Problem 1

Suppose the state price deflator is $m(t)$,

$$m(t) = e^{-\int_0^t r(u)du}$$

$$dm(t) = -e^{-\int_0^t r(u)du} r(t)dt = -m(t)r(t)dt$$

We know that:

$$dr(t) = (\mu - Kr(t))dt + \sqrt{\gamma r(t) + \sigma}\, dW(t)$$

From Ito's Lemma:

$$dr(t) = \frac{\partial r}{\partial t}dt + \frac{\partial r}{\partial W}dW(t) + \frac{1}{2}\cdot\frac{\partial^2 r}{\partial W^2}dt$$

$$= \left(\frac{\partial r}{\partial t} + \frac{1}{2}\frac{\partial^2 r}{\partial W^2}\right)dt + \frac{\partial r}{\partial W}dW(t)$$

$$\therefore \begin{cases} \frac{\partial r}{\partial t} + \frac{1}{2}\frac{\partial^2 r}{\partial W^2} = \mu - Kr(t) \\ \frac{\partial r}{\partial W} = \sqrt{\gamma r(t) + \sigma} \end{cases}$$

We can get: $\begin{cases} \frac{\partial r}{\partial t} = \mu - Kr(t) - \frac{\gamma}{4} \\ \frac{\partial r}{\partial W} = \sqrt{\gamma r(t) + \sigma} \\ \frac{\partial^2 r}{\partial W^2} = \frac{1}{2}\gamma \end{cases}$

Also suppose $p(t,T) = e^{A(t,T) - B(t,T)r(t)}$

if we fix T, the formula can be changed to:

$$p(t) = e^{A(t) - B(t)r(t)}$$

$$dp(t) = p(t)\cdot\left(\frac{dA}{dt} - \frac{dB}{dt}r(t) - B\frac{\partial r}{\partial t}\right)dt + p(t)\left(-B\cdot\frac{\partial r}{\partial W}\right)dW(t)$$
$$+ \frac{1}{2}\left(p(t)\left(-B\cdot\frac{\partial r}{\partial W}\right)^2 + p(t)\left(-B\cdot\frac{\partial^2 r}{\partial W^2}\right)\right)dt$$

$$= p(t)\left(\frac{dA}{dt} - \frac{dB}{dt}r(t) - B\cdot\frac{\partial r}{\partial t} + \frac{1}{2}B^2\left(\frac{\partial r}{\partial W}\right)^2 - \frac{1}{2}B\cdot\frac{\partial^2 r}{\partial W^2}\right)dt$$
$$- p(t)B\frac{\partial r}{\partial W}dW(t)$$

$$= p(t)\left(\frac{dA}{dt} - \frac{dB}{dt}r(t) - B\left(\mu - Kr(t) - \frac{\gamma}{4}\right) + \frac{1}{2}B^2\cdot(\gamma r(t) + \sigma)\right.$$
$$\left. - B\cdot\frac{1}{4}\gamma\right)dt - p(t)B\sqrt{\gamma r(t) + \sigma}\, dW(t)$$

$$= p(t)\left(\frac{dA}{dt} - \frac{dB}{dt}r(t) - B\mu + BKr(t) + \frac{1}{2}B^2\gamma r(t) + \frac{1}{2}B^2\sigma\right)dt$$
$$- p(t)B\sqrt{\gamma r(t) + \sigma}\, dW(t)$$

$\therefore \frac{dp(t)}{p(t)} = \left(\left(\frac{dA}{dt} - B\mu + \frac{1}{2}B^2\sigma\right) - r(t)\left(\frac{dB}{dt} - BK - \frac{1}{2}B^2\gamma\right)\right) dt$
$- p(t) B \sqrt{\gamma r(t) + \sigma}\, dW(t)$

We can get from $z(t) = p(t) m(t)$

$dz(t) = p(t) dm(t) + m(t) dp(t) + dp(t) dm(t)$

$= p(t)m(t)(-r(t)dt) + p(t)m(t)\left(\left(\frac{dA}{dt} - \frac{dB}{dt} r(t) - B\mu + BKr(t) + \frac{1}{2}B^2\gamma r(t) + \frac{1}{2}B^2\sigma\right) dt - B\sqrt{\gamma r(t) + \sigma}\, dW(t)\right)$

$dz(t) = z(t)\left(\left(\frac{dA}{dt} - \frac{dB}{dt} r(t) - B\mu + BKr(t) + \frac{1}{2}B^2\gamma r(t) + \frac{1}{2}B^2\sigma - r(t)\right) dt - B\sqrt{\gamma r(t) + \sigma}\, dW(t)\right)$

