Math 4543: Numerical Methods

Lecture 9 — Spline Interpolation Method

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Lecture Plan

The agenda for today

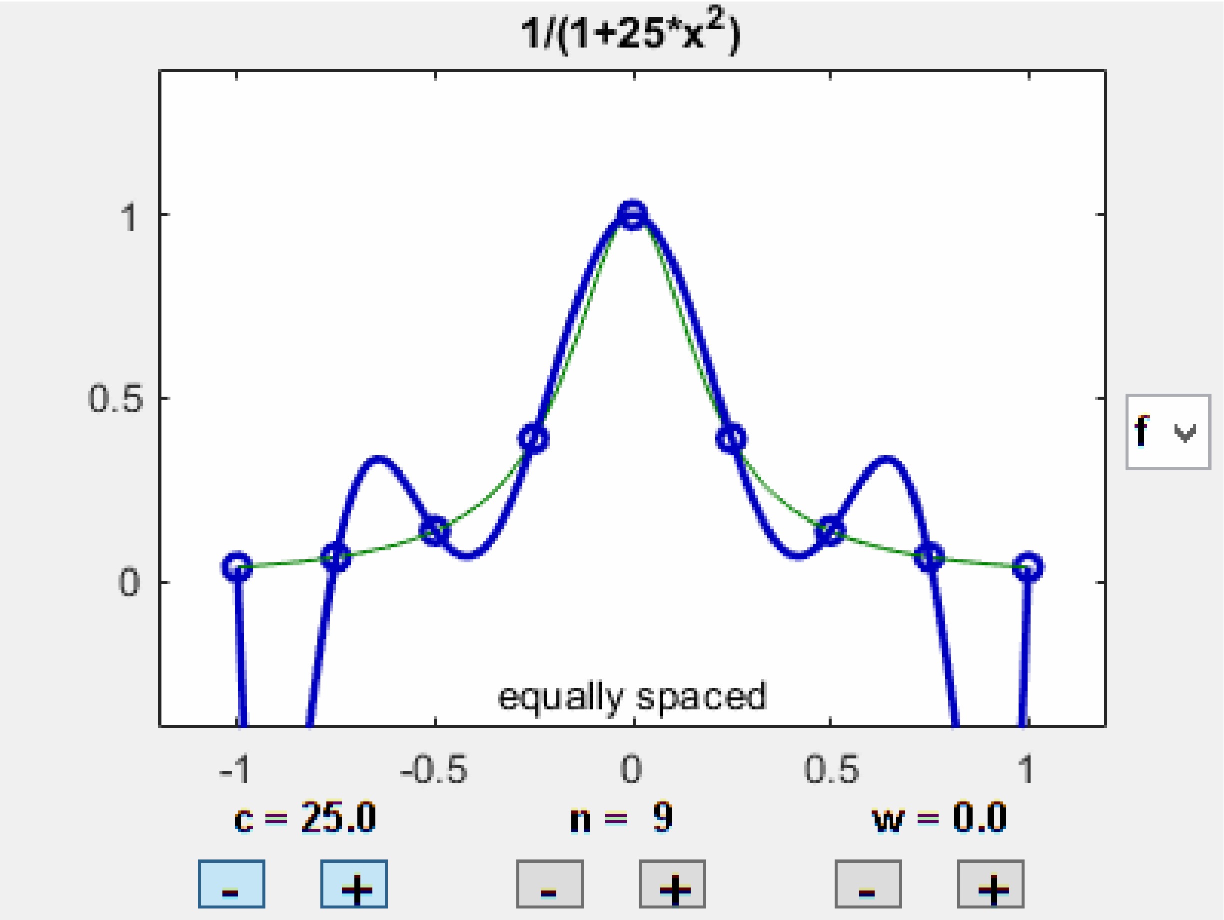
- A small recap of the Runge Phenomenon
- What is Spline Interpolation?
- Understand how Spline Interpolation is immune to the Runge Phenomenon
- Derive the equations for Linear Spline Interpolation
- Derive the equations for Quadratic Spline Interpolation

Runge Phenomenon

A small recapitulation

When n becomes ***large***, in many cases, one may get ***oscillatory behavior*** in the resulting polynomial. This was shown by Runge when he interpolated data based on a simple function on an interval of $[-1, 1]$.

