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A NEW METHOD TO OBTAIN UNIFORM DECAY RATES FOR MULTIDIMENSIONAL WAVE EQUATIONS WITH NONLINEAR ACOUSTIC BOUNDARY CONDITIONS*

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Abstract. In this paper, we investigate the uniform stability of a class of nonlinear acoustic wave motions with boundary and localized interior damping. Here the damping and potential in the boundary displacement equation are nonlinear. Moreover, the nonlinear system contains the localized interior damping term, which indicates that there is a thin absorption material and flow resistance on the endophragm of the boundary. Since some lower-order term in the nonlinear wave system is not below the energy level, the "compactness-uniqueness" method is not suitable for the problem. Our main purpose is to present a new method to obtain uniform decay rates for these damped wave equations with nonlinear acoustic boundary conditions.

Key words. wave equation, acoustic boundary condition, stability, uniform decay rates, localized interior damping

AMS subject classifications. 93D15, 35L20, 35L70, 35B35, 76Q05

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1. Introduction.

Since Beale and Rosencrans [6] introduced acoustic boundary conditions for wave equations in 1974, there has been a variety of research on wave or other kinds of equations with acoustic boundary conditions; see [3, 5–11, 14, 18] and references therein. Because of their applicability to many areas, such as noise reduction and vibration control, the studies about such boundary conditions have attracted lots of attention.

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary Γ. Let $\Gamma = \Gamma_0 \cup \Gamma_1$ with Γ_0 and Γ_1 closed, disjoint, and nonempty. Denote by $\nu(x)$ the outer unit vector normal to the boundary Γ_1. We consider the following acoustic boundary value problem

$$u_{tt}(x,t) - \Delta u(x,t) + w(x)u_t(x,t) = 0, \quad x \in \Omega,\ t > 0 \tag{1.1}$$

$$u(x,t) = 0, \quad x \in \Gamma_0,\ t > 0 \tag{1.2}$$

$$u_t(x,t) + z_{tt}(x,t) + f(z_t) + g(z) = 0, \quad x \in \Gamma_1,\ t > 0 \tag{1.3}$$

$$\frac{\partial u(x,t)}{\partial \nu} = z_t(x,t), \quad x \in \Gamma_1,\ t > 0 \tag{1.4}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \tag{1.5}$$

$$z(x,0) = z_0(x), \quad x \in \Gamma_1. \tag{1.6}$$

Here Δ is the Laplacian operator, f, g are given functions on $\mathbb{R}$, and $w(x) \in L^\infty(\Omega)$

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is a cutoff function defined by

$$w(x) = \begin{cases} 1, & \text{for } x \in \Omega_\eta; \\ 0, & \text{for } x \in \Omega/\Omega_\eta, \end{cases}$$

where $\eta > 0$ and $\Omega_\eta := \{x \in \Omega;\ d(x, \Gamma) \le \eta\}$.

This model (for $n = 3$) describes a fluid undergoing small irrotational perturbations from rest in the domain Ω. The portion Γ_1 of its surface is called interface; each point on Γ_1 reacts to the excess pressure from the fluid like a spring, and the "springs" are independent of each other; therefore, Γ_1 is locally reacting. The function u represents the velocity potential and z the normal displacement of Γ_1. The relation (1.4) expresses the impenetrability of the boundary Γ_1. See [6, 17] for more physical explanations. For the sake of simplicity, we set all the coefficients to be 1.

In 1976, Beale [5] pointed out for linear systems under the action of the boundary damping z_t (but without any interior damping) that the system energies have no uniform decay rates. Afterwards, Muñoz Rivera and Qin [18] obtained polynomial decay of the energies for smooth initial data. We refer the reader to Abbas and Nicaise [1, 2] for deep studies on asymptotic stability, nonuniform stability, and polynomial stability for related systems with generalized acoustic boundary conditions.

In [10], Graber considered porous acoustic boundary conditions, with the interface described by

$$\begin{aligned} &u_t + m(x)z_{tt} + f_1(x)z_t + g_1(x)z = 0, \quad x \in \Gamma_1,\ t > 0; \\ &\frac{\partial u}{\partial \nu} + \theta(u_t) = h(x)\eta(z_t), \quad x \in \Gamma_1,\ t > 0 \end{aligned} \tag{1.7}$$

(m, f_1, g_1, h, θ, and η being given functions), and obtained (among others) uniform decay rates thanks to the additional dissipative term $\theta(u_t)$; see also [11] and the references therein for related studies. Recently, Vicenti and Frota [19] took into account acoustic boundary conditions to a nonlocally reacting boundary, with the interface described by

$$\begin{aligned} &\rho_0 u_t(x,t) + m z_{tt}(x,t) - c^2\Delta_\Gamma z + f(z_t) + rz = 0, \quad x \in \Gamma_1,\ t > 0, \\ &\frac{\partial u(x,t)}{\partial \nu} = z_t(x,t), \quad x \in \Gamma_1,\ t > 0, \end{aligned} \tag{1.8}$$

where ρ_0, m, c are positive constants, $r \ge 0$, and Δ_Γ is the Laplace–Beltrami operator; they put a (nonlinear) internal localized damping term in the wave equation to achieve uniform stability successfully.

The aim of this paper is to obtain uniform stability of the system (1.1)–(1.6). The system contains an internal localized damping term $w(x)u_t$ in (1.1) as in [19] (for simplicity we here consider only linear dampings with coefficient 1 close to the boundary), which indicates that there is a thin absorption material with the thickness of $\eta > 0$ and flow resistance on the endophragm of Γ (cf. [17, section 6.2]). On the other hand, the interface Γ_1 is locally reacting (as in the original work [5, 6]), in contrast to that in [19]. That is, our system does not contain such a term as $-c^2\Delta_\Gamma z$ in (1.8). However, it is the lack of this term that brings about essential difficulty in doing stability analysis. Actually, the integral $\int_{\Gamma_1} z^2 d\Gamma$ is no longer a lower-order term below the energy level since the energy $E(t)$ (see (3.1) below) does not contain $\int_{\Gamma_1} |\nabla_T z|^2 d\Gamma$. This would result in the failure of the compactness-uniqueness argument (compare with the proof for [19, (3.36)], δ there representing z here). The argument is a general method used to absorb lower-order terms (cf. [10, 11, 13]). Therefore, we have to find some new ideas to deal with the problem. In this paper, we present a new method to obtain uniform decay rates for the damped wave equations with nonlinear acoustic boundary conditions.

This paper is organized as follows. In section 2, we give a well-posedness result by the theory of nonlinear semigroups. Section 3 is devoted to presenting our stability theorem; we will combine flexibly the method of convex functions (as in Lasiecka and Tataru [13] and Liu and Zuazua [15]), the method of Lyapunov functions (as in [15]), and the cutoff technique (as in Martinez [16]) to prove the uniform stability of the energy.

