Math 4543: Numerical Methods

Lecture 13 — Simpson's 1/3 Rule and Simpson's 3/8 Rule of Integration

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Lecture Plan

The agenda for today

- Understand the idea behind the Simpson's rules of integration
- Know about the derivation approaches of single-segment Simpson's 1/3 rule formula †
- Derive the Multiple-segment Simpson's 1/3 rule formula
- Know about the derivation approaches of single-segment Simpson's 3/8 rule formula †
- Derive the Multiple-segment Simpson's 3/8 rule formula
- Understand how the 1/3 and 3/8 rules can be fused or mixed together

† means that the topic is not important for exams.

Simpson's Rules

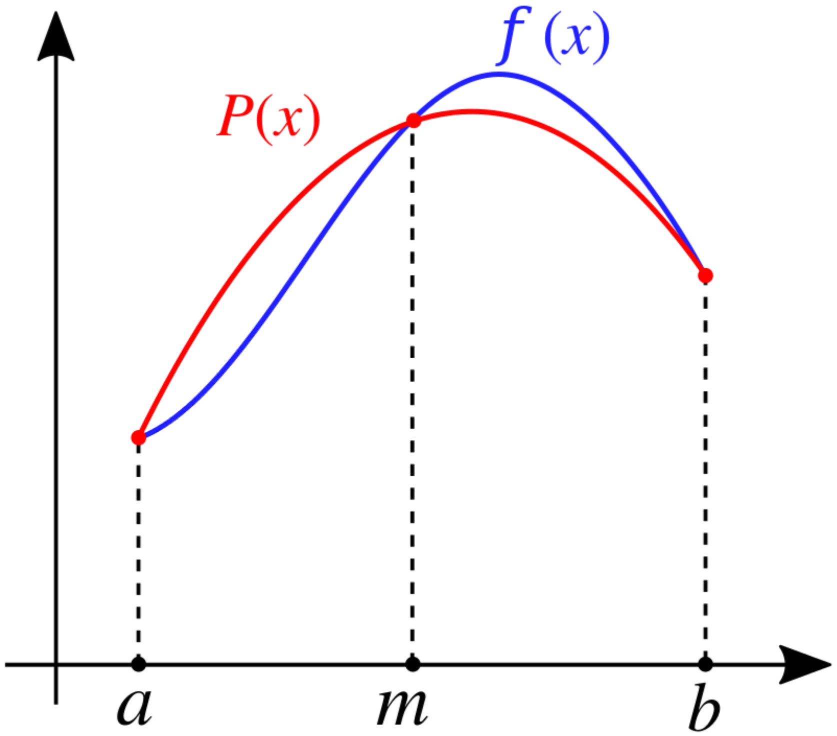
What are they?

Simpson's rules are methods of ***numerical integration*** invented by mathematician Thomas Simpson.

The idea is to ***approximate the definite integral*** of a function $f(x)$ by calculating the definite integral of an n^{th} order polynomial $f_n(x) = P(x)$ that ***approximates the function*** $f(x)$. $[n > 1]$

We'll mainly focus on **two** Simpson's rules —

- Simpson's **1/3** Rule
- Simpson's **3/8** Rule

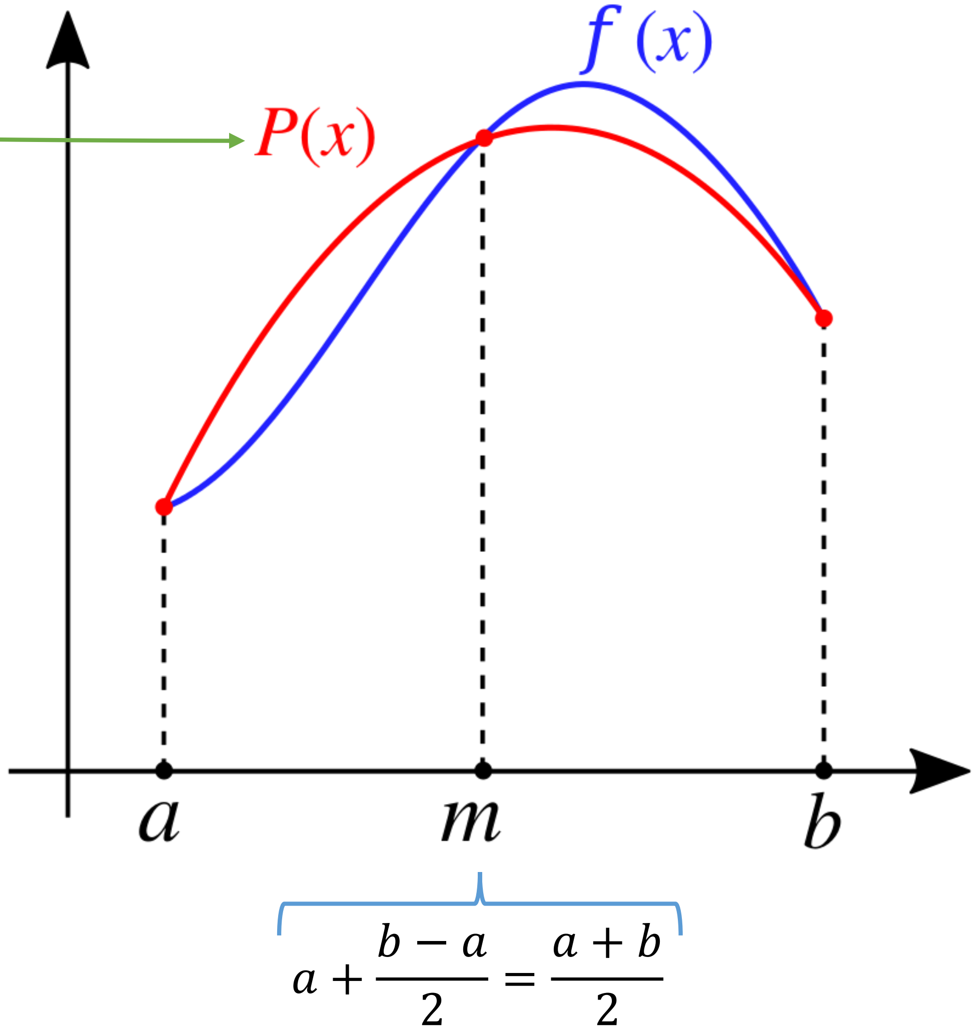


We will also solve an example using ***both of these rules*** together.