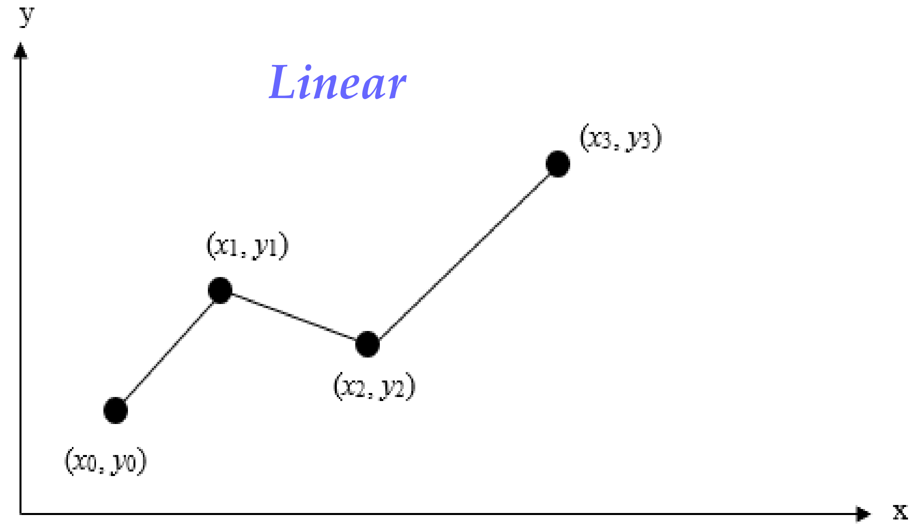
All the interpolation methods that we have covered so far — ***Direct*** method, ***NDD*** method, and ***Lagrangian*** method, all are *susceptible to this phenomenon*.



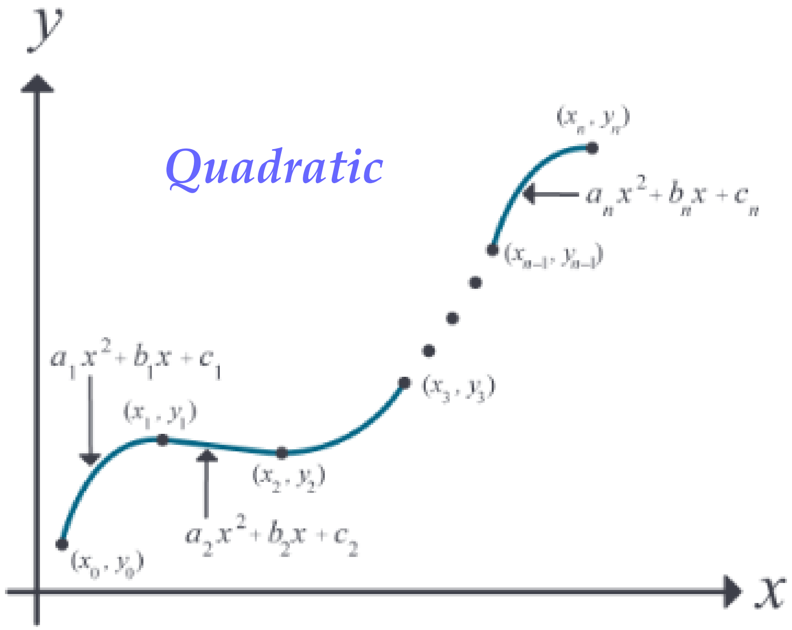
Spline Interpolation

What is it?

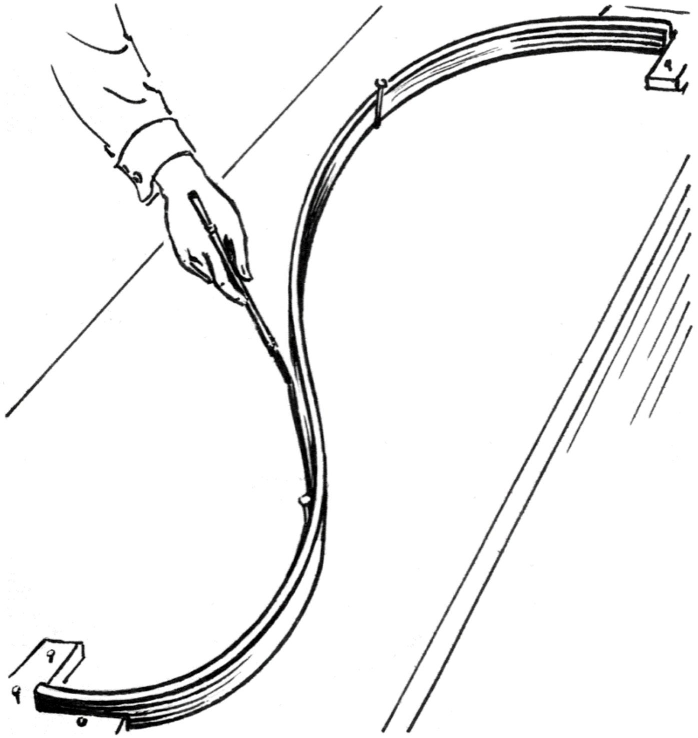
Spline — A function made up of polynomials that each have a specific interval. In other words a *"piecewise polynomial function/curve"*.



