Math 4543: Numerical Methods

Lecture 14 — Euler's Method for Ordinary Differential Equations

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Lecture Plan

The agenda for today

- Recap the concept of Ordinary Differential Equations (ODEs)
- Understand the idea behind Euler's method
- Derive the formula for Euler's method
- Approximate the error in the Euler's method using the Taylor Series
- Use Euler's method to solve definite intergrals

Euler’s Method

What is it?

Euler’s method is a numerical technique to solve ***first-order ordinary differential equations*** of the form

$$\frac{dy}{dx} = \underbrace{f(x, y)}_{\text{Bivariate Function}}, \underbrace{y(x_0) = y_0}_{\text{Initial Condition}} \qquad (1)$$

Only first-order ordinary differential equations of the form given by Equation (1) can be solved by using Euler’s method. So, whatever 1^{st} order differential equation we have, the preliminary step is to manipulate the equation to ***fit the aforementioned template***.

Euler's Method

Rewriting examples

Rewrite $\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$

in $\frac{dy}{dx} = f(x, y),\ y(0) = y_0$ form.

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Rewrite $e^y \frac{dy}{dx} + x^2y^2 = 2\sin(3x),\ y(0) = 5$

in $\frac{dy}{dx} = f(x, y),\ y(0) = y_0$ form.

Solution

$$e^y \frac{dy}{dx} + x^2y^2 = 2\sin(3x),\ y(0) = 5$$

$$\frac{dy}{dx} = \frac{2\sin(3x) - x^2y^2}{e^y},\ y(0) = 5$$

In this case

$$f(x, y) = \frac{2\sin(3x) - x^2y^2}{e^y}$$

Euler's Method

Deriving the formula

At $x = 0$, we are given the value of $y = y_0$. Let us call $x = 0$ as x_0. Now since we know the slope of y with respect to x, that is, $f(x, y)$, then at $x = x_0$, the slope is $f(x_0, y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0) = y_0$.

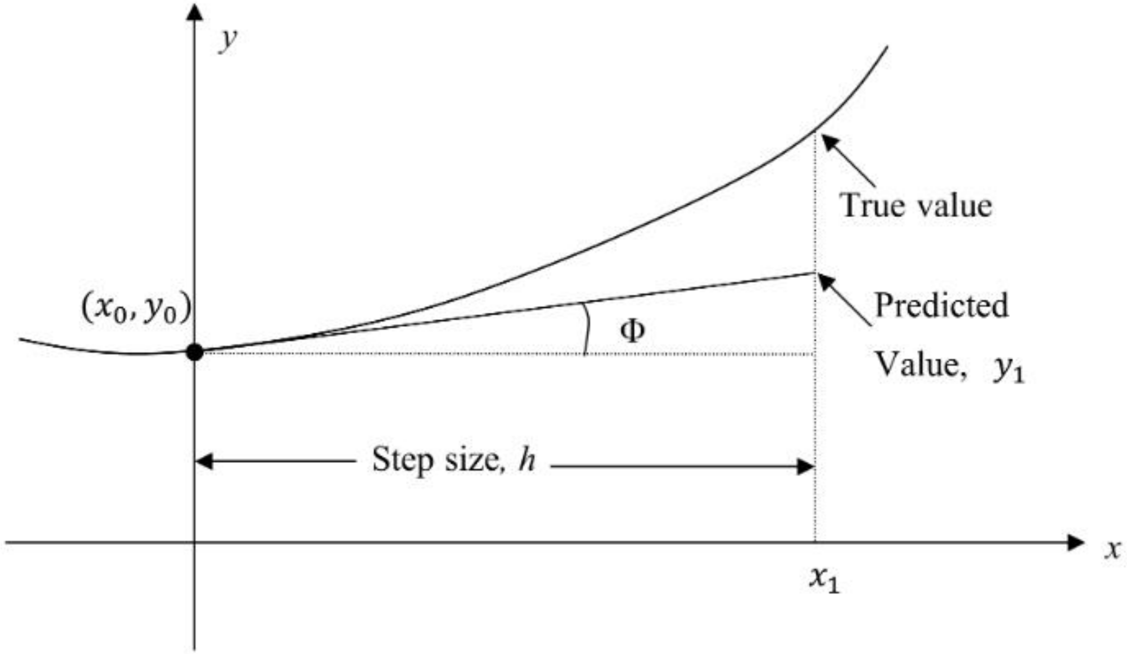


Figure 1 Graphical interpretation of the first step of Euler's method.

