Math 4543: Numerical Methods

Lecture 12 — Trapezoidal Rule of Integration

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Lecture Plan

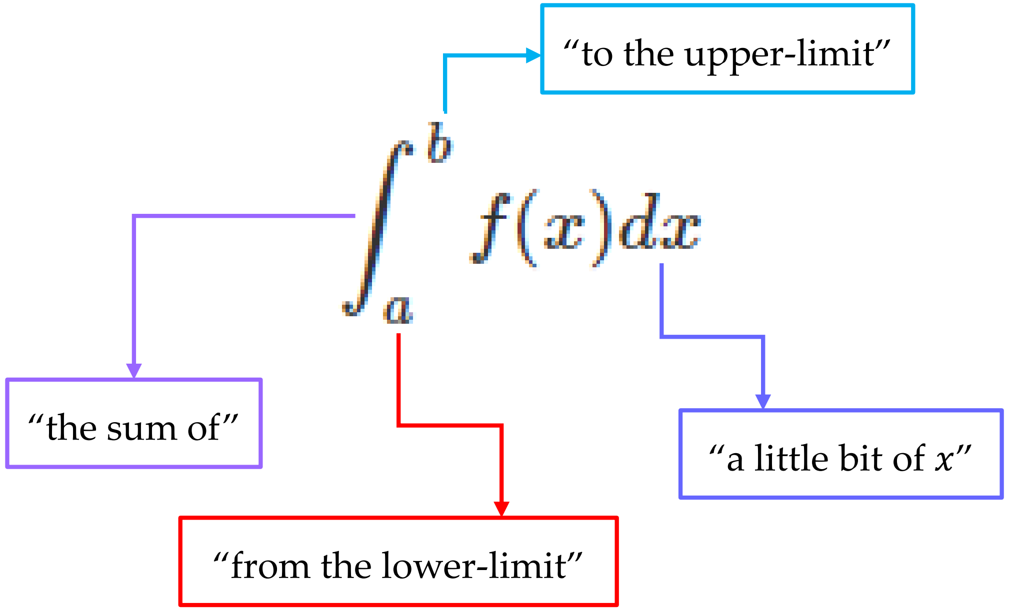
The agenda for today

- Understand the concept of Integration
- Recap the Riemann sum method of integration
- Single-segment Trapezoidal Rule
- Multiple-segment Trapezoidal Rule

Integration

What is it?

The dictionary definition of ***integration*** is combining parts so that they work together or form a whole.



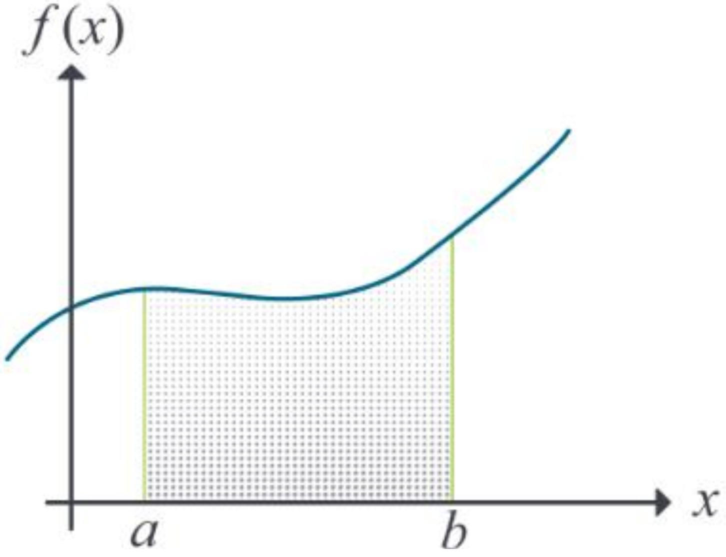


Figure 1 The definite integral as the area of a region under the curve, $\text{Area} = \int_a^b f(x)dx$.

The mean value $\bar{f}$ of a function f in an interval $[a, b]$ is given by

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$$

Mathematically, integration stands for finding the area under an integrand curve $f(x)$ from one point to another.

Riemann Sum Integration

Simplest way to perform numerical integration

We'll form rectangles of equal widths under the curve and *approximate* the area.

Let f be defined on the closed interval $[a, b]$, and let Δ be an arbitrary partition of $[a, b]$ such as:

$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$, where Δx_i is the length of the i^{th} subinterval (Figure 1).

If c_i is any point in the i^{th} subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \; x_{i-1} \le c_i \le x_i$$

is called a Riemann sum of the function f for the partition Δ on the interval $[a, b]$. For a given partition Δ, the length of the longest subinterval is called the norm of the partition. It is denoted by $\|\Delta\|$ (the norm of Δ). The following limit is used to define the definite integral.