Linear



Quadratic



Spline tool

We can now use information from *more data points*, but at the same time keeping the function *reasonably true to the data behavior*.

Spline Interpolation

Linear Spline Interpolation

Assuming that the data is given in ascending order, the interpolating linear spline, also called spline of degree 1, $f(x)$ is given by

$$
\begin{aligned}
f(x) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0),\ x_0 \le x \le x_1, \\
&= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1),\ x_1 \le x \le x_2, \\
&\ \vdots \\
&= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}),\ x_{n-1} \le x \le x_n.
\end{aligned}
$$

Note that $y_i = f(x_i)$ in the above expression and that the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \qquad (2)$$

are simply slopes between x_{i-1} and x_i.

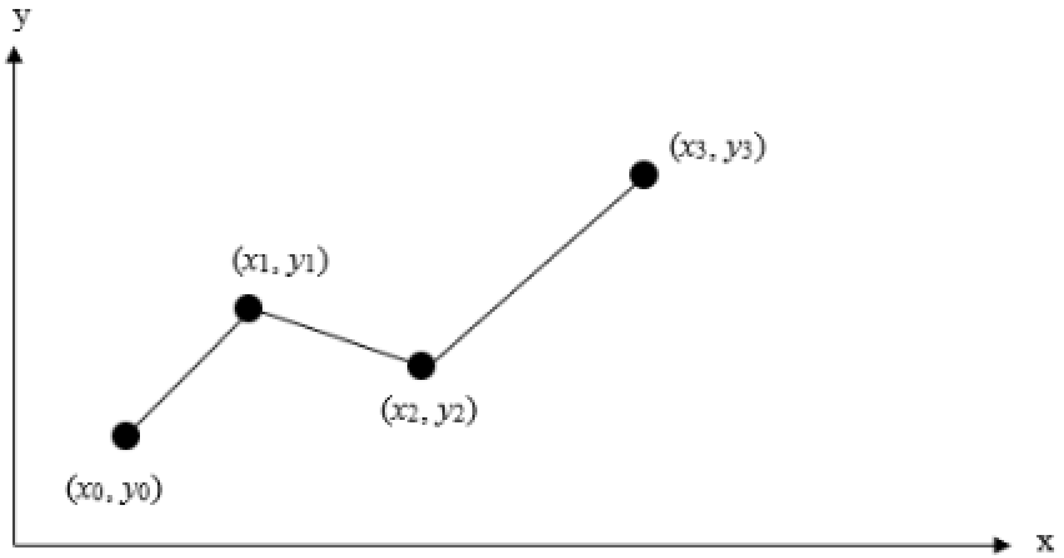


Figure 1. Linear splines.

Remember the equation for a straight line going through 2 points (x_1, y_1) and (x_2, y_2)?

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

Spline Interpolation

A linear spline example

The upward velocity of a rocket is given as a function of time in Table 1.

