Math 4543: Numerical Methods

Lecture 18 — *LU* Decomposition, Eigenvalues, and Eigenvectors

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Lecture Plan

The agenda for today

- Recap the concept of *LU* Decomposition from Linear Algebra

- Find the *LU* Decomposition of an example matrix

- Recap the concept of Eigenvalues and Eigenvectors from Linear Algebra

- Find the Eigenvalues and Eigenvectors of an example matrix

LU Decomposition

What is it and why do we need it?

The idea is to decompose a matrix A into an equivalent pair L and U, where L and U are *lower triangular* and *upper triangular* matrices respectively.

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} L_{11} & & \\ L_{21} & L_{22} & \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ & U_{22} & U_{23} \\ & & U_{33} \end{pmatrix}$$

✓ It helps us *memoize/record* the steps of *Gaussian Elimination*.

✓ Triangular system of equations are easy to solve using *Forward Substitution* and *Backward Substitution*.

✓ Facilitates efficient solving of $Ax = b$ for multiple column vectors b (especially for large A).

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{bmatrix} \xrightarrow{\begin{array}{c} R_2 = -2R_1 + R_2 \\ E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}} \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 2 & -10 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$R_3 = 2R_2 + R_3$$

$$E_{32} E_{21} A = U$$

$$\Rightarrow A = E_{21}^{-1} E_{32}^{-1} U$$

$$\underbrace{E_{21}^{-1} E_{32}^{-1}}_{L}$$

$$\Rightarrow A = LU$$

$$A = LU$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\left. \right\} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}_{L}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$U$$

7/11
...

$$A = LU$$

$$PA = LU$$

↓

pivot
matrix

$$A\vec{x} = \boxed{\vec{b}}$$

$$\Rightarrow PA\vec{x} = P\vec{b}$$

$$\Rightarrow LU\vec{x} = P\vec{b}$$

$$\Rightarrow LU\vec{x} = \vec{d}$$

$$\Rightarrow L\vec{y} = \vec{d}$$

✓ Fwd sub. → $\vec{y}$

$$U\vec{x} = \vec{y}$$

✓ Bckd sub. → $\vec{x}$

$$A\,\boxed{\vec{x}} = \vec{b}$$

?

$$\vec{b_1}, \vec{b_2}, ---, \vec{b_n}$$

LU Decomposition

An example

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 3 \\ (2) & -1 & 6 \\ 0 & 2 & -10 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 3 \\ (2) & -1 & 6 \\ (0) & (-2) & 2 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{pmatrix} = A.$$

Eigenvalues and Eigenvectors

What are they?

Suppose that A is a square $(n \times n)$ matrix. We say that a nonzero vector $\mathbf{v}$ is an eigenvector and a number λ is its eigenvalue if

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{14.1}$$

Geometrically this means that $A\mathbf{v}$ is in the same direction as $\mathbf{v}$, since multiplying a vector by a number changes its length, but not its direction.

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$

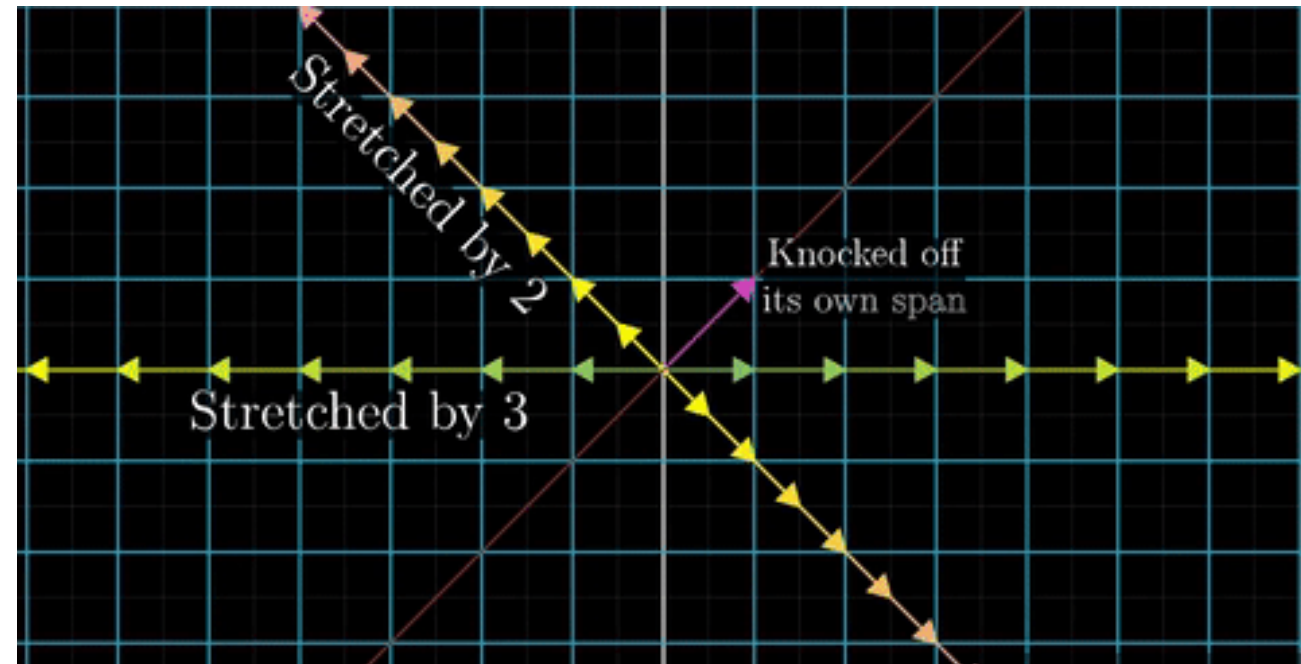$$A\mathbf{v} - \lambda\mathbf{v} = A\mathbf{v} - \lambda I\mathbf{v}$$

$$= (A - \lambda I)\mathbf{v}$$

$$= \mathbf{0}$$

Must be singular for non-zero $\vec{v}$.

$$\det(A - \lambda I) = 0.$$

This is called the *characteristic equation*.