2. Well-posedness. We let

$$V(\Omega) = \{u(x) \in H^1(\Omega), u|_{\Gamma_0} = 0\},$$

with the inner product and norm

$$(u, v)_V := \int_\Omega \nabla u(x) \cdot \nabla v(x) dx, \quad |u|_V := \int_\Omega |\nabla u(x)|^2 dx.$$

Note that Poincare's inequality holds in $V(\Omega)$,

$$\int_\Omega u^2 dx \le c \int_\Omega |\nabla u|^2 dx, \tag{2.1}$$

with some constant $c > 0$. For the spaces $L^2(\Omega)$ and $L^2(\Gamma_1)$, we define the inner products and norms by, as usual,

$$(u, v) = \int_\Omega u(x)v(x)dx, \quad |u| = \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}},$$
$$(\phi, \psi)_{\Gamma_1} = \int_{\Gamma_1} \phi(x)\psi(x)d\Gamma, \quad |\phi|_{\Gamma_1} = \left(\int_{\Gamma_1} |\phi(x)|^2 d\Gamma\right)^{\frac{1}{2}}.$$

Using the notations given above, we define the "finite energy space" that will be associated with "weak solutions" as

$$\mathcal{H} := V(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1),$$

where the norm on $\mathcal{H}$ is given by

$$|(u, v, z, \tilde{z})|^2_{\mathcal{H}} = |u|^2_V + |v|^2 + |z|^2_{\Gamma_1} + |\tilde{z}|^2_{\Gamma_1}.$$

Now we reduce (1.1)–(1.6) to an abstract Cauchy problem,

$$\begin{aligned} &\frac{d}{dt}U(t) = \mathcal{A}U(t),\\ &U(0) = U_0, \end{aligned}$$

where $U(t) = (u, u_t, z, z_t)^T$ is a vector in the Hilbert space $\mathcal{H}$ and $\mathcal{A} : D(A) \subset \mathcal{H} \to \mathcal{H}$ is an operator with the domain

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ v \\ z \\ \tilde{z} \end{pmatrix} \in \mathcal{H}; \quad v \in V(\Omega),\ \Delta u \in L^2(\Omega),\ \frac{\partial u}{\partial \nu}\Big|_{\Gamma_1} = \tilde{z} \right\}.$$

The action of the operator $\mathcal{A}$ is given by the matrix

$$\mathcal{A}\begin{pmatrix} u \\ v \\ z \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} v \\ \Delta u - wv \\ q \\ -v - f(\tilde{z}) - g(z) \end{pmatrix}.$$

Next we show the well-posedness of the nonlinear system (1.1)–(1.6) by the nonlinear monotone operator method. For this, we need some conditions on f and g.

Assumption (A1). The functions $f, g \in C(\mathbb{R})$ are monotone nondecreasing such that $g(0) = 0$, $f(0) = 0$, $g' \in L^\infty(\mathbb{R})$, and

$$|f(s)| \leq C|s| \quad \text{for } |s| \geq 1, \tag{2.2}$$

where $C > 0$ is a constant.

We note that $f(z) \in L^2(\Gamma_1)$ whenever $z \in L^2(\Gamma_1)$, under the condition (2.2).

THEOREM 2.1. *Suppose that Assumption (A1) holds. Then for $U(0) \in \mathcal{H}$, the system* (1.1)–(1.6) *has a unique mild solution $U(t; U_0) \in C([0,\infty); \mathcal{H})$ such that*

$$\|U(t;U_1) - U(t;U_2)\|_{\mathcal{H}} \leq e^{\omega t}\|U_1 - U_2\|_{\mathcal{H}} \quad \text{for } U_1, U_2 \in \mathcal{H},\ t \geq 0, \tag{2.3}$$

with some constant $\omega > 0$. Moreover, if $U(0) \in D(\mathcal{A})$, then $U(t;U_0) \in W^{1,1}_{\text{loc}}(0,\infty;\mathcal{H})$ is a strong solution of (1.1)–(1.6).

Proof. For $U_i = (u_i, v_i, z_i, \tilde{z}_i)^T \in D(\mathcal{A})$, $i = 1, 2$, we have

$$\begin{aligned} &(\mathcal{A}U_1 - \mathcal{A}U_2, U_1 - U_2) \\ &= \int_{\Gamma_1} (z_1 - z_2)(\tilde{z}_1 - \tilde{z}_2) - (\tilde{z}_1 - \tilde{z}_2)(g(z_1) - g(z_2)) \\ &\quad - (\tilde{z}_1 - \tilde{z}_2)(f(\tilde{z}_1) - f(\tilde{z}_2))d\Gamma - \int_\Omega w(v_1 - v_2)^2 dx \\ &\leq \omega |U_1 - U_2|^2_{\mathcal{H}} \end{aligned}$$

for some constant $\omega > 0$. Therefore, $\omega I - \mathcal{A}$ is monotone.

Next, we show that the range of $\lambda I - \mathcal{A}$ is all of $\mathcal{H}$ for $\lambda > \omega$. To the end, we let $(x_1, x_2, x_3, x_4) \in \mathcal{H}$ and try to find $U = (u_1, u_2, u_3, u_4) \in D(\mathcal{A})$ such that

$$\begin{cases} \lambda u_1 - u_2 = x_1, \\ \lambda u_2 - \Delta u_1 + w u_2 = x_2, \\ \lambda u_3 - u_4 = x_3, \\ \lambda u_4 + u_2 + f(u_4) + g(u_3) = x_4. \end{cases} \tag{2.4}$$

To prove the existence of solution U to (2.4), we apply the method of perturbation of maximal monotone operator as in [12, 13].

Define $\mathcal{N} : L^2(\Gamma_1) \to H^{\frac{3}{2}}(\Omega)$ such that for $\phi \in L^2(\Gamma_1)$, $\mathcal{N}\phi = \psi$ is the solution of the following equation:

$$\begin{cases} \Delta \psi = 0, \\ \psi|_{\Gamma_0} = 0, \quad \left.\dfrac{\partial \psi}{\partial \nu}\right|_{\Gamma_1} = \phi; \end{cases}$$

moreover, define

$$Au = -\Delta u, \quad D(A) = \left\{ u \in H^2(\Omega);\ \ u|_{\Gamma_0} = 0,\ \frac{\partial u}{\partial \nu}\Big|_{\Gamma_1} = 0 \right\},$$

which can be extended to a continuous operator from V to V' (with respect to the L^2 duality).