$\frac{dz(t)}{z(t)} = \left(\frac{dA}{dt} - B\mu + \frac{1}{2}B^2\sigma + r(t)\left(-\frac{dB}{dt} + BK + \frac{1}{2}B^2\gamma - 1\right)\right) dt$
$- B\sqrt{\gamma r(t) + \sigma}\, dW(t)$

Because $z(t)$ is a martingale, dt term should be 0

$\therefore \frac{dA}{dt} - B\mu + \frac{1}{2}B^2\sigma + r(t)\left(-\frac{dB}{dt} + BK + \frac{1}{2}B^2\gamma - 1\right) = 0$

$\therefore$ A and B should satisfy:

$$\begin{cases} \frac{dA}{dt} - \mu B + \frac{1}{2}\sigma B^2 = 0 \\ \frac{dB}{dt} - KB - \frac{1}{2}\gamma B^2 + 1 = 0 \end{cases}$$

Problem 2

1. For the given set of parameters, the system of Ricatti Equation becomes:

$$\begin{cases} \frac{dA}{dt} - 0.0025 B + \frac{1}{2} \times 0.01 \times B^2 = 0 & (1) \\ \frac{dB}{dt} - 0.05 B + 1 = 0 & (2) \end{cases}$$

Because (2) changes to a first-order linear ordinary differential equation, its solution is:

$p(t) = -0.05, \quad q(t) = -1$

$$B = C_1 e^{\int -p(t)dt} + e^{\int -p(t)dt}\left(\int q(t) e^{\int p(t)dt} dt\right)$$

$$= C_1 e^{\int 0.05 dt} + e^{\int 0.05 dx}\left(\int -e^{-\int 0.05t} dt\right)$$

$$= C_1 e^{0.05t} + e^{0.05t} \cdot \frac{e^{-0.05t}}{0.05}$$

$$= C_1 e^{0.05t} + 20$$

Let it into A:

$$\frac{dA}{dt} = 0.0025B - 0.005B^2$$

$$= 0.0025(C_1 e^{0.05t} + 20) - 0.005(C_1 e^{0.05t} + 20)^2$$

$$= 0.0025C_1 e^{0.05t} + 0.05 - 0.005(C_1^2 e^{0.1t} + 40C_1 e^{0.05t} + 400)$$

$$= 0.0025C_1 e^{0.05t} + 0.05 - 0.005C_1^2 e^{0.1t} - 0.2C_1 e^{0.05t} - 2$$

$$= -0.1975C_1 e^{0.05t} - 0.005C_1^2 e^{0.1t} - 1.95$$

$$A = -\frac{0.1975}{0.05} C_1 e^{0.05t} - \frac{0.005}{0.1} C_1^2 e^{0.1t} - 1.95t + C_2$$

$$= -3.95C_1 e^{0.05t} - 0.05C_1^2 e^{0.1t} - 1.95t + C_2$$

We know that for any $r(T)$,

$$P(T,T) = e^{A(T,T) - B(T,T)r(T)} = 1$$

$$\therefore \begin{cases} A(T,T) = 0 \\ B(T,T) = 0 \end{cases}$$

$$\begin{cases} -3.95C_1 e^{0.05T} - 0.05C_1^2 e^{0.1T} - 1.95T + C_2 = 0 \\ C_1 e^{0.05T} + 20 = 0 \end{cases}$$

The result is $\begin{cases} C_1 = -20e^{-0.05T} \\ C_2 = 1.95T - 59 \end{cases}$

$$\therefore \begin{cases} A = 79e^{0.05(t-T)} - 20e^{0.1(t-T)} + 1.95(T-t) - 59 \\ B = -20e^{0.05(t-T)} + 20 \end{cases}$$

2. We can notice that $\gamma=0$, then

$$dr(t) = (\mu - K r(t))\,dt + \sqrt{\sigma}\,dW(t)$$
$$= a(b - r(t))\,dt + \sqrt{\sigma}\,dW(t)$$

becomes the Vasicek model, where $a=K$, $b=\frac{\mu}{a}$.

