Math 4543: Numerical Methods

Lecture 15 — Runge-Kutta 2$^{\text{nd}}$ Order and Runge-Kutta 4$^{\text{th}}$ Order Method

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Lecture Plan

The agenda for today

- Understand the idea behind Runge-Kutta 2^{nd} order method
- Derive the formula of Runge-Kutta 2^{nd} order method
- Use 3 different variants of the Runge-Kutta 2^{nd} order method formula
- Compare the results obtained using each of these approaches
- Understand the idea behind Runge-Kutta 4^{th} order method
- Use 2 different variants of the Runge-Kutta 4^{th} order method formula

Runge-Kutta 2nd Order Method

What is it?

Runge-Kutta 2nd order method is a numerical technique to solve ***first-order ordinary differential equations*** of the form

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \qquad (1)$$

Bivariate Function

Initial Condition

Only first-order ODEs of the form given by Equation (1) can be solved by using Runge-Kutta 2nd order method. So, whatever 1st order differential equation we have, the preliminary step is to manipulate the equation to ***fit the aforementioned template***.

Runge-Kutta 2nd Order Method

Deriving the formula

Just consider the first 3 terms of the Taylor series!

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i,y_i}(x_{i+1}-x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i,y_i}(x_{i+1}-x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i,y_i}(x_{i+1}-x_i)^3 + \ldots$$

Since $\dfrac{dy}{dx} = f(x,y)$

$$y_{i+1} = y_i + f(x_i,y_i)(x_{i+1}-x_i) + \frac{1}{2!}f'(x_i,y_i)(x_{i+1}-x_i)^2 + \frac{1}{3!}f''(x_i,y_i)(x_{i+1}-x_i)^3 + \ldots$$

Euler's Method

$$y_{i+1} = y_i + f(x_i,y_i)\,h$$

Runge-Kutta 2nd Order Method

$$y_{i+1} = y_i + f(x_i,y_i)\,h + \frac{1}{2!}\boxed{f'(x_i,y_i)}\,h^2$$

An obvious **pitfall**!
Need to calculate $f'(x,y)$ symbolically using the chain-rule formula

$$f'(x,y) = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx}$$

Euler's method can also be dubbed as Runge-Kutta 1st Order method in that sense.

Runge-Kutta 2nd Order Method

Deriving the formula

Check out the **Appendix A** part of the lecture note for the proof.

To avoid finding $f'(x, y)$ symbolically, the RK2 formula approximates it as

$$y_{i+1} = y_i + \underbrace{f(x_i, y_i)\, h + \frac{1}{2!} f'(x_i, y_i)\, h^2}_{} \qquad (5)$$

$$y_{i+1} = y_i + \underbrace{(a_1 k_1 + a_2 k_2)\, h}_{} \qquad (7)$$

where

$$k_1 = f(x_i, y_i) \qquad (8a)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h) \qquad (8b)$$

We take a weighted sum of these two slope values. The weights are obviously a_1 and a_2.

So how do we find the unknowns a_1, a_2, p_1, and q_{11}? Without proof, equating Equation (5) and (7), gives three equations.

$$a_1 + a_2 = 1 \quad (9a) \qquad a_2 p_1 = \frac{1}{2} \quad (9b) \qquad a_2 q_{11} = \frac{1}{2} \quad (9c)$$

Since we have 3 equations and 4 unknowns, we can assume the value of one of the unknowns. The other three are then determined from the three equations. Generally, the value of a_2 is chosen to evaluate the other three constants. The three values used for a_2 are $\frac{1}{2}$, 1 and $\frac{2}{3}$, and are known as $\boxed{\text{Heun's Method}}$, the $\boxed{\text{midpoint method}}$, and $\boxed{\text{Ralston's method}}$, respectively.

Runge-Kutta 2nd Order Method

Variants of the formula

In the case of ***Heun's method,***

Here $a_2 = \dfrac{1}{2}$ is chosen, and from Equations (9a)-(9c),

$$a_1 = \frac{1}{2} \qquad p_1 = 1 \qquad q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \qquad (10)$$

where

$$k_1 = f\left(x_i, y_i\right) \qquad (11a)$$

$$k_2 = f\left(x_i + h, y_i + k_1 h\right) \qquad (11b)$$

This method is graphically explained in Figure 1.