So the slope at $x = x_0$, as shown in Figure 1, is

$$Slope,\ \left.\frac{dy}{dx}\right|_{x_0,y_0} = \frac{\text{Rise}}{\text{Run}} = \frac{y_1 - y_0}{x_1 - x_0}$$
$$= f(x_0, y_0)$$

gives

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

Calling $x_1 - x_0$ the step size h, we get

$$y_1 = y_0 + f(x_0, y_0)\, h \qquad (2)$$

One can now use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2, and that would be the predicted value at x_2, given by

$$y_2 = y_1 + f(x_1, y_1)\, h$$

$$x_2 = x_1 + h$$

Euler's Method

Deriving the formula

Based on the above equations, if we now know the estimated value of y at x_i as y_i, then

$$\boxed{y_{i+1} = y_i + f(x_i, y_i)\, h} \qquad (3)$$

Equation (3) is known as Euler's method and is illustrated graphically in Figure 2. In some textbooks, it is also called the Euler-Cauchy method.

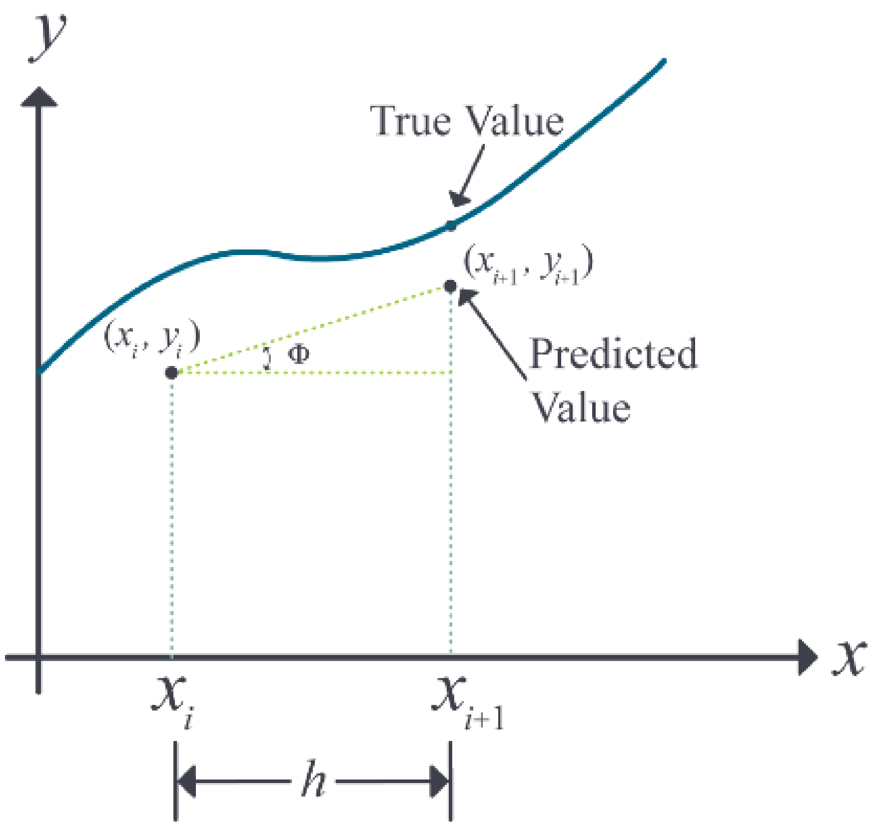
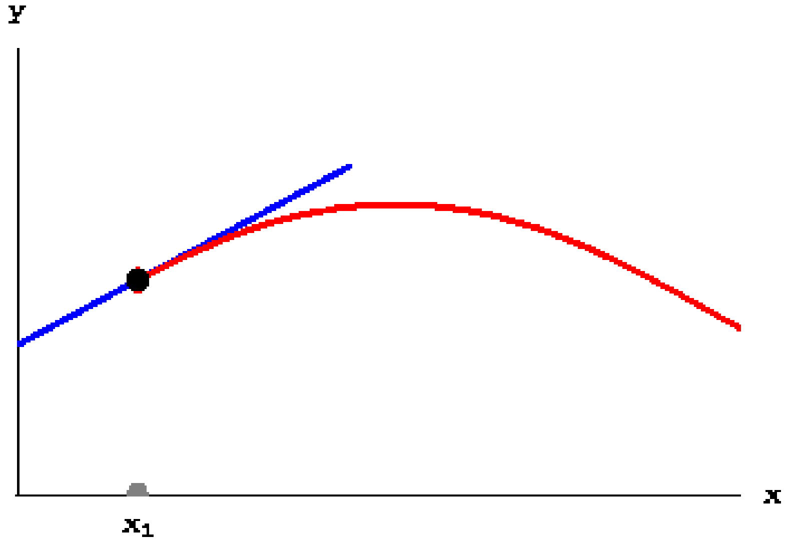


