Unit 3 LaTeX Project

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Example (USAMTS 4/1/33)

Let m, n, k be positive integers such that $k \leq mn$. Let S be the set consisting of a $(m+1)$-by-$(n+1)$ rectangular array of points on the Cartesian plane with coordinates (i, j) where i, j are integers satisfying $0 \leq i \leq m$ and $0 \leq j \leq n$. The diagram below shows the example where $m = 3$ and $n = 5$, with the points of S indicated by black dots:

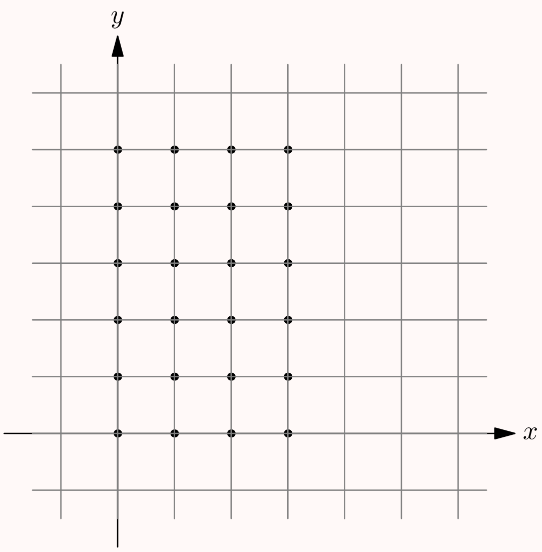

Figure 1: S with $m = 3$ and $n = 5$

Proof. Let $A = (0, 0)$, $B = (1, n)$, and $C = (x, y)$, for integral n, x, y.

Let D be the point on AB such that CD is parallel to the x-axis.

Let $[XYZ]$ denote the area of $\triangle XYZ$.

Suppose $k = mi - j$, where $i, j \in \mathbb{Z}$, $i, j \geq 0$, and $j < m$.

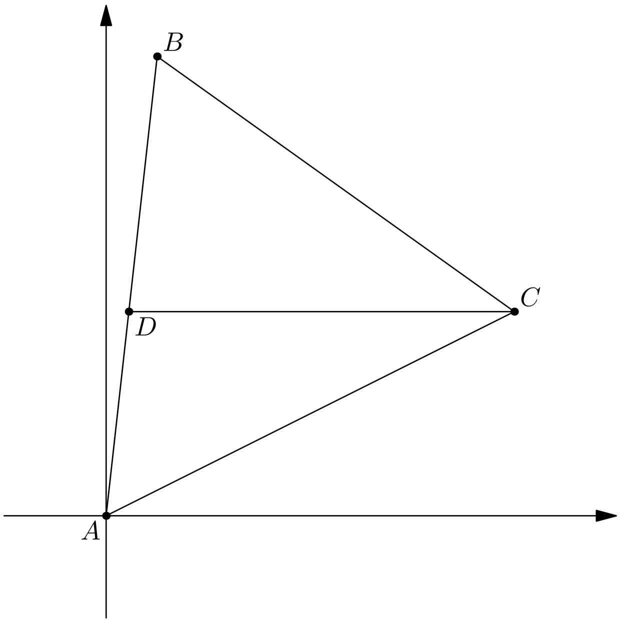

Figure 2: Triangle ABC

Observe that $[ABC] = [ACD] + [BCD] = \frac{1}{2}CD \cdot y + \frac{1}{2}CD \cdot (n - y) = \frac{1}{2}CD \cdot n$.

Since the slope of AB is n, $CD = x - \frac{y}{n}$.

So $[ABC] = \frac{1}{2}n\left(x - \frac{y}{n}\right) = \frac{1}{2}(xn - y)$.

Let $\lceil x \rceil$ denote the least integer greater than x.

Since the only thing we're bounded by is that $x, y \in \mathbb{Z}$ such that $0 \leq x \leq m$ and $0 \leq y \leq n$, let $x = \left\lceil \frac{k}{n} \right\rceil$ and $y = xn - k$. Then $[ABC] = \frac{1}{2}(xn - y) = k$, and we know that $x \leq m$ and $y \leq n$ since $x > m \implies k > mn$ and $y > n \implies \left\lceil \frac{k}{n} \right\rceil = x - 1$, both of which are contradictions, so such x, y exist, and we are done.

$\square$