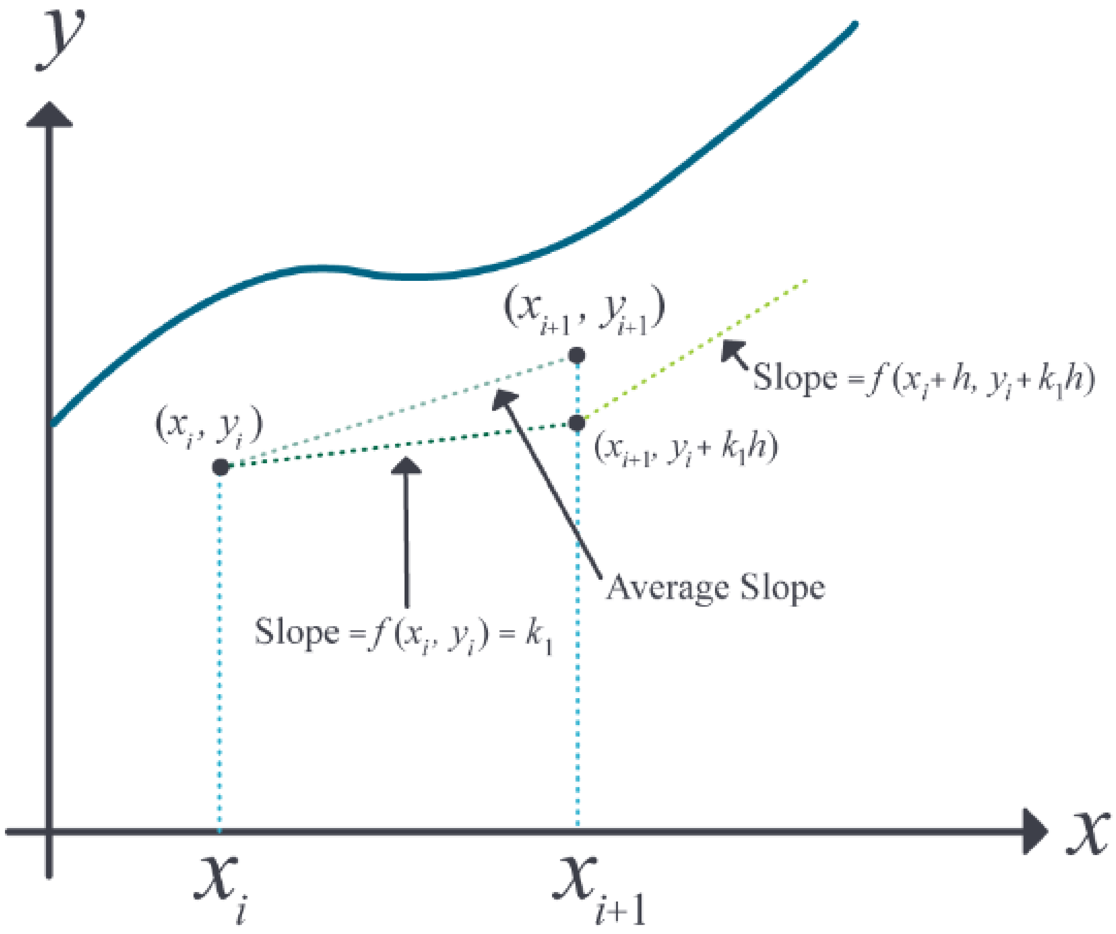


Figure 1 Runge-Kutta 2nd order method (Heun's method).

Runge-Kutta 2nd Order Method

Variants of the formula

In the case of ***Midpoint method,***

Here $a_2 = 1$ is chosen, and from Equations (9a)-(9c),

$$a_1 = 0 \qquad p_1 = \frac{1}{2} \qquad q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h \qquad (12)$$

where

$$k_1 = f(x_i, y_i) \qquad (13a)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \qquad (13b)$$

In the case of ***Ralston's method,***

Here $a_2 = \frac{2}{3}$ is chosen, and from Equations (9a)-(9c),

$$a_1 = \frac{1}{3} \qquad p_1 = \frac{3}{4} \qquad q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right) h \qquad (14)$$

where

$$k_1 = f(x_i, y_i) \qquad (15a)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right) \qquad (15b)$$

Runge-Kutta 2nd Order Method

An example

A ball at $1200\ \mathrm{K}$ is allowed to cool down in air at an ambient temperature of $300\ \mathrm{K}$. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

where θ is in K and t in seconds. Find the temperature at $t = 480$ seconds using Runge-Kutta 2nd order method. Assume a step size of $h = 240$ seconds.

Runge-Kutta 2nd Order Method

An example

Solution

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

$$f\left(t, \theta\right) = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

As per Heun's method given in the previous lesson for an ordinary differential equation,

$$\frac{d\theta}{dt} = f(t, \theta)$$

Heun's method formula is given by

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$k_1 = f\left(t_i, \theta_i\right)$$

$$k_2 = f\left(t_i + h, \theta_i + k_1 h\right)$$

For Step 1,

$$i = 0,\ t_0 = 0,\ \theta_0 = \theta(0) = 1200 \text{ K}$$

$$\begin{aligned} t_1 &= t_0 + h \\ &= 0 + 240 \\ &= 240 \text{ s} \end{aligned}$$

$$\begin{aligned} k_1 &= f\left(t_0, \theta_o\right) \\ &= f\left(0, 1200\right) \\ &= -2.2067 \times 10^{-12}\left(1200^4 - 81 \times 10^8\right) \\ &= -4.5579 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(t_0 + h, \theta_0 + k_1 h\right) \\ &= f\left(0 + 240, 1200 + \left(-4.5579\right)240\right) \\ &= f\left(240, 106.09\right) \\ &= -2.2067 \times 10^{-12}\left(106.09^4 - 81 \times 10^8\right) \\ &= 0.017595 \end{aligned}$$

