

AoPS Volume 2 Solutions

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August 22, 2021

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23 Number Theory 1

§23 Number Theory

Exercise 23.1. Show that if $a_1 \mid (a, b)$, then $\frac{b}{(a, b)} \mid \frac{b}{a_1}$

Solution. (a, b) is a multiple of a_1 , so it's pretty clear. □

Exercise 23.2. Compare our $2 \equiv 20 \pmod{6}$ example to equation (23.2)

Solution. It just states $1 \equiv 10 \pmod{3}$ □

Exercise 23.3. Divide out the common factors in the following congruences:

1. $6a \equiv 6b \pmod{20}$
2. $23 \equiv 138 \pmod{5}$
3. $12 \equiv 30 \pmod{9}$

Solution. Just regurgitate what we learned in the lesson:

1. $a \equiv b \pmod{10}$
2. $1 \equiv 6 \pmod{5}$
3. $2 \equiv 5 \pmod{3}$

□

Exercise 23.4. Solve the congruences.

1. $1235x + 45 \equiv 9090 \pmod{24}$
2. $1235x + 45 \equiv 9090 \pmod{11}$
3. $1235x + 45 \equiv 9087 \pmod{11}$
4. $1232x + 45 \equiv 9090 \pmod{24}$

Solution. Nothing special to see here.

1.

$$1235x + 45 \equiv 9090 \pmod{24}$$

$$1235x \equiv 9045$$

$$247x \equiv 1809$$

$$7x \equiv 9$$

$$7x \equiv 105$$

$$x \equiv 15 \pmod{24}$$

2.

$$1235x + 45 \equiv 9090 \pmod{11}$$

$$247x \equiv 1809$$

$$5x \equiv 5$$

$$x \equiv 1 \pmod{11}$$

3.

$$1235x + 45 \equiv 9087 \pmod{11}$$

$$3x + 1 \equiv 1$$

$$x \equiv 0 \pmod{11}$$

4.

$$1232x + 45 \equiv 9090 \pmod{24}$$

$$1232x \equiv 9045$$

$$8x \equiv 21 \pmod{24}$$

So there is no solution.

Remark. You can also just take everything mod 2 to get $1 \equiv 0 \pmod{2}$

□

Exercise 23.5. Solve simultaneously the three congruences $3x \equiv 4 \pmod{7}$, $4x \equiv 5 \pmod{8}$, and $5x \equiv 6 \pmod{9}$

Solution. In the first equation, $x \equiv 6 \pmod{7}$.

In the second, $4x \equiv 5 \pmod{8}$ is impossible.

Oops.

□

Definition. A **quadratic residue** \pmod{m} is a number n such that $\exists i$ such that $i^2 \equiv n \pmod{m}$

Exercise 23.6. Find all quadratic residues in mod 7, 8, and 9.

Solution. We're lazy so we'll just do mod 7. Well, we have $0^2, 1^2, 2^2$, and $3^2 \pmod{7}$, so 0, 1, 2, and 4.

□

Exercise 23.7. What is the most quadratic residues there can be \pmod{n} for $n = 2m + 1$?

Solution. Because every number k has the same (residue?) as $n - k$, our answer is $m + 1 + 1 = \boxed{m + 2}$ (since 0 goes to itself.) \square

Exercise 23.10. Write down and expand the product for $n = 28$

Solution. We have $(1 + 2 + 4)(1 + 7) = 56$ \square

Exercise 23.11. Why does this work?

Solution. It's like a genfunc! \square

Exercise 23.12. Make the product simpler with sum of geometric series

Solution.

$$\prod_{i=0}^k \frac{p_i^{e_i+1} - 1}{p_i - 1}$$

Honestly this isn't much simpler. \square

Exercise 23.13. These are pretty obviously correct, but I'm not particularly fond of either since they're not very helpful.

Exercise 23.15. Show that any perfect number of the form

$$2^k (2^{k+1} - 1)$$

is perfect if $(2^{k+1} - 1)$ is prime.

This was discovered by Euclid, and *all known perfect numbers have this form.* (In particular, no odd perfect numbers have ever been found.)