From the last two equations of (2.4), one has

$$\lambda u_4 + u_2 + f(u_4) + g\left(\frac{x_3 + u_4}{\lambda}\right) = x_4. \tag{2.5}$$

We define a scalar function J on $\mathbb{R}^4$ by

$$J(t, s, s_0, s_1) = \lambda t + s + f(t) + g\left(\frac{s_0 + t}{\lambda}\right) - s_1.$$

By Assumption(A1), we know that J is a continuous function, which is strictly increasing in t satisfying

$$\lim_{t \to \pm\infty} J(t, s, s_0, s_1) = \pm\infty.$$

Therefore, for any $(s, s_0, s_1) \in \mathbb{R}^3$, there exists a unique t, denoted by $-F(s, s_0, s_1)$, such that $J(t, s, s_0, s_1) = 0$. It is clear that the implicit function F is also a continuous function, being strictly increasing in the first variable. From (2.5) we obtain

$$u_4 = -F(u_2, x_3, x_4).$$

Substituting it and $u_1 = \frac{x_1 + u_2}{\lambda}$ into the second equation of (2.4), we get

$$\lambda u_2 + A\left(\frac{x_1 + u_2}{\lambda} + \mathcal{N} F(u_2, x_3, x_4)\right) + w u_2 = x_2. \tag{2.6}$$

Define two operators B_1, B_2 from V to V' by

$$B_1 v = A\mathcal{N} F(v, x_3, x_4), \quad B_2 v = \frac{1}{2}\lambda v + A\left(\frac{x_1 + v}{\lambda}\right) + wv.$$

Then B_1 is maximal monotone (it may be written as the subgradient of a convex functional as in [13, p. 514] and [12, p. 89]), and B_2 is monotone and continuous. From [4, Corollary 1.3, Chapter 2], we see that $B_1 + B_2$ is maximal monotone in $V \times V'$. Therefore, (2.6) has a solution $u_2 \in V$. Accordingly, we see that $\lambda I - \mathcal{A}$ is surjective, and so $\omega I - \mathcal{A}$ is maximal monotone. Therefore, from the nonlinear semigroup theory, we obtain the desired conclusion. □

3. A new method to obtain uniform decay rates. We define the energy of the system (1.1)–(1.6) as

$$E(t) = \frac{1}{2}\int_\Omega |\nabla u|^2 + u_t^2 dx + \frac{1}{2}\int_{\Gamma_1} z_t^2 d\Gamma + \int_{\Gamma_1} G(z) d\Gamma, \tag{3.1}$$

where $G(t) := \int_0^t g(s)ds$. Under Assumption (A1), it is easy to see that the following energy identity holds for strong solutions:

$$E(t) = E(0) - \int_0^t \int_\Omega w u_t^2 dx dt - \int_0^t \int_{\Gamma_1} f(z_t) z_t d\Gamma dt. \tag{3.2}$$

Thus, we see that $E(t)$ is decreasing for all finite energy solutions.

For studying the stability of the system energy, we need more conditions on f and g.

Assumption(A2). There exists a constant $c > 0$ such that

$$|f(s)|,\ |g(s)| \geq c|s| \quad \text{if } |s| \geq 1. \tag{3.3}$$

Assumption(A3).

(a) There exist two strictly increasing C^2-functions $h_1(s)$, $h_2(s) : [0, \infty) \to [0, \infty)$ such that

$$ch_1(|s|) \leq |f(s)| \leq Ch_1^{-1}(|s|), \quad |s| \leq 1 \tag{3.4}$$

$$h_2(|s|) \leq |g(s)|, \quad |s| \leq 1. \tag{3.5}$$

(b) The function h_1 satisfies

$$h_1(st) \leq Ch_1(s)h_1(t), \quad s, t \in [0, \infty), \tag{3.6}$$

and there exists an increasing and convex function $\phi(s) : [0, \infty) \to [0, \infty)$, with $\phi''(s)$ positive and bounded on $(0, s_0)$ for any $s_0 > 0$, such that

$$\phi(|s|^2) \leq h_i(|s|)|s|, \quad |s| \leq 1, \ \ i = 1, 2; \tag{3.7}$$

here, C, c are positive constants.

THEOREM 3.1. *Let f, g satisfy Assumptions* (A1), (A2), *and* (A3). *Then the energy of system* (1.1)–(1.6) *decays to zero, uniformly in initial data with* $E(0) \leq r$ (*for any fixed* $r > 0$). *More precisely, we have*

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right), \quad t > T_0,$$

for some positive constant $T_0 > 0$; here $S(\cdot)$ is the solution of the ODE

$$S'(t) + q(S(t)) = 0, \quad \textit{with the initial value } S(0) = E(0),$$

the function $q(\cdot)$ being given by $q(s) := s - (I + p)^{-1}(s)$ with $p(0) = 0$,

$$p(s) := \bar{C}_1\varepsilon(s)\phi(as), \quad s > 0,$$

$$\varepsilon(s) := \left[1 + \frac{\bar{C}_2}{\phi'(as)} + \frac{\bar{C}_3}{\sqrt{\phi'(as)}}h_1\left(\frac{C}{\sqrt{\phi'(as)}}\right)\right]^{-1}$$

($C > 0$ *is the constant in* (3.4), *and $a, \bar{C}_1, \bar{C}_2, \bar{C}_3$ are positive constants which depend continuously on* $E(0)$).

Proof. We fix $r > 0$ and assume $E(0) \leq r$; also, we may and do assume $E(t) > 0$ for $t \geq 0$ (in the other case, $E(t) \equiv 0$). Throughout the proof, C_1, C_2, C_2', and C_3 denote generic positive constants, which may depend on r and may vary from line to line. Moreover, we only need to deal with the strong solution case (the general case can be handled by a density argument owing to (2.3)).

Step 1. We first introduce an auxiliary function $V(t)$, and estimate $V'(t)$.

Define

$$V(t) = E(t) + \varepsilon\psi(E(t))\left[\int_\Omega uu_t dx + \int_{\Gamma_1} zz_t d\Gamma + \int_{\Gamma_1} zud\Gamma\right],$$

where

$$\psi(s) := \phi'(as)$$

and $a, \varepsilon \in (0,1)$, which will be specified later. Differentiating $V(t)$, we obtain

$$\begin{aligned} V'(t) = E'(t) + \varepsilon\psi'(E(t))E'(t)\left[\int_\Omega uu_t dx + \int_{\Gamma_1} zz_t d\Gamma + \int_{\Gamma_1} zud\Gamma\right] \\ + \varepsilon\psi(E(t))\left[\int_\Omega u_t^2 - |\nabla u|^2 - wu_t u dx + \int_{\Gamma_1} uz_t + z_t^2 + z_t u + zu_t + zz_{tt} d\Gamma\right]. \end{aligned}$$

From Assumptions (A1) and (A2), we have

$$\frac{z}{2} g\left(\frac{z}{2}\right) \le G(z) \le g(z)z \tag{3.8}$$

and

$$z^2 \le C_1\left[\frac{z}{2} g\left(\frac{z}{2}\right)\right] \le C_1 G(z) \quad \text{if } |z| \ge 2, \tag{3.9}$$

which implies

$$\left|\int_\Omega uu_t dx + \int_{\Gamma_1} zz_t d\Gamma + \int_{\Gamma_1} zud\Gamma\right| \le C_1$$

by the use of the Cauchy–Schwarz inequality and (2.1). Accordingly, from (1.3), we obtain