Runge-Kutta 2nd Order Method

An example

$$\begin{aligned}\theta_1 &= \theta_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\ &= 1200 + \left(\frac{1}{2}(-4.5579) + \frac{1}{2}(0.017595)\right)240 \\ &= 1200 + (-2.2702)\,240 \\ &= 655.16\ \mathrm{K} \\ &\approx \theta(240)\end{aligned}$$

For Step 2

$$i = 1, t_1 = 240\ \mathrm{s}, \theta_1 = 655.16\ \mathrm{K}$$

$$\begin{aligned} &= t_1 + h \\ &= 240 + 240 \\ &= 480\ \mathrm{s}\end{aligned}$$

$$\begin{aligned}k_1 &= f(t_1, \theta_1) \\ &= f(240, 655.16) \\ &= -2.2067 \times 10^{-12}\left(655.16^4 - 81 \times 10^8\right) \\ &= -0.38869\end{aligned}$$

$$\begin{aligned}k_2 &= f(t_1 + h, \theta_1 + k_1 h) \\ &= f(240 + 240, 655.16 + (-0.38869)\,240) \\ &= f(480, 561.87) \\ &= -2.2067 \times 10^{-12}\left(561.87^4 - 81 \times 10^8\right) \\ &= -0.20206\end{aligned}$$

$$\begin{aligned}\theta_2 &= \theta_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\ &= 655.16 + \left(\frac{1}{2}(-0.38869) + \frac{1}{2}(-0.20206)\right)240 \\ &= 655.16 + (-0.29538)\,240 \\ &= \boxed{584.27\ \mathrm{K}} \\ &\approx \theta(480)\end{aligned}$$

Runge-Kutta 2nd Order Method

An example

The results from Heun's method are compared with the exact results in Figure 1.

The exact solution of the ordinary differential equation is given by the solution of a nonlinear equation as

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.0033333\theta) = -0.22067 \times 10^{-3} t - 2.9282$$

The solution to this nonlinear equation at $t = 480$ s is

$$\theta(480) = 647.57 \text{ K}$$

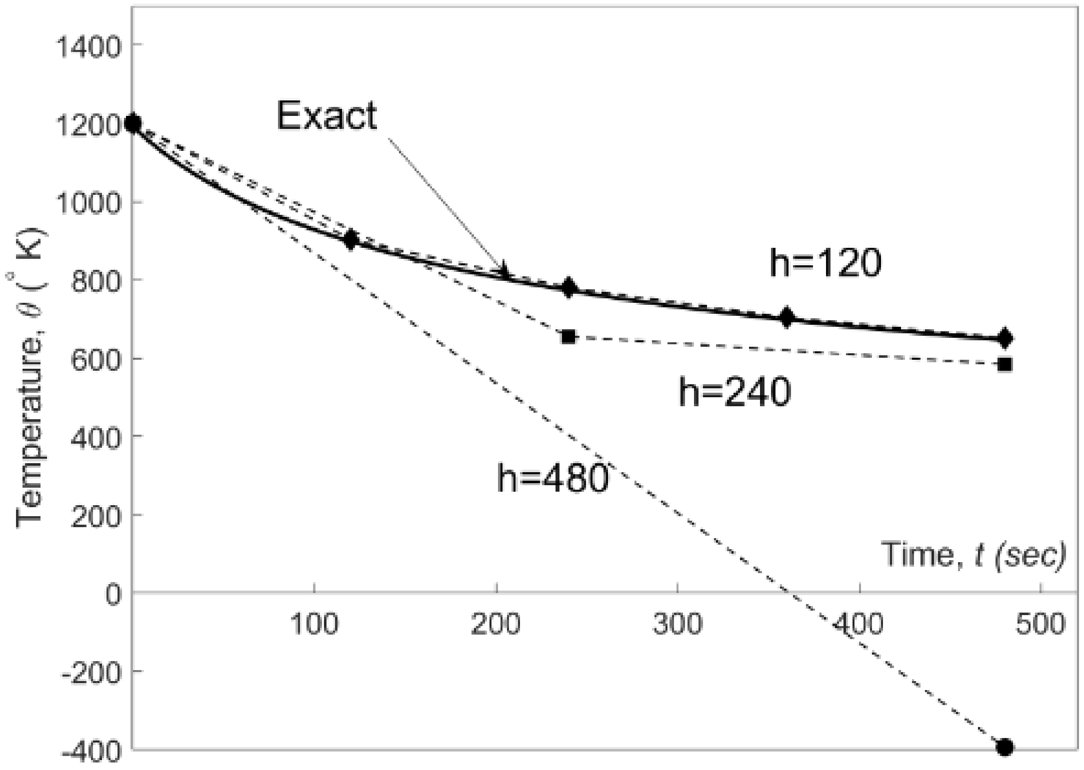


Figure 1 Heun's method results for different step sizes.

