An Introduction to Induction

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"I surrender."
— Brynn

Here's a handout on induction, since many people are really confused by it.

This is my first time doing something like this, so feedback would be appreciated!

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§1 What is induction?

Induction is a *very* useful technique in proofs. To use induction to prove something, we must prove the following two properties:

1. The desired result is true for the first value (the base case)
2. If the desired result is true for some result, then it is true for the next result (the inductive step)

This is a bit hard to parse, so let's restate that in terms of dominos. Suppose we have a row of dominos, and we want to prove that they will all be knocked over. To use induction we need to prove that:

1. The first domino gets knocked over (the base case)
2. If the $(k-1)$th domino is knocked over, then the kth domino will be knocked over[1]. (the inductive step)

[1]This is the most common form of induction, but there are variations such as strong induction that have a slightly different inductive step. We won't be going into those here.

So if the first domino is knocked over, then the second domino is also knocked over. If the second domino is knocked over, then the third domino is knocked over, and so on and so forth until every domino is knocked over.

This is all a bit abstract, so let's try some examples.

§2 Worked examples and exercises

Example 2.1

Prove that $2n$ is even for all $n \in \mathbb{Z}^+$.

Proof. Induction is completely unnecessary for this problem, but let's use it anyway.

For the base case, 2 is clearly even.

In addition, for any $k \in \mathbb{Z}^+$, if the hypothesis is true for $n = k - 1$ (i.e. $2(k-1)$ is even), then $2(k-1) + 2 = 2k$ is even, meaning the inductive hypothesis is true for $n = k$, so we are done. □

In the above proof, we proved the following two statements:

1. Our claim is true for $n = 1$
2. If the claim is true for $n = k - 1$ then it is true for $n = k$.

So because our claim is true for $n = 2 - 1$, we know our claim is true for $n = 2$. Because our claim is true for $n = 3 - 1$, we know our claim is true for $n = 3$, and so on and so forth, so our claim must be true for any arbitrary (positive integer) n.

Exercise 2.2. Prove that $2^n > n$ for all $n \in \mathbb{Z}^+$.

Let's try another example.

Example 2.3

Prove that

$$1 + 2 + \cdots + n = \frac{n\,(n+1)}{2}$$

Proof. We will use induction.

For the base case, clearly $1 = \frac{1(1+1)}{2}$. For the inductive step, suppose that the formula holds for $n = k - 1$. That is, suppose $1 + 2 + \cdots + (k-1) = \frac{(k-1)(k-1+1)}{2}$.

We want to show that $1 + 2 + \cdots (k-1) + k = \frac{k(k+1)}{2}$.

This is just algebra:

$$\begin{aligned} 1 + 2 + \cdots + (k-1) + k &= \frac{(k-1)k}{2} + k \\ &= \frac{k^2 - k}{2} + \frac{2k}{2} \\ &= \frac{k^2 + k}{2} \\ 1 + 2 + \cdots + (k-1) + k &= \frac{k(k+1)}{2} \end{aligned}$$

And we are done. □

Exercise 2.4. Prove that

$$1 + 2 + \cdots + (2n - 1) = n^2$$

Exercise 2.5. Prove that

$$1^2 + 2^2 + \cdots n^2 = \frac{n(n+1)(2n+1)}{6}$$

Example 2.6

Prove that $10 \mid 11^n - 1$ for any $n \in \mathbb{Z}^+$. (Recall that $a \mid b$ if a divides b.)

Proof. Once again, we will use induction. The base case is simple; 10 clearly divides $11 - 1$.

For the inductive step, suppose $10 \mid 11^{k-1} - 1$ for some integer $k \in \mathbb{Z}^+$. From this, we want to show that $10 \mid 11^k - 1$. Well,

$$10 \mid 11^{k-1} - 1 \implies 10 \mid 11 \cdot (11^{k-1} - 1) = 11^k - 11$$

And if $10 \mid 11^k - 11$ then $10 \mid 11^k - 1$ and we are done.

□

Remark. This class of problems can also be solved very quicky (and more satisfyingly) using modular arithmetic, which I may make a handout on soon.

If you want a taste of it, the idea is the following:
Note that 11^n always ends in a 1, so $11^n - 1$ always ends in a 0, and we're done.

Exercise 2.7. Prove that $3 \mid 4^n - 7$ for $n \geq 2$ (where $n \in \mathbb{Z}$)

Exercise 2.8. Prove that, in general, $a \mid (a+1)^n + (a-1)$

§3 Additional problems

Some of these problems (especially the later ones) are hard, don't worry if you can't solve them.

Problem 1. Prove that $(1 + 2 + \cdots n)^2 = 1^3 + 2^3 + \cdots + n^3$

Problem 2. Prove the formula for a (finite) geometric series:

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{r^n - 1}{r - 1}$$

Problem 3. Prove that the expansion of $(1+x)^n$ is

$$\binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n$$

Problem 4. Given that the sum of the interior angles of a triangle is 180 degrees, prove that the sum of the interior angles of a n-sided polygon is $180(n-2)$ degrees.

Problem 5. A plane is divided into regions by a finite number of lines. Show that it is possible to color the regions with two colors, white or black, so that any two bordering regions are opposite colors. (Two regions border each other if they share a side.)

Problem 6. Given that $\frac{a+b}{2} \geq \sqrt{ab}$ for positive reals a and b, prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

(This is known as the Arithmetic Mean – Geometric Mean inequality, or AM–GM.)
Hint: Regular induction won't work here. The solution uses something called Cauchy Induction, which involves showing that $n = k$ works implies that both $n = 2k$ works and $n = k - 1$ works, instead of $n = k + 1$.

§4 Sources (and helpful links)

https://brilliant.org/wiki/induction/
https://artofproblemsolving.com/wiki/index.php?title=Induction
Intermediate Counting and Probability by David Patrick