}{\sigma^2}$

4.2 Delta calculation of image method

Given the solution of the analytical formula of the double knock-out call option, we can calculate the Delta for that option:

$$\frac{\partial c}{\partial S} = \frac{\partial}{\partial S}\sum_{n=-\infty}^{\infty}\left\{\left(\frac{U^n}{L^n}\right)^{k+1}[\phi(d_1)-\phi(d_2)]-\left(\frac{L^{n+1}}{U^nS}\right)^{k+1}[\phi(d_3)-\phi(d_4)]\right\}$$
$$-Ke^{-rt}\sum_{n=-\infty}^{\infty}\left\{\left(\frac{U^n}{L^n}\right)^{k-1}\left[\phi\left(d_1-\sigma\sqrt{t}\right)-\phi\left(d_2-\sigma\sqrt{t}\right)\right]\right.$$
$$\left.-\left(\frac{L^{n+1}}{U^nS}\right)^{k-1}[\phi(d_3)-\phi(d_4)]\right\}$$
$$=\sum_{n=-\infty}^{\infty}\left\{\left(\frac{U^n}{L^n}\right)^{k+1}\left([\phi(d_1)-\phi(d_2)]+S[\phi'(d_1){d_1}'-\phi'(d_2){d_2}']\right)\right.$$
$$\left.-\left(\frac{L^{n+1}}{U^nS}\right)^{k+1}\left(-k[\phi(d_3)-\phi(d_4)]+S[\phi'(d_3){d_3}'-\phi'(d_4){d_4}']\right)\right\}$$
$$-Ke^{-rt}\sum_{n=-\infty}^{\infty}\left\{\left(\frac{U^n}{L^n}\right)^{k-1}\left[\phi'\left(d_1-\sigma\sqrt{t}\right){d_1}'-\phi'\left(d_2-\sigma\sqrt{t}\right){d_2}'\right]\right.$$
$$-\left(\frac{L^{n+1}}{U^nS}\right)^{k-1}\left(\frac{1-k}{S}\left[\phi\left(d_3-\sigma\sqrt{t}\right)-\phi\left(d_4-\sigma\sqrt{t}\right)\right]\right.$$
$$\left.\left.+\left[\phi'^{(d_3-\sigma\sqrt{t})}{d_3}'-\phi'\left(d_4-\sigma\sqrt{t}\right){d_4}'\right]\right)\right\}$$

where ${d_1}', {d_2}', {d_3}', {d_4}'$ are $\frac{\partial d_i}{\partial s}$, for $i = 1, 2, 3, 4$, $\phi'(\cdot)$ is the probability density function of standard normal distribution.

And $d_1{}', d_2{}', d_3{}', d_4{}'$ are

$$d1' = d2' = \frac{1}{S\sigma\sqrt{T}}$$

$$d3' = d4' = -\frac{1}{S\sigma\sqrt{T}}$$

Put them together and simplify the formula:

$$\begin{aligned}\Delta = \sum_{n=-\infty}^{\infty} \Bigg\{ \left(\frac{U^n}{L^n}\right)^{k+1} & \left([\phi(d_1) - \phi(d_2)] + \frac{\phi'(d_1) - \phi'(d_2)}{\sigma\sqrt{T}}\right) \\ & + \left(\frac{L^{n+1}}{U^n S}\right)^{k+1} \left(k[\phi(d_3) - \phi(d_4)] + \frac{\phi'(d_3) - \phi'(d_3)}{\sigma\sqrt{T}}\right)\Bigg\} \\ & - Ke^{-rt} \sum_{n=-\infty}^{\infty} \Bigg\{\left(\frac{U^n}{L^n}\right)^{k-1} \frac{\phi'\left(d_1 - \sigma\sqrt{t}\right) - \phi'\left(d_2 - \sigma\sqrt{t}\right)}{S\sigma\sqrt{T}} \\ & - \left(\frac{L^{n+1}}{U^n S}\right)^{k-1} \left(\frac{1-k}{S}\left[\phi\left(d_3 - \sigma\sqrt{t}\right) - \phi\left(d_4 - \sigma\sqrt{t}\right)\right]\right. \\ & \left. - \frac{\phi'\left(d_3 - \sigma\sqrt{t}\right) - \phi'\left(d_4 - \sigma\sqrt{t}\right)}{S\sigma\sqrt{T}}\right)\Bigg\}\end{aligned}$$

