Math 4543: Numerical Methods

Lecture 16 — Solving Higher-order Ordinary Differential Equations

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Lecture Plan

The agenda for today

- Understand the concept behind higher-order ODEs
- Order vs Degree of higher-order ODEs
- Recast higher-order ODEs as simultaneous 1st order ODEs
- Solve higher-order ODEs using numerical approaches meant for 1st order ODEs

Higher-order Derivatives

What are they?

Let f be a **differentiable** function. Conventionally, we denote the 1st derivative as f'. The 2nd derivative is f'', the 3rd derivative is f''', and so on.

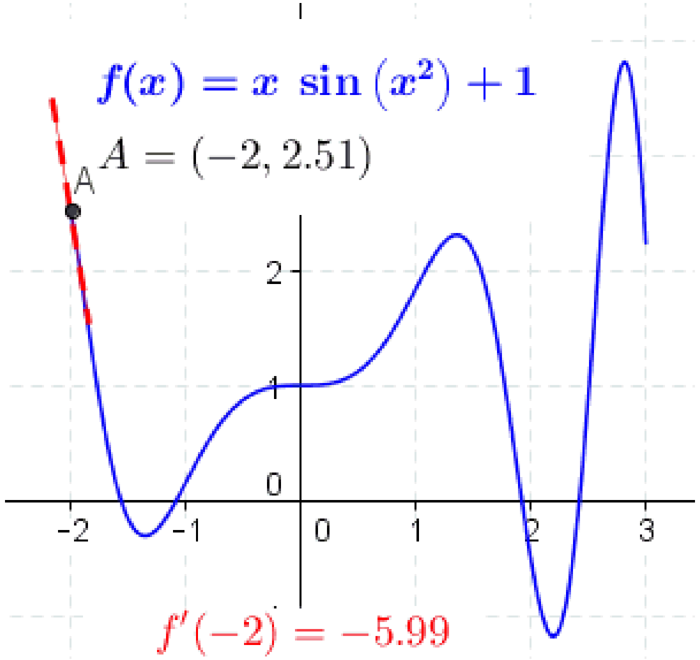

Figure: The derivative at different points of a differentiable function.

Link:
https://upload.wikimedia.org/wikipedia/commons/2/2d/Tangent_function_animation.gif

Continuing this process, one can define, if it exists, the nth derivative as the derivative of the $(n-1)$th derivative *i.e.* $f^n(x) = \frac{d}{dx}(f^{n-1}(x))$.

Higher-order Ordinary Differential Equations

What are they?

A **differential equation** is an equation that relates one or more unknown functions and their derivatives.

An **ordinary differential equation (ODE)** is an equation containing an unknown function of one real or complex variable x, its derivatives, and some given functions of x. The unknown function (often denoted y), depends on x. Thus x is often called the **independent variable** of the equation. The term *"ordinary"* is used in contrast with the term partial differential equation, which may be with respect to more than one independent variable.

An nth order ODE generally has the form,

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_o y = f(x)$$

Order vs Degree of ODEs

Understanding the difference

<u>Order:</u> The order of a differential equation is defined to be that of the ***highest order derivative*** it contains.

<u>Degree:</u> The degree of a differential equation is defined as the ***power*** to which the highest order derivative is raised.

$$\underbrace{\frac{d^2y}{dx^2}} - 3\left(\frac{dy}{dx}\right)^3 = x$$

Order: 2
Since, the highest order derivative is 2.

Degree: 1
Since, the power of the highest order derivative is 1.

$$x\frac{dy}{dx} - \underbrace{\left(\frac{d^2y}{dx^2}\right)^3} + 5 = 0$$

Order: 2
Since, the highest order derivative is 2.

Degree: 3
Since, the power of the highest order derivative is 3.

Solving Higher-order ODEs

Rewriting an higher-order ODE

What do we do to solve simultaneous (coupled) differential equations or differential equations higher than first order? For example, an n^{th}order differential equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_o y = f(x) \qquad (2)$$

with n initial conditions can be solved by assuming

$$y = z_1 \qquad (3.1)$$

$$\frac{dy}{dx} = \frac{dz_1}{dx} = z_2 \qquad (3.2)$$

$$\frac{d^2 y}{dx^2} = \frac{dz_2}{dx} = z_3 \qquad (3.3)$$

$$\vdots$$

$$\frac{d^{n-1} y}{dx^{n-1}} = \frac{dz_{n-1}}{dx} = z_n \qquad (3.n)$$

$$\begin{aligned} \frac{d^n y}{dx^n} &= \frac{dz_n}{dx} \\ &= \frac{1}{a_n}\left(-a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} \ldots - a_1 \frac{dy}{dx} - a_0 y + f(x)\right) \\ &= \frac{1}{a_n}\left(-a_{n-1} z_n \ldots - a_1 z_2 - a_0 z_1 + f(x)\right) \quad (3.n+1) \end{aligned}$$