$$\begin{aligned} V'(t) &\le E'(t) - C_1\varepsilon\psi'(E(t))E'(t) \\ &\quad + \varepsilon\psi(E(t))\left[\int_\Omega u_t^2 - \frac{3}{4}|\nabla u|^2 + C_1 w u_t^2 dx + \int_{\Gamma_1} C_1 z_t^2 - zg(z) - zf(z_t) d\Gamma\right] \\ &\le E'(t) - C_1\varepsilon E'(t) \\ &\quad + \varepsilon\psi(E(t))\left[\int_\Omega -u_t^2 - \frac{1}{4}|\nabla u|^2 dx + \int_{\Gamma_1} -G(z) - z_t^2 d\Gamma\right] \\ &\quad + \varepsilon\psi(E(t))\left[\int_\Omega 2u_t^2 + C_1 w u_t^2 - \frac{|\nabla u|^2}{2} dx + \int_{\Gamma_1} C_1 z_t^2 - zf(z_t) d\Gamma\right] \\ &\le E'(t) - C_1\varepsilon E'(t) - \frac{1}{2}\varepsilon\psi(E(t))E(t) \\ &\quad + \varepsilon\psi(E(t))\left[\int_\Omega 2u_t^2 + C_1 w u_t^2 dx + \int_{\Gamma_1} C_1 z_t^2 - zf(z_t) d\Gamma\right], \end{aligned}$$

noting

$$\psi'(E(t)) = a\phi''(aE(t)) \le \sup\{\phi''(s);\ s \in (0, r]\} < \infty$$

due to Assumptions (A3).

Step 2. For dealing with the term $\int_\Omega u_t^2$, we need the following lemma, which will be proved later.

LEMMA 3.2. *There exist two positive constants T_1 and C', depending on r and n, such that for any $T > T_1$,*

$$\begin{aligned}
&\int_0^T \psi(E(t)) \int_\Omega u_t^2 dx dt \\
&\leq \frac{C'}{T} \int_0^T \psi(E(t)) \int_{\Gamma_1} G(z) d\Gamma dt + C' \int_0^T -E'(t) dt \\
&\quad + C' \int_0^T \psi(E(t)) \left(\int_\Omega w u_t^2 dx + \int_{\Gamma_1} z_t^2 + f^2(z_t) d\Gamma \right) dt.
\end{aligned} \tag{3.10}$$

Making use of this lemma, we infer that for $T > T_1$,

$$\begin{aligned}
\int_0^T V'(t) dt \leq \int_0^T &\left[E'(t) - C_1 \varepsilon E'(t) - \frac{1}{2} \varepsilon \psi(E(t)) E(t) \right] dt \\
&+ C_1 \int_0^T \varepsilon \psi(E(t)) \int_\Omega w u_t^2 dx dt \\
&+ C_1 \int_0^T \varepsilon \psi(E(t)) \int_{\Gamma_1} \left[z_t^2 + f^2(z_t) + |z f(z_t)| + \frac{1}{T} G(z) \right] d\Gamma dt.
\end{aligned} \tag{3.11}$$

Step 3. Let $\phi^\star$ denote the Legendre transform of ϕ (see [15]) given by

$$\phi^\star(s) = s \phi'^{-1}(s) - \phi(\phi'^{-1}(s)), \quad s > 0. \tag{3.12}$$

Exploiting (3.11), we will control $\int_0^T V'(t) dt$ with the terms

$$\int_0^T E'(t) dt, \quad \int_0^T \phi^\star(\psi(E(t))) dt, \quad - \int_0^T \psi(E(t)) E(t) dt.$$

Clearly, by (3.2),

$$I_1 := C_1 \int_0^T \int_\Omega \varepsilon \psi(E(t)) w u_t^2 dx dt \leq \int_0^T -C_1 \varepsilon \psi(E(0)) E'(t) dt. \tag{3.13}$$

Using (3.3) and Young's inequality, we have

$$\begin{aligned}
I_2 &:= C_1 \int_0^T \int_{\Gamma_1} \varepsilon \psi(E(t)) z_t^2 d\Gamma dt \\
&\leq C_1 \int_0^T \int_{G_2} \varepsilon \psi(E(t)) z_t f(z_t) d\Gamma dt + C_1 \int_0^T \int_{G_1} \varepsilon \psi(E(t)) z_t^2 d\Gamma dt \\
&\leq C_1 \int_0^T \varepsilon \psi(E(t))(-E'(t)) dt + C_1 \int_0^T \left[\varepsilon \phi^\star(\psi(E(t))) + \varepsilon \int_{G_1} \phi(z_t^2) d\Gamma \right] dt \\
&\leq C_1 \left\{ \int_0^T \varepsilon \psi(E(0))(-E'(t)) dt + \int_0^T -\varepsilon E'(t) dt + \varepsilon \int_0^T \phi^\star(\psi(E(t))) dt \right\}
\end{aligned} \tag{3.14}$$

due to (3.2), (3.7), and (3.4), where

$$G_1 := \{x \in \Gamma_1; |z_t| \leq 1\}, \quad G_2 := \{x \in \Gamma_1; |z_t| \geq 1\}.$$

Moreover, for any $\tau > 0$ with $c(\tau) := \frac{1}{4\tau} + 1$,

$$\begin{aligned}
I_3 &:= C_1 \int_0^T \varepsilon\psi(E(t)) \int_{\Gamma_1} \left[f^2(z_t) + |zf(z_t)| + \frac{1}{T}G(z) \right] d\Gamma dt \\
&\le C_1 \int_0^T \varepsilon\psi(E(t)) \int_{\Gamma_1} \left[\frac{1}{T}G(z) + \tau z^2 \right] d\Gamma dt \\
&\quad + C_1 \int_0^T \varepsilon\psi(E(t)) \int_{\Gamma_1} c(\tau) f^2(z_t) d\Gamma dt \\
&=: I_4 + I_5.
\end{aligned}$$

Put

$$R_1 := \{x \in \Gamma_1;\, |z| \le 1\}, \quad R_2 := \{x \in \Gamma_1;\, |z| \ge 1\};$$

using Young's inequality and noting

$$G(z) \ge \frac{1}{2} c z^2 \text{ for } |z| \ge 1 \text{ (by (3.3)) and } \frac{z}{2} g\left(\frac{z}{2}\right) \le G(z),$$

we infer from (3.7) and (3.5) that

$$\begin{aligned}
I_4 &= C_1 \int_0^T \varepsilon\psi(E(t)) \int_{\Gamma_1} \left[\frac{1}{T}G(z) + \tau z^2 \right] d\Gamma dt \\
&\le \int_0^T \int_{R_2} C_1 \varepsilon \psi(E(t)) \tau G(z) d\Gamma dt + C_1 \int_0^T \int_{R_1} \tau\varepsilon\phi(z^2) d\Gamma dt \\
&\quad + C_1 \varepsilon\tau \int_0^T \phi^\star(\psi(E(t))) dt + \int_0^T \varepsilon\psi(E(t)) \int_{\Gamma_1} \frac{C_1}{T} G(z) d\Gamma dt \\
&\le C_1 \left\{ \int_0^T \left(\tau + \frac{1}{T}\right) \varepsilon\psi(E(t))E(t)dt + \int_0^T \tau\varepsilon E(t)dt + \int_0^T \tau\varepsilon\phi^\star(\psi(E(t)))dt \right\}.
\end{aligned}$$

