AoPS Volume 2 Solutions

Mckinley Xie

29 October 2023

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§7 Polynomials

§17 Counting in the Twilight Zone

Exercise 17.10. In how many ways can we get a sum of 25 when 10 distinct dice are rolled?

Solution. We will spam genfuncs.

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)^{10}$$
$$= \left(\frac{1 - x^7}{1 - x}\right)^{10}$$
$$= \left(1 - x^7\right)^{10} \left(1 - x\right)^{-10}$$
$$= \left[\binom{10}{0} - \binom{10}{1}x^7 + \cdots + \binom{10}{10}x^{70}\right]\left[\binom{-10}{0} - \binom{-10}{1}x + \cdots\right]$$

The contributing terms are

$\square$

Problem 293. Six people sat down along one side of a banquet table completely ignoring their name cards. In how many ways could this have been done so that no person was seated where his/her name card was placed?

Solution. Casework bash is actually feasible! The cases will be the number (and size of loops).

Case 1: 1 C_6: There are 5! ways.
Case 2: 3 C_2: There are $5 \cdot \binom{4}{2}$ ways.
Case 3: 1 C_4 + 1 C_2: There are $\binom{6}{2} \cdot 3!$ ways.
Case 4: 2 C_3: There are $\binom{5}{2} \cdot 2! \cdot 2!$ ways.
The sum is $120 + 30 + 90 + 40 = \boxed{260}$. $\square$

(This is wrong, the answer should be 265.)

§20 Find It and Make It

Exercise 20.1. Given lines l and m which intersect at O, prove that the locus of all points C such that the distnaces from C to l and m are the same is the pair of angle bisectors of the angles formed by l and m at O.

Solution. Equidistance $\implies$ Angle bisector by HL congruence,
Angle bisector $\implies$ Equidistance by sines. $\square$

§23 Number Theory

Exercise 23.1. Show that if $a_1 \mid (a, b)$, then $\frac{b}{(a,b)} \mid \frac{b}{a_1}$

Solution. (a, b) is a multiple of a_1, so it's pretty clear. $\square$

Exercise 23.2. Compare our $2 \equiv 20 \pmod{6}$ example to equation (23.2)

Solution. It just states $1 \equiv 10 \pmod{3}$ $\square$

Exercise 23.3. Divide out the common factors in the following congurences:

1. $6a \equiv 6b \pmod{20}$
2. $23 \equiv 138 \pmod{5}$
3. $12 \equiv 30 \pmod{9}$

Solution. Just regurgitate what we learned in the lesson:

1. $a \equiv b \pmod{10}$
2. $1 \equiv 6 \pmod{5}$
3. $2 \equiv 5 \pmod{3}$

     $\square$

Exercise 23.4. Solve the congurences.

1. $1235x + 45 \equiv 9090 \pmod{24}$
2. $1235x + 45 \equiv 9090 \pmod{11}$
3. $1235x + 45 \equiv 9087 \pmod{11}$
4. $1232x + 45 \equiv 9090 \pmod{24}$

Solution. Nothing special to see here.

1.

$$1235x + 45 \equiv 9090 \pmod{24}$$
$$1235x \equiv 9045$$
$$247x \equiv 1809$$
$$7x \equiv 9$$
$$7x \equiv 105$$
$$x \equiv 15 \pmod{24}$$

2.

$$1235x + 45 \equiv 9090 \pmod{11}$$
$$247x \equiv 1809$$
$$5x \equiv 5$$
$$x \equiv 1 \pmod{11}$$

3.

$$1235x + 45 \equiv 9087 \pmod{11}$$
$$3x + 1 \equiv 1$$
$$x \equiv 0 \pmod{11}$$

4.

$$1232x + 45 \equiv 9090 \pmod{24}$$
$$1232x \equiv 9045$$
$$8x \equiv 21 \pmod{24}$$

So there is no solution.