Figure 2 General graphical interpretation of Euler's method.



$$y'(x) = f(x, y)$$
$$y_{n+1} = y_n + f(x_n, y_n) \cdot (x_{n+1} - x_n)$$

Link:
https://phys23p.sl.psu.edu/~mrg3/mathanim/diff_equ/eulera.html

Euler’s Method

Deriving the formula (*from the Taylor Series*)

Euler’s method can be derived from the Taylor series as follows.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i,y_i} (x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i,y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i,y_i} (x_{i+1} - x_i)^3 + \ldots$$

Since $\frac{dy}{dx} = f(x, y)$

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots$$

As you can see, using only the first two terms of the Taylor series in Equation (6)

$$\boxed{y_{i+1} = y_i + f\left(x_i, y_i\right) h} \qquad (3 - repeated)$$

is the Euler’s method.

Euler's Method

Truncation error in Euler's method

There are ***two*** sources of error (not counting round-off) in Euler's method:

- The error committed in approximating the integral curve by the tangent line over the interval $[x_i, x_{i+1}]$.
- The error committed in using the approximation y_i instead of the exact value $y(x_i)$ to compute y_{i+1}.

The true error in the Euler's method hence is given by

$$E_t = \frac{f'(x_i, y_i)}{2!}h^2 + \frac{f''(x_i, y_i)}{3!}h^3 + \ldots$$

Local Truncation Error: The truncation error at the i^{th} step.

$$E_{t_i} = y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i))$$

Global Truncation Error: The cumulative effect of the local truncation errors E_{t_i} committed at each step.

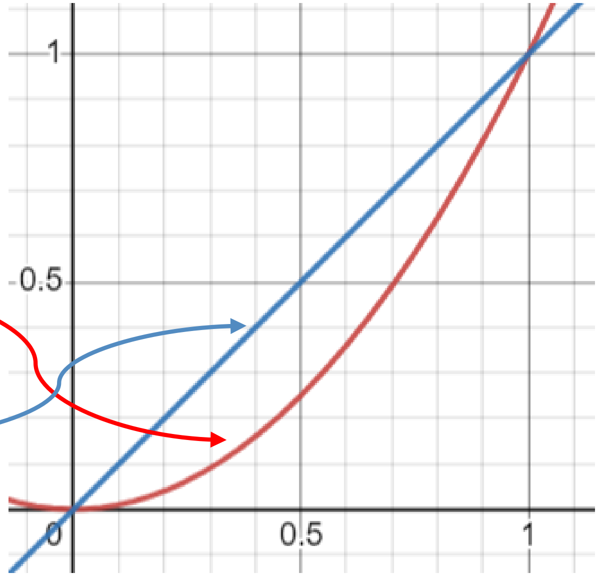
Local Truncation Error,

$$\boldsymbol{E_t \propto O(h^2)}$$

Global Truncation Error,

$$\boldsymbol{E_{t_{global}} \propto O(h)}$$

in general, $h \in (0,1]$



Click here if you are interested in the detailed analysis of the Truncation Error.

Euler's Method

An example

A ball at $1200\ \mathrm{K}$ is allowed to cool down in air at an ambient temperature of $300\ \mathrm{K}$. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right),\ \theta(0) = 1200\ \mathrm{K} \qquad (E1.1)$$

where θ is in K and t in seconds. Find the temperature at $t = 480$ seconds using Euler's method. Assume a step size of $h = 240$ seconds. Compare with the exact value.

