

L^AT_EX Practice

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Hello! I'm Mckinley! This is a bunch of words and two images and two problems.

I'll be using the `evan.sty` style file by Evan Chen since it looks really nice. You can find it [here](#).

You can find the the source to this [here](#).

Here goes!

I like competitive math. On last year's AMC 12A I got a score of 118.5 (though, somehow, my AMC 10B score was significantly worse) and on the AIME I got a score of 7. I'm hoping to qualify for USAMO this year. My favorite subject is (by far) combinatorics, and my weakest is probably geometry. (I'm not too sure what I want to do for my IA/EE, I think something about extremal graph theory, maybe Sperner's theorem or something, but that might be over my head.) I have an image of a banana here since... uhh... I dunno.



At the moment I'm also working on a couple of handouts for Math HL since some people seem to be confused, they can currently be found at <https://mckinleyx.github.io>. At the moment I'm trying to write something about modular arithmetic, but that may change on my whim.

In other news, I also do Linux! Both my laptop and my desktop are currently running bspwm on Arch Linux, and my current L^AT_EX setup is Vim + [VimTeX](#) + texlive.

Here are my two problems!

Example (2020 IMO Shortlist C1)

Let n be a positive integer. Find the number of permutations a_1, a_2, \dots, a_n of the sequence $1, 2, \dots, n$ satisfying

$$a_1 \leq 2a_2 \leq 3a_3 \leq \dots \leq na_n$$

Proof. We claim that the number of permutations is the n^{th} fibonacci number, F_n , defined as $F_0 = F_1 = 1, F_n = F_{n-2} + F_{n-1}$.

Claim — If $a_i = n$, then $i \geq n - 1$.

Proof. Suppose that $a_i = n$. Then $\{a_{i+1} \dots a_n\}$ must be a permutation of $\{i, i+1, \dots, n-1\}$. i must be at location n , which implies that $in = (i+1)a_{n+1} = \dots = ni$, which is clearly impossible for $i < n - 1$. \square

To finish we will use strong(-ish?) induction. Clearly the claim is true for $n = 1$ and $n = 2$. Now, suppose the claim is true for some $n - 2$ and $n - 1$. Now, we will use casework on the position of n .

Case 1. If $a_n = n$ the number of permutations satisfying the condition is equivalent to F_{n-1} since n^2 is greater than any other term in the inequality.

Case 2. If $a_{n-1} = n$, then $na_n \geq n(n-1) \implies a_n \geq n-1$, and now $n(n-1)$ is greater than or equal to any other terms in the inequality, so the number of permutations in this case is F_{n-2} .

The sum of these are $F_{n-1} + F_{n-2} = F_n$ and we are done. \square

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1 \begin{example*} %this is from evan.sty
2   [2020 IMO Shortlist C1]
3   Let $n$ be a positive integer.
4   Find the number of permutations
5   $a_1, a_2, \dots a_n$ of the sequence $1, 2, \dots n$
6   satisfying
7   $[a_1 \leqslant 2a_2 \leqslant 3a_3 \leqslant \dots \leqslant na_n]$
8 \end{example*}
9
10 \begin{proof} %also from evan.sty
11   We claim that the number of permutations is the $n^{\text{th}}$
12     fibonacci number, $F_n$, defined as $F_0 = F_1 = 1, F_n = F_{n-2}
13     + F_{n-1}$.
14   \begin{claim*} % evan.sty
15     If $a_i = n$, then $i \geq n-1$.
16   \end{claim*}
17   \begin{proof}
18     Suppose that $a_i = n$.
19     Then $\{a_{i+1} \dots a_n\}$ must be a permutation of $\{i, i+1,
20     \dots n-1\}$.
21     $i$ must be at location $n$,
22     which implies that $in = (i+1)a_{n+1} = \dots = ni$,
23     which is clearly impossible for $i < n-1$.
24   \end{proof}
25
26   To finish we will use strong(-ish?) induction.
27   Clearly the claim is true for $n=1$ and $n=2$.

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25     Now, suppose the claim is true for some  $n-2$  and  $n-1$ .
26     Now, we will use casework on the position of  $n$ .
27
28     \textbf{Case 1.} If  $a_n = n$  the number of permutations
29     satisfying the condition is equivalent to  $F_{n-1}$ 
30     since  $n^2$  is greater than any other term in the inequality.
31
32     \textbf{Case 2.} If  $a_{n-1} = n$ , then
33      $a_n \geq n(n-1)$  implies  $a_n \geq n-1$ ,
34     and now  $n(n-1)$  is greater than or equal to
35     any other terms in the inequality,
36     so the number of permutations in this case is  $F_{n-2}$ .
37
38     The sum of these are  $F_{n-1} + F_{n-2} = F_n$  and we are done.
39 \end{proof}

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Example (Erdős-Szekeres)

Prove that, in a sequence of $mn + 1$ distinct integers, \exists either an increasing subsequence of length $m + 1$ or a decreasing subsequence of length $n + 1$.

Proof. Let our sequence be $a_1, a_2, \dots, a_{mn+1}$.

Let $L(x)$ (similarly, $L'(x)$) denote the length of the longest increasing (similarly, decreasing) subsequence ending at a_x .

Now, consider x and y such that $1 \leq x < y \leq mn + 1$. Because a_y is either greater than or less than a_x , either $L(y) > L(x)$ or $L'(y) > L'(x)$, since we can always create a larger subsequence of one kind by appending a_y to a subsequence ending at a_x .

Because this is true $\forall x, y$ in the given range, then each a_x corresponds to a unique ordered pair $(L(x), L'(x))$.

Now, suppose our claim is not true, and that there do not exist such subsequences. Then $\forall x$ such that $1 \leq x \leq mn + 1$, $1 \leq L(x) \leq m$ and $1 \leq L'(x) \leq n$. However, observe that there are only mn ordered pairs $(L(x), L'(x))$ satisfying that condition, so by the pigeonhole principle such a subsequence exists, and we are done. \square

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1 \begin{example*}
2   [Erd\H{o}s-Szekeres]
3   Prove that, in a sequence of  $mn + 1$  distinct integers,
4    $\exists$  either an increasing subsequence of length  $m+1$ 
5   or a decreasing subsequence of  $n+1$ .
6 \end{example*}
7 \begin{proof}
8   Let our sequence be  $a_1, a_2, \dots a_{mn+1}$ .
9
10  Let  $L(x)$  (similarly,  $L'(x)$ ) denote the length of the
11  longest increasing (similarly, decreasing) subsequence ending at
12   $a_x$ .
13
14  Now, consider  $x$  and  $y$  such that  $1 \leq x < y \leq mn + 1$ .
15  Because  $a_y$  is either greater than or less than  $a_x$ ,
16  either  $L(y) > L(x)$  or  $L'(y) > L'(x)$ ,
17  since we can always create a larger subsequence of one kind
18  by appending  $a_y$  to a subsequence ending at  $a_x$ .
19
20  Because this is true  $\forall x, y$  in the given range,
21  each  $a_x$  corresponds to a unique ordered pair  $(L(x), L'(x))$ .
22
23  Now, suppose our claim is not true,

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23     and that there do not exist such subsequences.
24     Then  $\forall x$  such that  $1 \leq x \leq mn+1$ ,
25      $1 \leq L(x) \leq m$  and  $1 \leq L'(x) \leq n$ .
26     However, observe that there are only  $mn$  ordered pairs  $(L(x),$ 
         $L'(x))$ 
27     satisfying that condition,
28     so by the pigeonhole principle such a subsequence exists,
29     and we are done.
30 \end{proof}
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