Solving Higher-order ODEs

Rewriting an higher-order ODE

The above Equations from (3.2) to (3.n+1) represent n first-order differential equations as follows

$$\frac{dz_1}{dx} = z_2 = f_1\left(z_1, z_2, \ldots, x\right) \qquad (4.1)$$

$$\frac{dz_2}{dx} = z_3 = f_2\left(z_1, z_2, \ldots, x\right) \qquad (4.2)$$

$$\vdots$$

$$\frac{dz_n}{dx} = \frac{1}{a_n}\left(-a_{n-1}z_n \ldots - a_1 z_2 - a_0 z_1 + f\left(x\right)\right) \qquad (4.n)$$

Multivariate functions on the R.H.S.

Each of the n first-order ordinary differential equations should be accompanied by one initial condition. The initial condition should be on the corresponding dependent variable on the left-hand side of the ordinary differential equation. For example, Equation (4.1) would need an initial condition on z_1, Equation (4.n) would need an initial condition on z_n, etc. These first-order ordinary differential equations (Equations (4.1) to (4.n)) are simultaneous. Still, they can be solved by the methods used for solving first-order ordinary differential equations that we have already learned in the previous lessons.

Solving Higher-order ODEs

A rewriting example

Rewrite the following differential equation as a set of simultaneous first-order differential equations.

$$3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x}, y(0) = 5,\ y'(0) = 7$$

Solving Higher-order ODEs

A rewriting example

Solution

The ordinary differential equation

$$3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x}, y\left(0\right) = 5,\ y'\left(0\right) = 7 \qquad (E1.1)$$

would be rewritten as follows. Assume

$$\frac{dy}{dx} = z, \qquad (E1.2)$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} \qquad (E1.3)$$

Substituting Equations (E1.2) and (E1.3) in the given second-order ordinary differential equation gives

$$3\frac{dz}{dx} + 2z + 5y = e^{-x}$$

and rewritten as $\frac{dz}{dx} = \frac{1}{3}\left(e^{-x} - 2z - 5y\right)$

The set of two simultaneous first-order ordinary differential equations complete with the initial conditions then is

$$\boxed{\frac{dy}{dx} = z, y\left(0\right) = 5} \qquad (E1.5a)$$

$$\boxed{\frac{dz}{dx} = \frac{1}{3}\left(e^{-x} - 2z - 5y\right), z\left(0\right) = 7.} \qquad (E1.5b)$$

Now one can apply any of the numerical methods used for solving first-order ordinary differential equations.

Solving Higher-order ODEs

Another example

Given

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}, y(0) = 1, \frac{dy}{dt}(0) = 2,$$

estimate the following by Euler's method

a) $y(0.75)$

b) the absolute relative true error for part(a), if $y(0.75)|_{exact} = 1.668$

c) $\frac{dy}{dt}(0.75)$

Use a step size of $h = 0.25$.

Solving Higher-order ODEs

Another example

Solution

First, the second-order differential equation is rewritten as two simultaneous first-order differential equations as follows. Assume

$$\frac{dy}{dt} = z$$

then

$$\frac{dz}{dt} + 2z + y = e^{-t}$$

$$\frac{dz}{dt} = e^{-t} - 2z - y$$

So the two simultaneous first-order differential equations are

$$\frac{dy}{dt} = z = f_1(t, y, z), y(0) = 1 \qquad (E2.1)$$

$$\frac{dz}{dt} = e^{-t} - 2z - y = f_2(t, y, z), \ z(0) = 2 \qquad (E2.2)$$

Using Euler's method on Equations (E2.1) and (E2.2), we get

$$y_{i+1} = y_i + f_1(t_i, y_i, z_i)\, h \qquad (E2.3)$$

$$z_{i+1} = z_i + f_2(t_i, y_i, z_i)\, h \qquad (E2.4)$$

a) To find the value of $y(0.75)$ and since we are using a step size of 0.25 and starting at $t = 0$, we need to take three steps to find the value of $y(0.75)$.