> **Remark.** You can also just take everything mod 2 to get $1 \equiv 0 \pmod 2$

$\square$

Exercise 23.5. Solve simultaneously the three congruences $3x \equiv 4 \pmod 7$, $4x \equiv 5 \pmod 8$, and $5x \equiv 6 \pmod 9$

Solution. In the first equation, $x \equiv 6 \pmod 7$.
In the second, $4x \equiv 5 \pmod 8$ is impossible.
Oops. $\square$

Definition. A **quadratic residue** $\pmod m$ is a number n such that $\exists i$ such that $i^2 \equiv n \pmod m$

Exercise 23.6. Find all quadratic residues in mod 7, 8, and 9.

Solution. We're lazy so we'll just do mod 7. Well, we have $0^2, 1^2, 2^2$, and $3^2 \pmod 7$, so $\boxed{0, 1, 2, \text{ and } 4}$. $\square$

Exercise 23.7. What is the most quadratic residues there can be $\pmod{n}$ for $n = 2m + 1$?

Solution. Because every number k has the same (residue?) as $n - k$, our answer is $m + 1 + 1 = \boxed{m + 2}$ (since 0 goes to itself.) □

Exercise 23.10. Write down and expand the product for $n = 28$

Solution. We have $(1 + 2 + 4)(1 + 7) = 56$ □

Exercise 23.11. Why does this work?

Solution. It's like a genfunc! □

Exercise 23.12. Make the product simpler with sum of geometric series

Solution.

$$\prod_{i=0}^{k} \frac{p_i^{e_i+1} - 1}{p_i - 1}$$

Honestly this isn't much simpler. □

Exercise 23.13. These are pretty obviously correct, but I'm not particularly fond of either since they're not very helpful.

Exercise 23.15. Show that any perfect number of the form

$$2^k \left(2^{k+1} - 1\right)$$

is perfect if $(2^{k+1} - 1)$ is prime.

This was discovered by Euclid, and *all known perfect numbers have this form.* (In particular, no odd perfect numbers have ever been found.)

Solution. By our handy-dandy formula, the sum of divisors is equal to

$$(2^{k+1} - 1)(2^{k+1} - 1 + 1)$$

Which is twice our original number, and we're done. □

Exercise 23.19. Find $6^{1000} \pmod{23}$

Solution. $6^{22} \equiv 1 \pmod{23}$, so $6^{990} \equiv 1 \pmod{23}$. We then bash out $6^{10} \pmod{23}$ and find that it's equal to $\boxed{4}$ □

Exercise 23.20. Find all possible periods a number can have $\pmod{23}$

Solution. Well, a^{22} must be 1, so the period length must be 1, 2, 11, or 22. □

Definition. A **primitive root** $\pmod{p}$ is a number g with period $p - 1$.

Exercise 23.21. Let the divisors of $p - 1$ be $d_1, d_2, \ldots$ Prove that if we have a primitive root g $\pmod{p}$, then for each d_i there is an element with period d_i.

Solution. If g has period $p - 1$, then $g^{\frac{p-1}{d_i}}$ has period d_i. □

Exercise 23.24. It's just p^{k-1}

Exercise 23.27. We get $(m - 1)! \equiv 0 \pmod{a}$

Exercise 23.28. Look at the proof of Wilson's theorem

Proof. Consider some primitive root $g \pmod{p}$. By the definition of a primitive root, $\{g^k \mid k \in [p-1]\} = [p-1]$. So

$$(p - 1)! \equiv g^{\frac{p(p-1)}{2}} \pmod{p}$$

But because $g^p \equiv g\,(g^{p-1}) \equiv g \pmod{p}$, $g^{\frac{p(p-1)}{2}} \equiv (g^p)^{\frac{p-1}{2}} \equiv g^{\frac{p-1}{2}} \pmod{p}$. Let $t = g^{\frac{p-1}{2}}$. Note that
$$t^2 = g^{p-1} \equiv 1 \pmod{p}$$
So $t^2 - 1 \equiv 0 \pmod{p}$, and $t \equiv \pm 1 \pmod{p}$
But because $t = g^{\frac{p-1}{2}}$, we can't have $t \equiv 1$ because the period of g is $p - 1$ (and $\frac{p-1}{2} < p - 1$), so $t \equiv -1$, and $(p - 1)! \equiv -1 \pmod{p}$ □

Problem 373. Show that for all prime numbers p greater than 3, 24 divides $p^2 - 1$ evenly.