We can obtain:

$$r(t) = r(0)e^{-at} + b(1-e^{-at}) + \sqrt{\sigma}e^{-at}\int_0^t e^{as}\,dW(s)$$

The no-arbitrage condition:

$$m(t)P(t,T) = E_t[m(T)P(T,T)]$$

$$P(t,T) = \frac{E_t[m(T)]}{m(t)} = \frac{E_t(e^{-\int_0^T r(u)du})}{e^{-\int_0^t r(u)du}} = E_t(e^{-\int_t^T r(u)du})$$

$$= E_t(e^{-\int_t^T r(t)e^{-a(u-t)} + b(1-e^{-a(u-t)}) + \sqrt{\sigma}e^{-a(u-t)}\int_t^u e^{av}dW(v)du})$$

$$= e^{-\int_t^T r(t)e^{-a(u-t)} + b(1-e^{-a(u-t)})du}\, E_t(e^{-\sqrt{\sigma}\int_t^T e^{-a(u-t)}\int_t^u e^{av}dW(v)du})$$

$$= e^{\frac{r(t)}{a}(e^{-a(T-t)}-1) - b(T-t) - \frac{b}{a}(e^{-a(T-t)}-1)}\,E_t(e^{-\sqrt{\sigma}\int_t^T e^{-au}\int_t^u e^{av}dW(v)du})$$

Consider $\int_t^T e^{-au}\int_t^u e^{av}dW(v)\,du$,

interchange the order of integrations,

$$\int_t^T e^{-au}\int_t^u e^{av}dW(v) = \int_t^T e^{av}\int_v^T e^{-au}\,du\,dW(v)$$
$$= \int_t^T e^{av}\cdot\left(-\frac{1}{a}e^{-au}\Big|_v^T\right)dW(v)$$
$$= \frac{1}{a}\int_t^T (1-e^{a(v-T)})\,dW(v)$$

Because it is an Ito integral, it is subject to a normal distribution.

Suppose $X = -\sqrt{\sigma}\int_t^T e^{-au}\int_t^u e^{av}dW(v)du$,

$$\mu_X = E[-\sqrt{\sigma}\int_t^T e^{-au}\int_t^u e^{av}dW(v)du] = E[-\frac{\sqrt{\sigma}}{a}\int_0^{T-t}(1-e^{a(v-T)})dW(v)] = 0$$

$$\sigma_X^2 = Var[-\sqrt{\sigma}\int_t^T e^{-au}\int_t^u e^{av}dW(v)du]$$
$$= E[(-\frac{\sqrt{\sigma}}{a}\int_t^T (1-e^{a(v-T)})dW(v))^2]$$

$$= \frac{\sigma}{a^2}\int_0^{T-t}\left(1-e^{a(v-T)}\right)^2 dw$$

$$= \frac{\sigma}{a^2}(T-t) - \frac{2\sigma}{a^3} + \frac{2\sigma}{a^3}e^{a(t-T)} + \frac{\sigma}{2a^3} - \frac{\sigma}{2a^3}e^{2a(t-T)}$$

$\therefore E_t\left(e^{-\sqrt{\sigma}\int_t^T e^{-au}\int_0^{u-t} e^{av}\,dW(v)\,du}\right)$

$$= e^{\mu_x + \frac{1}{2}\sigma_x^2}$$

$$= e^{\frac{1}{2}\cdot\left(\frac{\sigma}{a^2}(T-t) - \frac{2\sigma}{a^3} + \frac{2\sigma}{a^3}e^{a(t-T)} + \frac{\sigma}{2a^3} - \frac{\sigma}{2a^3}e^{2a(t-T)}\right)}$$

$\therefore p(t,T) = \exp\Big(\frac{r(t)}{a}\left(e^{-a(T-t)}-1\right) - b(T-t) - \frac{b}{a}\left(e^{-a(T-t)}-1\right)$
$+ \frac{\sigma}{2a^2}(T-t) - \frac{\sigma}{a^3} + \frac{\sigma}{a^3}e^{a(t-T)} + \frac{\sigma}{4a^3} - \frac{\sigma}{4a^3}e^{2a(t-T)}\Big)$

$= \exp\Big(r(t)\left(\frac{e^{-a(T-t)}}{a} - \frac{1}{a}\right) + (T-t)\left(-b+\frac{\sigma}{2a^2}\right) + e^{-a(T-t)}\left(-\frac{b}{a}+\frac{\sigma}{a^3}\right)$
$+ e^{-2a(T-t)}\left(-\frac{\sigma}{4a^3}\right) + \frac{b}{a} - \frac{\sigma}{a^3} + \frac{\sigma}{4a^3}\Big)$

$= \exp\Big(r(t)\left(20e^{0.05(t-T)} - 20\right) + 79e^{0.05(t-T)} - 20e^{0.1(t-T)}$
$+ 1.95(T-t) - 59\Big)$

Problem 3.

