Math 4543: Numerical Methods

Lecture 18 — LU Decomposition, Eigenvalues, and Eigenvectors

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Lecture Plan

The agenda for today

- Recap the concept of LU Decomposition from Linear Algebra
- Find the LU Decomposition of an example matrix
- Recap the concept of Eigenvalues and Eigenvectors from Linear Algebra
- Find the Eigenvalues and Eigenvectors of an example matrix

LU Decomposition

What is it and why do we need it?

The idea is to decompose a matrix A into an equivalent pair L and U, where L and U are ***lower triangular*** and ***upper triangular*** matrices respectively.

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} L_{11} & & \\ L_{21} & L_{22} & \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ & U_{22} & U_{23} \\ & & U_{33} \end{pmatrix}$$

- ✓ It helps us ***memoize/record*** the steps of ***Gaussian Elimination***.
- ✓ Triangular system of equations are easy to solve using ***Forward Substitution*** and ***Backward Substitution***.
- ✓ Facilitates efficient solving of $Ax = b$ for multiple column vectors b (especially for large A).

LU Decomposition

An example

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 3 \\ (2) & -1 & 6 \\ 0 & 2 & -10 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 3 \\ (2) & -1 & 6 \\ (0) & (-2) & 2 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 12 \\ 0 & 2 & -10 \end{pmatrix} = A.$$

Eigenvalues and Eigenvectors

What are they?

Suppose that A is a square $(n \times n)$ matrix. We say that a nonzero vector $\mathbf{v}$ is an eigenvector and a number λ is its eigenvalue if

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{14.1}$$

Geometrically this means that $A\mathbf{v}$ is in the same direction as $\mathbf{v}$, since multiplying a vector by a number changes its length, but not its direction.

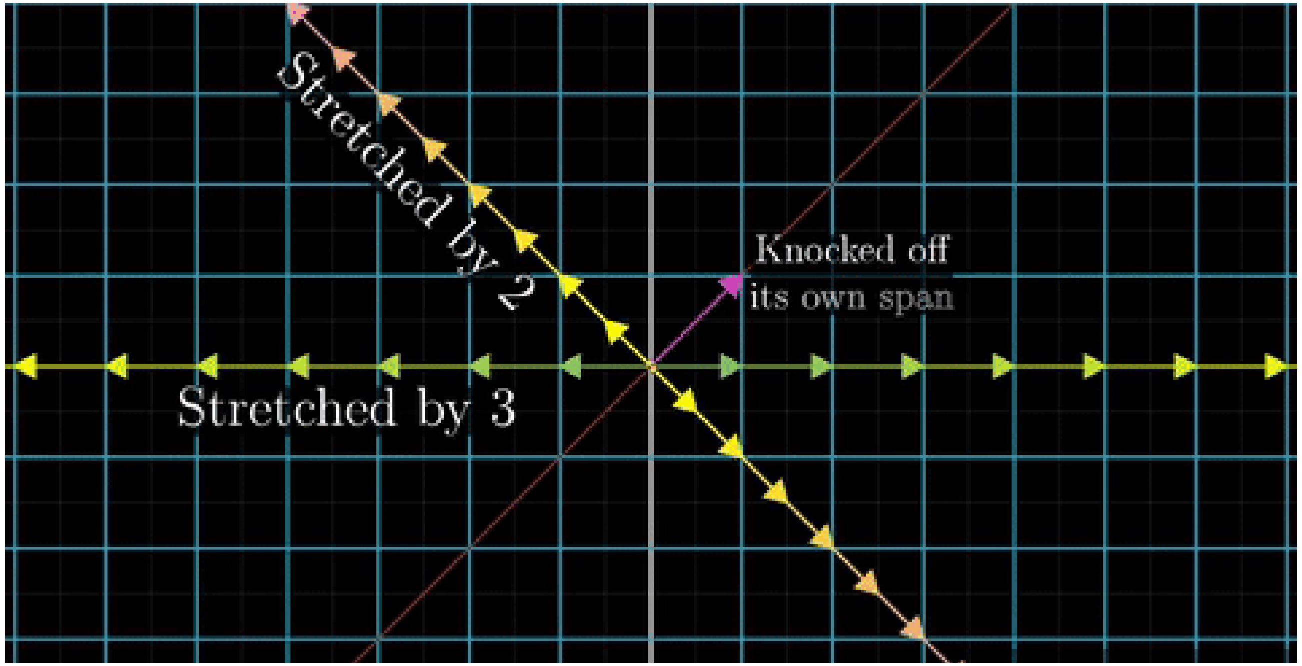
$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$

$$\begin{aligned} A\mathbf{v} - \lambda\mathbf{v} &= A\mathbf{v} - \lambda I\mathbf{v} \\ &= \underbrace{(A - \lambda I)}\mathbf{v} \\ &= \mathbf{0} \end{aligned}$$

Must be singular for non-zero $\vec{v}$.

$$\det(A - \lambda I) = 0.$$

This is called the *characteristic equation.*



Link: https://youtu.be/PFDu9oVAE-g?si=ARTE9IkmFDfikkrT

Eigenvalues and Eigenvectors

An example

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix}$$

$$\begin{aligned} A - \lambda I &= \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 1-\lambda & 4 \\ 3 & 5-\lambda \end{pmatrix}. \end{aligned}$$

The determinant of $A - \lambda I$ is then

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(5-\lambda) - 4 \cdot 3 \\ &= -7 - 6\lambda + \lambda^2. \end{aligned}$$

The characteristic equation $\det(A - \lambda I) = 0$ is simply a quadratic equation: $\lambda^2 - 6\lambda - 7 = 0$.

The roots of this equation are $\lambda_1 = 7$ and $\lambda_2 = -1$.

For $\lambda_1 = 7$, the equation for the eigenvector $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is equivalent to the augmented matrix

$$\left(\begin{array}{cc|c} -6 & 4 & 0 \\ 3 & -2 & 0 \end{array}\right)$$

$$3x - 2y = 0$$

we can let $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

This comes from noticing that $(x, y) = (2, 3)$ is a solution of $3x - 2y = 0$.

For $\lambda_2 = -1$,

$$\left(\begin{array}{cc|c} 2 & 4 & 0 \\ 3 & 6 & 0 \end{array}\right) \qquad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$A\mathbf{v}_1 = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 21 \end{pmatrix} = 7\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 7\mathbf{v}_1 \quad \text{and}$$

$$A\mathbf{v}_2 = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -1\begin{pmatrix} -2 \\ 1 \end{pmatrix} = -1\mathbf{v}_2\,.$$