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = I$$

$$I = \int_a^b f(x)dx$$

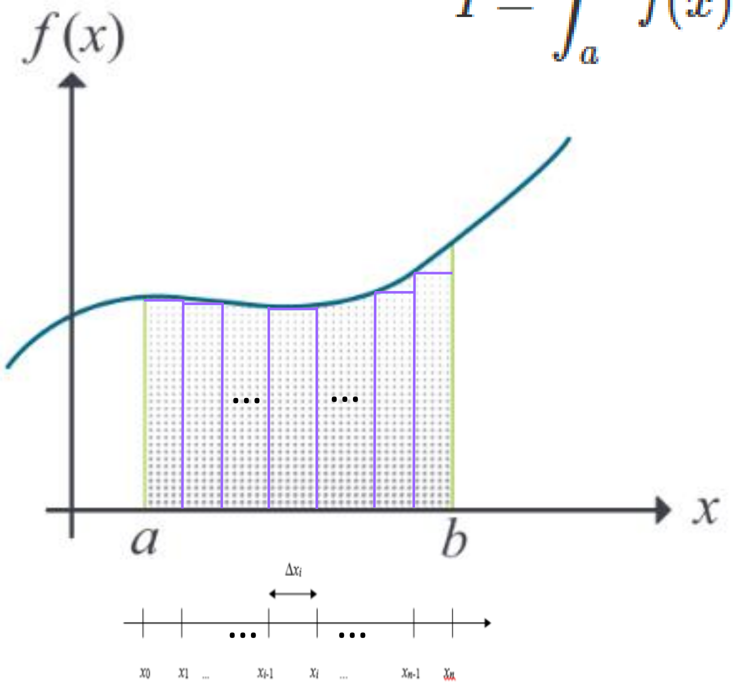


Figure 1 The definite integral as the area of a region under the curve, $\text{Area} = \int_a^b f(x)dx$.

Riemann Sum Integration

Simplest way to perform numerical integration

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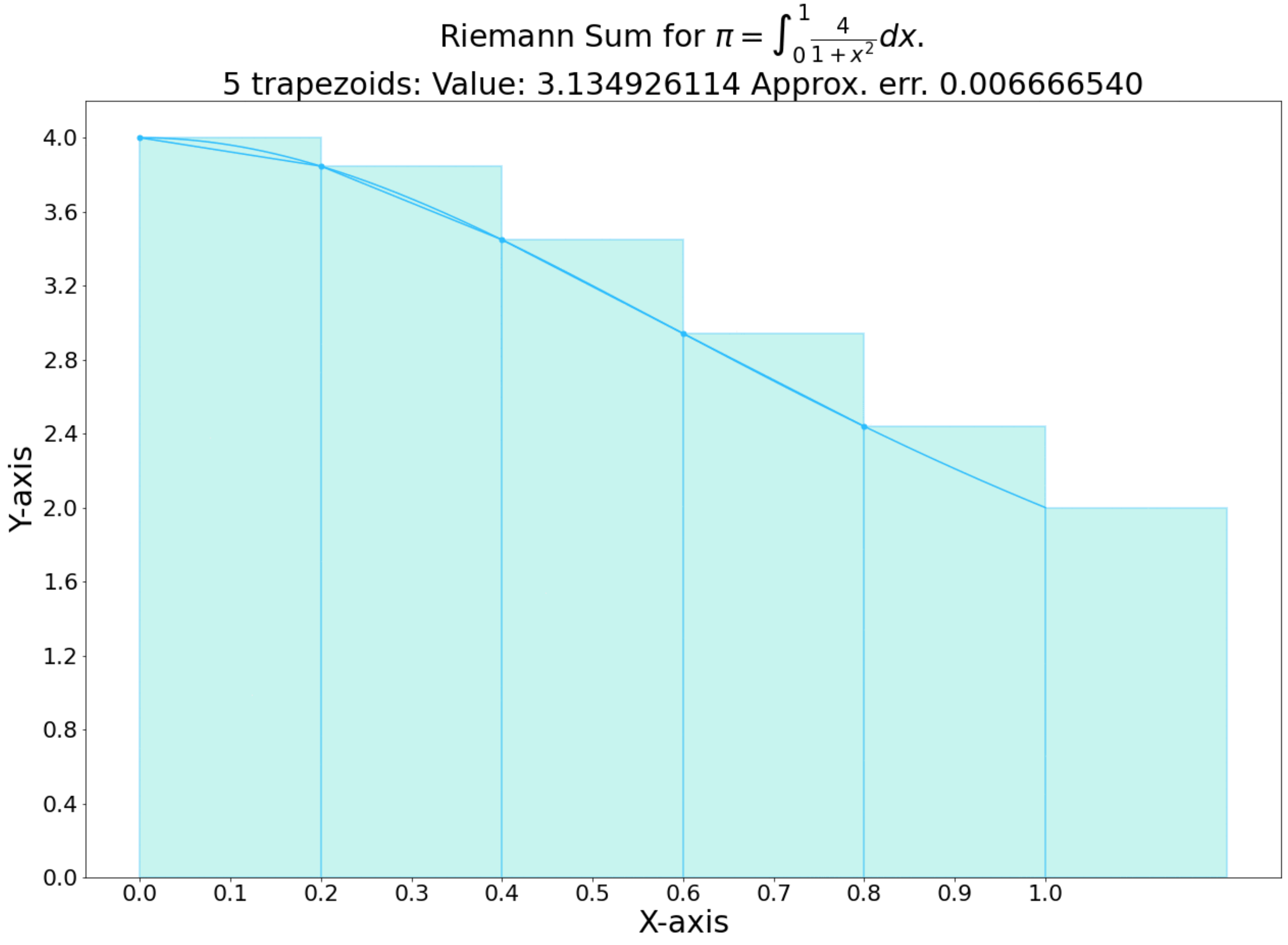