Solution. By our handy-dandy formula, the sum of divisors is equal to

$$(2^{k+1} - 1)(2^{k+1} - 1 + 1)$$

Which is twice our original number, and we're done. \square

Exercise 23.19. Find $6^{1000} \pmod{23}$

Solution. $6^{22} \equiv 1 \pmod{23}$, so $6^{990} \equiv 1 \pmod{23}$. We then bash out $6^{10} \pmod{23}$ and find that it's equal to $\boxed{4}$ \square

Exercise 23.20. Find all possible periods a number can have $\pmod{23}$

Solution. Well, a^{22} must be 1, so the period length must be 1, 2, 11, or 22. \square

Definition. A **primitive root** \pmod{p} is a number g with period $p - 1$.

Exercise 23.21. Let the divisors of $p - 1$ be d_1, d_2, \dots . Prove that if we have a primitive root $g \pmod{p}$, then for each d_i there is an element with period d_i .

Solution. If g has period $p - 1$, then $g^{\frac{p-1}{d_i}}$ has period d_i . \square

Exercise 23.24. It's just p^{k-1}

Exercise 23.27. We get $(m - 1)! \equiv 0 \pmod{a}$

Exercise 23.28. Look at the proof of Wilson's theorem

Proof. Consider some primitive root $g \pmod{p}$. By the definition of a primitive root, $\{g^k \mid k \in [p-1]\} = [p-1]$. So

$$(p-1)! \equiv g^{\frac{p(p-1)}{2}} \pmod{p}$$

But because $g^p \equiv g(g^{p-1}) \equiv g \pmod{p}$, $g^{\frac{p(p-1)}{2}} \equiv (g^p)^{\frac{p-1}{2}} \equiv g^{\frac{p-1}{2}} \pmod{p}$. Let $t = g^{\frac{p-1}{2}}$. Note that

$$t^2 = g^{p-1} \equiv 1 \pmod{p}$$

So $t^2 - 1 \equiv 0 \pmod{p}$, and $t \equiv \pm 1 \pmod{p}$

But because $t = g^{\frac{p-1}{2}}$, we can't have $t \equiv 1$ because the period of g is $p-1$ (and $\frac{p-1}{2} < p-1$), so $t \equiv -1$, and $(p-1)! \equiv -1 \pmod{p}$ \square

Problem 373. Show that for all prime numbers p greater than 3, 24 divides $p^2 - 1$ evenly.

Proof. $p^2 - 1 = (p+1)(p-1)$. If p is prime then $p \equiv 1$ or $p \equiv 3 \pmod{4}$ so $(p+1)(p-1)$ is divisible by 8. Similarly, $p \equiv 1$ or $2 \pmod{3}$ so one of the factors of $p^2 - 1$ is divisible by 3, and we're done. \square

Problem 374. Given that $n - 4$ is divisible by 5, list which of the following are also divisible by 5:

$$n^2 - 1, n^2 - 4, n^2 - 16, n + 4, n^4 - 1$$

(Mandelbrot #3)

Solution. Well, just take everything mod 5.

$n \equiv 1 \pmod{5}$ so $n^2 - 1$, $n^2 - 16$, $n + 4$, and $n^4 - 1$ are all divisible by 5. \square

Problem 375. If the same number r is the remainder when each of the numbers 1059, 1417, and 2312 is divided by d , where d is an integer greater than one, find $d - r$.

Solution. Two pairwise differences between these numbers are 358 and 895. These are both divisible by 179, so d is 179. Plugging in, r is 164, so our answer is 15 \square

Problem 376. Find the sum of all x , $1 \leq x \leq 100$, such that 7 divides $x^2 + 15x + 1$. (Mandelbrot #3)

Solution. The condition is equivalent to $x^2 + x + 1 \pmod{7}$. We just test all numbers $\pmod{7}$. 1 doesn't work, 2 does, 3 doesn't, 4 does, 5 doesn't, and 6 doesn't.

We can just bash this out. \square

Problem 377. Find the largest integer divisor of $n^5 - n$

Solution. This expression is equal to $n(n^2 + 1)(n + 1)(n - 1)$.

Note that if $n = 2$ then our expression is equal to 30, so our largest integer must be a divisor of 30.

Claim — 30 always divides $n^5 - n$ (for $n \in \mathbb{Z}$).

Clearly one of n and $n - 1$ are divisible by 2, so our expression is always divisible by 2. Similarly, one of $n - 1$, n , and $n + 1$ are divisible by 3.