Simpson's 1/3 Rule

The general idea

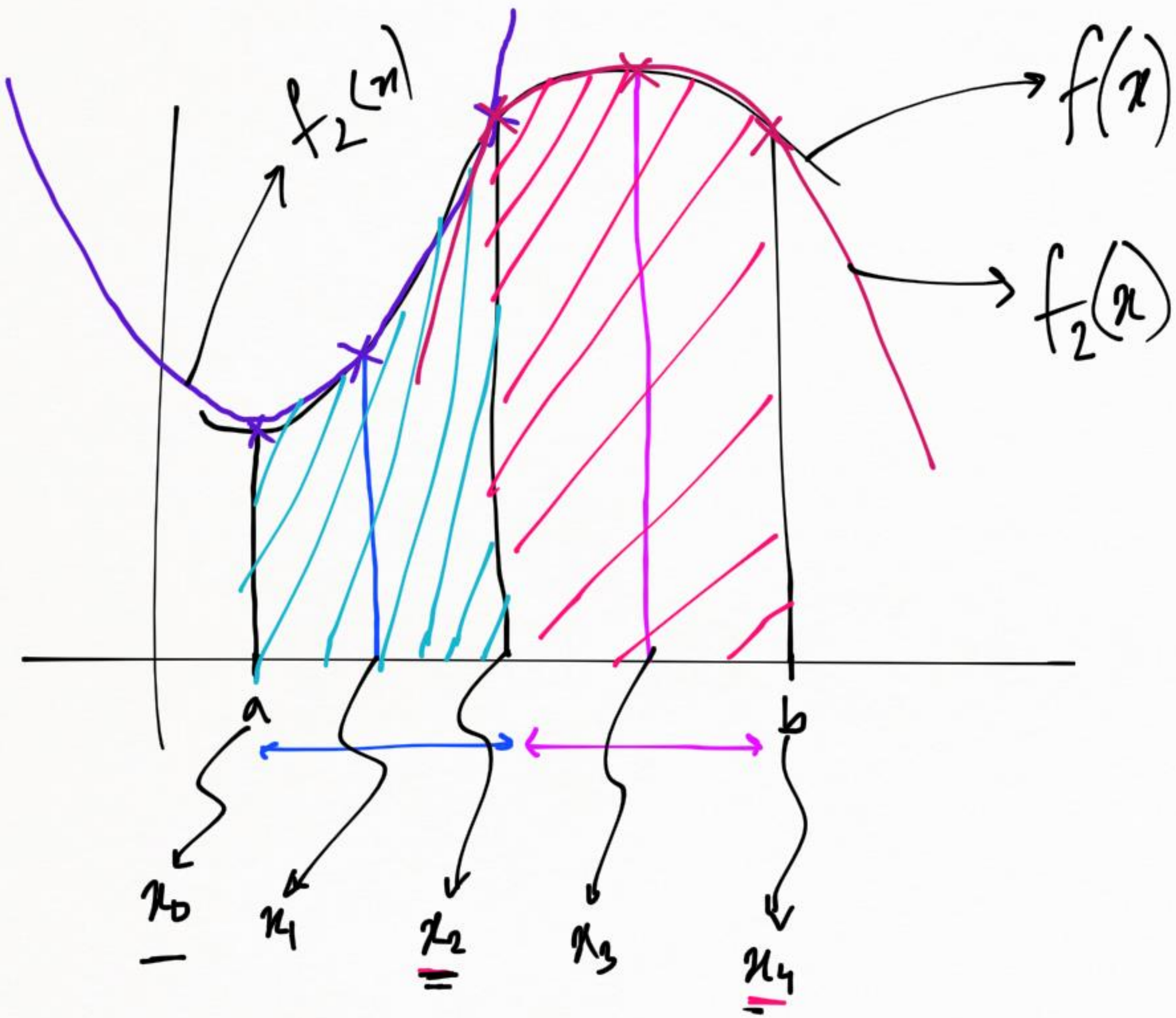
Simpson's 1/3 rule is an extension of Trapezoidal rule where the integrand is approximated by a ***second order*** polynomial $f_2(x)$.



$$I=\int_a^b f\left(x\right)dx \approx \int_a^b f_2\left(x\right)dx$$

where $f_2(x)$ is a second order polynomial given by

$$f_2(x)=a_0+a_1x+a_2x^2$$



$n \rightarrow$ seven

Simpson's 1/3 Rule

Deriving the formula †

Choose

$$(a, f(a)),\quad \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right),\ \text{and}\ (b, f(b))$$

as the three points of the function to evaluate a_1 and a_2.

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

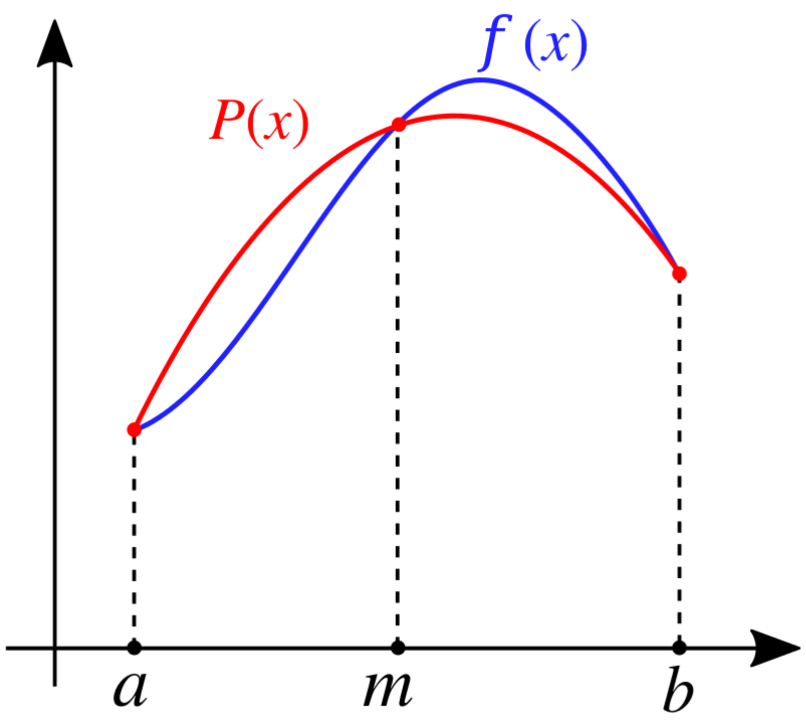
Solving the above three equations for unknowns, a_0, a_1 and a_2 give

Using symbolic solver software *e.g.* MATLAB, Octave, Maple, WolframAlpha etc.

$$\boxed{a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}}$$

$$\boxed{a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}}$$

$$\boxed{a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}}$$



$$\overbrace{a + \frac{b-a}{2} = \frac{a+b}{2}}^{m}$$

Simpson's 1/3 Rule

Deriving the formula †

Then

$$I \approx \int_a^b f_2(x)dx$$
$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right) dx$$
$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$
$$= a_0(b-a) + a_1 \frac{b^2-a^2}{2} + a_2 \frac{b^3-a^3}{3}$$

Substituting values of a_0, a_1 and a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since for the single application of Simpson 1/3 rule, the interval $[a, b]$ is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

The Simpson's 1/3 rule can be rewritten as

$$\boxed{\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]}$$

Since the above form has 1/3 in its formula, it is called Simpson's 1/3 rule.

Go through the lecture note PDF to know about the other approaches of deriving the same formula.

Simpson's 1/3 Rule

An example

The distance covered by a rocket in meters from $t = 8s$ to $t = 30s$ is given by

$$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use Simpson's 1/3 rule to find the approximate value of x.

b) Find the true error, E_t.

c) Find the absolute relative true error, $|\epsilon_t|$.

Simpson's 1/3 Rule

An example

Solution

a) $$x \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$a = 8$$

$$b = 30$$

$$\frac{a+b}{2} = 19$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

$$f(8) = 2000\ln\left[\frac{140000}{140000-2100(8)}\right] - 9.8(8) = 177.27m/s$$

$$f(30) = 2000\ln\left[\frac{140000}{140000-2100(30)}\right] - 9.8(30) = 901.67m/s$$

$$f(19) = 2000\ln\left(\frac{140000}{140000-2100(19)}\right) - 9.8(19) = 484.75m/s$$

$$x \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$
$$= \left(\frac{30-8}{6}\right)[f(8) + 4f(19) + f(30)]$$
$$= \frac{22}{6}[177.27 + 4 \times 484.75 + 901.67]$$
$$= 11065.72m$$

b) The exact value of the above integral is

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t\right)dt$$
$$= 11061.34m$$