4.3 Fourier method

4.3.1 Solving the Heat Equation

We assume that the solution to the heat equation will be separable: the temporal evolution will be independent from the spatial evolution, such that:

$$u(x,\tau) = U(x)T(\tau)$$

Thus, a solution that fulfills the condition

$$u(0,0) = u(x_+,0) = 0$$

will also hold for all τ.

We know that the solution to the heat equation will go to 0 at both boundaries. This naturally suggests that some combination of sine and cosine functions will provide a solution.

For the spatial solution $U(x)$ we try the ansatz:

$$U(x) = A\sin kx + B\cos kx$$

$$U(0) = B \to B = 0$$

$$U(x_+) = A\sin kx_+ = 0$$

$$\sin kx_+ = 0$$

$$kx_+ = n\pi$$

$$k = \frac{n\pi}{x_+}$$

$$\therefore U(x) = A\sin\frac{n\pi}{x_+}x$$

This is one solution to the heat equation, but by the superposition principal, we know that sums of equations of this form will also be solutions to the heat equation with the given boundary conditions. In this particular case, the solution to the heat equation given the boundary conditions will consist entirely of sines.

Now that we have a general form of the spatial evolution, we can consider the temporal evolution. We know that $u(x,\tau) = U(x)T(\tau)$, and that $u_\tau = u_{xx}$.

Thus:

$$u(x,\tau) = \sin\left(\frac{n\pi}{x_+}x\right)T(\tau)$$

$$u_\tau = \sin\left(\frac{n\pi}{x_+}x\right)T_\tau$$

$$u_{xx} = -T\left(\frac{n\pi}{x_+}\right)^2 \sin\left(\frac{n\pi}{x_+}x\right)$$

$$u_\tau = u_{xx}$$

$$\frac{\partial T}{\partial \tau} = -T\ \left(\frac{n\pi}{x_+}\right)^2$$

$$\frac{\partial T}{T}\ =\ -\ \left(\frac{n\pi}{x_+}\right)^2 \partial\tau$$

$$T = e^{-\left(\frac{n\pi}{x_+}\right)^2\tau}$$

Thus, we find that $u(x,\tau)$ will be some superposition of the form:

$$u(x,\tau) = A\sin\left(\frac{n\pi}{x_+}x\right)e^{-\left(\frac{n\pi}{x_+}\right)^{2\tau}}$$

We now must incorporate the initial condition.

4.3.2 Fourier Analysis

A Fourier Series is a way to approximate any function as an infinite sum of sines and cosines. In this case, our claim would be that:

$$U(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{x_+}x\right)$$

We note that sine functions form an orthogonal basis with each other. Therefore:

$$\int_0^{x_+} \sin\left(\frac{n\pi}{x_+}x\right)\sin\left(\frac{m\pi}{x_+}x\right)dx = \begin{cases} \frac{x_+}{2}, & if\ m = n \\ 0, & otherwise \end{cases}$$

We would use the following procedure to determine our coefficients.

First, integrate the multiplication of our solution $U(x)$ with a sine function as follows:

$$\int_0^{x_+} U(x)\sin\left(\frac{m\pi}{x_+}x\right)dx = \int_0^{x_+}\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{x_+}x\right)\sin\left(\frac{m\pi}{x_+}x\right)dx$$
$$= \sum_{n=1}^{\infty}\int_0^{x_+} A_n \sin\left(\frac{n\pi}{x_+}x\right)\sin\left(\frac{m\pi}{x_+}x\right)dx$$

But as we established earlier, the right-hand side becomes 0 for any cases where $n \neq m$. So we get the coefficients for every m as:

$$\int_0^{x_+} U(x) \sin\left(\frac{m\pi}{x_+}x\right) dx = A_m \frac{x_+}{2}$$

$$A_m = \frac{2}{x_+}\int_0^{x_+} U(x) \sin\left(\frac{m\pi}{x_+}x\right) dx \;\; for\; m \in Z^+$$

4.3.3 Financial Interpretation

We now have all the information to restore a financial interpretation of our terms.

