Math 4543: Numerical Methods

Lecture 4 — Secant Method & False Position Method

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Lecture Plan

The agenda for today

- What is the Secant method?
- Why do we need the Secant method? (instead of the Newton-Raphson method)
- Derivation of the approximation formula for the Secant method
- Pros and cons of Secant method
- Ditto for False Position method

Secant Method

What is it?

It is very similar to the Newton-Raphson method!

In the Secant method, we start the iterative process with **two initial guesses**.

The root **may or may not be bracketed** by these guesses.

The method hence falls in the category of ***open methods***.

Convergence in open methods is **not guaranteed**, but it does so ***much faster than the bracketing methods*** if the method does converge.

Secant Method

Why do we need it?

Remember the iterative formula for the Newton-Raphson method?

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

The symbolical evaluation of the derivative $f'(x_i)$ is very ***costly*** for computers.

So, we ***approximate*** the derivative!

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Secant Method

Deriving the formula using geometry

The secant method can also be derived from geometry, as shown in Figure 1. Taking two initial guesses, x_{i-1} and x_i, one draws a straight line between $f(x_i)$ and $f(x_{i-1})$ passing through the x -axis at x_{i+1}. *ABE* and *DCE* are similar triangles.

Hence

$$\frac{AB}{AE} = \frac{DC}{DE}$$

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

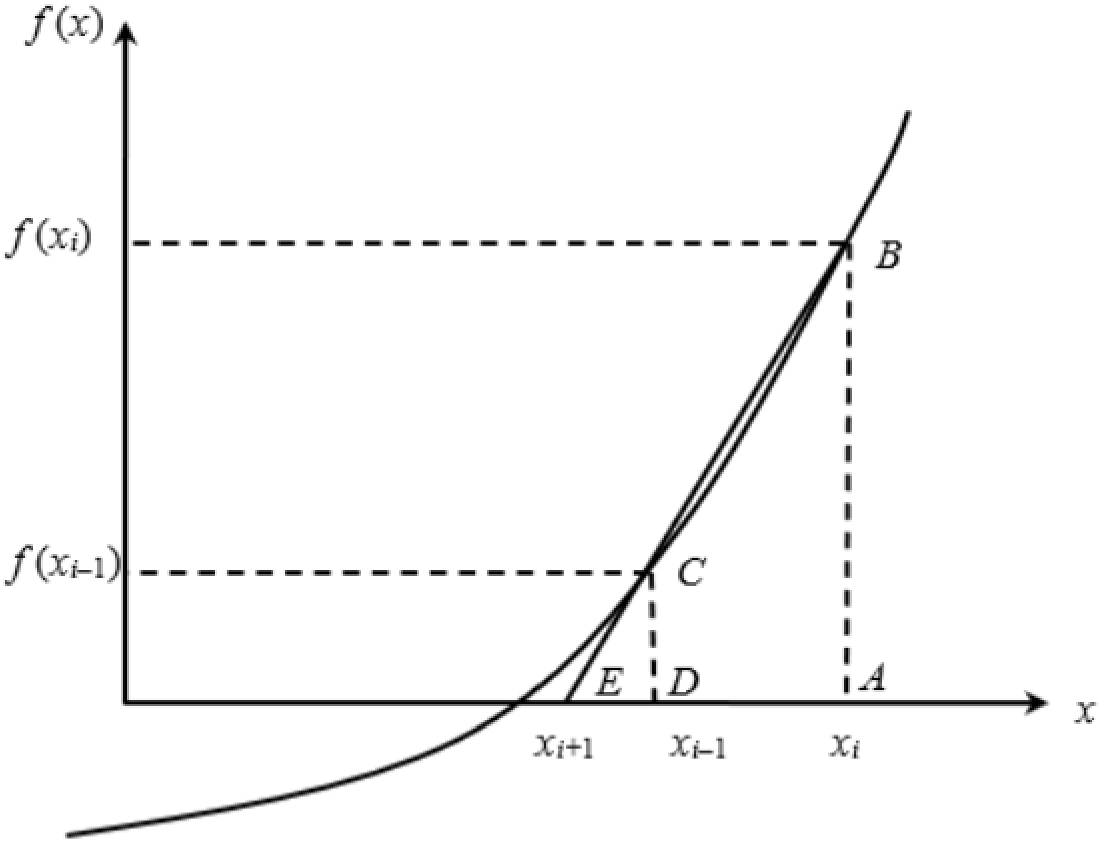


Figure 1 Geometrical representation of the secant method.