Runge-Kutta 2nd Order Method

An example

Using a smaller step size would increase the accuracy of the result, as given in Table 1 and Figure 2 below.

Table 1 Effect of step size for Heun's method

Step size, h	$\theta(480)$	E_t	$\lvert\epsilon_t\rvert\,\%$
480	-393.87	1041.4	160.82
240	584.27	63.304	9.7756
120	651.35	-3.7762	0.58313
60	649.91	-2.3406	0.36145
30	648.21	-0.63219	0.097625

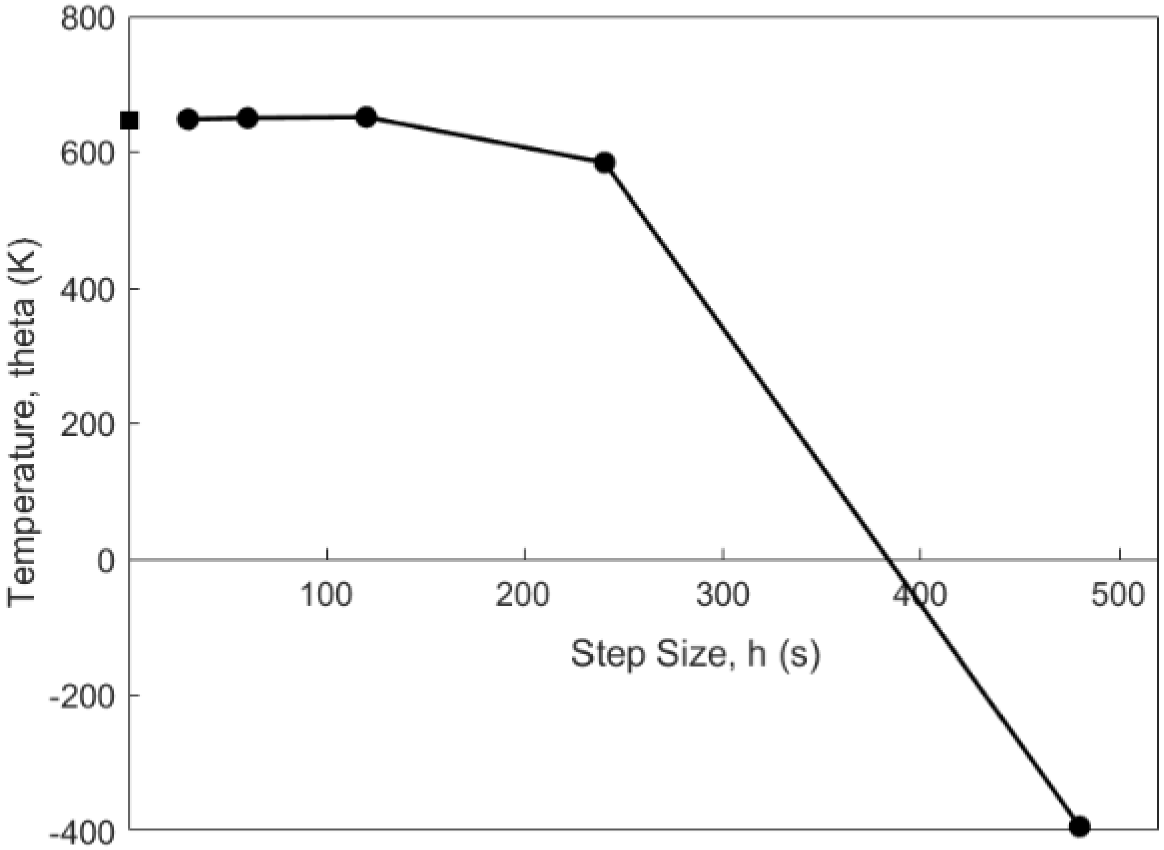


Figure 2 Effect of step size in Heun's method.

Runge-Kutta 2nd Order Method

An example

In Table 2, Euler's method and the Runge-Kutta 2nd order method results are shown as a function of step size,

while in Figure 3, the comparison is shown over time.

Table 2 Comparison of Euler and the Runge-Kutta methods

Step size, h	$\theta(480)$			
	Euler	*Heun*	*Midpoint*	*Ralston*
480	−987.84	−393.87	1208.4	449.78
240	110.32	584.27	976.87	690.01
120	546.77	651.35	690.20	667.71
60	614.97	649.91	654.85	652.25
30	632.77	648.21	649.02	648.61