Recall that $x = \log\frac{S}{L}$, $x_+ = \log\frac{U}{L}$, and $c \; = \; Lv$. Also recall that we define C as the vanilla call.

$$v(x,\tau) = e^{\alpha x + \beta\tau}\, u(x,\tau)$$

$$v = e^{\alpha x+\beta\tau}\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{x_+}x\right) e^{-\frac{n^2\pi^2}{x_+^2}\tau} \sin\left(\frac{n\pi}{x_+}x\right)$$

$$= e^{\alpha x}e^{\beta\tau}\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi \log\frac{S}{L}}{\log\frac{U}{L}}\right) e^{-\frac{n^2\pi^2}{\log^2\left(\frac{U}{L}\right)}\times\frac{\sigma^2}{2}(T-t)}$$

Now, consider the interpretation of A_m. First, consider the differentiation of x:

$$e^x = \frac{S}{L}$$

$$e^x dx = \frac{dS}{L}$$

$$dx = \frac{dS}{L}e^{-x} = \frac{dS}{L}\times\frac{L}{S}$$

$$dx = \frac{dS}{S}$$

This gives us the transformation of the infinitesimal.

For clarity's sake, we shall now discuss $e^{\alpha x}$ and $e^{-\alpha x}$.

$$e^{\alpha x} = \left(\frac{S}{L}\right)^{\alpha} = S^{\alpha}L^{-\alpha}$$

$$e^{-\alpha x} = \left(\frac{L}{S}\right)^{\alpha} = L^{\alpha}S^{-\alpha}$$

Now, we need to plug in the initial condition in a more specific form into our expression for A_m. Recall that we allow C to be the *payoff* of a vanilla European call option at maturity.

$$A_m = \frac{2}{\log\frac{U}{L}}\int_L^U U_0(x)\sin\left(\frac{m\pi\log\frac{S}{L}}{\log\frac{U}{L}}\right)\frac{dS}{S} = \frac{2}{\log\frac{U}{L}}\int_L^U \frac{e^{-\alpha x}C}{L}\sin\left(\frac{m\pi\log\frac{S}{L}}{\log\frac{U}{L}}\right)\frac{dS}{S}$$

$$= \frac{2}{\log\frac{U}{L}}\int_L^U \frac{L^{\alpha}S^{-\alpha}C}{L}\sin\left(\frac{m\pi\log\frac{S}{L}}{\log\frac{U}{L}}\right)\frac{dS}{S}$$

$$= \frac{2}{\log\frac{U}{L}}L^{\alpha-1}\int_L^U S^{-\alpha}C\sin\left(\frac{m\pi\log\frac{S}{L}}{\log\frac{U}{L}}\right)\frac{dS}{S}$$

Using these simplifications, we can also write the value of our option:

$$c = Lv = S^{\alpha}L^{1-\alpha}e^{\beta\frac{\sigma^2}{2}(T-t)}\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi\log\frac{S}{L}}{\log\frac{U}{L}}\right)e^{-\frac{n^2\pi^2}{\log^2\left(\frac{U}{L}\right)}\times\frac{\sigma^2}{2}(T-t)}$$