Now, we take $T_0 = T_1 + 12C_1$. Letting $T \ge T_0$ and $\tau = \tau(E(T))$ with

$$\tau(s) := \frac{1}{12} C_1^{-1} (1 + \psi(r))^{-1} \psi(s), \quad s > 0,$$

we then have

$$I_4 \le \frac{1}{4} \int_0^T \varepsilon\psi(E(t))E(t)dt + C_2 \int_0^T \varepsilon\phi^\star(\psi(E(t)))dt. \tag{3.15}$$

Next, taking $t_0 \in (0, 1]$ such that $\sqrt{c(\tau)}|f(s)| \le 1$ if $|s| \le t_0$, we define

$$F_1 = \{x \in \Gamma_1;\, |z_t| \le t_0\}, \quad F_2 = \{x \in \Gamma_1;\, |z_t| \ge t_0\}.$$

Making use of (2.2), (3.2), Young's inequality, (3.6), and (3.7) gives

$$\begin{aligned}
I_5 &= C_1 \int_0^T \varepsilon\psi(E(t)) \int_{\Gamma_1} c(\tau) f^2(z_t) d\Gamma dt \\
&\le C_2 \int_0^T \varepsilon c(\tau)\psi(E(t)) \int_{F_2} f(z_t) z_t d\Gamma dt \\
&\quad + C_2 \int_0^T \varepsilon c(\tau)\psi(E(t)) \int_{F_1} f^2(z_t) d\Gamma dt
\end{aligned}$$

$$
\begin{aligned}
&\le C_2 \int_0^T -\varepsilon c(\tau)\psi(E(t))E'(t)dt + C_2 \int_0^T \int_{F_1} \varepsilon \phi(c(\tau)f^2(z_t))d\Gamma dt \\
&\quad + C_2 \int_0^T \varepsilon \phi^\star(\psi(E(t)))dt \\
&\le C_2 \int_0^T \int_{F_1} \varepsilon \sqrt{c(\tau)}\, |f(z_t)|\, h_1(\sqrt{c(\tau)}\, |f(z_t)|)d\Gamma dt \\
&\quad + C_2 \int_0^T -\varepsilon c(\tau)\psi(E(t))E'(t)dt + C_2 \int_0^T \varepsilon \phi^\star(\psi(E(t)))dt \\
&\le C_2 \int_0^T \int_{F_1} \varepsilon \sqrt{c(\tau)}\, |f(z_t)|\, h_1(C\sqrt{c(\tau)})h_1(C^{-1}\, |f(z_t)|)d\Gamma dt \\
&\quad + C_2 \int_0^T -\varepsilon c(\tau)\psi(E(t))E'(t)dt + C_2 \int_0^T \varepsilon \phi^\star(\psi(E(t)))dt.
\end{aligned}
$$

Observe

$$
\begin{aligned}
&\int_{F_1} |f(z_t)|\, h_1(C^{-1}\, |f(z_t)|)d\Gamma \le \int_{F_1} |f(z_t)|\, h_1(h_1^{-1}(|z_t|))d\Gamma \\
&= \int_{F_1} f(z_t)z_t d\Gamma \le -E'(t)
\end{aligned}
$$

by (3.4) and (3.2). Accordingly,

$$
\begin{aligned}
I_5 &\le C_2 \int_0^T -\varepsilon c(\tau)\psi(E(0))E'(t)dt + C_2 \int_0^T -\varepsilon \sqrt{c(\tau)} h_1(C\sqrt{c(\tau)})E'(t)dt \\
&\quad + C_2 \int_0^T \varepsilon \phi^\star(\psi(E(t)))dt.
\end{aligned}
$$

Combining this and (3.11)–(3.15) together, we obtain

$$
\begin{aligned}
\int_0^T V'(t)dt &\le \int_0^T \left\{ C_2'\varepsilon\phi^\star(\psi(E(t))) - \frac{\varepsilon}{4}\psi(E(t))E(t) \right\} dt \\
&\quad + \int_0^T \left\{ 1 - C_2'\varepsilon \left[1 + c(\tau) + \sqrt{c(\tau)} h_1 \left(C\sqrt{c(\tau)} \right) \right] \right\} E'(t)dt.
\end{aligned} \tag{3.16}
$$

Step 4. Taking a suitable ε, we will drop the controlling term in row 2 of (3.16), and show that $V(t)$ is equivalent to the energy $E(t)$.

Observe, by Young's inequality, (3.12), (3.7), (3.5), and (3.8),

$$
\begin{aligned}
\varepsilon\psi(E(t)) \int_{\Gamma_{12}} z^2 dx &\le C_2'\varepsilon\phi^\star(\psi(E(t))) + C_2'\varepsilon \int_{\Gamma_{12}} \phi\left(\left(\frac{z}{2} \right)^2 \right) d\Gamma \\
&\le C_2'\varepsilon[\phi'(aE(t))aE(t) - \phi(aE(t))] + C_2'\varepsilon \int_{\Gamma_{12}} G(z)d\Gamma \\
&\le C_2'\varepsilon\phi'(aE(0))aE(t) + C_2'\varepsilon \int_{\Gamma_1} G(z)d\Gamma \\
&\le C_2'\varepsilon E(t),
\end{aligned}
$$

where $\Gamma_{12} := \{x \in \Gamma_1;\ |z| \le 2\}$. From this and (3.9), we have

$$\varepsilon\psi(E(t))\left|\int_\Omega uu_t dx + \int_{\Gamma_1} zz_t d\Gamma + \int_{\Gamma_1} zud\Gamma\right| \le C_2'\varepsilon E(t).$$

Let

$$\varepsilon = \varepsilon(E(T))$$

with

$$\varepsilon(s) := \frac{1}{2C_2'}\left[1 + c(\tau(s)) + \sqrt{c(\tau(s))}h_1\left(C\sqrt{c(\tau(s))}\right)\right]^{-1}, \quad s > 0$$

(an increasing function). Then

$$\varepsilon\psi(E(t))\left|\int_\Omega uu_t dx + \int_{\Gamma_1} zz_t d\Gamma + \int_{\Gamma_1} zud\Gamma\right| \le \frac{1}{2}E(t),$$

so that

$$\frac{1}{2}E(t) \le V(t) \le \frac{3}{2}E(t). \tag{3.17}$$

Also, it follows from (3.16) that

$$\int_0^T V'(t)dt \le \int_0^T \left\{C_2'\varepsilon\phi^\star(\psi(E(t))) - \frac{\varepsilon}{4}\psi(E(t))E(t)\right\} dt. \tag{3.18}$$

Step 5. From (3.18), we will derive a discrete inequality for $V(t)$ which enables us to obtain the energy decay rate by means of [13, Lemma 3.3].