For $i = 0, t_0 = 0, y_0 = 1, z_0 = 2,$

From Equation (E2.3)

$$\begin{aligned} y_1 &= y_0 + f_1(t_0, y_0, z_0)\, h \\ &= 1 + f_1(0, 1, 2)(0.25) \\ &= 1 + 2(0.25) \\ &= 1.5 \end{aligned}$$

Solving Higher-order ODEs

Another example

y_1 is the approximate value of y at

$$t = t_1 = t_0 + h = 0 + 0.25 = 0.25$$

$$y_1 = y\left(0.25\right) \approx 1.5$$

From Equation (E2.4)

$$\begin{aligned} z_1 &= z_0 + f_2\left(t_0, y_0, z_0\right) h \\ &= 2 + f_2\left(0, 1, 2\right)\left(0.25\right) \\ &= 2 + \left(e^{-0} - 2\left(2\right) - 1\right)\left(0.25\right) \\ &= 1 \end{aligned}$$

z_1 is the approximate value of z (same as $\frac{dy}{dt}$) at $t = 0.25$

$$z_1 = z\left(0.25\right) \approx 1$$

For $i = 1, t_1 = 0.25, y_1 = 1.5, z_1 = 1,$

From Equation (E2.3)

$$\begin{aligned} y_2 &= y_1 + f_1\left(t_1, y_1, z_1\right) h \\ &= 1.5 + f_1\left(0.25, 1.5, 1\right)\left(0.25\right) \\ &= 1.5 + \left(1\right)\left(0.25\right) \\ &= 1.75 \end{aligned}$$

y_2 is the approximate value of y at

$$t = t_2 = t_1 + h = 0.25 + 0.25 = 0.50$$

$$y_2 = y\left(0.5\right) \approx 1.75$$

From Equation (E2.4)

$$\begin{aligned} z_2 &= z_1 + f_2\left(t_1, y_1, z_1\right) h \\ &= 1 + f_2\left(0.25, 1.5, 1\right)\left(0.25\right) \\ &= 1 + \left(e^{-0.25} - 2\left(1\right) - 1.5\right)\left(0.25\right) \\ &= 1 + \left(-2.7211\right)\left(0.25\right) \\ &= 0.31970 \end{aligned}$$

Solving Higher-order ODEs

Another example

z_2 is the approximate value of z at

$$t = t_2 = 0.5$$

$$z_2 = z(0.5) \approx 0.31970$$

For $i = 2, t_2 = 0.5, y_2 = 1.75, z_2 = 0.31970,$

From Equation (E2.3)

$$\begin{aligned} y_3 &= y_2 + f_1(t_2, y_2, z_2)\, h \\ &= 1.75 + f_1(0.50, 1.75, 0.31970)(0.25) \\ &= 1.75 + (0.31970)(0.25) \\ &= 1.8299 \end{aligned}$$

y_3 is the approximate value of y at

$$t = t_3 = t_2 + h = 0.5 + 0.25 = 0.75$$

$$y_3 = y(0.75) \approx 1.8299$$

From Equation (E2.4)

$$\begin{aligned} z_3 &= z_2 + f_2(t_2, y_2, z_2)\, h \\ &= 0.31972 + f_2(0.50, 1.75, 0.31970)(0.25) \\ &= 0.31972 + \left(e^{-0.50} - 2(0.31970) - 1.75\right)(0.25) \\ &= 0.31972 + (-1.7829)(0.25) \\ &= -0.1260 \end{aligned}$$

z_3 is the approximate value of z at

$$t = t_3 = 0.75$$

$$z_3 = z(0.75) \approx -0.12601$$

$$\boxed{y(0.75) \approx y_3 = 1.8299}$$

Solving Higher-order ODEs

Another example

b) The exact value of $y\left(0.75\right)$ is

$$y\left(0.75\right)\Big|_{exact} = 1.668$$

The absolute relative true error in the result from part (a) is

$$\begin{aligned} |\in_t| &= \left|\frac{1.668 - 1.8299}{1.668}\right| \times 100 \\ &= 9.7062\% \end{aligned}$$

c)

$$\begin{aligned} \frac{dy}{dt}(0.75) &= z_3 \\ &\approx -0.12601 \end{aligned}$$

Check out the Runge-Kutta 2nd order example using Heun's method from the provided lecture note.
(Too big to include in the lecture slide!)