Single-segment Trapezoidal Rule

Using trapezoids instead of rectangles

The idea is to form a ***single trapezoid*** under the curve and ***approximate*** the area.

The trapezoidal rule is based on the Newton-Cotes formula that if one approximates the integrand by an n^{th} order polynomial, then the integral of the function is approximated by the integral of that n^{th} order polynomial.

$$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n.$$

where $f_n(x)$ is a n^{th} order polynomial. The trapezoidal rule assumes $n = 1$, that is, approximating the integral by a linear polynomial (straight line),

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx$$

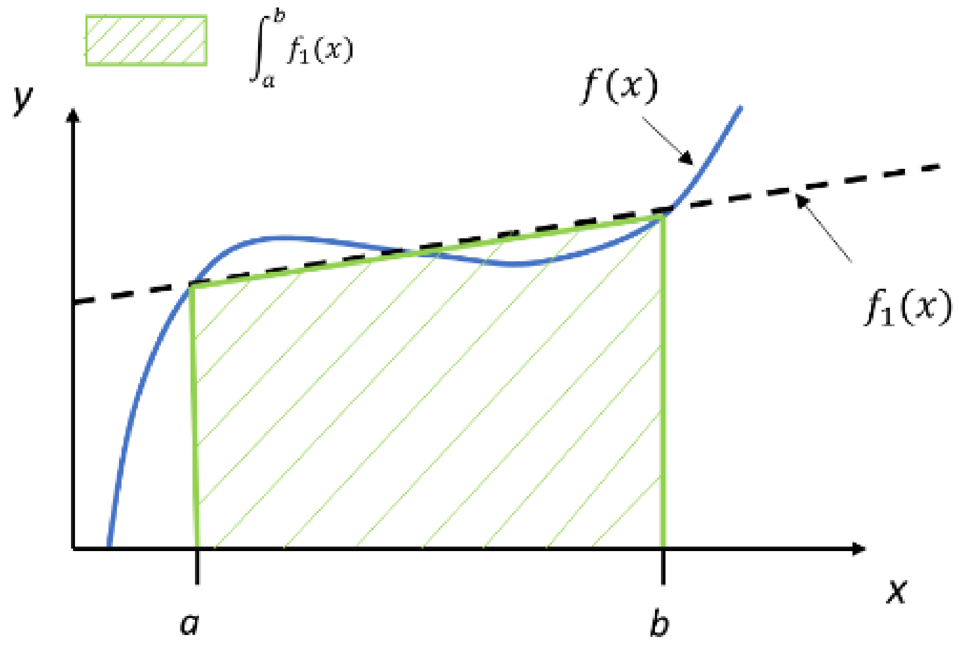

Figure 2. Approximating the function by a first-order polynomial to derive the trapezoidal rule

Single-segment Trapezoidal Rule

Method 1: Derivation from Calculus

Approximating the integrand $f(x)$ by a first-order polynomial $f_1(x)$, that is, $f_1(x) = a_0 + a_1 x$,

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx = \int_a^b (a_0 + a_1 x)dx$$

$$= a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) \qquad (5)$$

But what are a_0 and a_1? Now if one chooses, $(a, f(a))$ and $(b, f(b))$ as the two points to approximate $f(x)$ by a straight line from a to b,

$$f(a) = f_1(a) = a_0 + a_1 a$$
$$f(b) = f_1(b) = a_0 + a_1 b \qquad (6a, b)$$

Solving Equations (6a) and (6b) for a_0 and a_1, we get

$$a_1 = \frac{f(b)-f(a)}{b-a} \qquad a_0 = \frac{f(a)b - f(b)a}{b-a} \qquad (7a, b)$$