Table 1. Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

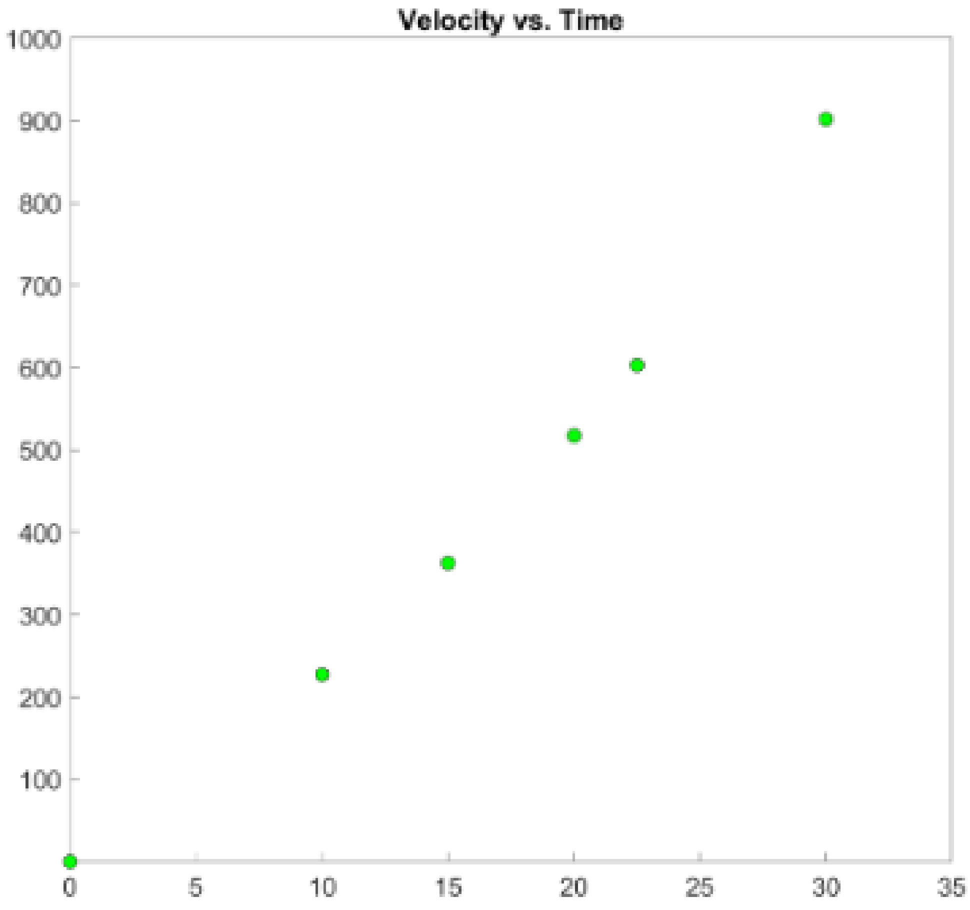


Figure 2. Graph of velocity vs. time data for the rocket example.

Determine the value of the velocity at $t = 16$ seconds using an interpolating linear spline.

Spline Interpolation

A linear spline example

Solution

Since we want to evaluate the velocity at $t = 16$ and use linear spline interpolation, we need to choose the two data points closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The two points are $t_0 = 15$ and $t_1 = 20$.

Then

$$t_0 = 15,\ v(t_0) = 362.78$$

$$t_1 = 20,\ v(t_1) = 517.35$$

gives

$$\begin{aligned} v(t) &= v(t_0) + \frac{v(t_1) - v(t_0)}{t_1 - t_0}(t - t_0) \\ &= 362.78 + \frac{517.35 - 362.78}{20 - 15}(t - 15) \\ &= 362.78 + 30.913(t - 15),\ 15 \le t \le 20 \end{aligned}$$

At $t = 16$,

$$\begin{aligned} v(16) &= 362.78 + 30.913(16 - 15) \\ &= 393.7 \text{ m/s} \end{aligned}$$

Spline Interpolation

Some caveats about Linear Spline Interpolation

- Linear spline interpolation is ***no different*** from linear polynomial interpolation.
- It still uses data only from the ***two consecutive data points***, and data from other points is not used at all.
- At the interior points of the data, the slope of the spline changes abruptly, which implies that the first derivative is *"artificially"* ***not continuous*** at these points.

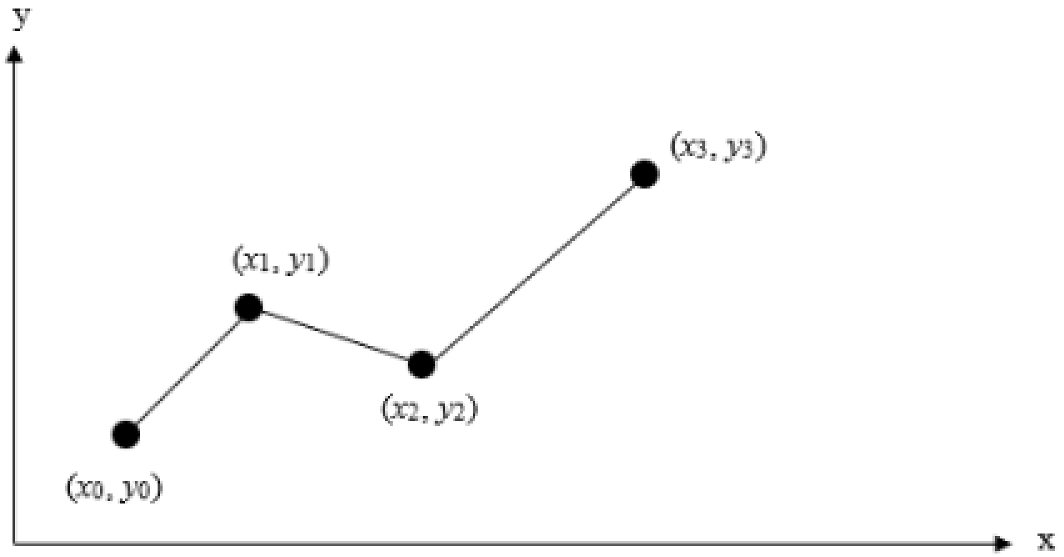


Figure 1. Linear splines.

We need ***Quadratic Splines*** and above for continuity!

