

Seminar - Markowitz Portfolio Optimization

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December 5, 2023

Theoretical Part

Problem Description

In the realm of financial portfolio management, the Markowitz portfolio optimization problem is a classical and essential topic. The primary objective is to allocate weights to different assets in a portfolio to maximize the expected return while minimizing the overall portfolio risk. Let's consider a portfolio with n assets. The goal is to find the optimal set of weights for these assets.

Formalization

Let:

r_i : Expected return of asset i
 σ_i : Volatility (risk) of asset i
 w_i : Weight of asset i in the portfolio

The objective is to find the vector of weights $\mathbf{w} = [w_1, w_2, \dots, w_n]$ that maximizes the expected portfolio return μ while minimizing the portfolio risk σ_p :

$$\begin{aligned}
& \text{Maximize} && \mu = \sum_{i=1}^n r_i w_i \\
& \text{Subject to} && \sigma_p = \sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_i \sigma_j \rho_{ij}} \quad (\text{Portfolio risk}) \\
& && \sum_{i=1}^n w_i = 1 \quad (\text{Sum of weights equals 1}) \\
& && w_i \geq 0 \quad (\text{Non-negativity constraint})
\end{aligned} \tag{1}$$

Where:

ρ_{ij} : Correlation coefficient between assets i and j

An other formulation of the problem is to minimize the portfolio risk σ_p while achieving a target expected return μ :

The objective is to find the vector of weights $\mathbf{w} = [w_1, w_2, \dots, w_n]$ that minimizes the portfolio risk σ_p while achieving a given expected portfolio return μ :

$$\begin{aligned}
& \text{Minimize} && \sigma_p = \sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_i \sigma_j \rho_{ij}} \quad (\text{Portfolio risk}) \\
& \text{Subject to} && \mu = \sum_{i=1}^n r_i w_i \quad (\text{Expected portfolio return}) \\
& && \sum_{i=1}^n w_i = 1 \quad (\text{Sum of weights equals 1}) \\
& && w_i \geq 0 \quad (\text{Non-negativity constraint})
\end{aligned} \tag{2}$$

Numerical Part

Selected Optimization Methods

For solving the Markowitz portfolio optimization problem, we have chosen two numerical optimization methods:

In order to simplify the problem, we will first forget about the expected return constraint. We will only focus on minimizing the portfolio risk σ_p .

$$\begin{aligned}
& \text{Minimize} \quad \sigma_p = \sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_i \sigma_j \rho_{ij}} \quad (\text{Portfolio risk}) \\
& \text{Subject to} \quad \sum_{i=1}^n w_i = 1 \quad (\text{Sum of weights equals 1}) \\
& \quad \quad \quad w_i \geq 0 \quad (\text{Non-negativity constraint})
\end{aligned} \tag{3}$$

To restruct the weight vector to be positive, we can use a variable change:

$$w_i = e^{x_i} \quad \text{with} \quad x_i \in \mathbb{R} \tag{4}$$

The problem becomes:

$$\begin{aligned}
& \text{Minimize} \quad \sigma_p = \sqrt{\sum_{i=1}^n \sum_{j=1}^n e^{x_i+x_j} \sigma_i \sigma_j \rho_{ij}} \quad (\text{Portfolio risk}) \\
& \text{Subject to} \quad \sum_{i=1}^n e^{x_i} = 1 \quad (\text{Sum of weights equals 1})
\end{aligned} \tag{5}$$

We can also forget about the square root in the objective function, as it does not change the optimal solution.

$$\begin{aligned}
& \text{Minimize} \quad \sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n e^{x_i+x_j} \sigma_i \sigma_j \rho_{ij} \quad (\text{Portfolio risk}) \\
& \text{Subject to} \quad \sum_{i=1}^n e^{x_i} = 1 \quad (\text{Sum of weights equals 1})
\end{aligned} \tag{6}$$

Finally, we can inject the sum constraint into the objective function using a variable change: $x_{n+1} = 1 - \sum_{i=1}^n x_i$.

Or we can also use the softmax function to ensure that the sum of the weights equals 1, such as:

$$w_i = \frac{e^{x_i}}{\sum_{j=1}^n e^{x_j}} \tag{7}$$

Leading to the following problem formulation (with softmax):

$$\begin{aligned}
&\text{Minimize} \quad \sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n e^{x_i+x_j} \sigma_i \sigma_j \rho_{ij} \quad (\text{Portfolio risk}) \\
&\text{Subject to} \quad x_i \in \mathbb{R} \quad \forall i \in \{1, \dots, n\}
\end{aligned} \tag{8}$$

1. Method 1: [Insert Method 1 Name]
2. Method 2: [Insert Method 2 Name]

Algorithm Implementation

Below are the basic functions describing the two chosen algorithms:

Method 1: [Insert Method 1 Name]

[Insert code or pseudocode for Method 1 implementation]

Method 2: [Insert Method 2 Name]

[Insert code or pseudocode for Method 2 implementation]

Results and Analysis

We have applied both methods to the Markowitz portfolio optimization problem and obtained the following results:

[Insert results, tables, or graphs]

Interpretation

[Provide interpretation of the results]

Comparison

To compare the two methods, we analyze factors such as computational time and the number of iterations:

[Insert comparison results]