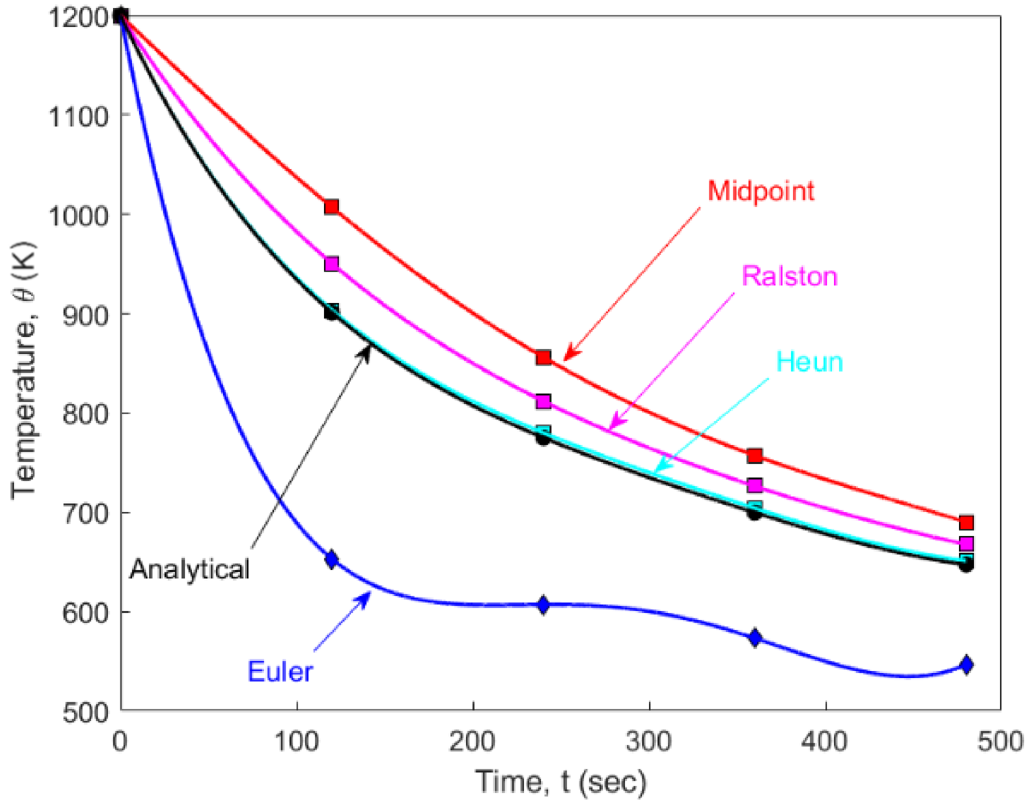


Figure 3 Comparison of Euler and Runge Kutta methods with exact results over time.

Runge-Kutta 4th Order Method

What is it?

Runge-Kutta 4th order method is yet another numerical technique to solve ***first-order ordinary differential equations*** of the form

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \qquad (1)$$

Bivariate Function

Initial Condition

Only first-order ODEs of the form given by Equation (1) can be solved by using Runge-Kutta 4th order method. So, whatever 1st order differential equation we have, the preliminary step is to manipulate the equation to ***fit the aforementioned template***.

Runge-Kutta 4th Order Method

Deriving the formula

This time, consider the first 5 terms of the Taylor series!

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i,y_i}(x_{i+1}-x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i,y_i}(x_{i+1}-x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i,y_i}(x_{i+1}-x_i)^3 + \frac{1}{4!}\left.\frac{d^4y}{dx^4}\right|_{x_i,y_i}(x_{i+1}-x_i)^4 \ldots$$

Since $\frac{dy}{dx} = f(x,y)$

$$y_{i+1} = y_i + f(x_i,y_i)(x_{i+1}-x_i) + \frac{1}{2!}f'(x_i,y_i)(x_{i+1}-x_i)^2 + \frac{1}{3!}f''(x_i,y_i)(x_{i+1}-x_i)^3 + \frac{1}{4!}f'''(x_i,y_i)(x_{i+1}-x_i)^4 \ldots$$

Euler's Method

$$y_{i+1} = y_i + f(x_i,y_i)\,h$$

Runge-Kutta 2nd Order Method

$$y_{i+1} = y_i + f(x_i,y_i)\,h + \frac{1}{2!}f'(x_i,y_i)\,h^2$$

Runge-Kutta 4th Order Method

$$y_{i+1} = y_i + f(x_i,y_i)h + \frac{1}{2!}f'(x_i,y_i)h^2 + \frac{1}{3!}f''(x_i,y_i)h^3 + \frac{1}{4!}f'''(x_i,y_i)h^4$$