1. From Chapter 3 of the Fisher & Gilles paper,

We know: $\quad dX(t) = (\mu - KX^2(t))dt + \sqrt{\gamma X(t)+\sigma}\,dW(t)$

$$\mu_X(X(t)) = \mu - KX^2(t)$$

$$\sigma_X(X(t)) = \sqrt{\gamma X(t)+\sigma}$$

$r(t) = R(X(t)) = X(t)$

$\because \quad m(t) = e^{-\int_0^t r(u)\,du}$

$\therefore \quad dm(t) = -e^{-\int_0^t r(u)\,du}\, r(t)\,dt = -m(t)r(t)\,dt$

$$\frac{dm(t)}{m(t)} = -r(t)\,dt$$

$\therefore \quad \Lambda(X(t)) = 0$

$M(x) = \begin{pmatrix} R(x) \\ \mu_X(x)\end{pmatrix} = \begin{pmatrix} x \\ \mu - Kx^2\end{pmatrix}$ is not affine in x.

$S(x) = \left(\Lambda(x) \quad \sigma_X(x)\right) = \left(0 \quad \sqrt{\gamma x+\sigma}\right)$

$S(X)^T S(X) = \begin{pmatrix} 0 \\ \sqrt{\gamma X+\sigma} \end{pmatrix} \begin{pmatrix} 0 & \sqrt{\gamma X+\sigma} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma X+\sigma \end{pmatrix}$ is affine in X

$\therefore$ That model $dr(t) = (\mu - Kr^2(t))dt + \sqrt{\gamma r(t)+\sigma}\, dW(t)$ is not from the Affine class.

2. Assume $P(x,\tau) = e^{-A(\tau)-B(\tau)^T x} = e^{-A(\tau)-B(\tau)x}$

For no-arbitrage condition:

$$A'(\tau) + B'(\tau)x - B(\tau)\hat{\mu}_X(x) + \tfrac{1}{2}B(\tau)\sigma_X^2(x)B(\tau) - R(X) = 0$$

$$A'(\tau) + B'(\tau)x - B(\tau)\mu_X(x) + \tfrac{1}{2}\sigma_X^2(x)B^2(\tau) - x = 0$$

$$A'(\tau) + B'(\tau)x - (\mu - Kx^2)B(\tau) + \tfrac{1}{2}(\gamma x+\sigma)B^2(\tau) - x = 0$$

$$A'(\tau) - \mu B(\tau) + \tfrac{1}{2}\sigma B^2(\tau) + x(B'(\tau) + \tfrac{1}{2}\gamma B^2(\tau) - 1) + x^2(kB(\tau)) = 0$$

For any x, if the equation above holds:

$$\begin{cases} A'(\tau) - \mu B(\tau) + \tfrac{1}{2}\sigma B^2(\tau) = 0 & ① \\ B'(\tau) + \tfrac{1}{2}\gamma B^2(\tau) - 1 = 0 & ② \\ KB(\tau) = 0 & ③ \end{cases}$$

Then if ③ holds, $B(\tau) = 0$

②: $B'(\tau) + \tfrac{1}{2}\gamma B^2(t) - 1 = 0 + \tfrac{1}{2}\cdot\gamma\cdot 0^2 - 1 = -1 \neq 0$

doesn't hold

So the solutions to the system of ODE don't exist.

$\therefore$ The assumption $P(x,\tau) = e^{-A(\tau)-B(\tau)^T x}$ is wrong, and the zero-coupon bond price $p(t,T)$ is not a solution to a system of ODEs.

Actually, from Fisher & Gilles paper, it should be a solution to a PDE:

$$R(x) = \hat{\mu}(x)^T \left(\frac{P_x(x,\tau)}{P(x,\tau)}\right) + \frac{1}{2} tr\left[\left(\frac{P_{xx}(x,\tau)}{P(x,\tau)}\right)\sigma_X^T(x)\,\sigma_X(x)\right] - \left(\frac{P_\tau(x,\tau)}{P(x,\tau)}\right)$$

where $\hat{\mu}(x) = \mu(x) - \sigma_x(x)^T \Lambda(x)$,

$R(x) = x$,

$\mu(x) = \mu - kx^2$

$\sigma_x(x) = \sqrt{\gamma x}$

$\Lambda(x) = 0$

It becomes:

$$x = (\mu - kx^2)\frac{P_x(x,\tau)}{P(x,\tau)} + \frac{1}{2}\frac{P_{xx}(x,\tau)}{P(x,\tau)} \cdot (\gamma x + \sigma) - \frac{P_\tau(x,\tau)}{P(x,\tau)}$$