Observe that we can cancel out the $L^{\alpha-1}$ terms in the expression of the price and the coefficients, so our formulas become:

$$c = S^{\alpha}e^{\beta\frac{\sigma^2}{2}(T-t)}\sum_{n=1}^{\infty}\hat{A}_n\sin\left(\frac{n\pi\log\frac{S}{L}}{\log\frac{U}{L}}\right)e^{-\frac{n^2\pi^2}{\log^2\left(\frac{U}{L}\right)}\times\frac{\sigma^2}{2}(T-t)}$$

where:

$$\hat{A}_m = \frac{2}{\log\frac{U}{L}}\int_L^U S^{-\alpha}C\sin\left(\frac{m\pi\log\frac{S}{L}}{\log\frac{U}{L}}\right)\frac{dS}{S}$$

Next, we restate α and β in terms of k.

$$c = S^{-\frac{1}{2}(k-1)}e^{-\frac{1}{4}(k+1)^2\frac{\sigma^2}{2}(T-t)}\sum_{n=1}^{\infty}\hat{A}_n\sin\left(\frac{n\pi\log\frac{S}{L}}{\log\frac{U}{L}}\right)e^{-\frac{n^2\pi^2}{\log^2\left(\frac{U}{L}\right)}\times\frac{\sigma^2}{2}(T-t)}$$

where:

$$\hat{A}_m = \frac{2}{\log\frac{U}{L}}\int_L^U S^{-\frac{1}{2}(k-1)}C\sin\left(\frac{m\pi\log\frac{S}{L}}{\log\frac{U}{L}}\right)\frac{dS}{S}$$

Recall that $k = \frac{r}{\frac{1}{2}\sigma^2}$. While this could be simplified further, there isn't going to be much more analytical insight to be gleaned from doing so.

5. Model testing

We both apply Image Method and Fourier Method to price double knock-out option, the latter one performs numerically unstable, so we use **Image Method** in later part.

5.1 Case 1

When the lower barrier is extremely small and the upper barrier is large enough, the double barrier knock-out call option price should be equal with the vanilla call option price.

Suppose S=100,K=100,L=0.1,U=1000,t=0.5,sigma=0.25,r=0.05, the double barrier knock out option's price is 8.2600, while the vanilla call option with S=100,K=100,t=0.5,sigma=0.25,r=0.05 is 8.2600. The result makes sense.

```
BSC(S=100,K=100,t=0.5,sigma=0.25,r=0.05)
```

```
8.26001519934323
```

```
DoubleBarrierC(S=100,K=100,L=0.1,U=1000,t=0.5,sigma=0.25,r=0.05).Price()
```

```
8.26001519934323
```

5.2 Case 2

If the upper barrier is large enough, then the double barrier knock-out call option price should be equal with the single barrier knock-out call option price.

Suppose S=100,K=100,L=90,U=1000,t=0.5,sigma=0.25,r=0.05, the double barrier knock out option's price is 7.1479, while the single barrier knock-out call option with S=100,K=100,X=90,t=0.5,sigma=0.25,r=0.05 is 7.1479. The result makes sense.

```
DoubleBarrierC(S=100,K=100,L=90,U=1000,t=0.5,sigma=0.25,r=0.05).Price()
```

```
7.1478509863158095
```

```
SingleBarrierC(S=100,K=100,X=90,t=0.5,sigma=0.25,r=0.05)
```

```
7.147850986315813
```

5.3 Case 3

If time goes infinity, the knock-out probability will approach 1, so the price will be nearly zero.

Suppose S=100, K=100, L=90, U=110, t=100, sigma=0.25, r=0.05, the double barrier knock-out option's price is $1.6497*10^{-5}$. The result makes sense.

```
DoubleBarrierC(S=100,K=100,L=90,U=110,t=100,sigma=0.25,r=0.05).Price()
```

```
1.6497064312162422e-05
```

5.4 Case 4

Check the analytical delta function. If the lower barrier is extremely small and the upper barrier is large enough, the delta for the double barrier option will be equal with a vanilla call option's delta.