Now, suppose that this expression is not divisible by 5 for some n . That implies that each of $n - 1, n, n + 1 \not\equiv 0 \pmod{5}$. That means that n must be either 2 or 3 $\pmod{5}$. But $2^2 + 1 \equiv 3^2 + 1 \equiv 5 \pmod{5}$, which is a contradiction since n is not divisible by 5. Because 2, 3, 5 all divide $n^5 - n$ then $30 \mid n^5 - n$ and we are done. \square

Remark 23.29. Another solution could be to use induction and show that

$$30 \mid ((n+1)^5 - (n+1)) - (n^5 - n)$$

However this is terribly messy and sad.

Problem 378. What is the units digit of $7^{(7^7)}$?

Solution. By Fermat's theorem we know that $7^4 \equiv 1 \pmod{10}$.

Now we need to find $7^7 \pmod{4}$.

Well, we know that $7^2 \equiv 1 \pmod{4}$ so $7^7 \equiv 3 \pmod{4}$

So our answer is $\boxed{3}$. □

Problem 379. What is the size of the largest subset S of $[50]$ such that no pair of distinct elements of S has a sum divisible by 7?

Solution. Clearly if we have a number $a \in S$, we cannot have $b \equiv -a$, where $b \in S$.

So we just take all $x \equiv 1, 2, 3 \pmod{7}$.

We can also throw in a 7, so our answer is $22 + 1 = \boxed{23}$ □

Problem 381. For any integer n greater than 1, how many prime numbers are there greater than $n! + 1$ and less than $n! + n$?

Solution. $k \mid n! + k$ for $k \leq n$, so our answer is $\boxed{0}$. □

Problem 382. Find the last three digits of 9^{10^5} .

Solution. This is equivalent to $3^{2^{10}}$.

$3^{400} \equiv 1 \pmod{1000}$ so $3^{200} \equiv -1 \pmod{1000}$.

We then bash out the last three digits of 3^{10} , which are 049, so our answer is $\boxed{951}$ □

Problem 383. What is the least possible value of n such that

$$\sqrt{\frac{3}{1} \cdot \frac{4}{2} \cdot \frac{5}{3} \cdots \frac{n+2}{n}}$$

is an integer?

Solution. This just cancels out to

$$\sqrt{\frac{(n+2) \cdot (n+1)}{2}}$$

Assuming $n \in \mathbb{Z}$, then one factor is a perfect square while the other is twice a perfect square. We test small perfect squares until we stumble upon $\boxed{n = 7}$ □

Problem 387. Let x and y be integers such that $2x + 3y$ is a multiple of 17. Show that $9x + 5y$ must also be a multiple of 17. (USAMTS 1)

Solution. Note first that $-4(2x + 3y) \equiv -8x - 12y \equiv 0 \pmod{17}$.

So $17x - 8x + 17y - 12y \equiv 0 \pmod{17}$, and the rest is trivial. □

Problem 388. Note that 1990 can be “turned into a square” by adding a digit on its right, and some digits on its left, i.e., $419904 = 648^2$. Prove that 1991 cannot be turned into a square by the same procedure; i.e., there are no digits d, x, y, \dots such that $\dots yx1991d$ is a perfect square. (USAMTS 3)

Solution. Suppose there exists a number whose square satisfies the desired property. Let its last two digits be a and b . Clearly the second-to-last digit of $20ab + b^2$ must be a 1. That implies that the tens digit of b^2 is odd, so $b = 4$ or $b = 6$. So we need to find a such that $8a + 1 \equiv 1 \pmod{10}$ or $12a + 3 \equiv 1 \pmod{10}$, so we have $10a + b = 04, 46, 54, \text{ or } 96$. Uhh I dunno COME BACK HERE LATER \square

Problem 393. Let n be an integer. If the tens digit of n^2 is 7, what is the units digit of n^2 ?

Solution. Consider $n^2 \pmod{100}$.

We want to find $10a + b$ such that $2ab + (\text{the tens digit of } b^2) = 7$.

Clearly the tens digit of b^2 must be odd, so $b = 4$ or $b = 6$, so our units digit must be $\boxed{6}$. \square

Problem 394. Prove that none of the numbers $a_n = 1001001 \cdots 1001$ is prime, where $n = 2, 3, 4, \dots$ denotes the number of occurrences of the digit 1 in a_n .

Solution. Define a function $f(x, n)$ as

$$f(x, n) = \sum_{k=0}^{n-1} x^k = \frac{x^{n+1} - 1}{x - 1}$$

Conveniently, $a_n = f(1000, n) = \frac{10^{3n+1} - 1}{(10-1)(111)}$.

Now, consider only odd n , since $2 \mid n \implies a_2 \mid a_n$.