Proof. $p^2 - 1 = (p+1)(p-1)$. If p is prime then $p \equiv 1$ or $p \equiv 3 \pmod 4$ so $(p+1)(p-1)$ is divisible by 8. Similarly, $p \equiv 1$ or $2 \pmod 3$ so one of the factors of $p^2 - 1$ is divisible by 3, and we're done. □

Problem 374. Given that $n - 4$ is divisible by 5, list which of the following are also divisible by 5:
$$n^2 - 1, n^2 - 4, n^2 - 16, n + 4, n^4 - 1$$

(Mandelbrot #3)

Solution. Well, just take everything mod 5.
$n \equiv 1 \pmod 5$ so $n^2 - 1$, $n^2 - 16$, $n + 4$, and $n^4 - 1$ are all divisible by 5. □

Problem 375. If the same number r is the remainder when each of the numbers 1059, 1417, and 2312 is divided by d, where d is an integer greater than one, find $d - r$.

Solution. Two pairwise differences between these numbers are 358 and 895. These are both divisible by 179, so d is 179. Plugging in, r is 164, so our answer is $\boxed{15}$ □

Problem 376. Find the sum of all x, $1 \le x \le 100$, such that 7 divides $x^2 + 15x + 1$. (Mandelbrot #3)

Solution. The condition is equivalent to $x^2 + x + 1$. We just test all numbers $\pmod 7$. 1 doesn't work, 2 does, 3 doesn't, 4 does, 5 doesn't, and 6 doesn't.
We can just bash this out. □

Problem 377. Find the largest integer divisor of $n^5 - n$

Solution. This expression is equal to $n(n^2 + 1)(n + 1)(n - 1)$.

Note that if $n = 2$ then our expression is equal to 30, so our largest integer must be a divisor of 30.

> **Claim** — 30 always divides $n^5 - n$ (for $n \in \mathbb{Z}$).

Clearly one of n and $n - 1$ are divisible by 2, so our expression is always divisible by 2.

Similarly, one of $n - 1$, n, and $n + 1$ are divisible by 3.

Now, suppose that this expression is not divisible by 5 for some n. That implies that each of $n - 1, n, n + 1 \not\equiv 0 \pmod 5$. That means that n must be either 2 or 3 $\pmod 5$. But $2^2 + 1 \equiv 3^2 + 1 \equiv 5 \pmod 5$, which is a contradiction since n is not divisible by 5.

Because $2, 3, 5$ all divide $n^5 - n$ then $\mid n^5 - n$ and we are done. $\qquad\square$

> **Remark 23.29.** Another solution could be to use induction and show that
>
> $$30 \mid ((n + 1)^5 - (n + 1)) - (n^5 - n)$$
>
> However this is terribly messy and sad.

Problem 378. What is the units digit of $7^{\left(7^7\right)}$?

Solution. By Fermat's theorem we know that $7^4 \equiv 1 \pmod{10}$.

Now we need to find $7^7 \pmod 4$.

Well, we know that $7^2 \equiv 1 \pmod 4$ so $7^7 \equiv 3 \pmod 4$

So our answer is $\boxed{3}$. $\qquad\square$

Problem 379. What is the size of the largest subset S of $[50]$ such that no pair of distinct elements of S has a sum divisible by 7?

Solution. Clearly if we have a number $a \in S$, we cannot have $b \equiv -a$, where $b \in S$.

So we just take all $x \equiv 1, 2, 3 \pmod 7$.