Suppose S=100,K=100,L=0.1,U=1000,t=0.5,sigma=0.25,r=0.05, the double barrier knock-out option's delta is 0.5909, while the vanilla call option's delta is 0.5909 with S=100, K=100, t=0.5, sigma=0.25, r=0.05. The result makes sense.

```
DoubleBarrierC(S=100,K=100,L=0.1,U=1000,t=0.5,sigma=0.25,r=0.05).Delta()
```

```
0.5908801780443125
```

```
norm.cdf((np.log(100/100)+(0.05+0.5*0.25**2)*0.5)/(0.25*np.sqrt(0.5)))
```

```
0.5908801780443125
```

5.5 Monte Carlo Method

The double barrier options in Home Assignment Barrier Option Problem 4 is tested by both analytical solution and Monte Carlo Method. The confidence level is 99%.

Suppose S = 100, K = 100, t = 0.5, sigma = 0.25, r =0.05;

For L = 90, U = 110, the analytical price is 0.0442 in the range.

```
DoubleBarrierC(S=100,K=100,L=90,U=110,t=0.5,sigma=0.25,r=0.05).Price()
```

```
0.04419118799893145
```

```
DoubleBarrierMC(S=100,K=100,L=90,U=110,t=0.5,sigma=0.25,r=0.05,n=10000)
```

```
(0.03613393334515445, 0.05876520844390295)
```

For L = 80, U = 120, the analytical price is 1.4583 in the range.

```
DoubleBarrierC(S=100,K=100,L=80,U=120,t=0.5,sigma=0.25,r=0.05).Price()
```

```
1.4583205347007344
```

```
DoubleBarrierMC(S=100,K=100,L=80,U=120,t=0.5,sigma=0.25,r=0.05,n=10000)
```

```
(1.3858990474701622, 1.5512721892701964)
```

For L = 70, U = 130, the analytical price is 3.7320 in the range.

```
DoubleBarrierC(S=100,K=100,L=70,U=130,t=0.5,sigma=0.25,r=0.05).Price()
```

```
3.7319849143551265
```

```
DoubleBarrierMC(S=100,K=100,L=70,U=130,t=0.5,sigma=0.25,r=0.05,n=10000)
```

```
(3.617406696018445, 3.9203091080471695)
```

6. Empirical Study

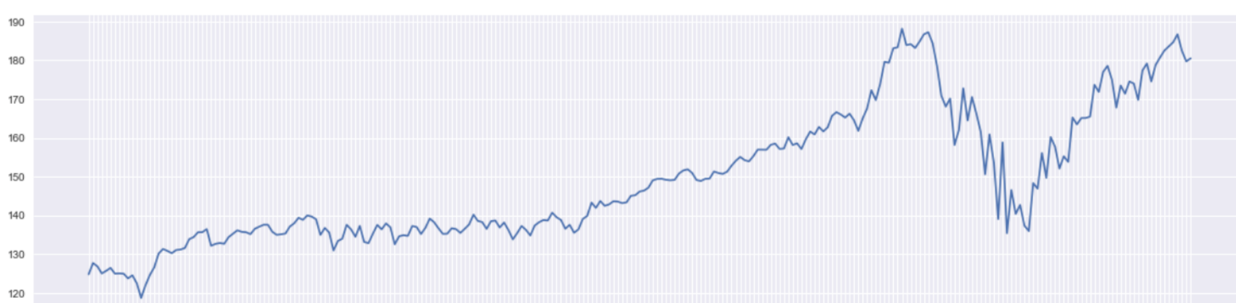
All 5 stocks "MSFT"," IBM"," CSCO"," DELL"," SNY" have been back-tested by using the last year of history: May 15, 2019 - May 14, 2020. Volatilities of log returns are calculated. The strike is chosen to be ATM at the first day of the period and the expiration of our options is in one year, so May 14, 2020. Up and down barriers are chosen to be close to maximum and minimum of the stock price in that period.

For each day i in the history, the derivative's price Vi and its Delta Δi is calculated. The current one-year Libor rate is 2.51%. All options is hedged with the daily Delta and the daily residual PnL is computed and plotted below:

$$PnL_i = (V(S_i, t_i, T) - V(S_{i-1}, t_{i-1}, T)) - \Delta_{i-1}(S_i - S_{i-1})$$

To further analysis, the pictures of all stocks' price series are also provided.