Spline Interpolation

Quadratic Spline Interpolation

Quadratic spline interpolation is a method to curve fit data. For quadratic spline interpolation, piecewise quadratics approximates the data between two consecutive data points (Figure 1). Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit an interpolating quadratic spline through the data. The quadratics of the spline are given by

$$
\begin{aligned}
f(x) &= a_1x^2 + b_1x + c_1, \; x_0 \le x \le x_1 \\
&= a_2x^2 + b_2x + c_2, \; x_1 \le x \le x_2 \\
&\vdots \\
&= a_nx^2 + b_nx + c_n, \; x_{n-1} \le x \le x_n
\end{aligned}
$$

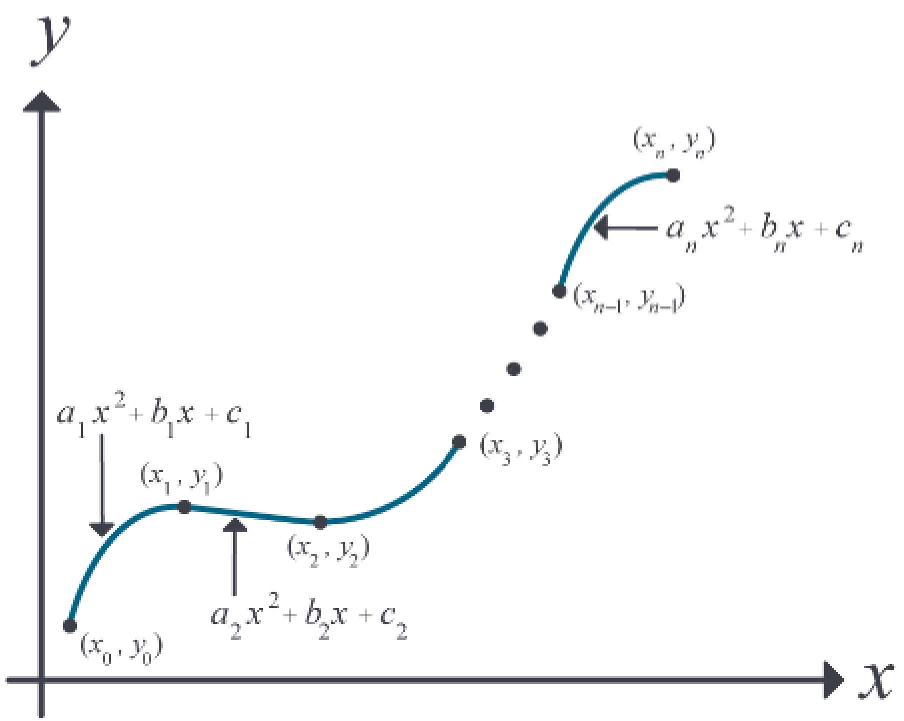


Figure 1. Quadratic spline interpolation

So how does one find the coefficients of these quadratics? There are $3n$ such coefficients

$$a_i, \; i = 1, 2, \ldots\ldots, n$$

$$b_i, \; i = 1, 2, \ldots\ldots, n$$

$$c_i, \; i = 1, 2, \ldots\ldots, n$$

To find $3n$ unknowns, one needs to set up $3n$ equations and then simultaneously solve them. These $3n$ equations are found as follows.

Quadratic Spline Interpolation

The first $2n$ equations

1. Each quadratic goes through two consecutive data points

$$
\begin{aligned}
&a_1{x_0}^2 + b_1x_0 + c_1 = f(x_0)\\
&a_1{x_1}^2 + b_1x_1 + c_1 = f(x_1)\\
&\vdots\\
&a_i{x_{i-1}}^2 + b_ix_{i-1} + c_i = f(x_{i-1})\\
&a_i{x_i}^2 + b_ix_i + c_i = f(x_i)\\
&\vdots\\
&a_n{x_{n-1}}^2 + b_nx_{n-1} + c_n = f(x_{n-1})\\
&a_n{x_n}^2 + b_nx_n + c_n = f(x_n)
\end{aligned}
$$

This condition gives $2n$ equations as there are n quadratics going through two consecutive data points.