We can also throw in a 7, so our answer is $22 + 1 = \boxed{23}$ $\qquad\square$

Problem 381. For any integer n greater than 1, how many prime numbers are there greater than $n! + 1$ and less than $n! + n$?

Solution. $k \mid n! + k$ for $k \leq n$, so our answer is $\boxed{0}$. $\qquad\square$

Problem 382. Find the last three digits of 9^{105}.

Solution. This is equivalent to 3^{210}.

$3^{400} \equiv 1 \pmod{1000}$ so $3^{200} \equiv -1 \pmod{1000}$.

We then bash out the last three digits of 3^{10}, which are 049, so our answer is $\boxed{951}$

$\qquad\square$

Problem 383. What is the least possible value of n such that

$$\sqrt{\frac{3}{1} \cdot \frac{4}{2} \cdot \frac{5}{3} \cdots \frac{n + 2}{n}}$$

is an integer?

Solution. This just cancels out to

$$\sqrt{\frac{(n+2)\cdot(n+1)}{2}}$$

Assuming $n \in \mathbb{Z}$, then one factor is a perfect square while the other is twice a perfect square. We test small perfect squares until we stumble upon $\boxed{n=7}$ $\square$

Problem 387. Let x and y be integers such that $2x+3y$ is a multiple of 17. Show that $9x+5y$ must also be a multiple of 17. (USAMTS 1)

Solution. Note first that $-4(2x+3y) \equiv -8x - 12y \equiv 0 \pmod{17}$.
So $17x - 8x + 17y - 12y \equiv 0 \pmod{17}$, and the rest is trivial. $\square$

Problem 388. Note that 1990 can be "turned into a square" by adding a digit on its right, and some digits on its left, i.e., $419904 = 648^2$. Prove that 1991 cannot be turned into a square by the same procedure; i.e., there are no digits $d, x, y, \dots$ such that $\dots yx1991d$ is a perfect square. (USAMTS 3)

Solution. Suppose there exists a number whose square satisfies the desired property. Let its last two digits be a and b. Clearly the second-to-last digit of $20ab + b^2$ must be a 1. That implies that the tens digit of b^2 is odd, so $b = 4$ or $b = 6$. So we need to find a such that $8a + 1 \equiv 1 \pmod{10}$ or $12a + 3 \equiv 1 \pmod{10}$, so we have $10a + b =$ 04, 46, 54, or 96. Uhh I dunno COME BACK HERE LATER $\square$

Problem 393. Let n be an integer. If the tens digit of n^2 is 7, what is the units digit of n^2?

Solution. Consider $n^2 \pmod{100}$.
We want to find $10a + b$ such that $2ab + (\text{the tens digit of } b^2) = 7$.
Clearly the tens digit of b^2 must be odd, so $b = 4$ or $b = 6$, so our units digit must be $\boxed{6}$. $\square$

Problem 394. Prove that none of the numbers $a_n = 1001001\cdots 1001$ is prime, where $n = 2, 3, 4\dots$ denotes the number of occurrences of the digit 1 in a_n.

Solution. Define a function $f(x, n)$ as

$$f(x, n) = \sum_{k=0}^{n-1} x^k = \frac{x^{k+1} - 1}{x - 1}$$

Conveniently, $a_n = f(1000, n) = \frac{10^{3n+1} - 1}{(10-1)(111)}$.
Now, consider only odd n, since $2 \mid n \implies a_2 \mid a_n$.
If n is odd then $a_n = \frac{1000^{n+1} - 1}{999} = \frac{(1000^{\frac{n+1}{2}} + 1)(1000^{\frac{n+1}{2}} - 1)}{999}$, and for $n > 1$ this is clearly not prime, and we're done. $\square$

> **Remark.** I hate my proof. Hopefully there's something better.

Problem 395. Let p be a prime number. Prove that there exists an integer a such that
$p \mid a^2 - a + 3 \iff \exists b$ such that $p \mid b^2 - b + 25$.