Stretched by 2

Stretched by 3

Knocked off its own span

Link: https://youtu.be/PFDu9oVAE-g?si=ARTE9IkmFDfikkrT

$$A\vec{v} = \lambda\vec{v}$$

eigenvector   eigenvalue

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A\vec{x} = 0$$

non-zero
$$\Downarrow$$
singular

$$\det(A) = 0$$

$$A\vec{v} = \lambda I \vec{v}$$

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\Rightarrow A\vec{v} - \lambda I \vec{v} = 0$$

$$\Rightarrow (A - \lambda I)\vec{v} = 0$$

$$\Rightarrow \det(A - \lambda I) = 0 \Rightarrow |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

<u>characteristic eqn</u>

eigen
$$\downarrow$$
property
characteristic

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\Rightarrow |A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(5-\lambda) - 3 \times 4 = 0$$

$$\Rightarrow 5 - \lambda - 5\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda - 7 = 0$$

$$\Rightarrow \lambda = 7, -1$$

$$\lambda_1 \qquad \lambda_2$$

for $\lambda_1 = 7$:

$$\begin{bmatrix} 1-7 & 4 \\ 3 & 5-7 \end{bmatrix} \vec{v} = 0$$

$$\Rightarrow \begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix} \vec{v} = 0$$

$$\Rightarrow \left[\begin{array}{cc|c} -6 & 4 & 0 \\ 3 & -2 & 0 \end{array} \right]$$

$$-6x + 4y = 0$$

$$\vec{v_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

for $\lambda_2 = -1$:

$$\begin{bmatrix} 1+1 & 4 \\ 3 & 5+1 \end{bmatrix} \vec{v} = 0$$

$$\xrightarrow{R_2 = \frac{1}{2}R_1 + R_2} \left[\begin{array}{cc|c} -6 & 4 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\begin{bmatrix} 2 & 4 & \vdots & 0 \\ 3 & 6 & \vdots & 0 \end{bmatrix} \xrightarrow{R_2 = \frac{-3}{2}R_1 + R_2} \begin{bmatrix} 2 & 4 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$$

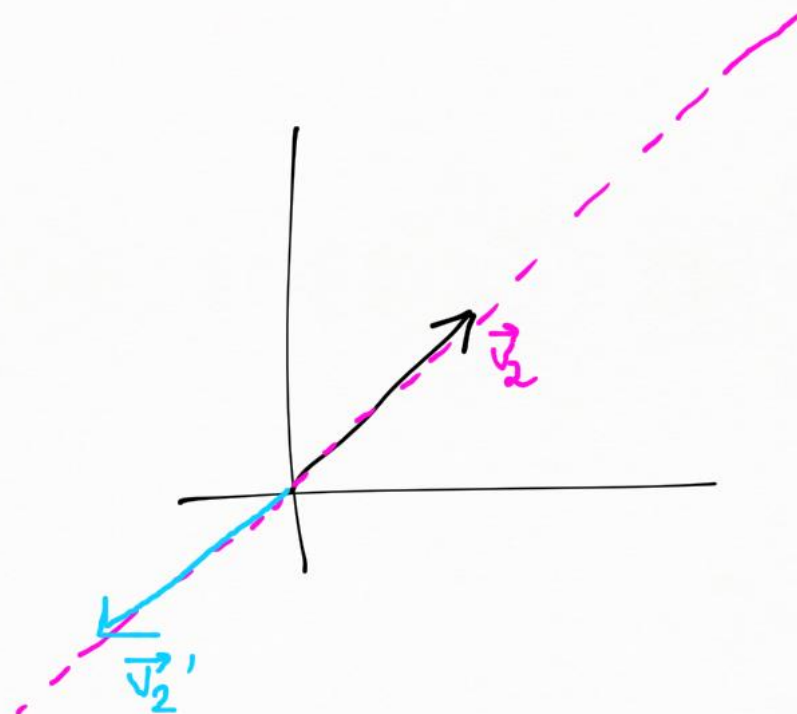$$2x + 4y = 0$$

$$\vec{V_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Eigenvalues and Eigenvectors

An example

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 4 \\ 3 & 5-\lambda \end{pmatrix}.$$

The determinant of $A - \lambda I$ is then

$$\det(A - \lambda I) = (1-\lambda)(5-\lambda) - 4 \cdot 3$$
$$= -7 - 6\lambda + \lambda^2.$$

The characteristic equation $\det(A - \lambda I) = 0$ is simply a quadratic equation: $\lambda^2 - 6\lambda - 7 = 0$.

The roots of this equation are $\lambda_1 = 7$ and $\lambda_2 = -1$.

For $\lambda_1 = 7$, the equation for the eigenvector $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is equivalent to the augmented matrix

$$\begin{pmatrix} -6 & 4 & \big| & 0 \\ 3 & -2 & \big| & 0 \end{pmatrix}$$

$$3x - 2y = 0$$

we can let $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

This comes from noticing that $(x, y) = (2, 3)$ is a solution of $3x - 2y = 0$.

For $\lambda_2 = -1$,

$$\begin{pmatrix} 2 & 4 & \big| & 0 \\ 3 & 6 & \big| & 0 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$A\mathbf{v}_1 = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 21 \end{pmatrix} = 7 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 7\mathbf{v}_1 \quad \text{and}$$

$$A\mathbf{v}_2 = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -1\mathbf{v}_2.$$