Euler's Method

An example

Solution

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

$$f(t,\theta) = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

Euler's method formula is

$$\theta_{i+1} = \theta_i + f(t_i, \theta_i)\, h$$

For $i = 0$, $t_0 = 0$, $\theta_0 = 1200$

$$\begin{aligned}
\theta_1 &= \theta_0 + f(t_0, \theta_0)\, h \\
&= 1200 + f(0, 1200) \times 240 \\
&= 1200 + \left(-2.2067 \times 10^{-12}\left(1200^4 - 81 \times 10^8\right)\right) \times 240 \\
&= 1200 + (-4.5579) \times 240 \\
&= 106.09 \ \text{K}
\end{aligned}$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 240 = 240 \text{ s} \qquad \theta_1 = \theta(240) \approx 106.09 \text{ K}$$

For $i = 1$, $t_1 = 240$, $\theta_1 = 106.09$

$$\begin{aligned}
\theta_2 &= \theta_1 + f(t_1, \theta_1)\, h \\
&= 106.09 + f(240, 106.09) \times 240 \\
&= 106.09 + \left(-2.2067 \times 10^{-12}\left(106.09^4 - 81 \times 10^8\right)\right) \times 240 \\
&= 106.09 + (0.017595) \times 240 \\
&= 110.32 \ \text{K}
\end{aligned}$$

θ_2 is the approximate temperature at

$$t = t_2 = t_1 + h = 240 + 240 = 480 \text{ s}$$

$$\theta_2 = \theta(480) \approx 110.32 \text{ K}$$

The exact solution of the ordinary differential Equation (E1.1) is given without proof as the solution of the nonlinear equation

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}\left(0.333 \times 10^{-2}\theta\right)$$
$$= -0.22067 \times 10^{-3} t - 2.9282$$

$$\theta_{exact} = 647.57 \text{ K} \ \text{ at } t = 480 \text{ s}$$

Euler's Method

An example

Figure 1 compares the exact solution (647.57 K) with the numerical solution from Euler's method for the step size of $h = 240$.

