Unit 3 LaTeX Project

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Example (USAMTS 4/1/33)

Let m, n, k be positive integers such that $k \leq mn$. Let S be the set consisting of a $(m+1)$-by-$(n+1)$ rectangular array of points on the Cartesian plane with coordinates (i, j) where i, j are integers satisfying $0 \leq i \leq m$ and $0 \leq j \leq n$. The diagram below shows the example where $m = 3$ and $n = 5$, with the points of S indicated by black dots:

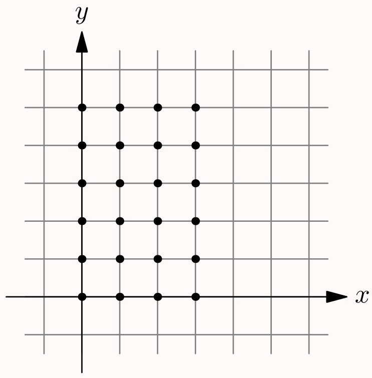

Figure 1: S with $m = 3$ and $n = 5$

Proof. Let $A = (0, 0)$, $B = (1, n)$, and $C = (x, y)$, for integral n, x, y.

Let D be the point on AB such that CD is parallel to the x-axis.

Let $[XYZ]$ denote the area of $\triangle XYZ$.

Suppose $k = mi - j$, where $i, j \in \mathbb{Z}$, $i, j \geq 0$, and $j < m$.

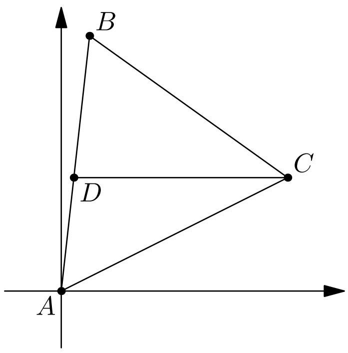

Figure 2: Triangle ABC

Observe that $[ABC] = [ACD] + [BCD] = \frac{1}{2}CD \cdot y + \frac{1}{2}CD \cdot (n - y) = \frac{1}{2}CD \cdot n$.
Since the slope of AB is n, $CD = x - \frac{y}{n}$.
So $[ABC] = \frac{1}{2}n\left(x - \frac{y}{n}\right) = \frac{1}{2}(xn - y)$.
Let $\lceil x \rceil$ denote the least integer greater than x.
Since the only thing we're bounded by is that $x, y \in \mathbb{Z}$ such that $0 \leq x \leq m$ and $0 \leq y \leq n$, let $x = \left\lceil \frac{k}{n} \right\rceil$ and $y = xn - k$. Then $[ABC] = \frac{1}{2}(xn - y) = k$, and we know that $x \leq m$ and $y \leq n$ since $x > m \implies k > mn$ and $y > n \implies \left\lceil \frac{k}{n} \right\rceil = x - 1$, both of which are contradictions, so such x, y exist, and we are done.

□

Example (USAMTS 2/1/33)

Find, with proof, the minimum number positive integer n with the following property: for any coloring of the integers $\{1, 2 \ldots, n\}$ using the colors red and blue (that is, assigning the color "red" or "blue" to each integer in the set), there exist distinct integers a, b, c between 1 and n, inclusive, all of the same color, such that $2a + b = c$.

Proof. Let S be a coloring of $\{1, 2 \ldots n\}$ with the required property. Let n be the size of S. We claim that $n = 14$ is maximal and achievable.
First, we have the following lemma:

Lemma

If a and b are the same color ($a < b$, $a, b \in S$), then $2a + b$, $a + 2b$ and $b - 2a$ are the opposite color.

The proof of the lemma is trivial, but it is very useful.
Without loss of generality, let 1 be colored red. Then we will break our problem into the following cases:
Case (1) 2 is red.
Case (2) 2 is blue, 3 is red
Case (3) 2 is blue, 3 is blue

We claim that:
In case (1) $n \leq 14$, and $n = 14 is achievable.$
In case (2) $n \leq 10$
In case (3) $n \leq 8$

Case (1)

In this case, 4 and 5 are both blue (for $a = 1, b = 2$) so 13 and 14 are both red ($(a, b) = (4, 5)$) so 11 and 12 are both blue ($(1, 13)$ and $(1, 14)$). However, because $3 + 2(4) = 11$, 3 must be red. This means that 7 and 8 are both blue ($(1, 3)$ and $(2, 3)$) We will now take a moment and look at our new list of numbers:

1 2 3 4 5 6 7 8 9 10 11 12 13 14

Now, $2(4) + 7 = 15$ must be red, but that is impossible since $2(1) + 13$ must be blue, so $n \leq 14$. $14 - 2(2)$ and $13 - 2(2)$ must also both be blue, and 6 can be whatever, so here is our final set:

1 2 3 4 5 6 7 8 9 10 11 12 13 14

Case (2)
In this case, we know that 5 and 7 are both blue, so 9 and 11 are both red. However, $2(1) + 9 = 11$ so $n \leq 10$ in case (2).

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10$$

We do not need to show that $n = 10$ is achievable since by case (1) $n = 14$ is achievable.
Case (3)
In this case, we know that both 7 and 8 are red, so 5 and 6 are blue (by subtracting by 2(1)) as are 9 and 10 (by adding 2(1)).

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10$$

However, this is impossible, as 9 must be red since 2 and 5 are both blue. So $n \leq 8$ in case (3).
Once again, we don't actually need to show that $n = 8$ is achievable since $n = 14$ is achievable.
In every case $n \leq 14$, and since $n = 14$ is achievable our answer is $\boxed{16}$ and we are done.

□

The International Baccalaureate Organization (IBO) has a number of assessment criteria identified for the internal assessment in mathematics, with each criterion having level descriptors describing specific achievement levels [Org21, p. 80]. For the criterion on mathematical communication, it is very important for students to understand that appropriate notation, symbols, and terminology must be used in their written work.
As Dr. Annalisa Crannell explains in her paper "Writing in Mathematics", all variables used should be defined with words describing what the variables represent. In particular, "variables in text are italicized to tell them apart from regular letters" [Cra94, p. 5]

References

[Cra94] Annalisa Crannell. *Writing in Mathematics*. Franklin and Marshall College. https://www.fandm.edu/uploads/files/107682389602454187-guide-to-writing.pdf, [accessed September 25, 2021]. (1994). URL: https://www.fandm.edu/uploads/files/107682389602454187-guide-to-writing.pdf.

[Org21] International Baccalaureate Organization. *Mathematics: analysis and approaches guide (first assessment 2021)*. International Baccalaureate Organization. 2021 [accessed 25 September 2021].