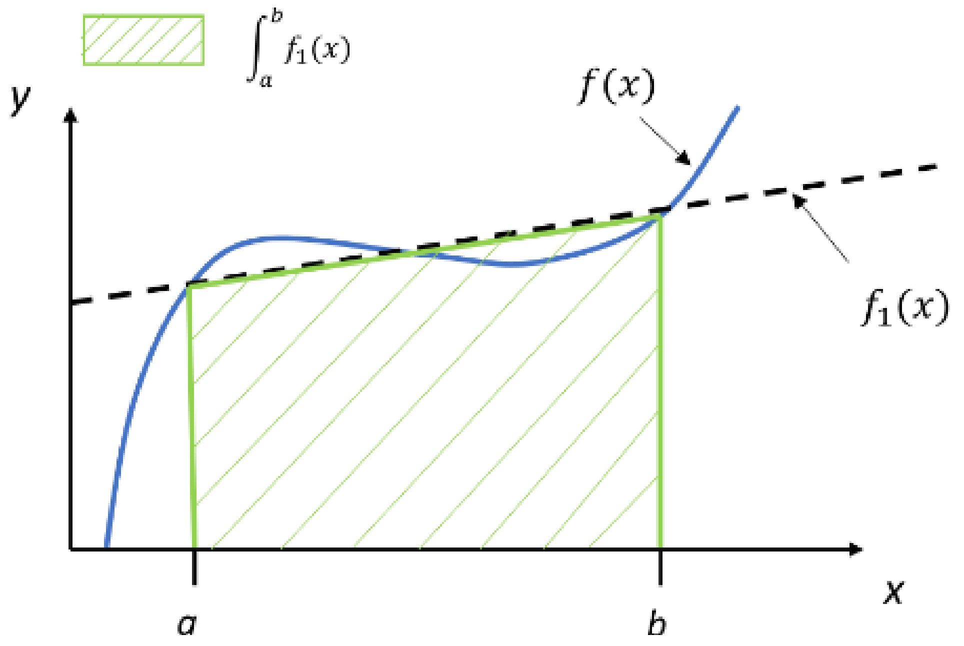

Figure 2. Approximating the function by a first-order polynomial to derive the trapezoidal rule

Substituting values of a_0 and a_1 from Equations (7a) and (7b) in Equation (5) gives,

$$\int_a^b f(x)dx \approx \frac{f(a)b - f(b)a}{b-a}(b-a) + \frac{f(b)-f(a)}{b-a}\frac{b^2-a^2}{2}$$

$$= \boxed{(b-a)\left[\frac{f(a)+f(b)}{2}\right]} \qquad (8)$$

Single-segment Trapezoidal Rule

Method 2: Derivation from Calculus (using NDD polynomial)

$f_1(x)$ can also be approximated by using Newton's divided difference polynomial as

$$f_1(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \qquad (A.1)$$

Hence

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx = \int_a^b \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right] dx$$

$$= \left[f(a)x + \frac{f(b)-f(a)}{b-a}\left(\frac{x^2}{2} - ax\right)\right]_a^b$$

$$= f(a)b - f(a)a + \left(\frac{f(b)-f(a)}{b-a}\right)\left(\frac{b^2}{2} - ab - \frac{a^2}{2} + a^2\right)$$

$$= f(a)b - f(a)a + \left(\frac{f(b)-f(a)}{b-a}\right)\left(\frac{b^2}{2} - ab + \frac{a^2}{2}\right)$$

$$= f(a)b - f(a)a + \left(\frac{f(b)-f(a)}{b-a}\right)\frac{1}{2}(b-a)^2$$

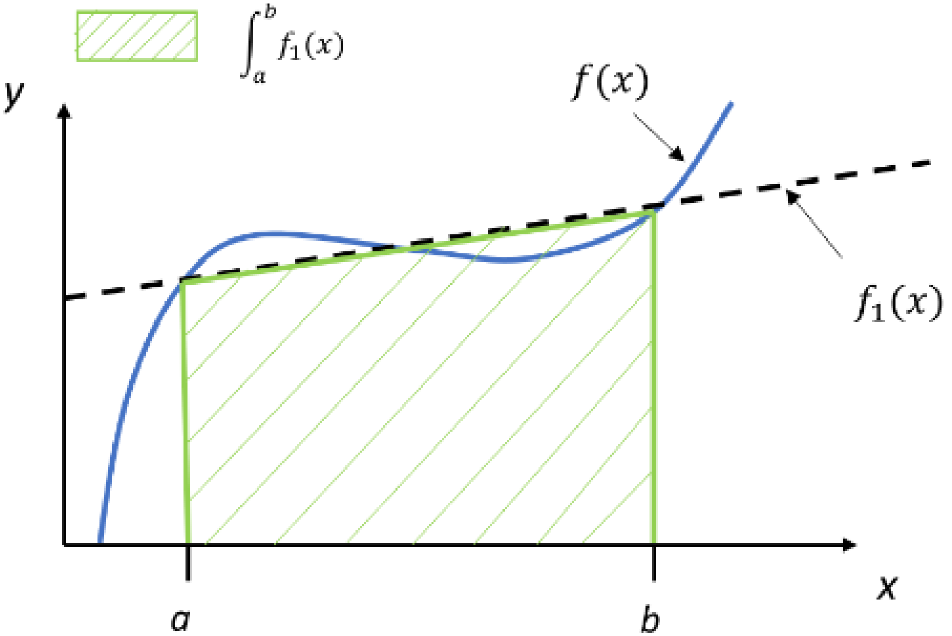

Figure 2. Approximating the function by a first-order polynomial to derive the trapezoidal rule