Take $a > 0$ small enough such that $aC_2' \le \frac{1}{4}$, and notice

$$\phi^\star(\psi(E(t))) = \phi'(aE(t))aE(t) - \phi(aE(t)).$$

Then (3.18) becomes

$$\begin{aligned}\int_0^T V'(t)dt &\le \varepsilon\int_0^T \left(C_2'a - \frac{1}{4}\right)\phi'(aE(t))E(t) - C_2'\phi(aE(t))dt\\ &\le \int_0^T -\varepsilon C_2'\phi(aE(t))dt.\end{aligned}$$

From this and (3.17), we get

$$C_2'\varepsilon\left(\frac{2}{3}V(T)\right)\phi\left(\frac{2a}{3}V(T)\right)T \le V(0) - V(T).$$

As in [13], we fix $T = T_0$ and define

$$p(x) = C_2'\varepsilon\left(\frac{2}{3}x\right)\phi\left(\frac{2a}{3}x\right)T_0,$$

and it is easy to see that $p(x)$ is a positive, increasing function with $p(0) = 0$.

Also, the same arguments apply to the time interval $[mT_0, (m+1)T_0]$ for any positive integer m. Thus, we have

$$V(mT_0 + T_0) + p(V(mT_0 + T_0)) \le V(mT_0), \quad m = 0, 1, \ldots$$

Using Lemma 3.3 in [13], we get

$$V(t) \le S\left(\frac{t}{T_0} - 1\right), \quad t > T_0,$$

where $S(t)$ is the solution of

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = V(0),$$

with $q(s) := s - (I + p)^{-1}(s)$. Therefore, by (3.17), we obtain $E(t) \leq 2V(t)$, which ends the proof. ∎

Remark 3.3.

(1). In the above proof, we used a Lyapunov function $V(t)$, which is equivalent to the energy $E(t)$ and somewhat similar to the one in [15], but estimating $V(t)$ here turns out to be much more difficult. Indeed, the inner damping is only local (in contrast to that in [15]), and it takes time to control the whole energy of the system, so an estimate with time integral is definitely involved (see Lemma 3.2). Thus, we are unable to derive a nonlinear differential relation for $V(t)$ (as in [15, (2.9)]). We ended up obtaining the decay rate of $V(t)$ from a discrete inequality for $V(t)$ with the help of a lemma in [13]. In our case, another new difficulty is caused by the lack of compactness in the dynamics on the boundary; this means that a compactness-uniqueness type of argument would fail (as explained in section 1). We then adopted the convex function ϕ (also used in [15] to control the inner damping term) to subtly control the boundary term $g(z)$ as well as the boundary damping term $f(z_t)$.

(2). We would like to say that the local inner damping is important for the proof of Theorem 3.1. One of the key points is how to control the boundary term $\int_{\Gamma_1} u_t^2 d\Gamma$ (see the proof of Lemma 3.2). Due to the existence of term z_{tt}, one cannot obtain the estimate for $\int_{\Gamma_1} u_t^2 d\Gamma$ only from the boundary damping $f(z_t)$. In fact, without the inner damping, the energy would not decay in a uniform way (as indicated in [6]).

Example 3.4. Let Assumptions (A1) and (A2) hold. Suppose that there are $\alpha \geq \beta \geq 1$ and $C, c > 0$ such that for $|s| \leq 1$,

$$C|s|^{\frac{1}{\alpha}} \geq |f(s)| \geq c|s|^{\alpha}, \quad |g(s)| \geq c|s|^{\beta}.$$

Then we have

$$E(t) \leq \begin{cases} C_0 e^{-c_0 t}, & \text{if} \quad \alpha = 1, \\ C_0(t+1)^{-\frac{1}{3}}, & \text{if} \quad 1 < \alpha < 3, \\ C_0(t+1)^{-\frac{4}{(\alpha+1)^2-4}}, & \text{if} \quad \alpha \geq 3, \end{cases} \tag{3.19}$$

where C_0, c_0 are positive constants depending on $E(0)$.

Indeed, take

$$h_1(s) = cs^{\alpha}, \quad h_2(s) = cs^{\beta},$$
$$\phi(s) = cs^{\frac{\alpha+1}{2}} \text{ if } \alpha = 1 \text{ or } \alpha \geq 3, \quad \phi(s) = cs^2 \text{ if } 1 < \alpha < 3.$$

It follows that for s small,

$$\varepsilon(s) \sim c_1 s^{\frac{\alpha^2-1}{4}}, \quad p(s) \sim c_2 s^{\frac{(\alpha+1)^2}{4}}, \quad q(s) \sim c_3 s^{\frac{(\alpha+1)^2}{4}}, \quad \text{if } \alpha = 1, \text{ or } \alpha \geq 3,$$
$$\varepsilon(s) \sim c_1 s^2, \quad p(s) \sim c_2 s^4, \quad q(s) \sim c_3 s^4, \quad \text{if } 1 < \alpha < 3$$

(c_1, c_2, and c_3 are positive constants depending on $E(0)$). This leads to the estimate (3.19) in view of Theorem 3.1

Now it remains to prove Lemma 3.2.

Proof of Lemma 3.2. We use the cutoff technique (cf. [16]) to overcome the problem caused by the term u_t on Γ.

Step 1. Write

$$\Gamma_b = \bigcup_{x\in\Gamma} \{y \in \mathbb{R}^n; |y - x| < b\},$$

with $b > 0$ a small constant. We then construct a cutoff function $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$\begin{aligned}&0 \le \varphi(x) \le 1,\\ &\varphi(x) = 1,\, x \in (\mathbb{R}^n \setminus Q_1) \cap \Omega \triangleq \Omega \setminus Q_1,\\ &\varphi(x) = 0,\, x \in Q_0 \cup (\mathbb{R}^n \setminus \Omega);\end{aligned}$$

here Q_0, Q_1 are two open domains with

$$\Gamma_b \subset Q_0 \subset Q_1 \subset \mathbb{R}^n \setminus (\Omega \setminus \Omega_\eta)$$

and

$$\overline{Q_0} \cap (\overline{\Omega \setminus Q_1}) = \emptyset, \quad \overline{Q_1} \cap (\overline{\Omega \setminus \Omega_\eta}) = \emptyset.$$

Multiplying (1.1) by $\varphi x \cdot \nabla u \psi(E)$ and integrating in time and space, we deduce

(3.20)
$$\begin{aligned}&\int_0^T \psi(E) \int_{\Omega\setminus Q_1} \frac{n}{2} u_t^2 + \Big(1 - \frac{n}{2}\Big) |\nabla u|^2 dxdt\\
&\le \int_0^T \psi(E) \int_\Gamma \varphi x \cdot \nabla u \frac{\partial u}{\partial \nu} + \frac{\varphi x \cdot \nu}{2}(u_t^2 - |\nabla u|^2) d\Gamma dt\\
&\quad - \int_\Omega \varphi x \cdot \nabla u u_t \psi(E)|_0^T dx + \int_0^T \int_\Omega \psi'(E) E' \varphi x \cdot \nabla u u_t dxdt\\
&\quad + \int_0^T \psi(E) \int_\Omega \tau_1 |\nabla u|^2 + \tilde{c}(\tau_1) w u_t^2 dxdt + C_3 \int_0^T \psi(E) \int_{\Omega\cap Q_1} u_t^2 + |\nabla u|^2 dxdt\\
&\le C_3 \psi(E(0))E(0) - C_3 \int_0^T \psi'(E) E' E dt\\
&\quad + \int_0^T \psi(E) \int_\Omega \tau_1 |\nabla u|^2 + \tilde{c}(\tau_1) w u_t^2 dxdt + C_3 \int_0^T \psi(E) \int_{\Omega\cap Q_1} u_t^2 + |\nabla u|^2 dxdt\\
&\le C_3 \psi(E(0))E(0) + C_3 \int_0^T \psi(E) \int_{\Omega\cap Q_1} |\nabla u|^2 dxdt\\
&\quad + \int_0^T \psi(E) \int_\Omega \tau_1 |\nabla u|^2 + \tilde{c}(\tau_1) w u_t^2 dxdt\end{aligned}$$