6.1 MSFT

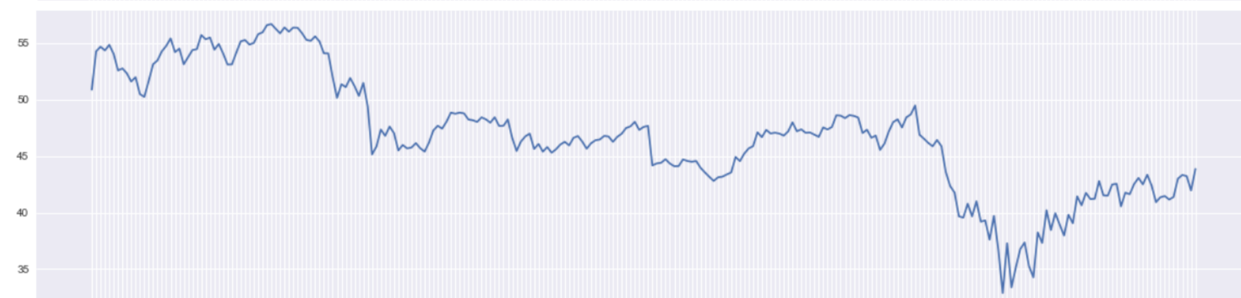


We can see that when the price nearly reaches the upper barrier, the PnL residuals becomes extremely large. When the price moves up and down dramatically, the PnL moves quickly accordingly. Also, as time goes to the expiration day, the PnL will be more volatile.

6.2 IBM

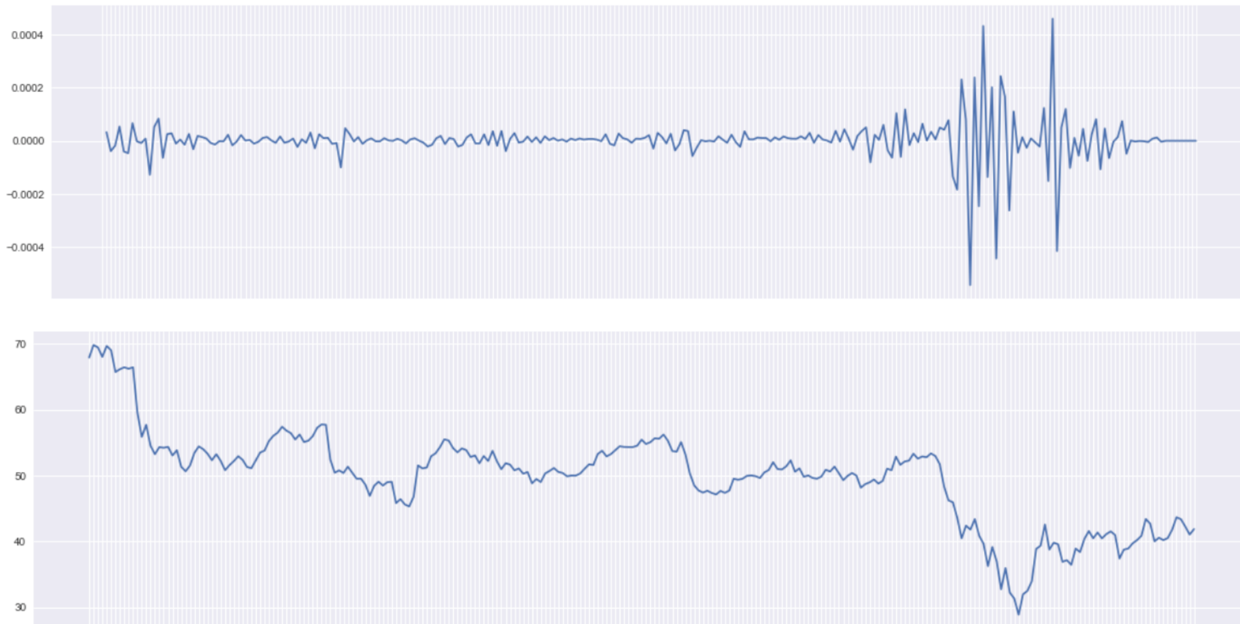
We can see that when the price nearly reaches the upper barrier, the PnL residuals becomes extremely large. When the price moves up and down dramatically, the PnL moves quickly accordingly. Also, as time goes to the expiration day, the PnL will be more volatile.