$$= f(a)b - f(a)a + \frac{1}{2}(f(b) - f(a))(b-a)$$

$$= f(a)b - f(a)a + \frac{1}{2}f(b)b - \frac{1}{2}f(b)a - \frac{1}{2}f(a)b + \frac{1}{2}f(a)a$$

$$= \frac{1}{2}f(a)b - \frac{1}{2}f(a)a + \frac{1}{2}f(b)b - \frac{1}{2}f(b)a$$

$$= \boxed{(b-a)\left[\frac{f(a)+f(b)}{2}\right]} \qquad (A.2)$$

Single-segment Trapezoidal Rule

Method 3: Derivation from Geometry

The trapezoidal rule can also be derived from geometry. Look at Figure 2. The area under the curve $f_1(x)$ is the area of a trapezoid. The integral

$$\int_a^b f(x)dx \approx \text{Area of trapezoid}$$
$$= \frac{1}{2}(\text{Sum of the length of parallel sides})$$
$$\times (\text{Perpendicular distance between the parallel sides})$$
$$= \frac{1}{2}(f(b) + f(a))\,(b - a)$$
$$= \boxed{(b - a)\left[\frac{f(a) + f(b)}{2}\right]} \qquad (A.3)$$

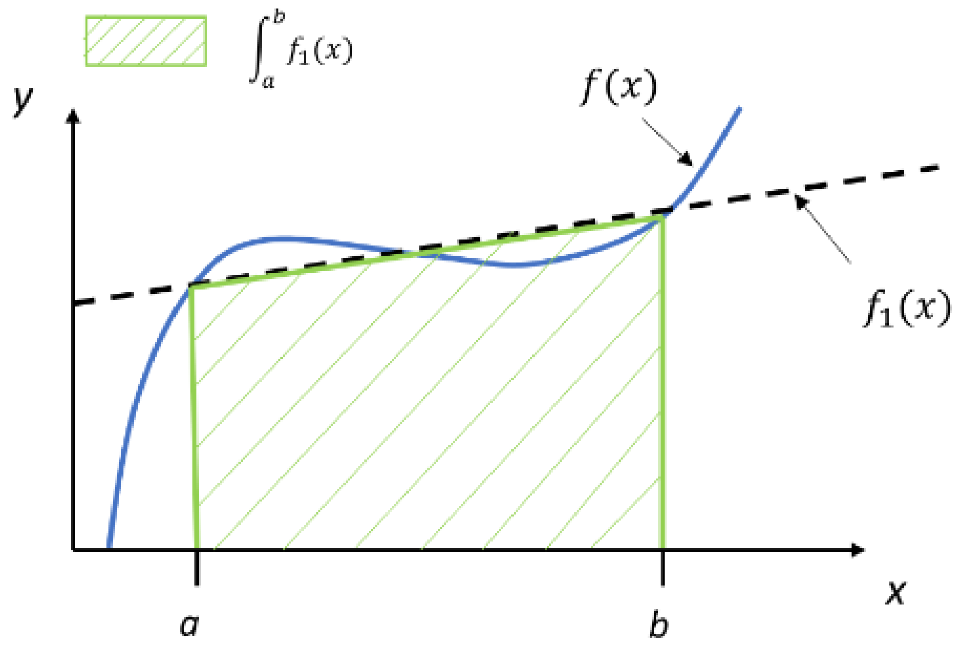

Figure 2. Approximating the function by a first-order polynomial to derive the trapezoidal rule

Single-segment Trapezoidal Rule

Method 4: Derivation from Method of Coefficients

Choose the integral $\int_a^b f(x)dx$ approximated as follows.

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b) \qquad (A.4)$$

The coefficients c_1 and c_2 are undetermined. We will find these coefficients such that the right-hand side is exact for integrals of a straight line $a_0 + a_1 x$.

So from exact integration

$$\int_a^b (a_0 + a_1 x)\,dx = \left[a_0 x + a_1 \frac{x^2}{2}\right]_a^b$$
$$= a_0 (b-a) + a_1 \left(\frac{b^2 - a^2}{2}\right) \qquad (A.5)$$

But we want the right-hand side formula to give the same result as Equation (A.5) for $f(x) = a_0 + a_1 x$ which is

$$c_1 f(a) + c_2 f(b) = c_1(a_0 + a_1 a) + c_2(a_0 + a_1 b)$$
$$= a_0 (c_1 + c_2) + a_1 (c_1 a + c_2 b) \qquad (A.6)$$