So the true error is $E_t = \text{True Value} - \text{Approximate Value}$

$$= 11061.34 - 11065.72 = -4.38m$$

c) The absolute relative true error is

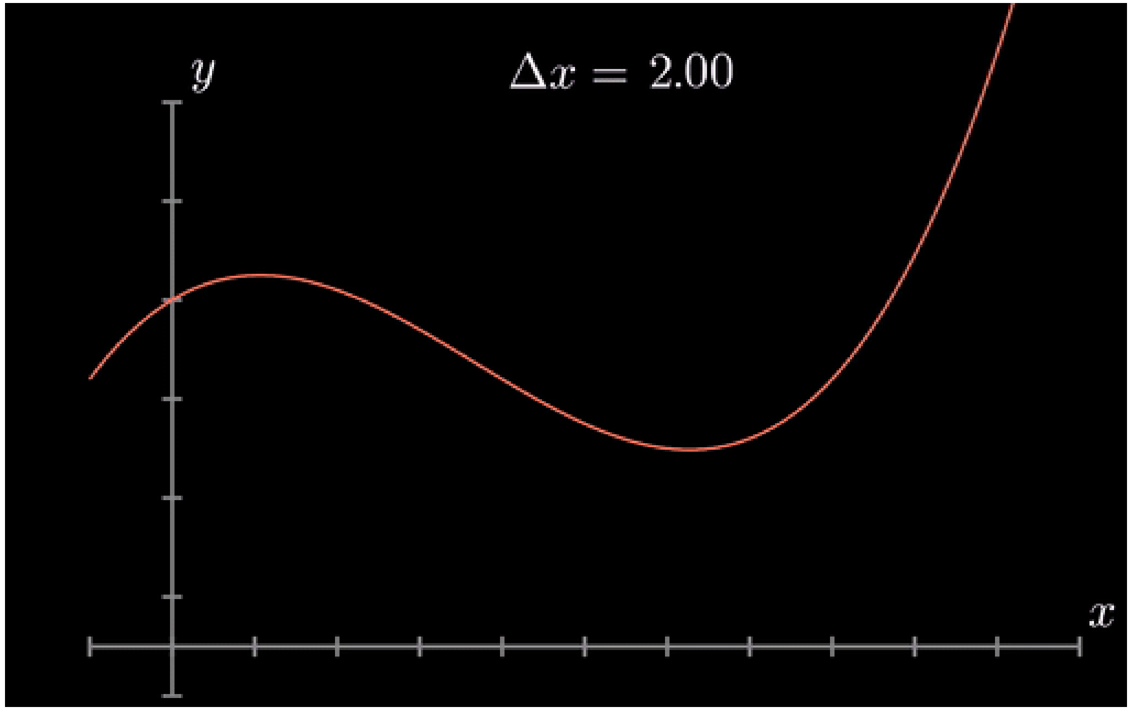
$$|\epsilon_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$
$$= \left|\frac{-4.38}{11061.34}\right| \times 100$$
$$= 0.0396\%$$

Multi-segment Simpson's 1/3 Rule

The composite version

Just like in composite trapezoidal rule, one can subdivide the interval $[a, b]$ into n segments and apply Simpson's 1/3 rule repeatedly over every two segments. Note that n needs to be even. Divide interval $[a, b]$ into n equal segments, so that the segment width is given by

$$h = \frac{b-a}{n}$$



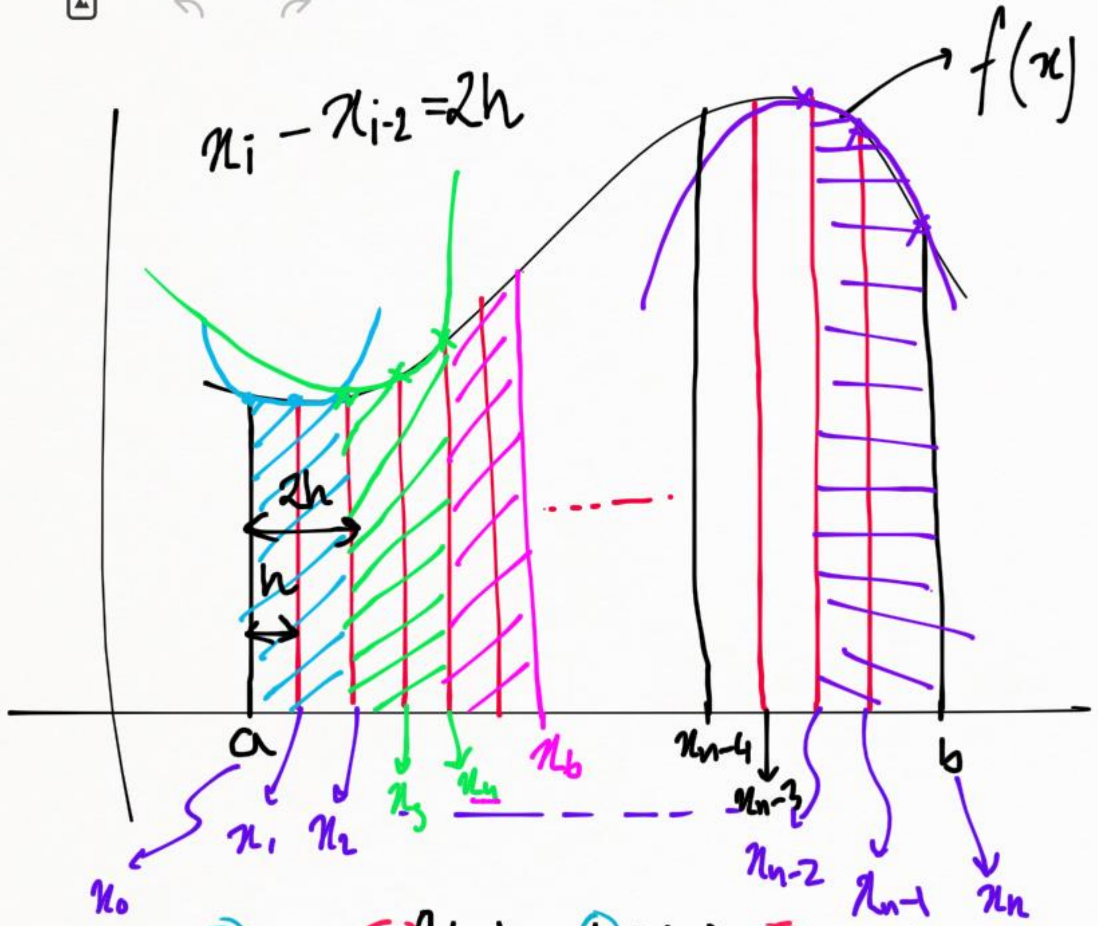
Adapted from www.youtube.com/watch?v=DdNAcv_rezc

Now

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

where $x_0 = a \quad x_n = b$

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots\ldots + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$



$$I = \int_a^b f(x)\,dx \approx \int_{x_0=a}^{x_2} f_2(x)\,dx + \int_{x_2}^{x_4} f_2(x)\,dx + \cdots$$

$$+ \int_{x_{n-4}}^{x_{n-2}} f_2(x)\,dx + \int_{x_{n-2}}^{x_n=b} f_2(x)\,dx$$

$$= \frac{\cancel{x_2 - x_0}^{\cancel{2h}\ h}}{\cancel{6}\ 3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] +$$

$$\frac{\cancel{x_4 - x_2}^{\cancel{2h}\ h}}{\cancel{6}\ 3}\left[f(x_2) + 4f(x_3) + f(x_4)\right] + \cdots$$

$$+ \frac{\cancel{x_{n-2} - x_{n-4}}^{\cancel{2h}\ h}}{3\ \cancel{6}}\left[f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})\right] +$$

$$+ \frac{\cancel{x_n - x_{n-2}}^{\cancel{2h}\ h}}{\cancel{6}\ 3}\left[f(x_{n-2}) + 4(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots$$

$$+ 2f(x_{n-4}) + 4f(x_{n-3}) + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$$

$$= \boxed{\frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1\\ \text{odd}}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\ \text{even}}}^{n-2} f(x_i) + f(x_n)\right]}$$

Multi-segment Simpson's 1/3 Rule

Deriving the formula

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots\ldots + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$

Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx \cong (x_2 - x_0)\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + (x_4 - x_2)\left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}\right] + \ldots +$$

$$(x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6}\right] + (x_n - x_{n-2})\left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right]$$

Since

$$x_i - x_{i-2} = 2h$$

$$i = 2, 4, \ldots, n$$

then

$$\int_a^b f(x)dx \cong 2h\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right] + 2h\left[\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}\right] + \ldots$$

$$+2h\left[\frac{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})}{6}\right] + 2h\left[\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right]$$

$$= \frac{h}{3}[f(x_0) + 4(f(x_1) + f(x_3) + \ldots + f(x_{n-1})) + 2(f(x_2) + f(x_4) + \ldots + f(x_{n-2})) + f(x_n)]$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(x_i) + f(x_n)\right] \cong \boxed{\frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(x_i) + f(x_n)\right]}$$

Multi-segment Simpson's 1/3 Rule

The same example (now with *multiple segments*)

Use composite Simpson's 1/3 rule with 4 segments to approximate the distance covered by a rocket in meters from $t = 8$s to $t = 30$s as given by

$$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use composite Simpson's 1/3rd rule with 4 segments to estimate x.

b) Find the true error, E_t for part (a).

c) Find the absolute relative true error, $|\epsilon_t|$ for part (a).