6.3 CSCO



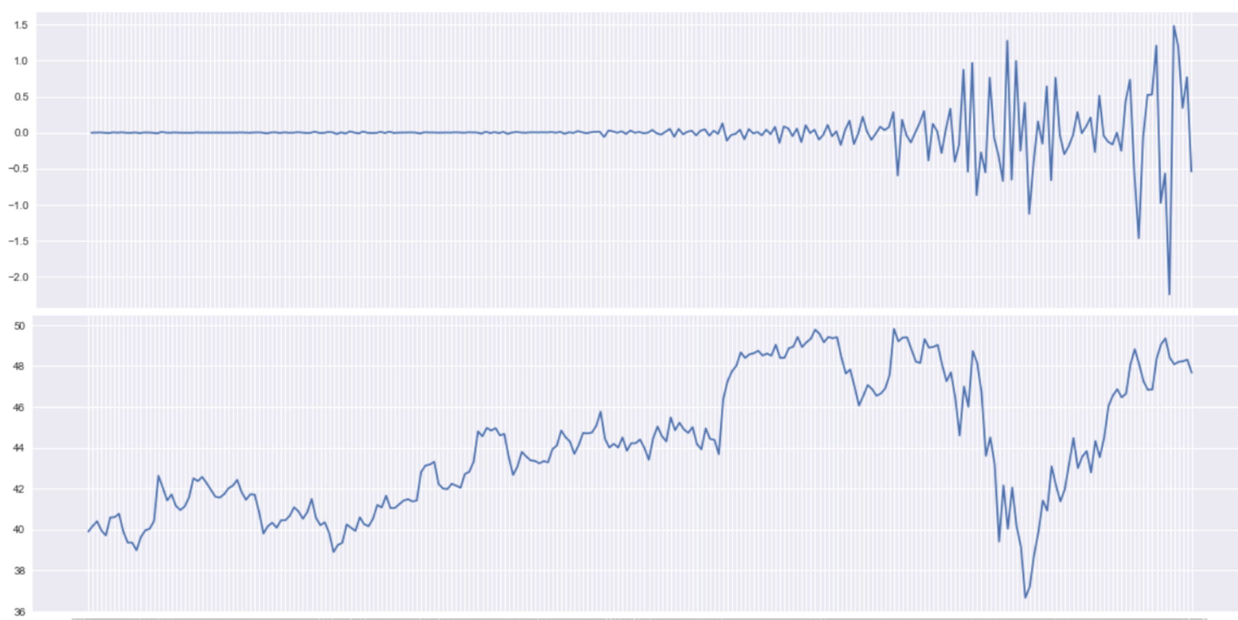
We can see that when the price moves up and down dramatically, the PnL moves quickly accordingly. Also, as time goes to the expiration day, the PnL will be more volatile.

6.4 DELL



We can see that the initial price is already at the upper barrier price, when the price moves up and down dramatically, the PnL moves quickly accordingly. But as time goes to the expiration day, since the option price is far out the money, the PnL does not change much.

6.5 SNY



We can see that when the price nearly reaches the upper barrier, the PnL residuals becomes extremely large. When the price moves up and down dramatically, the PnL moves quickly accordingly. Also, as time goes to the expiration day, the PnL will be more volatile.

6.6 Conclusions for empirical study

The PnL residual often changes rapidly when the price nearly reaches the upper barrier, moves up and down dramatically or time approaches expiration unless it is deeply out the money. If the option is deeply out the money and also near expiration, its price won't change rapidly and thus the PnL will be trivial.

7. Reference

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