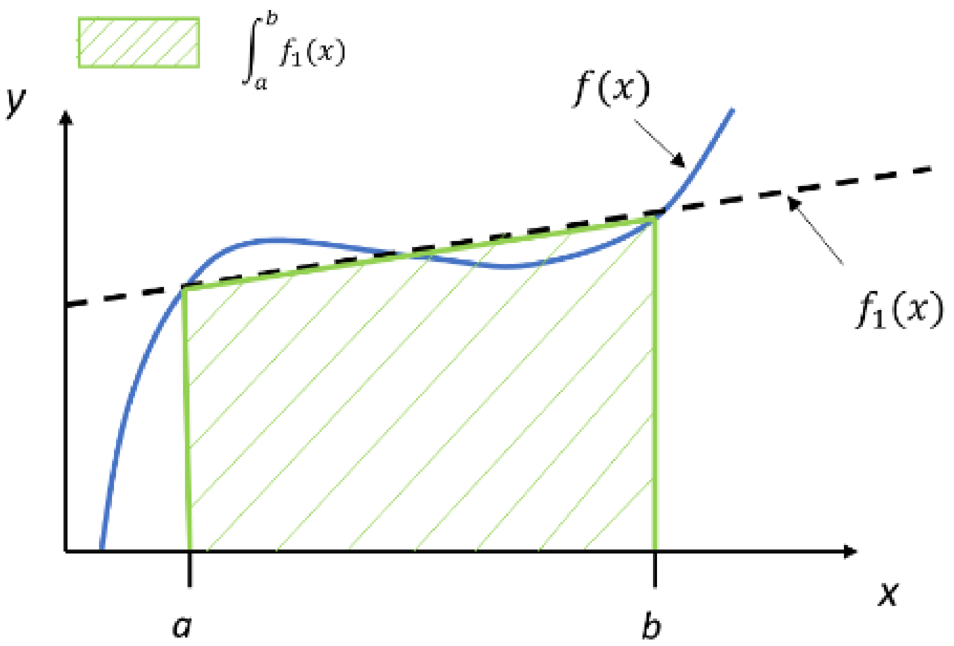

Figure 2. Approximating the function by a first-order polynomial to derive the trapezoidal rule

Hence from Equations (A.5) and (A.6),

$$a_0 (b-a) + a_1 \left(\frac{b^2 - a^2}{2}\right) = a_0 (c_1 + c_2) + a_1 (c_1 a + c_2 b) \qquad (A.7)$$

Single-segment Trapezoidal Rule

Method 4: Derivation from Method of Coefficients

Since a_0 and a_1 are arbitrary constants for the chosen general straight line, the coefficients of a_0 and a_1 need to be equal. That gives

$$c_1 + c_2 = b - a \qquad (A.8a)$$

$$c_1 a + c_2 b = \frac{b^2 - a^2}{2} \qquad (A.8b)$$

Multiplying Equation (A.8a) by a and subtracting from Equation (A.8b) gives

$$c_2 = \frac{b-a}{2} \qquad (A.9a)$$

Substituting the value of c_2 from Equation (A.9a) in Equation (A.8a) gives

$$c_1 = \frac{b-a}{2} \qquad (A.9b)$$

Therefore, from Equation (A.4), (A.9a), and (A.9b),

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$
$$= \boxed{\frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)} \qquad (A.10)$$

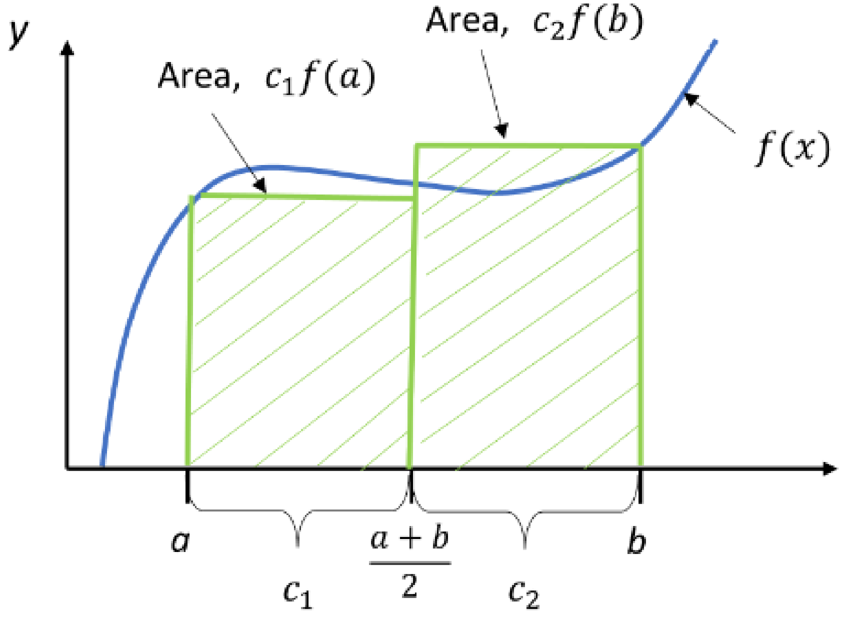

Figure 3 Area by the method of coefficients.