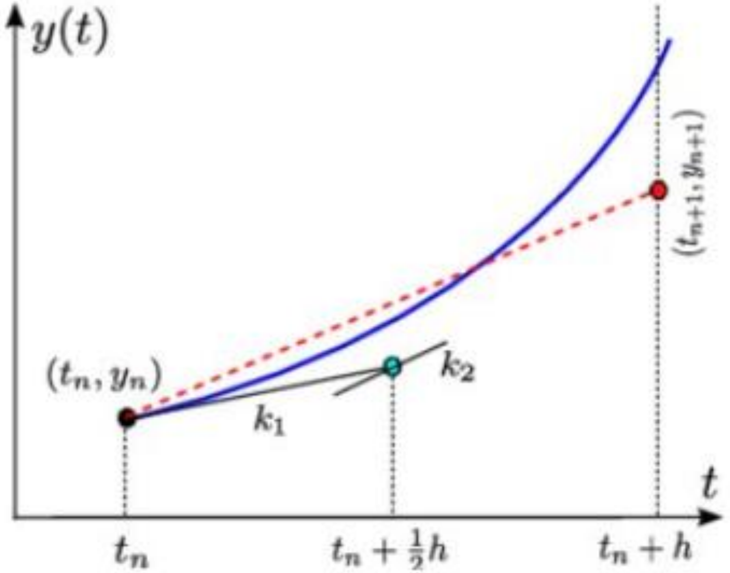
Runge-Kutta 4th Order Method

Deriving the formula

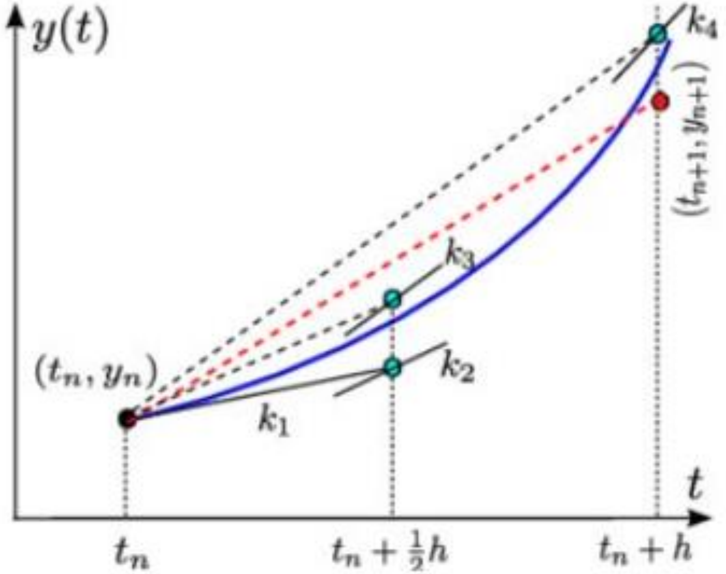
To avoid finding $f'(x,y)$, $f''(x,y)$, and $f'''(x,y)$ symbolically, the RK4 formula approximates it as

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 + \frac{1}{3!}f''(x_i, y_i)h^3 + \frac{1}{4!}f'''(x_i, y_i)h^4$$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2 + a_3 k_3 + a_4 k_4)h$$



Runge-Kutta 2nd Order (Midpoint method)



Runge-Kutta 4th Order

Runge-Kutta 4th Order Method

Deriving the formula

To avoid finding $f'(x,y)$, $f''(x,y)$, and $f'''(x,y)$ symbolically, the RK4 formula approximates it as

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 + \frac{1}{3!}f''(x_i, y_i)h^3 + \frac{1}{4!}f'''(x_i, y_i)h^4$$

$$\boxed{y_{i+1} = y_i + (a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4)h}$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h\right)$$

$$k_4 = f(x_i + h, y_i + k_3h)$$

Runge's approach (same as Simpson's 1/3 rule)

If $f(x,y)$ was a univariate function of x.

$$y_{i+1} = y_i + \frac{1}{8}(k_1 + 3k_2 + 3k_3 + k_4)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{3}h, y_i + \frac{1}{3}hk_1\right)$$

$$k_3 = f\left(x_i + \frac{2}{3}h, y_i - \frac{1}{3}hk_1 + hk_2\right)$$

$$k_4 = f(x_i + h, y_i + hk_1 - hk_2 + hk_3)$$

Kutta's approach (same as Simpson's 3/8 rule)

Runge-Kutta 4^{th} Order Method

An example

A ball at $1200\ \mathrm{K}$ is allowed to cool down in air at an ambient temperature of $300\ \mathrm{K}$. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12} \left(\theta^4 - 81 \times 10^8\right)$$

where θ is in K and t in seconds. Find the temperature at $t = 480$ seconds using Runge-Kutta 4th order method. Assume a step size of $h = 240$ seconds.

Runge-Kutta 4th Order Method

An example

Solution

$$\frac{d\theta}{dt} = -2.2067\times 10^{-12}\left(\theta^4 - 81\times 10^8\right)$$

$$f(t,\theta) = -2.2067\times 10^{-12}\left(\theta^4 - 81\times 10^8\right)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $\theta_0 = 1200\text{K}$

$$\begin{aligned} k_1 &= f(t_0, \theta_0) \\ &= f(0, 1200) \\ &= -2.2067\times 10^{-12}\left(1200^4 - 81\times 10^8\right) \\ &= -4.5579 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_1 h\right) \\ &= f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-4.5579)\times 240\right) \\ &= f(120, 653.05) \\ &= -2.2067\times 10^{-12}\left(653.05^4 - 81\times 10^8\right) \\ &= -0.38347 \end{aligned}$$

$$\begin{aligned} k_3 &= f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_2 h\right) \\ &= f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-0.38347)\times 240\right) \\ &= f(120, 1154.0) \\ &= -2.2067\times 10^{-12}\left(1154.0^4 - 81\times 10^8\right) \\ &= -3.8954 \end{aligned}$$