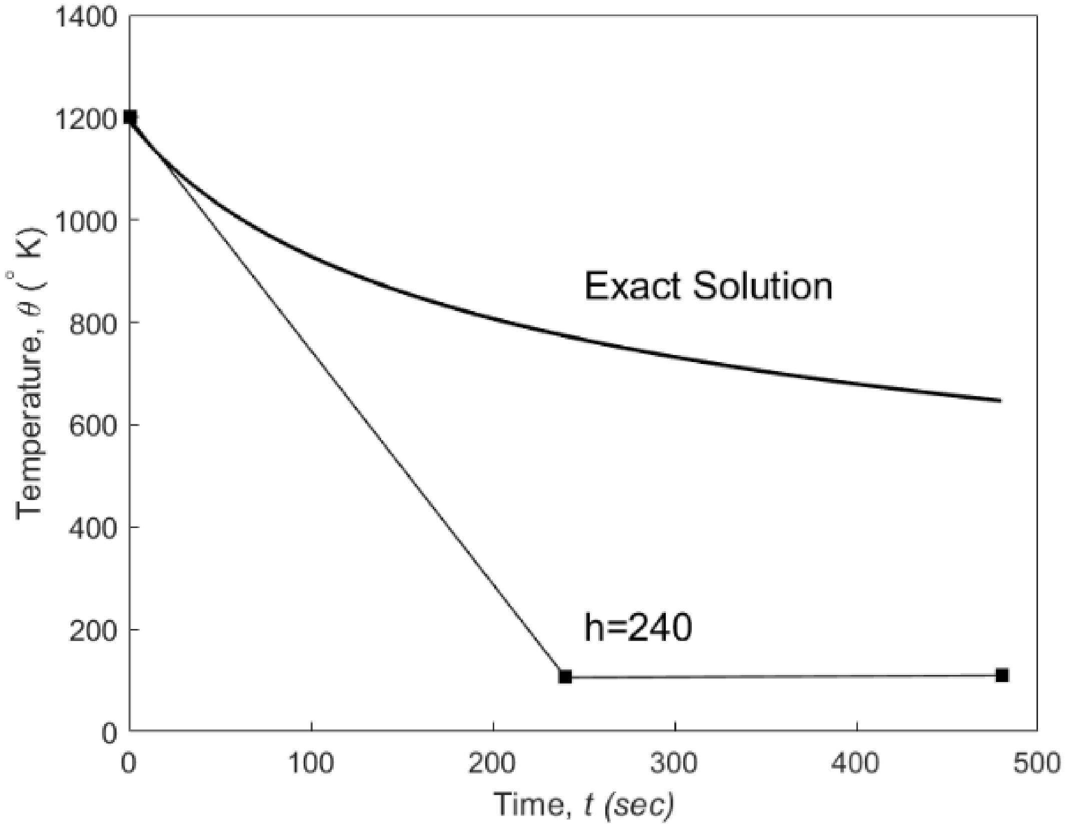


Figure 1 Comparing the exact and Euler's method solutions.

The problem was solved again using a smaller step size. The results are given below in Table 1.

Table 1 Temperature at 480 seconds as a function of step size, h.

Step size, h	$\theta\,(480)$	E_t	$\lvert\epsilon_t\rvert\%$
480	-987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60	614.97	32.607	5.0352
30	632.77	14.806	2.2864

Euler's Method

An example

Figure 2 shows how the temperature varies as a function of time for different step sizes.

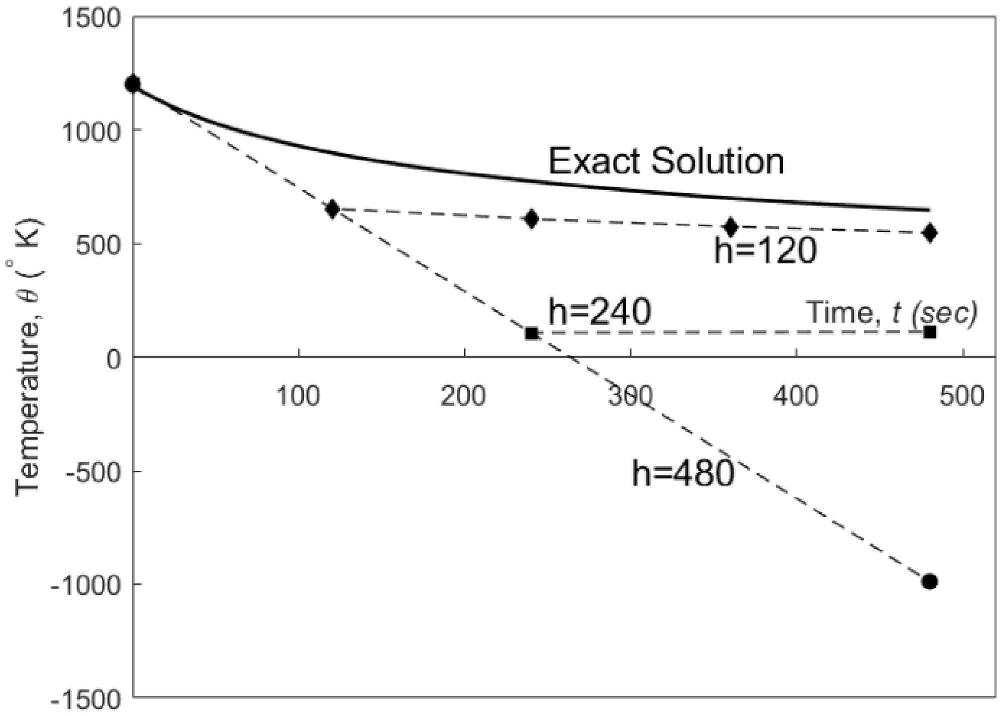


Figure 2 Comparison of Euler's method with the exact solution for different step sizes.

The values of the calculated temperature at $t = 480$ s as a function of step size are plotted in Figure 3.