Single-segment Trapezoidal Rule

An example

The following integral is given

$$\int_{0.1}^{1.3} 5xe^{-2x}dx$$

a) Use the trapezoidal rule to estimate the value of the integral.

b) Find the true error, E_t for part (a).

c) Find the absolute relative true error, $|\varepsilon_t|$ for part (a).

Solution

a) where

$$\int_b^a f(x)dx \approx \frac{(b-a)}{2}[f(a)+f(b)],$$

$a = 0.1$

$b = 1.3$

Then

$$\begin{aligned}\int_{0.1}^{1.3} f(x)dx &\approx \frac{(1.3-0.1)}{2}[f(0.1)+f(1.3)] \\ &= 0.6\,[f(0.1)+f(1.3)] \\ &= 0.6[5(0.1)e^{-2(0.1)}+5(1.3)e^{-2(1.3)}] \\ &= 0.6(0.4094+0.4828) \\ &= 0.5353\end{aligned}$$

b) The true error is

$$\begin{aligned}E_t &= \text{True Value} - \text{Approximate Value} \\ &= 0.8939 - 0.5353 \\ &= 0.3586\end{aligned}$$

c) The absolute relative true error, $|\varepsilon_t|$ would then be

$$\begin{aligned}|\varepsilon_t| &= \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100 \\ &= \left|\frac{0.3586}{0.8939}\right| \times 100 \\ &= \boxed{40.12\%}\end{aligned}$$

Now that's a lotta error!

Multi-segment Trapezoidal Rule

The general idea

A single segment trapezoidal rule seldom gives you acceptable results for an integral. Instead for higher accuracy and its control, we can use the composite (also called multiple-segment) trapezoidal rule where the integral is broken into segments, and the single-segment trapezoidal rule is applied over each segment.

Divide $(b-a)$ into n equal segments, as shown in Figure 1. Then the width of each segment is

$$h = \frac{b-a}{n} \quad (1)$$

The integral I can be broken into n integrals as

$$I = \int_a^b f(x)dx$$

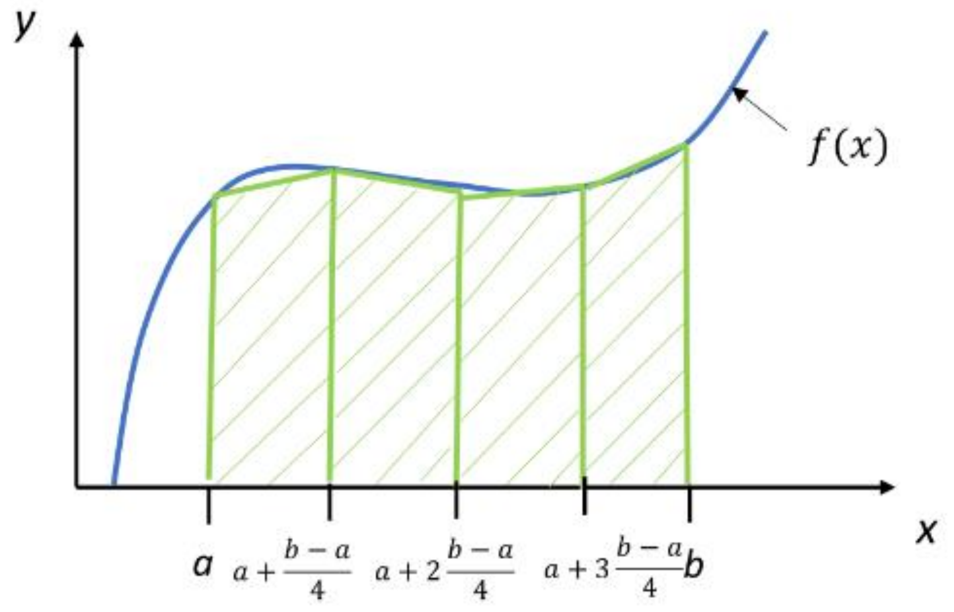

Figure 1 Composite ($n = 4$) trapezoidal rule

$$= \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \ldots + \int_{a+(n-2)h}^{a+(n-1)h} f(x)dx + \int_{a+(n-1)h}^{b} f(x)dx \quad (2)$$

