Unit 2 LaTeX Project

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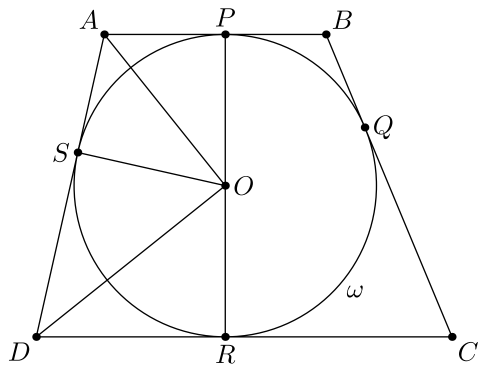
Example (Komal Problem B. 5188)
Prove that the height of a circumscribed trapezium cannot be greater than the geometric mean of the bases.

Proof. Consider the left-hand side of the following diagram:



Observe that $AP = AS$ and $PO = OS$, and $\angle OSA = \angle OPA = 90°$ [2], so $\triangle APO \cong \triangle ASO$. Similarly, $\triangle ORD \cong \triangle ODS$.

Because AO bisects $\angle POS$ and DO bisects $\angle SOR$, $\angle AOD = \frac{180°}{2} = 90°$. Because $\angle SAO = \angle OAS$, $\triangle AOD \sim \triangle ASO$. Similarly, $\triangle OSD \sim \triangle AOD$.

Let r be the radius of circle ω and $c = \frac{SO}{AS}$. Then $AS = \frac{r}{c}$, and by similar trinagles $SD = rc$.

Repeating this argument for the right-hand side we have $BQ = \frac{r}{d}$, $OQ = r$, and $OC = rd$ for some constant d.

Let g be the geometric mean of the bases.

$$\begin{aligned} g &= \sqrt{\left(\frac{r}{c} + \frac{r}{d}\right)(rc + rd)} \\ &= r\sqrt{\left(\frac{1}{c} + \frac{1}{d}\right)(c + d)} \\ &= r\sqrt{1 + \frac{c}{d} + \frac{d}{c} + 1} \end{aligned}$$

Finally, by AM-GM[1] on $\frac{c}{d} + \frac{d}{c}$,

$$g \geq r\sqrt{1 + 2 + 1} = 2r$$

And since $2r$ is equal to the height of the trapezium, we are done. $\square$

References

[1] Eric W. Weisstein. *Arithmetic-Geometric Mean*. URL: https://mathworld.wolfram.com/Arithmetic-GeometricMean.html.

[2] Eric W. Weisstein. *Circle Tangent Line*. URL: https://mathworld.wolfram.com/CircleTangentLine.html.

[3] Eric W. Weisstein. *Similar Triangles*. URL: https://mathworld.wolfram.com/SimilarTriangles.html.