$\LaTeX$ Practice

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Hello! I'm Mckinley! This is a bunch of words and two images and two problems.

I'll be using the `evan.sty` style file by Evan Chen since it looks really nice. You can find it here.

You can find the the source to this here.

Here goes!

I like competitive math. On last year's AMC 12A I got a score of 118.5 (though, somehow, my AMC 10B score was significantly worse) and on the AIME I got a score of 7. I'm hoping to qualify for USAMO this year. My favorite subject is (by far) combinatorics, and my weakest is probably geometry. (I'm not too sure what I want to do for my IA/EE, I think something about extremal graph theory, maybe Sperner's theorem or something, but that might be over my head.) I have an image of a banana here since... uhh... I dunno.

At the moment I'm also working on a couple of handouts for Math HL since some people seem to be confused, they can currently be found at `https://mckinleyx.github.io`. At the moment I'm trying to write something about modular arithmetic, but that may change on my whim.

In other news, I also do Linux! Both my laptop and my desktop are currently running bspwm on Arch Linux, and my current $\LaTeX$ setup is Vim + VimTeX + texlive.

Here are my two problems!

> **Example** (2020 IMO Shortlist C1)
> Let n be a positive integer. Find the number of permutations $a_1, a_2, \dots a_n$ of the sequence $1, 2, \dots n$ satisfying
> $$a_1 \leqslant 2a_2 \leqslant 3a_3 \leqslant \dots \leqslant na_n$$

Proof. We claim that the number of permutations is the n^{th} fibonacci number, F_n, defined as $F_0 = F_1 = 1, F_n = F_{n-2} + F_{n-1}$.

> **Claim** — If $a_i = n$, then $i \geq n-1$.

Proof. Suppose that $a_i = n$. Then $\{a_{i+1} \dots a_n\}$ must be a permutation of $\{i, i+1, \dots n-1\}$. i must be at location n, which implies that $in = (i+1)a_{n+1} = \cdots = ni$, which is clearly impossible for $i < n-1$. □

To finish we will use strong(-ish?) induction. Clearly the claim is true for $n = 1$ and $n = 2$. Now, suppose the claim is true for some $n-2$ and $n-1$. Now, we will use caeswork on the position of n.

Case 1. If $a_n = n$ the number of permutations satisfying the condition is equivalent to F_{n-1} since n^2 is greater than any other term in the inequality.

Case 2. If $a_{n-1} = n$, then $na_n \geq n(n-1) \implies a_n \geq n-1$, and now $n(n-1)$ is greater than or equal to any other terms in the inequality, so the number of permutations in this case is F_{n-2}.

The sum of these are $F_{n-1} + F_{n-2} = F_n$ and we are done.

□

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\begin{example*} %this is from evan.sty
[2020 IMO Shortlist C1]
Let $n$ be a positive integer.
Find the number of permutations
$a_1, a_2, \dots a_n$ of the sequence $1,2,\dots n$
satisfying
\[a_1 \leqslant 2a_2 \leqslant 3a_3 \leqslant \dots \leqslant na_n\]
\end{example*}

\begin{proof} %also from evan.sty
We claim that the number of permutations
is the $n^\text{th}$ fibonacci number, $F_n$,
defined as $F_0 = F_1 = 1, F_n = F_{n-2} + F_{n-1}$.
\begin{claim*} % evan.sty
If $a_i = n$, then $i \geq n-1$.
\end{claim*}
\begin{proof}
Suppose that $a_i = n$.
Then $\{a_{i+1} \dots a_n\}$ must be a permutation of
$\{ i, i+1, \dots n-1\}$.
$i$ must be at location $n$,
which implies that $in = (i+1)a_{n+1} = \cdots = ni$,
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which is clearly impossible for $i < n-1$.
\end{proof}

To finish we will use strong(-ish?) induction.
Clearly the claim is true for $n=1$ and $n=2$.
Now, suppose the claim is true for some $n-2$ and $n-1$.
Now, we will use casework on the position of $n$.

\textbf{Case 1.} If $a_n = n$ the number of permutations
satisfying the condition is equivalent to $F_{n-1}$
since $n^2$ is greater than any other term in the inequality.

\textbf{Case 2.} If $a_{n-1} = n$, then
$na_n \geq n(n-1) \implies a_{n} \geq n-1$,
and now $n(n-1)$ is greater than or equal to
any other terms in the inequality,
so the number of permutations in this case is $F_{n-2}$.

The sum of these are $F_{n-1} + F_{n-2} = F_n$ and we are done.
\end{proof}
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Example (Erdős-Szekeres)
Prove that, in a sequence of $mn+1$ distinct integers, $\exists$ either an increasing subsequence of length $m+1$ or a decreasing subsequence of $n+1$.

Proof. Let our sequence be $a_1, a_2, \cdots a_{mn+1}$.

Let $L(x)$ (similarly, $L'(x)$) denote the length of the longest increasing (similarly, decreasing) subsequence ending at a_x.

Now, consider x and y such that $1 \leq x < y \leq mn+1$. Because a_y is either greater than or less than a_x, either $L(y) > L(x)$ or $L'(y) > L'(x)$, since we can always create a larger subsequence of one kind by appending a_y to a subsequence ending at a_x.

Because this is true $\forall x, y$ in the given range, then each a_x corresponds to a unique ordered pair $(L(x), L'(x))$.

Now, suppose our claim is not true, and that there do not exist such subsequences. Then $\forall x$ such that $1 \leq x \leq mn+1$, $1 <= L(x) <= m$ and $1 <= L'(x) <= n$. However, observe that there are only mn ordered pairs $(L(x), L'(x))$ satisfying that condition, so by the pigeonhole principle such a subsequence exists, and we are done. $\square$

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\begin{example*}
[Erd\H{o}s-Szekeres]
Prove that, in a sequence of $mn + 1$ distinct integers,
$\exists$ either an increasing subsequence of length $m+1$
or a decreasing subsequence of $n+1$.
\end{example*}
\begin{proof}
Let our sequence be $a_1, a_2, \cdots a_{mn+1}$.

Let $L(x)$ (similarly, $L'(x)$) denote the length of the
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longest increasing (similarly, decreasing) subsequence ending at $a_x$.

Now, consider $x$ and $y$ such that $1 \leq x < y \leq mn + 1$.
Because $a_y$ is either greater than or less than $a_x$,
either $L(y) > L(x)$ or $L'(y) > L'(x)$,
since we can always create a larger subsequence of one kind
by appending $a_y$ to a subsequence ending at $a_x$.

Because this is true $\forall x,y$ in the given range,
each $a_x$ corresponds to a unique ordered pair $(L(x), L'(x))$.

Now, suppose our claim is not true,
and that there do not exist such subsequences.
Then $\forall x $ such that $1 \leq x \leq mn+1$,
$1 <= L(x) <= m$ and $1 <= L'(x) <= n$.
However, observe that there are only $mn$ ordered pairs $(L(x), L'(x))$
satisfying that condition,
so by the pigeonhole principle such a subsequence exists,
and we are done.
\end{proof}
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