Solution. The problem statement is equivalent to proving that
$\exists a$ such that $a(a - 1) \equiv -3 \pmod{p} \iff \exists b$ such that $b(b - 1) \equiv -25 \pmod{p}$
I'm not sure how to progress. $\qquad\square$

Problem 396. Each of the numbers $x_1, x_2, \ldots x_n$ equals 1 or -1, and

$$x_1 x_2 x_3 x_4 + x_2 x_3 x_4 x_5 + \cdots + x_{n-2} x_{n-1} x_n x_1 + x_{n-1} x_n x_1 x_2 + x_n x_1 x_2 x_3 = 0$$

Prove that $4 \mid n$.

Solution. Let term number k be y_k. So each y_i is either 1 or -1, and the sum of all
$y_1 + \cdots + y_n = 0 \implies 2 \mid n$. Let $n = 2m$. $y_1 y_2 \cdots y_n = (-1)^m$, but each x_j appears 4
times in this product, so $2 \mid m$ and we are done. $\qquad\square$

Problem 398. Find the positive integer m such that the polynomial $p^3 + 2p + m$ divides
$p^{12} - p^{11} + 3p^{10} + 11p^3 - p^2 + 23p + 10$.

Solution. We plug in 0 and find that $m \mid 30$.
We plug in 1 and find that $m + 3 \mid 66$, so $m = 3, 8, 19, 30, 63$
So clearly $m = 3$ or $m = 30$. We plug in $p = 2$ to get $2048 + 3072 + 88 - 4 + 46 + 30 = 5120 + 84 + 46 + 30 = 5280$, and our possible divisors are 15 or 42.
42 doesn't work so our answer is $\boxed{m = 3}$. $\qquad\square$

Problem 399. Prove that, for all positive integer pairs (a, b) where $b > 2, 2^b - 1$ does
not evenly divide $2^a + 1$.

Solution. Suppose $\exists k \in \mathbb{Z}$ such that $k(2^b - 1) = 2^a + 1$.
Note that $2^{b-a}(2^a + 1) = 2^b + 2^{b-a} > 2^b - 1$, so $k < 2^{b-a}$.
But $(2^{b-a} - 1)(2^a + 1) = 2^b - 2^a - 1 < 2^b - 1$, so $k > 2^{b-a} - 1$
This is absurd since k cannot be between two integers, so we're done.

$\qquad\square$

Problem 400. Let d be any positive integer not equal to 2, 5, or 13. Show that one can
find distinct (a, b) in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

Solution. Suppose there exist positive integers x, y, z such that

$$x^2 = 2d + 1$$
$$y^2 = 5d + 1$$
$$z^2 = 13d + 1$$

Then $y^2 - x^2 = 3d$, $z^2 - y^2 = 8d$, and $z^2 - x^2 = 11d$. Clearly x is odd, so $x^2 \equiv 1 \pmod{8}$,
and $2d \equiv x^2 + 1 \equiv 2 \pmod{8}$, so d is odd. This forces y and z to be even, so let $y = 2u$
and $z = 2v$. Since $z^2 - y^2 = 8d$, we have $(v + u)(v - u) = 2d$. But because $u + v$ and
$u - v$ must have the same parity, then both must be even, so $4 \mid 2d \implies 2 \mid d$, which is
a contradiction.

$\qquad\square$

Problem 401. Let a and b be integers and n a positive integer. Prove that

$$n! \mid b^{n-1}a(a+b)(a+2b)\cdots(a+(n-1)b)$$

Solution. We will use a proof by contradiction.

Let $x = b^{n-1}a(a+b)(a+2b)\cdots(a+(n-1)b)$

Suppose that, for some prime p, $v_p(x) < v_p(n!)$ (where $v_p(x)$ is the greatest integer n such that $p^n \mid x$).

> **Claim —** $(b, p) = 1$.