$$\begin{aligned} k_4 &= f(t_0 + h, \theta_0 + k_3 h) \\ &= f(0 + 240, 1200 + (-3.894)\times 240) \\ &= f(240, 265.10) \\ &= -2.2067\times 10^{-12}\left(265.10^4 - 81\times 10^8\right) \\ &= 0.0069750 \end{aligned}$$

$$\begin{aligned} \theta_1 &= \theta_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\ &= 1200 + \frac{1}{6}(-4.5579 + 2(-0.38347) + 2(-3.8954) + (0.069750))240 \\ &= 1200 + (-2.1848)\times 240 \\ &= 675.65\,\text{K} \end{aligned}$$

Runge-Kutta 4th Order Method

An example

θ_1 is the approximate temperature at

$$
\begin{aligned}
t &= t_1 \\
&= t_0 + h \\
&= 0 + 240 \\
&= 240 \\
\theta_1 &= \theta(240) \\
&\approx 675.65\,\text{K}
\end{aligned}
$$

For $i = 1, t_1 = 240, \theta_1 = 675.65\,\text{K}$

$$
\begin{aligned}
k_1 &= f(t_1, \theta_1) \\
&= f(240, 675.65) \\
&= -2.2067\times 10^{-12}\left(675.65^4 - 81\times 10^8\right) \\
&= -0.44199
\end{aligned}
$$

$$
\begin{aligned}
k_2 &= f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_1 h\right) \\
&= f\left(240 + \frac{1}{2}(240), 675.65 + \frac{1}{2}(-0.44199)240\right) \\
&= f(360, 622.61) \\
&= -2.2067\times 10^{-12}\left(622.61^4 - 81\times 10^8\right) \\
&= -0.31372
\end{aligned}
$$

$$
\begin{aligned}
k_3 &= f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_2 h\right) \\
&= f\left(240 + \frac{1}{2}(240), 675.65 + \frac{1}{2}(-0.31372)\times 240\right) \\
&= f(360, 638.00) \\
&= -2.2067\times 10^{-12}\left(638.00^4 - 81\times 10^8\right) \\
&= -0.34775
\end{aligned}
$$

$$
\begin{aligned}
k_4 &= f(t_1 + h, \theta_1 + k_3 h) \\
&= f(240 + 240, 675.65 + (-0.34775)\times 240) \\
&= f(480, 592.19) \\
&= 2.2067\times 10^{-12}\left(592.19^4 - 81\times 10^8\right) \\
&= -0.25351
\end{aligned}
$$

$$
\begin{aligned}
\theta_2 &= \theta_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\
&= 675.65 + \frac{1}{6}(-0.44199 + 2(-0.31372) + 2(-0.34775) + (-0.25351))\times 240 \\
&= 675.65 + \frac{1}{6}(-2.0184)\times 240 \\
&= \boxed{594.91\,\text{K}}
\end{aligned}
$$

Runge-Kutta 4th Order Method

An example

θ_2 is the approximate temperature at

$$t = t_2$$
$$= t_1 + h$$
$$= 240 + 240$$
$$= 480$$

$$\theta_2 = \theta(480)$$
$$\approx 594.91\,\text{K}$$

The exact solution of the ordinary differential equation is given by the solution of a nonlinear equation as

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.0033333\theta) = -0.22067 \times 10^{-3} t - 2.9282$$

The solution to this nonlinear equation at $t = 480$ s is

$$\theta(480) = 647.57\ \text{K}$$

Figure 1 compares the exact solution with the numerical solution using the Runge-Kutta 4th order method with different step sizes.

