

# L<sup>A</sup>T<sub>E</sub>X Practice

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Hello! I'm McKinley! This is a bunch of words and two images and two problems.

I'll be using the `evan.sty` style file by Evan Chen since it looks really nice.

You can find the the code to this at

<https://github.com/MckinleyX/ib-math-handouts/>

Here goes!

I like competitive math. On last year's AMC 12A I got a score of 118.5 (though, somehow, my AMC 10B score was significantly worse) and on the AIME I got a score of 7. I'm hoping to qualify for USAMO this year. My favorite subject is (by far) combinatorics, and my weakest is probably geometry. (I'm not too sure what I want to do for my IA/EE, I think something about extremal graph theory, maybe Sperner's theorem or something, but that might be over my head.) I have an image of a banana here since... uhh... I dunno.

At the moment I'm also working on a couple of handouts for Math HL since some people seem to be confused, they can currently be found at <https://mckinleyx.github.io>. At the moment I'm trying to write something about modular arithmetic, but that may change on my whim.

In other news, I also do Linux! Both my laptop and my desktop are currently running bspwm on Arch Linux, and my current L<sup>A</sup>T<sub>E</sub>X setup is Vim + **VimTeX** + texlive.

Here are my two problems!

**Example (2020 IMO Shortlist C1)**

Let  $n$  be a positive integer. Find the number of permutations  $a_1, a_2, \dots, a_n$  of the sequence  $1, 2, \dots, n$  satisfying

$$a_1 \leq 2a_2 \leq 3a_3 \leq \dots \leq na_n$$

*Proof.* We claim that the number of permutations is the  $n^{\text{th}}$  fibonacci number,  $F_n$ , defined as  $F_0 = F_1 = 1, F_n = F_{n-2} + F_{n-1}$ .

**Claim —** If  $a_i = n$ , then  $i \geq n - 1$ .

*Proof.* Suppose that  $a_i = n$ . Then  $\{a_{i+1} \dots a_n\}$  must be a permutation of  $\{i, i+1, \dots, n-1\}$ .  $i$  must be at location  $n$ , which implies that  $in = (i+1)a_{n+1} = \dots = ni$ , which is clearly impossible for  $i < n - 1$ .  $\square$

To finish we will use strong(-ish?) induction. Clearly the claim is true for  $n = 1$  and  $n = 2$ . Now, suppose the claim is true for some  $n - 2$  and  $n - 1$ . Now, we will use casework on the position of  $n$ .

**Case 1.** If  $a_n = n$  the number of permutations satisfying the condition is equivalent to  $F_{n-1}$  since  $n^2$  is greater than any other term in the inequality.

**Case 2.** If  $a_{n-1} = n$ , then  $na_n \geq n(n-1) \implies a_n \geq n-1$ , and now  $n(n-1)$  is greater than or equal to any other terms in the inequality, so the number of permutations in this case is  $F_{n-2}$ .

The sum of these are  $F_{n-1} + F_{n-2} = F_n$  and we are done.  $\square$

**Example (Erdős-Szekeres)**

Prove that, in a sequence of  $mn + 1$  distinct integers,  $\exists$  either an increasing subsequence of length  $m + 1$  or a decreasing subsequence of length  $n + 1$ .

*Proof.* Let our sequence be  $a_1, a_2, \dots, a_{mn+1}$ .

Let  $L(x)$  (similarly,  $L'(x)$ ) denote the length of the longest increasing (similarly, decreasing) subsequence ending at  $a_x$ .

Now, consider  $x$  and  $y$  such that  $1 \leq x < y \leq mn + 1$ . Because  $a_y$  is either greater than or less than  $a_x$ , either  $L(y) > L(x)$  or  $L'(y) > L'(x)$ , since we can always create a larger subsequence of one kind by appending  $a_y$  to a subsequence ending at  $a_x$ .

Because this is true  $\forall x, y$  in the given range, then each  $a_x$  corresponds to a unique ordered pair  $(L(x), L'(x))$ .

Now, suppose our claim is not true, and that there do not exist such subsequences. Then  $\forall x$  such that  $1 \leq x \leq mn + 1$ ,  $1 \leq L(x) \leq m$  and  $1 \leq L'(x) \leq n$ . However, observe that there are only  $mn$  ordered pairs  $(L(x), L'(x))$  satisfying that condition, so by the pigeonhole principle such a subsequence exists, and we are done.  $\square$