Multi-segment Simpson's 1/3 Rule

The same example (now with *multiple segments*)

Solution:

a) Using composite Simpson's 1/3 rule,

$$x \approx \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1 \\ \text{i=odd}}}^{n-1} f(t_i) + 2\sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f(t_i) + f(t_n)\right]$$

$$n = 4 \qquad h = \frac{b-a}{n}$$

$$a = 8 \qquad = \frac{30-8}{4}$$

$$b = 30 \qquad = 5.5$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

So $f(t_0) = f(8)$

$$f(8) = 2000\ln\left[\frac{140000}{140000-2100(8)}\right] - 9.8(8)$$

$$= 177.27 m/s$$

$$f(t_1) = f(8+5.5) = f(13.5)$$

$$f(13.5) = 2000\ln\left[\frac{140000}{140000-2100(13.5)}\right] - 9.8(13.5)$$

$$= 320.25 m/s$$

$$f(t_2) = f(13.5+5.5) = f(19)$$

$$f(19) = 2000\ln\left(\frac{140000}{140000-2100(19)}\right) - 9.8(19)$$

$$= 484.75 m/s$$

$$f(t_3) = f(19+5.5) = f(24.5)$$

$$f(24.5) = 2000\ln\left[\frac{140000}{140000-2100(24.5)}\right] - 9.8(24.5)$$

$$= 676.05 m/s$$

$$f(t_4) = f(t_n) = f(30)$$

$$f(30) = 2000\ln\left[\frac{140000}{140000-2100(30)}\right] - 9.8(30)$$

$$= 901.67 m/s$$

Multi-segment Simpson's 1/3 Rule

The same example (now with *multiple segments*)

$$
\begin{aligned}
x &= \frac{b-a}{3n}\left[f\left(t_0\right)+4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f\left(t_i\right)+2 \sum_{\substack{i=2 \\ i=\text{even}}}^{n-2} f\left(t_i\right)+f\left(t_n\right)\right] \\
&= \frac{30-8}{3(4)}\left[f(8)+4 \sum_{\substack{i=1 \\ i=\text{odd}}}^{3} f\left(t_i\right)+2 \sum_{\substack{i=2 \\ i=\text{even}}}^{2} f\left(t_i\right)+f(30)\right] \\
&= \frac{22}{12}\left[f(8)+4 f\left(t_1\right)+4 f\left(t_3\right)+2 f\left(t_2\right)+f(30)\right] \\
&= \frac{11}{6}[f(8)+4 f(13.5)+4 f(24.5)+2 f(19)+f(30)] \\
&= \frac{11}{6}[177.27+4(320.25)+4(676.05)+2(484.75)+901.67] \\
&= 11061.64 m
\end{aligned}
$$

b) The exact value of the above integral is

$$
\begin{aligned}
x &= \int_8^{30}\left(2000 \ln \left[\frac{140000}{140000-2100 t}\right]-9.8 t\right) dt \\
&= 11061.34 m
\end{aligned}
$$

So the true error is

$$
\begin{aligned}
E_t &= \text{True Value} - \text{Approximate Value} \\
E_t &= 11061.34-11061.64 \\
&= -0.30 m
\end{aligned}
$$

c) The absolute relative true error is

$$
\begin{aligned}
\left|\epsilon_t\right| &= \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100 \\
&= \left|\frac{-0.3}{11061.34}\right| \times 100 \\
&= 0.0027\%
\end{aligned}
$$

Multi-segment Simpson's 1/3 Rule

The same example (now with *multiple segments*)

Table 1 Values of composite Simpson's 1/3 rule for Example 1

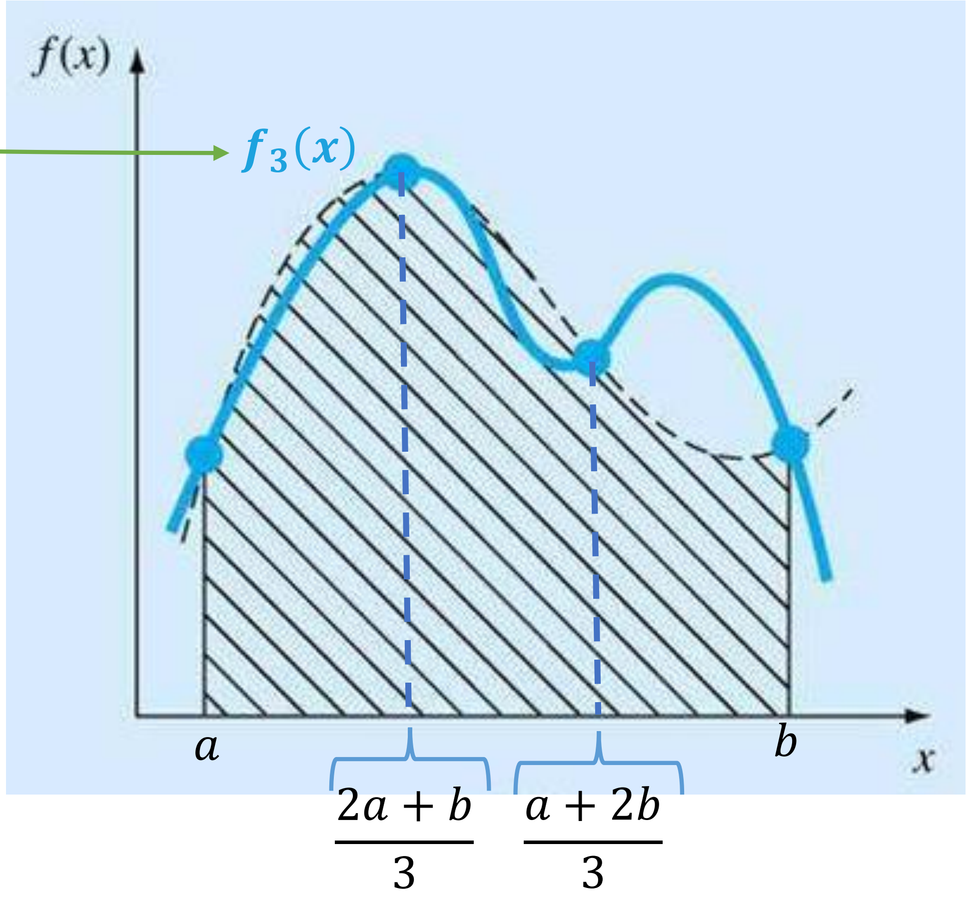
n	Approximate Value	E_t	$\lvert\varepsilon_t\rvert$
2	11065.72	-4.38	0.0396%
4	11061.64	-0.30	0.0027%
6	11061.40	-0.06	0.0005%
8	11061.35	-0.02	0.0002%
10	11061.34	-0.01	0.0001%

The trend is that the *error values consistently decrease* the *more we increase the number of segments*.