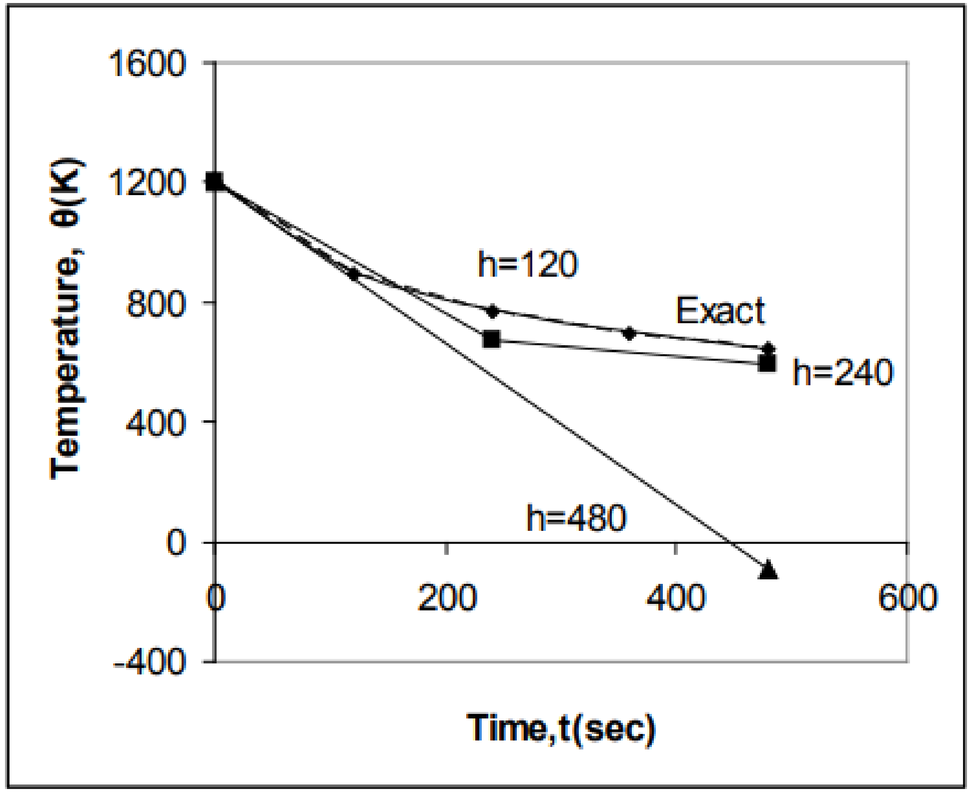


Figure 1 Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

Runge-Kutta 4th Order Method

An example

Table 1 and Figure 2 show the effect of step size on the value of the calculated temperature at $t = 480$ seconds.

Table 1 Value of temperature at time, $t = 480\,\text{s}$ for different step sizes

Step size, h	$\theta(480)$	E_t	$\lvert\varepsilon_t\rvert\%$
480	-90.278	737.85	113.94
240	594.91	52.660	8.1319
120	646.16	1.4122	0.21807
60	647.54	0.033626	0.0051926
30	647.57	0.00086900	0.00013419

So much better than the RK2 method!

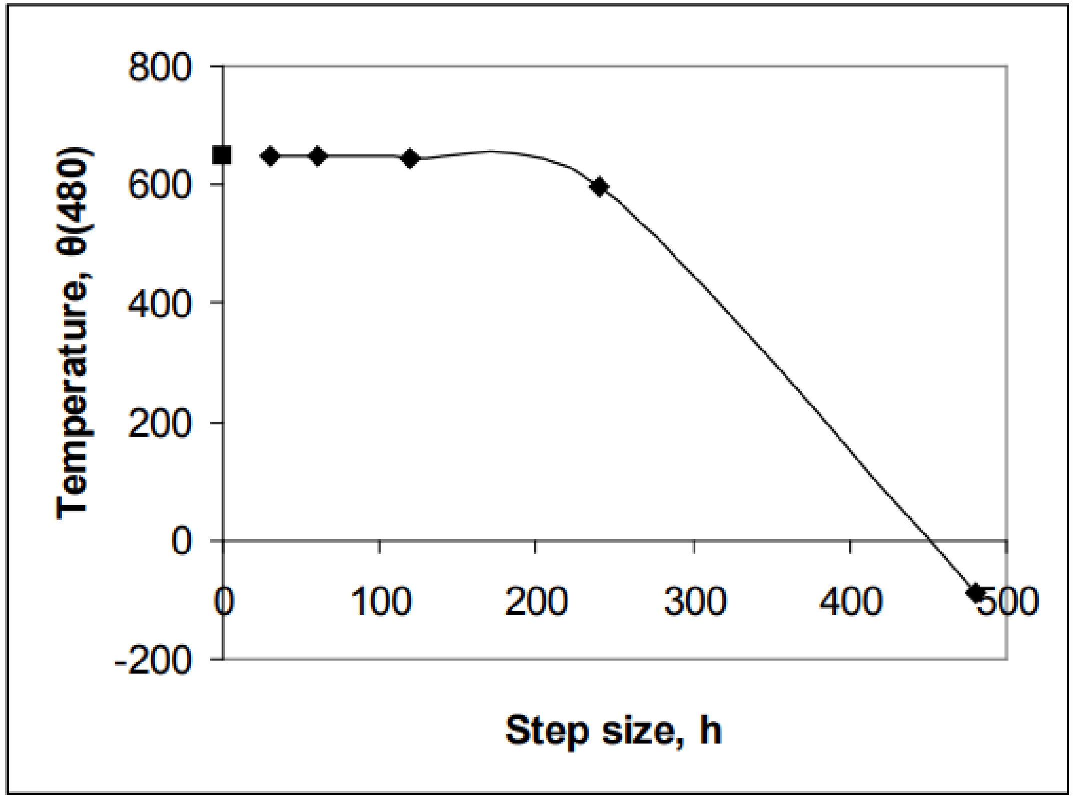


Figure 2 Effect of step size in Runge-Kutta 4th order method.

Mini Quiz

Establishing upper bounds of Truncation Errors

What are the growth rates of the ***Local Truncation Error*** and the ***Global Truncation Error*** in the case of the Runge-Kutta 2$^{\text{nd}}$ order method?

What about the Runge-Kutta 4$^{\text{th}}$ order method?