Multi-segment Trapezoidal Rule

Deriving the formula

Applying single-segment trapezoidal rule on Equation (2) on each segment gives

$$\int_a^b f(x)dx \approx [(a+h)-a]\left[\frac{f(a)+f(a+h)}{2}\right]$$

$$+[(a+2h)-(a+h)]\left[\frac{f(a+h)+f(a+2h)}{2}\right]$$

$$+\dots\dots\dots\dots\dots$$

$$+[(a+(n-1)h)-(a+(n-2)h)]\left[\frac{f(a+(n-2)h)+f(a+(n-1)h)}{2}\right]$$

$$+[b-(a+(n-1)h)]\left[\frac{f(a+(n-1)h)+f(b)}{2}\right]$$

$$=h\left[\frac{f(a)+f(a+h)}{2}\right]+h\left[\frac{f(a+h)+f(a+2h)}{2}\right]$$

$$+\dots\dots\dots\dots\dots$$

$$+h\left[\frac{f(a+(n-2)h)+f(a+(n-1)h)}{2}\right]$$

$$+h\left[\frac{f(a+(n-1)h)+f(b)}{2}\right]$$

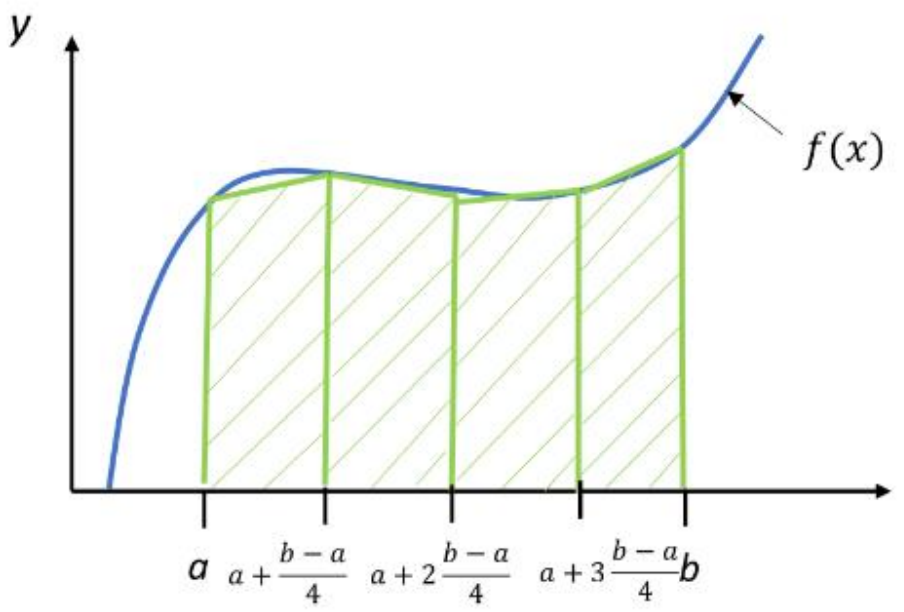

Figure 1 Composite ($n=4$) trapezoidal rule

$$=h\left[\frac{f(a)+2f(a+h)+2f(a+2h)+\dots+2f(a+(n-1)h)+f(b)}{2}\right]$$

$$=\frac{h}{2}\left[f(a)+2\left\{\sum_{i=1}^{n-1}f(a+ih)\right\}+f(b)\right]$$

$$=\boxed{\frac{b-a}{2n}\left[f(a)+2\left\{\sum_{i=1}^{n-1}f(a+ih)\right\}+f(b)\right]} \qquad (3)$$

Multi-segment Trapezoidal Rule

The same example (now with *multiple segments*)

The following integral is given:

$$\int_{0.1}^{1.3} 5xe^{-2x}\,dx$$

a) Use the composite trapezoidal rule to estimate the value of this integral. Use three segments.

b) Find the true error E_t for part (a).

c) Find the absolute relative true error $|\varepsilon_t|$ for part (a).

Multi-segment Trapezoidal Rule

The same example (now with *multiple segments*)

Solution

a) The solution using the composite trapezoidal rule with 3 segments is applied as follows.

$$I \approx \frac{b-a}{2n}\left[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)\right]$$

$n = 3$

$a = 0.1$

$b = 1.3$

$$h = \frac{b-a}{n} = \frac{1.3-0.1}{3} = 0.4$$

From Equation (3),

$$I \approx \frac{1.3-0.1}{6}\left[f(0.1) + 2\sum_{i=1}^{3-1} f(0.1+0.4i) + f(1.3)\right]$$

$$I \approx \frac{1.3-0.1}{6}\left[f(0.1) + 2\sum_{i=1}^{2} f(0.1+0.4i) + f(1.3)\right]$$

$$= 0.2[f(0.1) + 2f(0.5) + 2f(0.9) + f(1.3)]$$

$$= 0.2[5 \times 0.1 \times e^{-2(0.1)} + 2(5 \times 0.5 \times e^{-2(0.5)}) + 2(5 \times 0.9 \times e^{-2(0.9)}) + 5 \times 1.3 \times e^{-2(1.3)}]$$

$$= 0.84385$$

Multi-segment Trapezoidal Rule

The same example (now with *multiple segments*)

b) The exact value of the above integral can be found by integration by parts and is

$$\int_{0.1}^{1.3} 5xe^{-2x}dx = 0.89387$$

So the true error is

$$\begin{aligned} E_t &= \text{True Value} - \text{Approximate Value} \\ &= 0.89387 - 0.84385 \\ &= 0.05002 \end{aligned}$$

c) The absolute relative true error is

$$\begin{aligned} |\varepsilon_t| &= \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100 \\ &= \left| \frac{0.05002}{0.89387} \right| \times 100 \\ &= \boxed{5.5959\%} \end{aligned}$$

Much better!