Simpson's 3/8 Rule

The general idea

Simpson's 3/8 rule is a numerical integration method where the integrand is approximated by a ***third order*** polynomial $f_3(x)$.



$$I = \int_a^b f(x)\,dx \approx \int_a^b f_3(x)\,dx \quad (1)$$

where $f_3(x)$ is a 3^{rd} order polynomial given by

$$f_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \qquad (2)$$

Simpson's 3/8 Rule

Deriving the formula †

In a similar fashion, Simpson 3/8 rule for integration can be derived by approximating the given function $f(x)$ with the 3rd order (cubic) polynomial $f_3(x)$

$$\left.\begin{aligned} f_3(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ &= \{1, x, x^2, x^3\} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{aligned}\right\} \quad (3)$$

The unknown coefficients $a_0,\ a_1,\ a_2$ and a_3 in Equation (3) can be obtained by substituting 4 known coordinate data points $\{x_0, f(x_0)\}, \{x_1, f(x_1)\}, \{x_2, f(x_2)\}$ and $\{x_3, f(x_3)\}$ into Equation (3) as follows.

$$\left.\begin{aligned} f(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^3 \\ f(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 \\ f(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3 \\ f(x_3) &= a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^3 \end{aligned}\right\} \quad (4)$$

Equation (4) can be expressed in matrix notation as

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} \quad (5)$$

The above Equation (5) can symbolically be represented as

$$[A]_{4\times 4} \vec{a}_{4\times 1} = \vec{f}_{4\times 1} \quad (6)$$

The above Equation (5) can symbolically be represented as

$$[A]_{4\times 4} \vec{a}_{4\times 1} = \vec{f}_{4\times 1} \quad (6)$$

Thus,

$$\vec{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = [A]^{-1} \times \vec{f} \quad (7)$$

Simpson's 3/8 Rule

Deriving the formula †

Substituting Equation (7) into Equation (3), one gets

$$f_3(x) = \{1, x, x^2, x^3\} \times [A]^{-1} \times \vec{f} \qquad (8)$$

$$\begin{aligned} x_0 &= a \\ x_1 &= a + h \\ &= a + \tfrac{b-a}{3} \\ &= \tfrac{2a+b}{3} \\ x_2 &= a + 2h \\ &= a + \tfrac{2b-2a}{3} \\ &= \tfrac{a+2b}{3} \\ x_3 &= a + 3h \\ &= a + \tfrac{3b-3a}{3} \\ &= b \end{aligned} \qquad (9)$$

With the help from MATLAB, the unknown vector $\vec{a}$ (shown in Equation 7) can be solved for symbolically.

$$\begin{aligned} I &= \int_a^b f(x)\,dx \\ &= \int_a^b f_3(x)\,dx \\ &= (b-a) \times \frac{\{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\}}{8} \end{aligned} \qquad (11)$$

Since

$$h = \frac{b-a}{3}$$

$$b - a = 3h$$

and Equation (11) becomes

$$\boxed{I \approx \frac{3h}{8} \times \{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\}} \qquad (12)$$

Note the 3/8 in the formula, and hence the name of method as the Simpson's 3/8 rule.

Go through the lecture note PDF to know about the other approaches of deriving the same formula.

Simpson's 3/8 Rule

The same example

The vertical distance in meters covered by a rocket from $t = 8$ to $t = 30$ seconds is given by

$$s = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Use Simpson 3/8 rule to find the approximate value of the integral.

Simpson's 3/8 Rule

The same example

Solution

$$h = \frac{b-a}{n} = \frac{b-a}{3} = \frac{30-8}{3} = 7.3333$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

$$I \approx \frac{3h}{8} \times \{f(t_0) + 3f(t_1) + 3f(t_2) + f(t_3)\}$$

$$t_0 = 8$$

$$f(t_0) = 2000\ln\left(\frac{140000}{140000-2100\times 8}\right) - 9.8\times 8$$
$$= 177.2667$$

$$t_1 = t_0 + h$$
$$= 8 + 7.3333$$
$$= 15.3333$$

$$f(t_1) = 2000\ln\left(\frac{140000}{140000-2100\times 15.3333}\right) - 9.8\times 15.3333$$
$$= 372.4629$$

$$t_2 = t_0 + 2h$$
$$= 8 + 2(7.3333)$$
$$= 22.6666$$

$$f(t_2) = 2000\ln\left(\frac{140000}{140000-2100\times 22.6666}\right) - 9.8\times 22.6666$$
$$= 608.8976$$

$$t_3 = t_0 + 3h$$
$$= 8 + 3(7.3333)$$
$$= 30$$

$$f(t_3) = 2000\ln\left(\frac{140000}{140000-2100\times 30}\right) - 9.8\times 30$$
$$= 901.6740$$

Applying Equation (12), one has

$$I = \frac{3}{8} \times 7.3333 \times \{177.2667 + 3\times 372.4629 + 3\times 608.8976 + 901.6740\}$$
$$= 11063.3104m$$