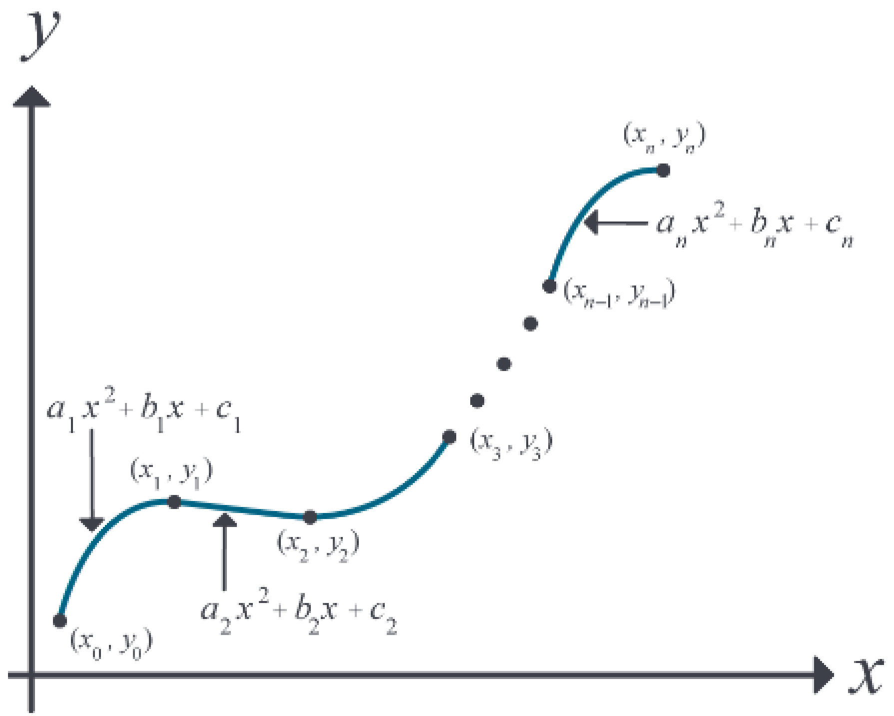


Figure 1. Quadratic spline interpolation

We need to find n more equations!

Quadratic Spline Interpolation

The next $(n-1)$ equations

2. The first derivatives of two consecutive quadratics are continuous at the common interior points. For example, the derivative of the first quadratic $a_1x^2 + b_1x + c_1$ is $2a_1x + b_1$

The derivative of the second quadratic $a_2x^2 + b_2x + c_2$ is $2a_2x + b_2$

and the two are equal at the common interior point $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$

Similarly, at the other interior points, $x_2, \ldots, x_{n-1}$,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

$$\vdots$$

$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

$$\vdots$$

$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$

For $(n-1)$ interior points, we get $(n-1)$ such equations.

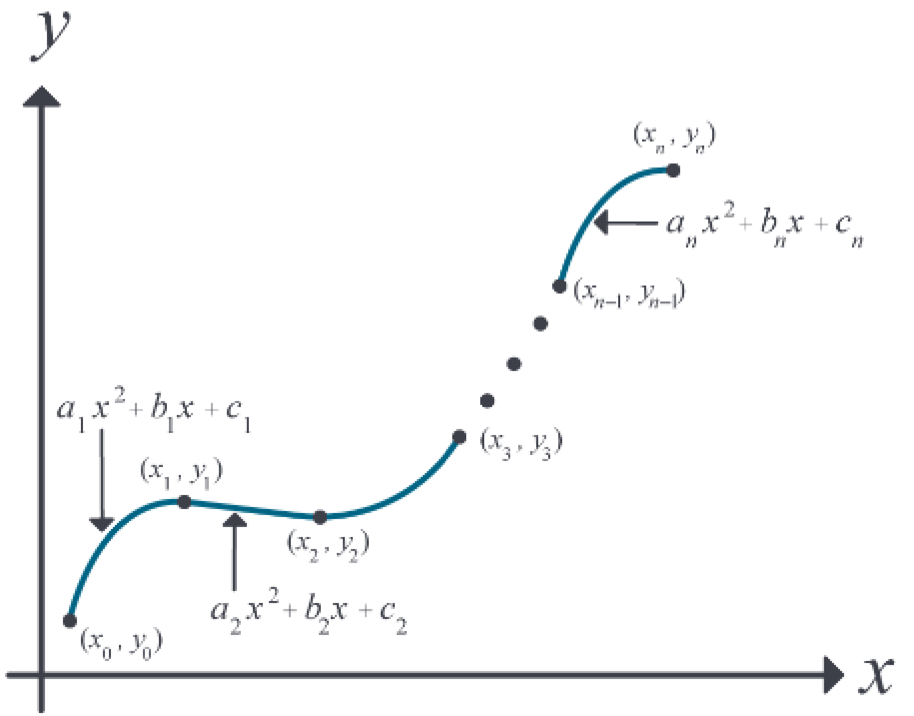


Figure 1. Quadratic spline interpolation

We still need to find

1 *more equation!*

Quadratic Spline Interpolation

The final equation

So far, the total number of equations obtained is $2n + (n - 1) = (3n - 1)$ equations.

We still then need one more equation.

We can assume the first quadratic is linear, that is,

$$a_1 = 0$$

Some assume the last quadratic is linear, that is,

$$a_n = 0$$

Others rightly base it on which interval is smaller, $[x_0, x_1]$ or $[x_{n-1}, x_n]$. If $|x_1 - x_0| \leq |x_n - x_{n-1}|$, then one would choose $a_1 = 0$, else choose $a_n = 0$.