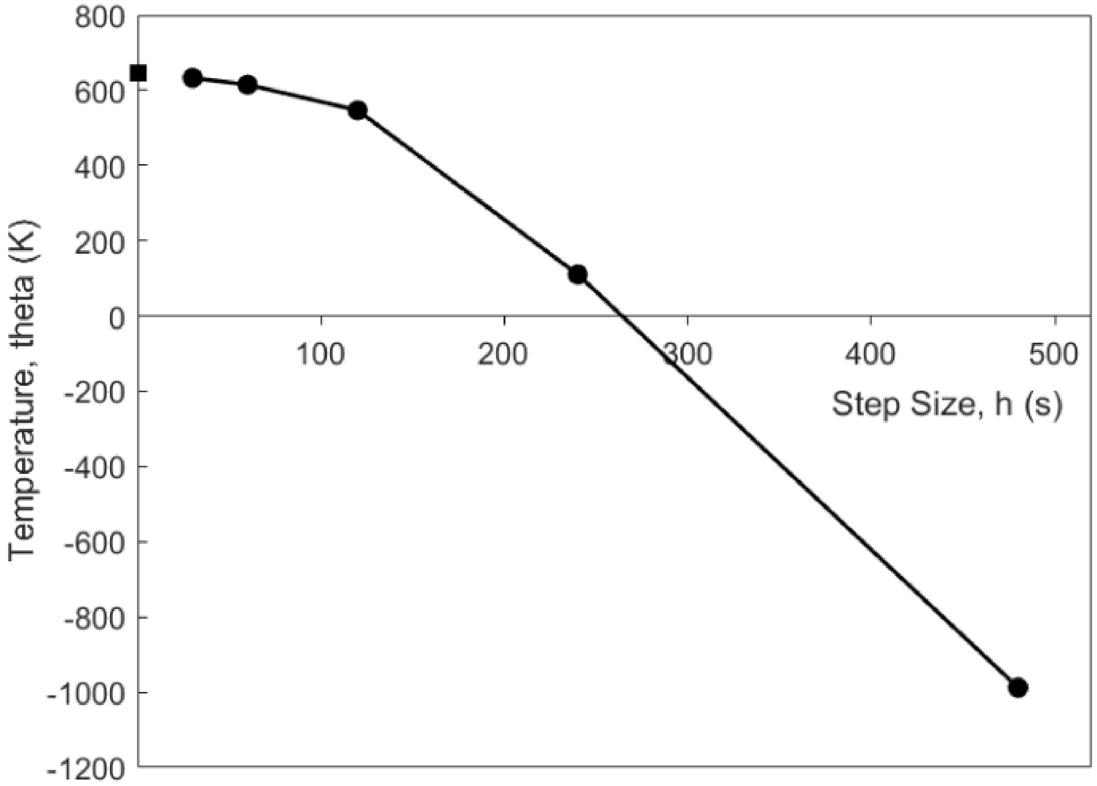


Figure 3 Effect of step size in Euler's method.

Euler’s Method

Approximating the value of a Definite Integral

We need to invoke the fundamental theorems of calculus for this —

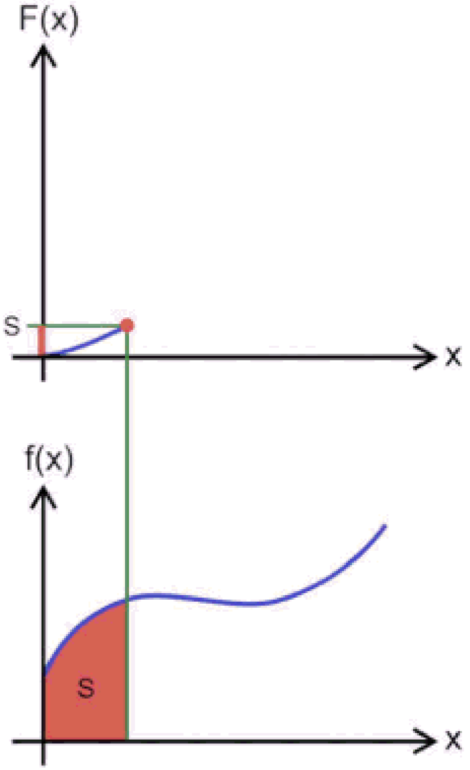
- **First Fundamental Theorem of Calculus:** Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all x in $[a, b]$ by

$$F(x) = \int_a^x f(t)dt$$

Then F is uniformly continuous on $[a, b]$ and differentiable on the open interval (a, b), and $F'(x) = f(x)$ for all x in (a, b) so F is an antiderivative of f.