$$I_{\text{exact}} = 11061.34m$$

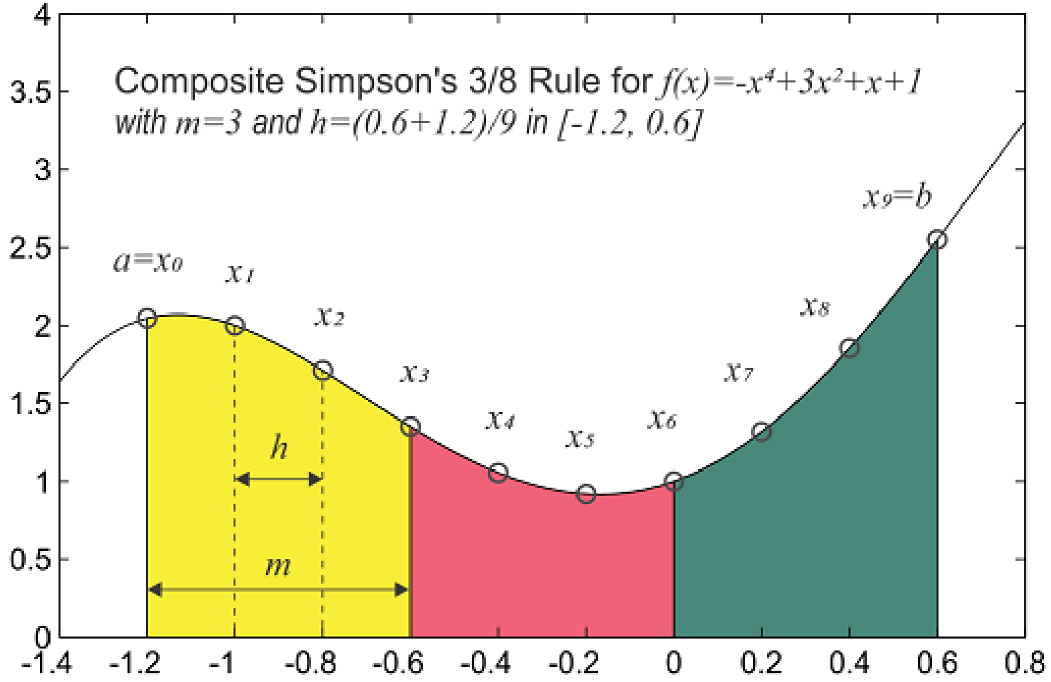
Multi-segment Simpson's 3/8 Rule

The composite version

Using n = number of equal segments, the width h can be defined as

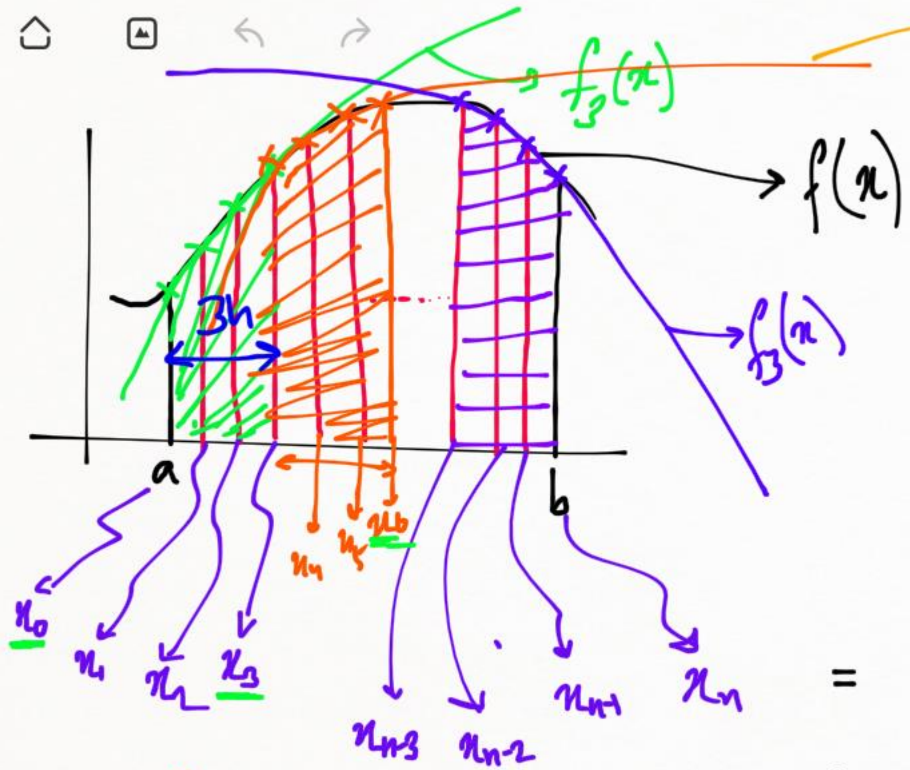
$$h = \frac{b-a}{n}$$

The number of segments need to be an integer multiple of 3 as a single application of Simpson 3/8 rule requires 3 segments.



The integral shown in Equation (1) can be expressed as

$$\begin{aligned} I &= \int_a^b f(x)\,dx \\ &\approx \int_a^b f_3(x)\,dx \\ &\approx \int_{x_0=a}^{x_3} f_3(x)\,dx + \int_{x_3}^{x_6} f_3(x)\,dx + \ldots\ldots\ldots + \int_{x_{n-3}}^{x_n=b} f_3(x)\,dx \quad (15) \end{aligned}$$



$f_3(x)$

$$I = \int_a^b f(x)\,dx \approx \int_{x_0=a}^{x_3} f_3(x)\,dx + \int_{x_3}^{x_6} f_3(x)\,dx + \cdots + \int_{x_{n-6}}^{x_{n-3}} f_3(x)\,dx + \int_{x_{n-3}}^{x_n=b} f_3(x)\,dx$$

$$x_i - x_{i-3} = 3h$$

$$= \frac{\cancelto{3h}{x_3 - x_0}}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] + \frac{\cancelto{3h}{x_6 - x_3}}{8}\left[f(x_3) + 3f(x_4) + 3f(x_5) + f(x_6)\right] + \cdots + \frac{\cancelto{3h}{x_{n-3} - x_{n-6}}}{8}\left[f(x_{n-6}) + 3f(x_{n-5}) + 3f(x_{n-4}) + f(x_{n-3})\right] + \frac{\cancelto{3h}{x_n - x_{n-3}}}{8}\left[f(x_{n-3}) + 3f(x_{n-2}) + 3f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3f(x_4) + 3f(x_5) + 2f(x_6) + \cdots + 2f(x_{n-6}) + 3f(x_{n-5}) + 3f(x_{n-4}) + 2f(x_{n-3}) + 3f(x_{n-2}) + 3f(x_{n-1}) + f(x_n)\right]$$

$$\boxed{= \frac{3h}{8}\left[f(x_0) + 3\sum_{\substack{i=1\\ i\%3=1}}^{n-2} f(x_i) + 3\sum_{\substack{i=2\\ i\%3=2}}^{n-1} f(x_i) + 2\sum_{\substack{i=3\\ i\%3=0}}^{n-3} f(x_i) + f(x_n)\right]}$$

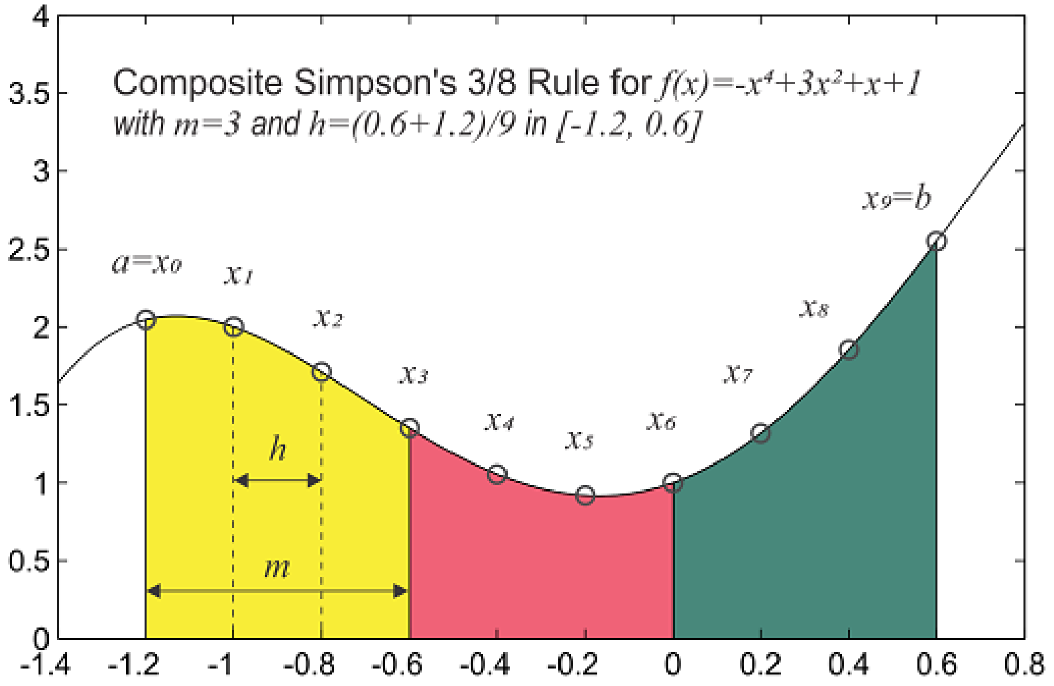
Multi-segment Simpson's 3/8 Rule

Deriving the formula

Using Simpson 3/8 rule (See Equation 12) into Equation (15), one gets

$$I = \frac{3h}{8}\left\{\begin{array}{c} f(x_0)+3f(x_1)+3f(x_2)+f(x_3)+f(x_3)+3f(x_4)+3f(x_5)+f(x_6) \\ +\ldots\ldots+f(x_{n-3})+3f(x_{n-2})+3f(x_{n-1})+f(x_n) \end{array}\right\} \quad (16)$$

$$= \boxed{\frac{3h}{8}\left\{f(x_0)+3\sum_{i=1,4,7,..}^{n-2} f(x_i)+3\sum_{i=2,5,8,..}^{n-1} f(x_i)+2\sum_{i=3,6,9,..}^{n-3} f(x_i)+f(x_n)\right\}} \quad (17)$$



Multi-segment Simpson's 3/8 Rule

The same example (now with *multiple segments*)

The vertical distance in meters covered by a rocket from $t = 8$ to $t = 30$ seconds is given by

$$s = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Use composite Simpson 3/8 rule with six segments to estimate the vertical distance.