This gives us $3n$ simultaneous linear equations and $3n$ unknowns. These can be solved by several techniques used to solve a general set of simultaneous linear equations.

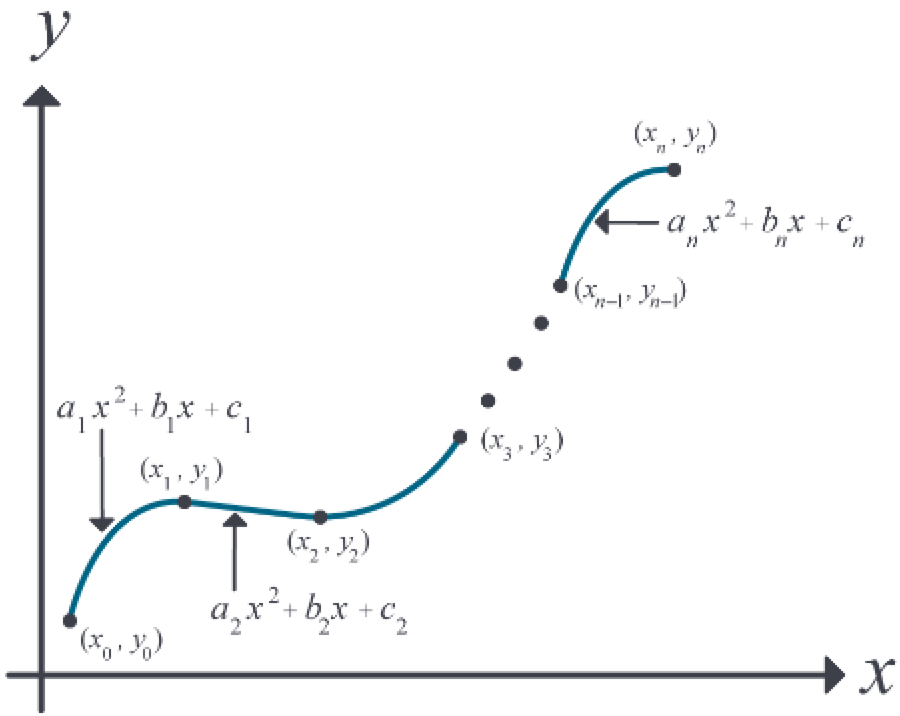


Figure 1. Quadratic spline interpolation

Mini Quiz

What about Cubic Spline Interpolation?

What is the total number of equations we need for ***Cubic Splines***?

How can we obtain them?

Spline Interpolation

A quadratic spline example

The upward velocity of a rocket is given as a function of time in Table 1.