- **Second Fundamental Theorem of Calculus:** If f is Riemann integrable on $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$



Link: https://upload.wikimedia.org/wikipedia/commons/2/2f/Fundamental_theorem_of_calculus_%28animation%29.gif?20200314134940

Euler's Method

Approximating the value of a Definite Integral

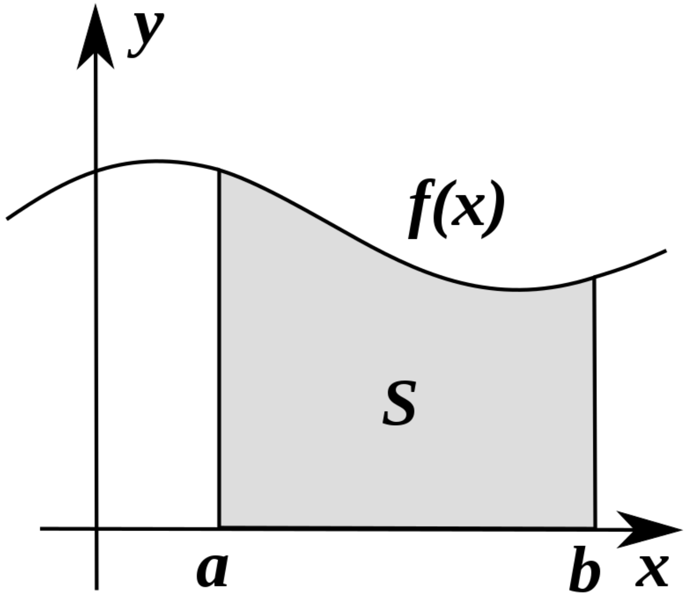
The idea is to rewrite the integral as the solution of an ODE,

$$\underbrace{\frac{dy}{dx} = f(x)}_{\text{1st Fundamental Theorem}},\ y(a) = 0,$$

1st Fundamental Theorem

where then $\underbrace{y(b) \text{ gives us the value of the integral } \int_a^b f(x)\,dx}$.

2nd Fundamental Theorem



Euler's Method

A definite integral approximation example

Find an approximate value of

$$\int_{5}^{8} 6x^3 dx$$

using Euler's method of solving an ordinary differential equation. Use a step size of $h = 1.5$.

Euler's Method

A definite integral approximation example

Solution

Given $\int_5^8 6x^3 dx$, we can rewrite the integral as the solution of an ordinary differential equation

$$\frac{dy}{dx} = 6x^3, y(5) = 0$$

where $y(8)$ gives the value of the integral $\int_5^8 6x^3 dx$.

$$\frac{dy}{dx} = 6x^3 = f(x, y), \; y(5) = 0$$

The Euler's method formula is

$$y_{i+1} = y_i + f(x_i, y_i)\, h$$

Step 1

$$i = 0, \; x_0 = 5, y_0 = 0$$

$$h = 1.5$$

$$\begin{aligned} x_1 &= x_0 + h \\ &= 5 + 1.5 \\ &= 6.5 \end{aligned}$$

$$y_1 = y_0 + f(x_0, y_0)\, h$$

$$\begin{aligned} y_1 &= 0 + f(5, 0) \times 1.5 \\ &= 0 + f\left(6 \times 5^3\right) \times 1.5 \\ &= 1125 \\ &\approx y(6.5) \end{aligned}$$

Step 2

$$i = 1, x_1 = 6.5, y_1 = 1125$$

$$\begin{aligned} x_2 &= x_1 + h \\ &= 6.5 + 1.5 \\ &= 8 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1)\, h \\ &= 1125 + f(6.5, 1125) \times 1.5 \\ &= 1125 + f\left(6 \times 6.5^3\right) \times 1.5 \\ &= 3596.625 \\ &\approx y(8) \end{aligned}$$

Hence

$$\begin{aligned} \int_5^8 6x^3 dx &= y(8) - y(5) \\ &\approx 3596.625 - 0 \\ &= 3596.625 \end{aligned}$$