Proof. Note that $v_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor \cdots < \left\lfloor \frac{n}{p} + \frac{n}{p^2} + \cdots \right\rfloor = \left\lfloor \frac{n}{p-1} \right\rfloor$

Because $b^{n-1} \mid x$, and $n - 1 > \left\lfloor \frac{n}{p-1} \right\rfloor$, $(b, p) = 1$. $\square$

But then $v_p(n!) = v_p(x)$ since both are equal to

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor \cdots$$

and we are done. $\square$

Problem 402. Prove that a positive integer is a sum of at least two consecutive positive integers if and only if it is not a power of two.

Solution. For some $n = 2^a \cdot b$ where b is odd, we split this up into two cases.

Case 1. $2^a \geq \frac{b+1}{2}$ (and $b > 1$)
We just take 2^a, the $\frac{b-1}{2}$ numbers below it, and the $\frac{b-1}{2}$ numbers above it. Because $2^a > \frac{b+1}{2}$ all of these numbers are positive, and there are $b > 1$ terms in this sequence.

Case 2. $2^a < \frac{b+1}{2}$
Consider the 2^a consecutive integers ending at $\frac{b-1}{2}$ and the 2^a consecutive integers beginning at $\frac{b+1}{2}$. The sum of these integers is obviously $2^a b = n$, and the minimum integer is $\frac{b+1}{2} - 2^a \geq 1$, so we're done. $\square$

Problem 437. If $\ln x^4 = (\ln x)^3$, find x.

Solution. Well, that implies that $4\ln x = (\ln x)^3$, so either $\ln x = 2$ or $\ln x = 0$, or $x = e^2$ or $x = 1$. $\square$

Problem 438. Given A, a set of 13 (distinct elements), what fraction of the functions from A to A have a well-defined inverse function?

Solution. A must be injective, so in this case it's bijective. That means that there are 13! ways. $\square$

Problem 439. With $1000 a racher is to buy steers at $25 each and cows at $26 each. If he has no moneyleft over, and he bought at least one of each animal, how many cows did he buy?

Solution. 25 cows works. $\square$

Problem 441 (ARML 1986)**.** If x is real, compute the maximum integral value of
$$\frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}.$$

Solution. Well, we want to minimize the value of $3x^2 + 9x + 7$ (while keeping it above zero). Plugging in $x = -\frac{3}{2}$ gives us $\frac{\frac{81}{4}}{\frac{41}{4}} = \frac{81}{41}$. □

Problem 505 (Canada 1984)**.** Given any 7 real numbers, prove that there are two of them, say x and y, such that
$$0 \le \frac{x - y}{1 + xy} \le \frac{1}{\sqrt{3}}.$$

Solution. Note that $\frac{x-y}{1+xy} = \tan(\arctan x - \arctan y)$. Since the domain of arctan is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, by pigeonhole there must exist x, y such that $0 \le \arctan x - \arctan y < \frac{\pi}{6}$, which implies that $0 \le \tan(\arctan x - \arctan y) < \frac{1}{\sqrt{3}}$, and we are done. □

Problem 506 (Mandlebrot #1)**.** In acute triangle ABC, we are given $AD \perp BC, DF \perp AB$, and $DE \perp AC$. The circumradius of a triangle is the radius of the circle circumscribed about the triangle. Find $\angle A$ if the product of the circumradii of $\triangle ABC$ and $\triangle AEF$ is the area of $\triangle ABC$.

Solution. AD must be the diameter of the circumcircle of $\triangle AEF$, so the circumradius of $\triangle AEF$ is one half of AD.

That means that the circumradius of $\triangle ABC$ must be equal to BC, which implies that $A = \frac{\pi}{3}$. □

Problem 507. An "unfair" coin has a 2/3 probability of turning up heads. If this coin is tossed 50 times, what is the probability that the total number of heads is even?

Solution. With generating functions and a roots of unity filter we find that the answer is
$$\boxed{\frac{1 + 3^{-50}}{2}}$$
□