Table 1. Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

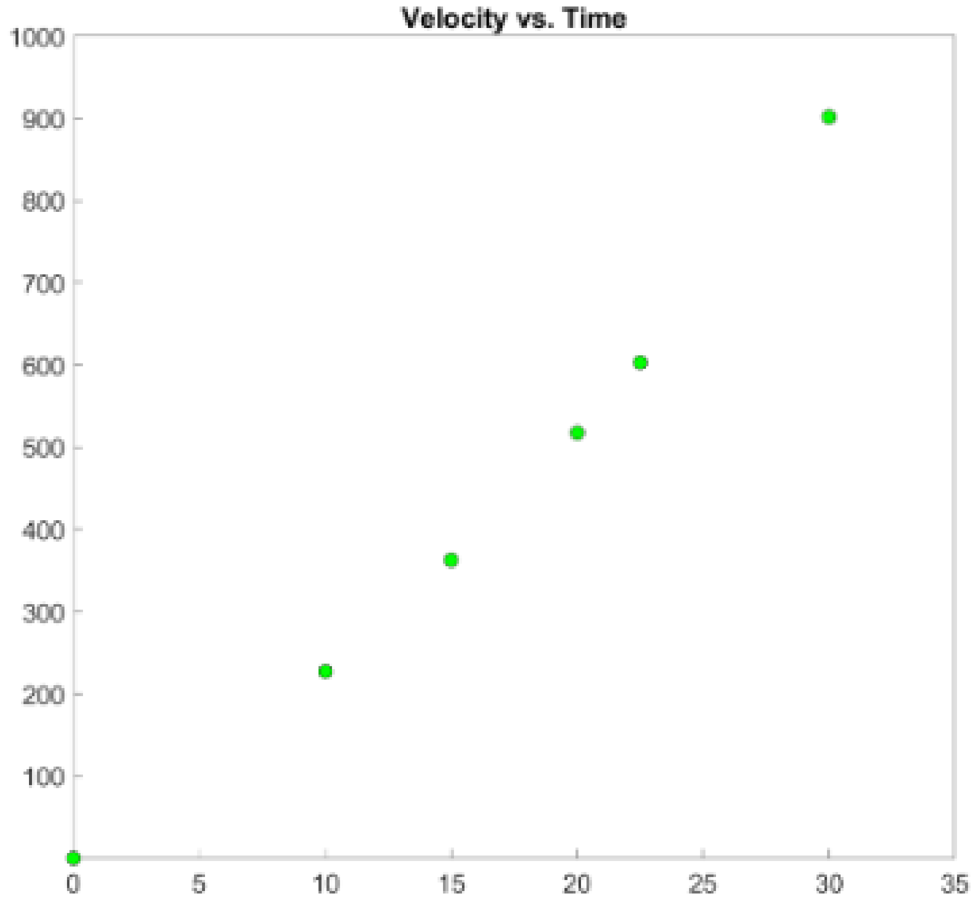


Figure 2. Graph of velocity vs. time data for the rocket example.

Determine the value of the velocity at $t = 16$ seconds using quadratic spline interpolation.

Spline Interpolation

A quadratic spline example

Solution

a) Since there are six data points, five quadratics pass through them.

$$
\begin{aligned}
v(t) &= a_1t^2 + b_1t + c_1, \ 0 \le t \le 10 \\
&= a_2t^2 + b_2t + c_2, \ 10 \le t \le 15 \\
&= a_3t^2 + b_3t + c_3, \ 15 \le t \le 20 \\
&= a_4t^2 + b_4t + c_4, \ 20 \le t \le 22.5 \\
&= a_5t^2 + b_5t + c_5, \ 22.5 \le t \le 30
\end{aligned}
$$

The equations are found as follows.

1. Each quadratic passes through two consecutive data points.

Quadratic $a_1t^2 + b_1t + c_1$ passes through $t = 0$ and $t = 10$.

$$a_1(0)^2 + b_1(0) + c_1 = 0$$

$$a_1(10)^2 + b_1(10) + c_1 = 227.04$$

Quadratic $a_2t^2 + b_2t + c_2$ passes through $t = 10$ and $t = 15$.

$$a_2(10)^2 + b_2(10) + c_2 = 227.04$$

$$a_2(15)^2 + b_2(15) + c_2 = 362.78$$

Quadratic $a_3t^2 + b_3t + c_3$ passes through $t = 15$ and $t = 20$.

$$a_3(15)^2 + b_3(15) + c_3 = 362.78$$

$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$

Quadratic $a_4t^2 + b_4t + c_4$passes through $t = 20$ and $t = 22.5$.

$$a_4(20)^2 + b_4(20) + c_4 = 517.35$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97$$

Quadratic $a_5t^2 + b_5t + c_5$ passes through $t = 22.5$ and $t = 30$.

$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

Spline Interpolation

A quadratic spline example

2. The quadratics have continuous derivatives at the common interior data points.

At $t = 10$

$$2a_1(10) + b_1 - 2a_2(10) - b_2 = 0$$

At $t = 15$

$$2a_2(15) + b_2 - 2a_3(15) - b_3 = 0$$

At $t = 20$

$$2a_3(20) + b_3 - 2a_4(20) - b_4 = 0$$

At $t = 22.5$

$$2a_4(22.5) + b_4 - 2a_5(22.5) - b_5 = 0$$

3. Assuming the last quadratic $a_5t^2 + b_5t + c_5$ is linear

$$a_5 = 0$$

Spline Interpolation

A quadratic spline example

Think about the drawbacks.

$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\
20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Spline Interpolation

A quadratic spline example

Solving the above 15 simultaneous linear equations for the 15 unknowns gives

i	a_i	b_i	c_i
1	-0.15667	24.271	0
2	1.2021	-2.9053	135.88
3	-0.44893	46.627	-235.61
4	2.2315	-60.589	836.55
5	$\boxed{0}$ Assumed	39.827	-293.13

Therefore, the interpolating quadratic spline is given by

$$\begin{aligned}
v(t) &= -0.15667t^2 + 24.271t,\ 0 \le t \le 10 \\
&= 1.2021t^2 - 2.9053t + 135.88,\ 10 \le t \le 15 \\
&= -0.44893t^2 + 46.627t - 235.61,\ \boxed{15 \le t \le 20} \\
&= 2.2315t^2 - 60.589t + 836.55,\ 20 \le t \le 22.5 \\
&= 39.827t - 293.13,\ 22.5 \le t \le 30
\end{aligned}$$

At $t = 16$ s

$$\begin{aligned}
v(16) &= -0.44893(16)^2 + 46.627(16) - 235.61 \\
&= \boxed{395.50 \text{ m/s}}
\end{aligned}$$

Not the same answer as before!