for any $\tau_1 > 0$ ($\tilde{c}(\tau_1)$ is a positive constant depending on τ_1). Here, we used the equalities

$$\begin{aligned}&\int_\Omega \varphi x \cdot \nabla u u_{tt} dx\\
&\quad = \frac{d}{dt} \int_\Omega \varphi x \cdot \nabla u u_t dx - \frac{1}{2} \int_\Gamma \varphi x \cdot \nu |u_t|^2 d\Gamma + \frac{1}{2} \int_\Omega \operatorname{div}(\varphi x) |u_t|^2 dx,\end{aligned}$$

$$\begin{aligned}&\int_\Omega \varphi x\cdot\nabla u\triangle u dx\\&\quad=\int_\Gamma \varphi x\cdot\nabla u\frac{du}{d\nu}d\Gamma-\frac{1}{2}\int_\Gamma\varphi x\cdot\nu|\nabla u|^2d\Gamma\\&\qquad+\frac{1}{2}\int_\Omega \operatorname{div}(\varphi x)|\nabla u|^2dx-\int_\Omega\sum_{j,\ k}\frac{\partial(\phi x_k)}{\partial x_j}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_j}dx;\end{aligned}$$

moreover, we used the Cauchy–Schwar inequality, the increasing property of $\psi(s)$, the decreasing property of $E(s)$, and the estimate

$$-\int_0^T\psi'(E)E'Edt=-\int_0^T Ed\psi(E)\le E(0)\psi(E(0)). \tag{3.21}$$

On the other hand, multiplying (1.1) by $\psi(E)u$ and integrating in time and space, we also have

$$\begin{aligned}\left|\int_0^T\psi(E)\int_\Omega u_t^2-|\nabla u|^2dxdt\right|&\le C_3\psi(E(0))E(0)+C_3\int_0^T\psi(E)\int_{\Gamma_1}z_t^2d\Gamma dt\\&\quad+\int_0^T\psi(E)\int_\Omega\tau_1|\nabla u|^2+\tilde{c}(\tau_1)wu_t^2dxdt.\end{aligned}\tag{3.22}$$

Next, we try to obtain an estimate in terms of $|\nabla u|$. Combining (3.20) and (3.22), we have

$$\begin{aligned}&\int_0^T\psi(E)\int_{\Omega\setminus Q_1}|\nabla u|^2dxdt\\&=\int_0^T\psi(E)\int_{\Omega\setminus Q_1}\frac{n}{2}u_t^2+\left(1-\frac{n}{2}\right)|\nabla u|^2-\frac{n}{2}(u_t^2-|\nabla u|^2)dxdt\\&\le\int_0^T\psi(E)\int_{\Omega\setminus Q_1}\frac{n}{2}u_t^2+\left(1-\frac{n}{2}\right)|\nabla u|^2dxdt+\frac{n}{2}\left|\int_0^T\psi(E)\int_{\Omega\setminus Q_1}(u_t^2-|\nabla u|^2)dxdt\right|\\&\le C_3\psi(E(0))E(0)+C_3\int_0^T\psi(E)\int_{\Gamma_1}z_t^2d\Gamma dt\\&\quad+C_3\int_0^T\psi(E)\int_\Omega\tau_1|\nabla u|^2+\tilde{c}(\tau_1)wu_t^2dxdt+C_3\int_0^T\psi(E)\int_{\Omega\cap Q_1}|\nabla u|^2dxdt.\end{aligned}\tag{3.23}$$

From now on, the general constant C_3 also depends on n. Multiplying $(n-1)$ by (3.23), together with (3.20), leads to

$$\begin{aligned}&\int_0^T\psi(E)\int_{\Omega\setminus Q_1}u_t^2+|\nabla u|^2dxdt\\&\le C_3\psi(E(0))E(0)+C_3\int_0^T\psi(E)\int_{\Gamma_1}z_t^2d\Gamma dt\\&\quad+C_3\int_0^T\psi(E)\int_\Omega\tau_1|\nabla u|^2+\tilde{c}(\tau_1)wu_t^2dxdt+C_3\int_0^T\psi(E)\int_{\Omega\cap Q_1}|\nabla u|^2dxdt.\end{aligned}$$

This, together with (3.20), implies

(3.24)

$$\begin{aligned}
&\int_0^T \psi(E)\int_\Omega u_t^2+|\nabla u|^2dxdt\\
&=\int_0^T \psi(E)\Big(\int_{\Omega\setminus Q_1}+\int_{\Omega\cap Q_1}\Big)u_t^2+|\nabla u|^2dxdt\\
&\le C_3\psi(E(0))E(0)+C_3\int_0^T\psi(E)\int_{\Gamma_1}z_t^2d\Gamma dt\\
&\quad+C_3\int_0^T\psi(E)\int_\Omega \tau_1|\nabla u|^2+\tilde c(\tau_1)wu_t^2dxdt+C_3\int_0^T\psi(E)\int_{\Omega\cap Q_1}|\nabla u|^2dxdt.
\end{aligned}$$

Step 2. In order to deal with $\int_0^T\psi(E)\int_{\Omega\cap Q_1}|\nabla u|^2dxdt$, we define a cutoff function $\xi(x)\in C_0^\infty(\mathbb{R}^n)$ that satisfies

$$\begin{aligned}
&0\le\xi(x)\le 1,\\
&\xi(x)=1,\ x\in Q_1,\\
&\xi(x)=0,\ x\in\Omega\setminus Q_2.
\end{aligned}$$

Here Q_2 is an open domain with

$$Q_1\subset Q_2\subset\mathbb{R}^n\setminus(\Omega\setminus\Omega_\eta)$$

and

$$\overline{Q_1}\cap(\overline{\Omega\setminus Q_2})=\emptyset,\quad \overline{Q_2}\cap(\overline{\Omega\setminus\Omega_\eta})=\emptyset.$$

Multiplying (1.1) by $\xi u\psi(E)$ and integrating in time and space, we then obtain

$$\begin{aligned}
0=&\int_\Omega \xi uu_t\psi(E)|_0^Tdx-\int_0^T\psi(E)\int_\Omega\xi u_t^2dxdt-\int_0^T\psi(E)\int_{\Gamma_1}\xi uz_td\Gamma dt\\
&+\int_0^T\psi(E)\int_\Omega\xi wuu_tdxdt-\int_0^T\int_\Omega\psi'(E)E'\xi uu_tdxdt\\
&+\int_0^T\psi(E)\int_\Gamma\frac{u^2}{2}\nu\cdot\nabla\xi dxdt-\int_0^T\psi(E)\int_\Omega\frac{u^2}{2}\Delta\xi dxdt\\
&+\int_0^T\psi(E)\int_\Omega\xi|\nabla u|^2dxdt.
\end{aligned}$$