If n is odd then $a_n = \frac{1000^{n+1} - 1}{999} = \frac{(1000^{\frac{n+1}{2}} + 1)(1000^{\frac{n+1}{2}} - 1)}{999}$, and for $n > 1$ this is clearly not prime, and we're done. \square

Remark. I hate my proof. Hopefully there's something better.

Problem 395. Let p be a prime number. Prove that there exists an integer a such that $p \mid a^2 - a + 3 \iff \exists b$ such that $p \mid b^2 - b + 25$.

Solution. The problem statement is equivalent to proving that

$\exists a$ such that $a(a-1) \equiv -3 \pmod{p} \iff \exists b$ such that $b(b-1) \equiv -25 \pmod{p}$

I'm not sure how to progress. \square

Problem 396. Each of the numbers x_1, x_2, \dots, x_n equals 1 or -1, and

$$x_1x_2x_3x_4 + x_2x_3x_4x_5 + \cdots + x_{n-2}x_{n-1}x_nx_1 + x_{n-1}x_nx_1x_2 + x_nx_1x_2x_3 = 0$$

Prove that $4 \mid n$.

Solution. Split each term up into $x_i x_{i+1}$ and $x_{i+2} x_{i+3}$. Note that \square

Problem 398. Find the positive integer m such that the polynomial $p^3 + 2p + m$ divides $p^{12} - p^{11} + 3p^{10} + 11p^3 - p^2 + 23p + 10$.

Solution. We plug in 0 and find that $m \mid 30$.

We plug in 1 and find that $m + 3 \mid 66$, so $m = 3, 8, 19, 30, 63$

So clearly $m = 3$ or $m = 30$. We plug in $p = 2$ to get $2048 + 3072 + 88 - 4 + 46 + 30 = 5120 + 84 + 46 + 30 = 5280$, and our possible divisors are 15 or 42.

42 doesn't work so our answer is $\boxed{m = 3}$. \square

Problem 399. Prove that, for all positive integer pairs (a, b) where $b > 2$, $2^b - 1$ does not evenly divide $2^a + 1$.

Solution. Suppose $\exists k \in \mathbb{Z}$ such that $k(2^b - 1) = 2^a + 1$.

Note that $2^{b-a}(2^a + 1) = 2^b + 2^{b-a} > 2^b - 1$, so $k < 2^{b-a}$.

But $(2^{b-a} - 1)(2^a + 1) = 2^b - 2^a - 1 < 2^b - 1$, so $k > 2^{b-a} - 1$.

This is absurd since k cannot be between two integers, so we're done. \square

Problem 400. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct (a, b) in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

Solution. Suppose there exist positive integers x, y, z such that

$$x^2 = 2d + 1$$

$$y^2 = 5d + 1$$

$$z^2 = 13d + 1$$

Then $y^2 - x^2 = 3d$, $z^2 - y^2 = 8d$, and $z^2 - x^2 = 11d$. Clearly x is odd, so $x^2 \equiv 1 \pmod{8}$, and $2d \equiv x^2 + 1 \equiv 2 \pmod{8}$, so d is odd. This forces y and z to be even, so let $y = 2u$ and $z = 2v$. Since $z^2 - y^2 = 8d$, we have $(v + u)(v - u) = 2d$. But because $u + v$ and $u - v$ must have the same parity, then both must be even, so $4 \mid 2d \implies 2 \mid d$, which is a contradiction. \square

Problem 401. Let a and b be integers and n a positive integer. Prove that

$$n! \mid b^{n-1}a(a+b)(a+2b) \cdots (a+(n-1)b)$$

Problem 402. Prove that a positive integer is a sum of at least two consecutive positive integers if and only if it is not a power of two.

Solution. For some $n = 2^a \cdot b$ where b is odd, we split this up into two cases.

Case 1. $2^a \geq \frac{b+1}{2}$ (and $b > 1$)

We just take 2^a , the $\frac{b-1}{2}$ numbers below it, and the $\frac{b-1}{2}$ numbers above it. Because $2^a > \frac{b+1}{2}$ all of these numbers are positive, and there are $b > 1$ terms in this sequence.

Case 2. $2^a < \frac{b+1}{2}$

Consider the 2^a consecutive integers ending at $\frac{b-1}{2}$ and the 2^a consecutive integers beginning at $\frac{b+1}{2}$. The sum of these integers is obviously $2^a b = n$, and the minimum integer is $\frac{b+1}{2} - 2^a \geq 1$, so we're done. \square