Multi-segment Simpson's 3/8 Rule

The same example (now with *multiple segments*)

Solution

In this example, one has (see Equation 14):

$$f(t) = 2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t$$

$$h = \frac{30-8}{6} = 3.6666$$

$$\{t_0, f(t_0)\} = \{8, 177.2667\}$$

$$\{t_1, f(t_1)\} = \{11.6666, 270.4104\} \text{ where } t_1 = t_0 + h = 8 + 3.6666 = 11.6666$$

$$\{t_2, f(t_2)\} = \{15.3333, 372.4629\} \text{ where } t_2 = t_0 + 2h = 15.3333$$

$$\{t_3, f(t_3)\} = \{19, 484.7455\} \text{ where } t_3 = t_0 + 3h = 19$$

$$\{t_4, f(t_4)\} = \{22.6666, 608.8976\} \text{ where } t_4 = t_0 + 4h = 22.6666$$

$$\{t_5, f(t_5)\} = \{26.3333, 746.9870\} \text{ where } t_5 = t_0 + 5h = 26.3333$$

$$\{t_6, f(t_6)\} = \{30{,}901.6740\} \text{ where } t_6 = t_0 + 6h = 30$$

Applying Equation (17), one obtains:

$$\begin{aligned}
I &= \frac{3}{8}(3.6666)\left\{177.2667 + 3\sum_{i=1,4,..}^{n-2=4} f(t_i) + 3\sum_{i=2,5,..}^{n-1=5} f(t_i) + 2\sum_{i=3,6,..}^{n-3=3} f(t_i) + 901.6740\right\} \\
&= (1.3750)\left\{\begin{array}{c} 177.2667 + 3(270.4104 + 608.8976) \\ +3(372.4629 + 746.9870) + 2(484.7455) + 901.6740 \end{array}\right\} \\
&= 11601.4696m
\end{aligned}$$

Multi-segment Mixed Simpson's Rule

Combining the 1/3 and 3/8 rule

Based on the earlier discussion on (single and composite) Simpson 1/3 and 3/8 rules, the following "pseudo" step-by-step mixed Simpson rules for estimating

$$I = \int_a^b f(x)dx$$

can be given as

Step 1

$n_1 =$ number of segments in conjunction with Simpson 1/3 rule (a multiple of 2)

$n_2 =$ number of segments in conjunction with Simpson 3/8 rule (a multiple of 3)

Step 2

Compute

$$n = n_1 + n_2$$

$$h = \frac{b-a}{n}$$

$$x_0 = a$$

$$x_1 = a + 1h$$

$$x_2 = a + 2h$$

$$\vdots$$

$$x_i = a + ih$$

$$\vdots$$

$$x_n = a + nh = b$$

Step 3

Compute result from composite Simpson 1/3 rule

$$I_1 = \left(\frac{h}{3}\right)\left\{f(x_0) + 4\sum_{i=1,3,\ldots}^{n_1-1} f(x_i) + 2\sum_{i=2,4,6\ldots}^{n_1-2} f(x_i) + f(x_{n_1})\right\}$$

Step 4

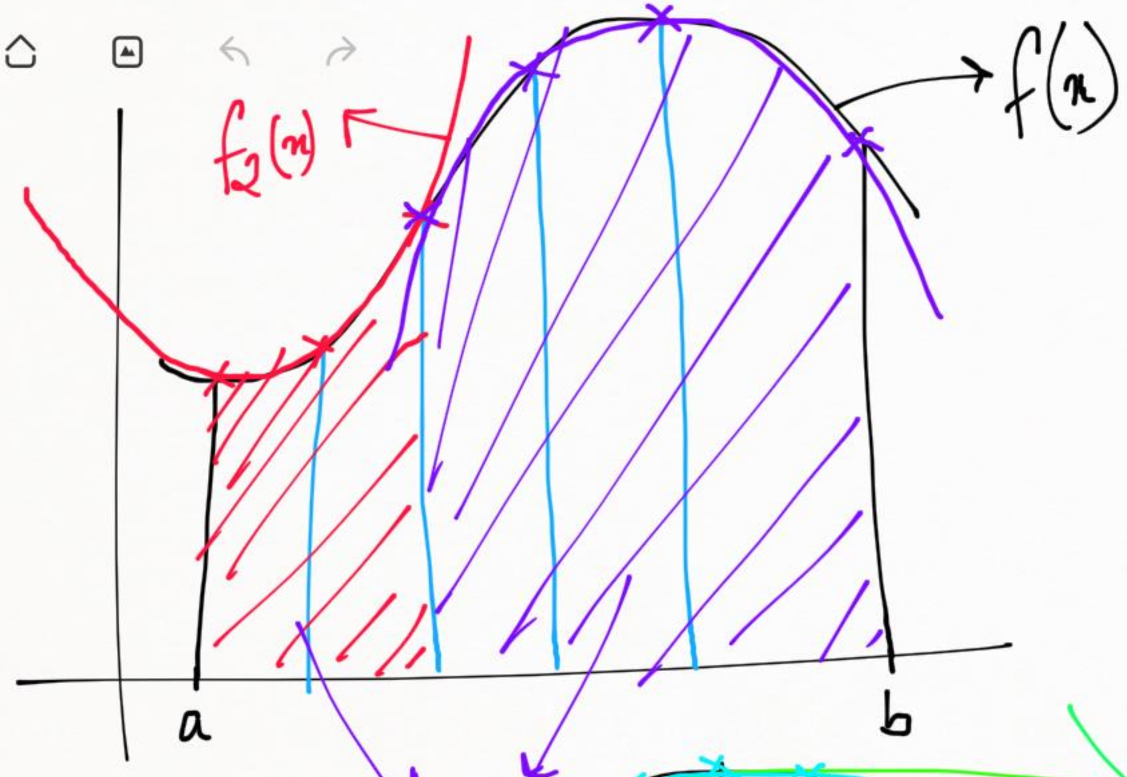
Compute result from composite Simpson 3/8 rule

$$I_2 = \left(\frac{3h}{8}\right)\left\{f(x_0) + 3\sum_{i=1,4,7\ldots}^{n_2-2} f(x_i) + 3\sum_{i=2,5,8\ldots}^{n_2-1} f(x_i) + 2\sum_{i=3,6,9,\ldots}^{n_2-3} f(x_i) + f(x_{n_2})\right\}$$

Step 5

$$I \approx I_1 + I_2$$

the final approximated answer for I.