Due to the construction of ξ, we have

$$\begin{aligned}
&\int_0^T\psi(E)\int_{\Omega\cap Q_1}|\nabla u|^2dxdt\\
&\le C_3\psi(E(0))E(0)+C_3\int_0^T\psi(E)\int_{\Gamma_1}z_t^2d\Gamma dt \qquad (3.25)\\
&\quad+C_3\int_0^T\psi(E)\int_\Omega\tau_2|\nabla u|^2+\tilde c(\tau_2)wu_t^2dxdt+C_3\int_0^T\psi(E)\int_{\Omega\cap Q_2}u^2dxdt
\end{aligned}$$

for any $\tau_2>0$; here, we used the Cauchy–Schwarz inequality, (2.1), and (3.21).

Step 3. To estimate $\int_0^T\psi(E)\int_{\Omega\cap Q_2}u^2dxdt$, we construct another cutoff function $\beta(x)\in C_0^\infty(\mathbb{R}^n)$ satisfying

$$\begin{aligned} &0 \le \beta(x) \le 1, \\ &\beta(x) = 1,\, x \in Q_2, \\ &\beta(x) = 0,\, x \in \Omega \setminus \Omega_\eta. \end{aligned}$$

Let $v(x)$ be the solution of the following elliptic problem:

$$\begin{aligned} &\Delta v = \beta(x) u,\, x \in \Omega, \\ &v = 0, \quad x \in \Gamma_0, \\ &\frac{\partial v}{\partial \nu} = 0,\, x \in \Gamma_1. \end{aligned} \tag{3.26}$$

We multiply (3.26) by $v(x)$ to deduce that

$$\int_\Omega |\nabla v|^2 dx = -\int_\Omega \beta u v dx \le C_3 \left(\int_\Omega \beta u^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega v^2 dx \right)^{\frac{1}{2}}.$$

Thus, using Poincare's inequality gives

$$\int_\Omega v^2 dx \le C_3 \int_\Omega \beta u^2 dx. \tag{3.27}$$

Similarly, we have

$$\int_\Omega v_t^2 dx \le C_3 \int_\Omega \beta u_t^2 dx. \tag{3.28}$$

Next, multiplying (1.1) by $\psi(E)v$ and integrating in time and space yields

$$\begin{aligned} 0 = &\int_\Omega v u_t \psi(E)|_0^T dx - \int_0^T \psi(E) \int_\Omega v_t u_t dx dt - \int_0^T \psi(E) \int_{\Gamma_1} v z_t d\Gamma dt \\ &+ \int_0^T \psi(E) \int_\Omega w v u_t dx dt - \int_0^T \int_\Omega \psi'(E) E' v u_t dx dt - \int_0^T \psi(E) \int_\Omega u \Delta v dx dt. \end{aligned}$$

Thus, employing Young's inequality, (3.27), (3.28), and (3.21), we obtain

$$\begin{aligned} &\int_0^T \psi(E) \int_{\Omega \cap Q_2} u^2 dx dt \\ &\le C_3 \psi(E(0)) E(0) + C_3 \int_0^T \psi(E) \int_{\Gamma_1} z_t^2 d\Gamma dt \\ &\quad + C_3 \int_0^T \psi(E) \int_\Omega \tau_3 u_t^2 + \tilde{c}(\tau_3) w u_t^2 dx dt \end{aligned} \tag{3.29}$$

for any $\tau_3 > 0$. Now, combining the above estimates (3.24), (3.25), and (3.29) and choosing τ_1, τ_2, τ_3 small enough, we see

$$\begin{aligned} \int_0^T \psi(E) \int_\Omega |\nabla u|^2 + u_t^2 dx dt \le C_3 \psi(E(0)) E(0) + C_3 \int_0^T \psi(E) \int_{\Gamma_1} z_t^2 d\Gamma dt \\ + C_3 \int_0^T \psi(E) \int_\Omega w u_t^2 dx dt. \end{aligned} \tag{3.30}$$

Step 4. It remains to estimate $\psi(E(0))E(0)$. By adding $\int_0^T \psi(E)\int_{\Gamma_1}(z_t^2+G(z))d\Gamma dt$ to (3.30), we get

$$\int_0^T \psi(E)Edt \leq C_3\psi(E(0))E(0) + C_3\int_0^T \psi(E)\int_\Omega wu_t^2 dxdt + C_3\int_0^T \psi(E)\int_{\Gamma_1} z_t^2 + G(z)d\Gamma dt. \tag{3.31}$$

Notice from (3.2) that

$$\psi(E(t))E(t) = \psi(E(0))E(0) + \int_0^t \psi'(E)E'Ed\tau - \int_0^t \psi(E)\left(\int_\Omega wu_t^2dx + \int_{\Gamma_1} z_t f(z_t)d\Gamma\right)d\tau$$

and

$$\psi'(E(t))) = \phi''(aE(t))a.$$

Combining this with (3.31), we obtain

$$\begin{aligned}\psi(E(0))E(0) \leq{}& \frac{rT\sup\{\phi''(s);\ s\in(0,r]\}}{T-C_3}\int_0^T -E'dt\\ &+ \frac{T}{T-C_3}\int_0^T \psi(E)\left(\int_\Omega wu_t^2dx + \int_{\Gamma_1} z_t f(z_t)d\Gamma\right)dt\\ &+ \frac{C_3}{T-C_3}\int_0^T \psi(E)\left(\int_{\Gamma_1} z_t^2 + G(z)d\Gamma + \int_\Omega wu_t^2dx\right)dt\end{aligned}$$

for $T > C_3$, This and (3.30) together yield

$$\begin{aligned}&\int_0^T \psi(E)\int_\Omega u_t^2dxdt\\ &\leq \frac{C_3^2}{T-C_3}\int_0^T \psi(E)\int_{\Gamma_1} G(z)d\Gamma dt + \frac{C_3 rT\sup\{\phi''(s);\ s\in(0,r]\}}{T-C_3}\int_0^T -E'dt\\ &\quad+ \int_0^T \psi(E)\Bigg[\left(\frac{C_3^2}{T-C_3}+\frac{C_3T}{T-C_3}\right)\int_\Omega wu_t^2dx + \frac{C_3T}{T-C_3}\int_{\Gamma_1} f^2(z_t)d\Gamma\\ &\quad+ \left(C_3 + \frac{C_3T}{T-C_3}\right)\int_{\Gamma_1} z_t^2 d\Gamma\Bigg]dt.\end{aligned}$$

Taking

$$T_1 = 2C_3, \quad C' = 2C_3^2 + 3C_3 + 2C_3 r\sup\{\phi''(s);\ s\in(0,r]\},$$

we then obtain the estimate (3.10).

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