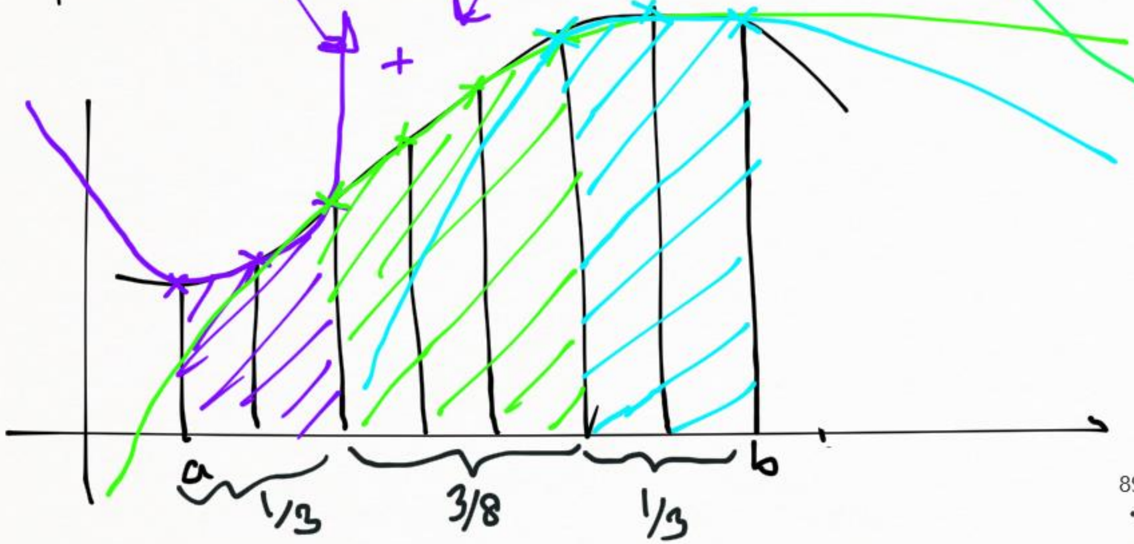


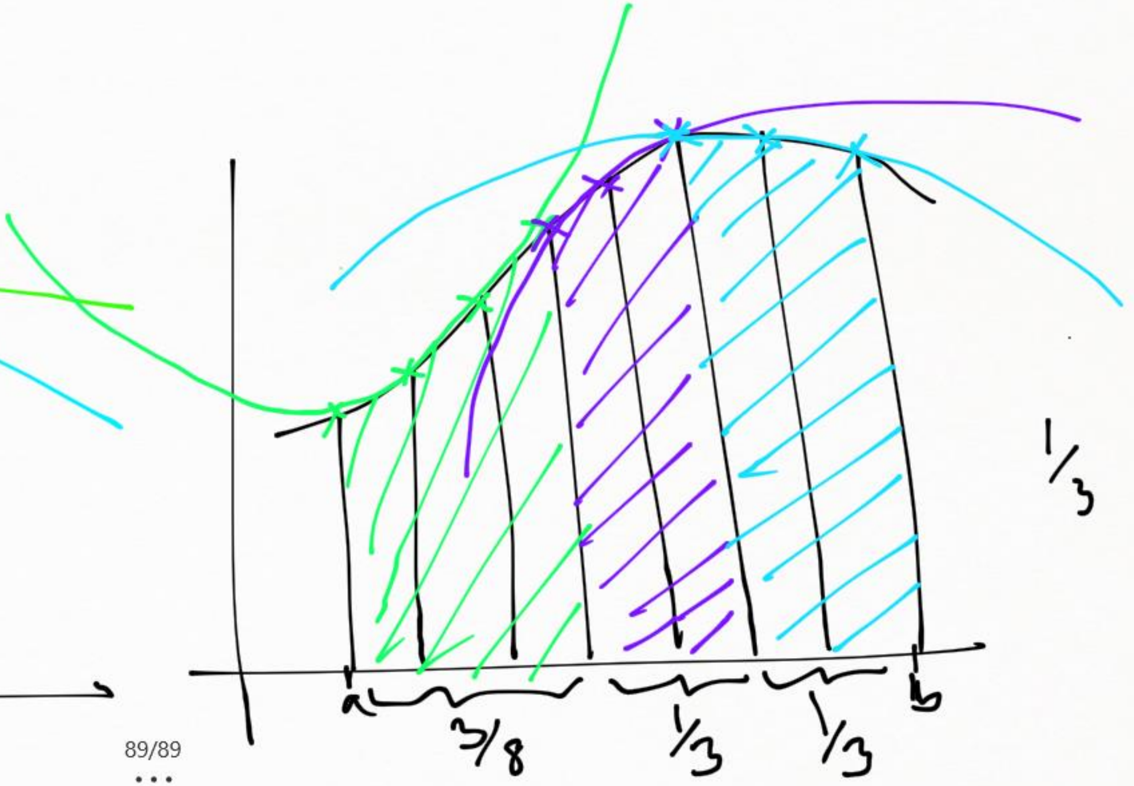
$n = 5$

$\times 2$ $\times 3$

$n = 7$

$$\frac{\lfloor 3}{\lfloor 2} = 3$$





1/3 1/3 3/8

Multi-segment Mixed Simpson's Rule

The same example (now with *mixed multiple segments*)

Compute

$$I = \int_{8}^{30} \left\{ 2000 \ln\left(\frac{140000}{140000 - 2100t} \right) - 9.8t \right\} dt,$$

using Simpson 1/3 rule (with $n_1 = 4$), and Simpson 3/8 rule (with $n_2 = 3$).

Multi-segment Mixed Simpson's Rule

The same example (now with *mixed multiple segments*)

Solution

The segment width is

$$h = \frac{b-a}{n} = \frac{b-a}{n_1+n_2} = \frac{30-8}{(4+3)} = 3.1429$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

$$\left.\begin{aligned} t_0 &= a = 8 \\ t_1 &= x_0 + 1h = 8 + 3.1429 = 11.1429 \\ t_2 &= t_0 + 2h = 8 + 2\,(3.1429) = 14.2857 \\ t_3 &= t_0 + 3h = 8 + 3\,(3.1429) = 17.4286 \\ t_4 &= t_0 + 4h = 8 + 4\,(3.1429) = 20.5714 \end{aligned}\right\} \text{Simpson's } 1/3\text{rule}$$

$$\left.\begin{aligned} t_4 &= t_0 + 4h = 8 + 4\,(3.1429) = 20.5714 \\ t_5 &= t_0 + 5h = 8 + 5\,(3.1429) = 23.7143 \\ t_6 &= t_0 + 6h = 8 + 6\,(3.1429) = 26.8571 \\ t_7 &= t_0 + 7h = 8 + 7\,(3.1429) = 30 \end{aligned}\right\} \text{Simpson's } 3/8\text{rule}$$

$$f(t_0 = 8) = 2000\ln\left(\frac{140,000}{140,000 - 2100\times 8}\right) - 9.8\times 8$$
$$= 177.2667$$

$$f(t_1) = 256.5863$$

$$f(t_2) = 342.3241$$

$$f(t_3) = 435.2749$$

$$f(t_4) = 536.3909$$

$$f(t_5) = 646.8260$$

$$f(t_6) = 767.9978$$

$$f(t_7) = 901.6740$$

Multi-segment Mixed Simpson's Rule

The same example (now with *mixed multiple segments*)

For several segments ($n_1 =$ first 4 segments), using composite Simpson 1/3 rule, one obtains

$$I_1 = \left(\frac{h}{3}\right)\left\{f(t_0) + 4\sum_{i=1,3,\ldots}^{n_1-1=3} f(t_i) + 2\sum_{i=2,\ldots}^{n_1-2=2} f(t_i) + f(t_{n_1})\right\}$$

$$= \left(\frac{h}{3}\right)\{f(t_0) + 4(f(t_1) + f(t_3)) + 2f(t_2) + f(t_4)\}$$

$$= \left(\frac{3.1429}{3}\right)\{177.2667 + 4(256.5863 + 435.2749) + 2(342.3241) + 536.3909\}$$

$$= 4364.1197$$

For several segments ($n_2 =$ last 3 segments), using single application Simpson 3/8 rule, one obtains

$$I_2 = \left(\frac{3h}{8}\right)\left\{f(t_0) + 3\sum_{i=1,3,\ldots}^{n_2-2=1} f(t_i) + 3\sum_{i=2,\ldots}^{n_2-1=2} f(t_i) + 2\sum_{i=3,6,..}^{n_2-3=0} f(t_i) + f(t_{n_1})\right\}$$

$$= \left(\frac{3h}{8}\right)\{f(t_0) + 3f(t_1) + 3f(t_2) + 2(\text{no contribution}) + f(t_3)\}$$

$$= \left(\frac{3h}{8}\right)\{f(t_4) + 3f(t_5) + 3f(t_6) + f(t_7)\}$$

$$= \left(\frac{3}{8} \times 3.1429\right)\{536.3909 + 3(646.8260) + 3(767.9978) + 901.6740\}$$

$$= 6697.3663$$

The mixed (combined) Simpson 1/3 and 3/8 rules give

$$I = I_1 + I_2$$
$$= 4364.1197 + 6697.3663$$
$$= 11061m$$