Secant Method

The Algorithm

The steps of the secant method to find the root of an equation $f(x) = 0$ are

1. Evaluate $f'(x)$ approximately $f'(x_i) = \dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$
2. Estimate the new value of the root, x_{i+1}, as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

3. Find the absolute relative approximate error $|\epsilon_a|$ as

$$|\epsilon_a| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100 \qquad (2)$$

4. Compare the absolute relative approximate error with the pre-specified relative error tolerance, ϵ_s. If $|\epsilon_a| > \epsilon_s$, then go to Step 2, else stop the algorithm. Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

Secant Method

Pros and Cons

Advantages

- ***Avoids the cost*** of symbolically finding the derivative, unlike Newton-Raphson.
- Typically ***converges faster*** than bracketed methods like the Bisection method.

Disadvantages

- ***Division by zero*** if functional values at both guesses are equal.
- ***Converges slower*** compared to Newton-Raphson, since derivative is approximated.
- ***No guarantee*** of converging.

Mini Quiz

Bad initial guesses for $\boldsymbol{sin(x)}$

For finding the root of $sin(x) = 0$ using the **Secant method**,

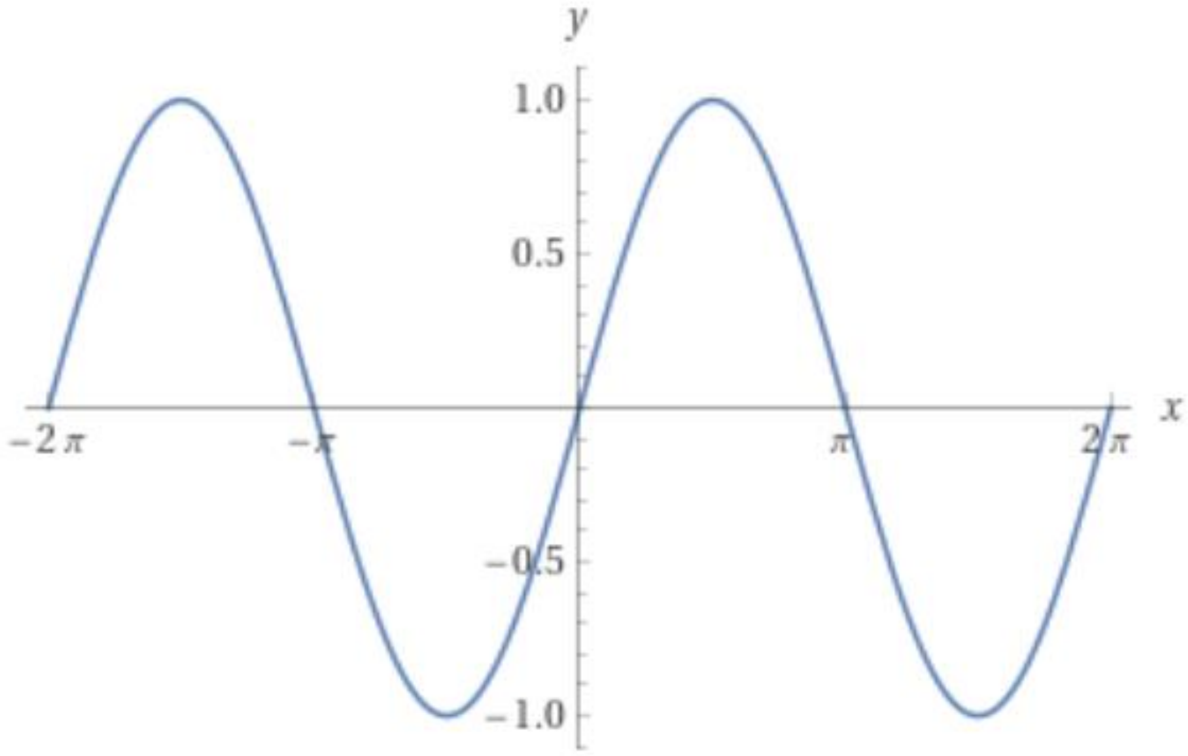
which pair of initial guesses are **inappropriate**?

(A) $\frac{\pi}{4}$ and $\frac{\pi}{2}$

(B) $\frac{\pi}{4}$ and $\frac{3\pi}{4}$

(C) $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

(D) $\frac{\pi}{3}$ and $\frac{\pi}{2}$



y = sin(x) graph | Computed by Wolfram|Alpha

False Position Method

What is it?

It's a bracketed version of the Secant method!

In the False Position method a.k.a *Regula Falsi*, we start the iterative process with **two initial guesses**.

The root **must be bracketed** by these guesses.

The method hence falls in the category of ***bracketed methods***.

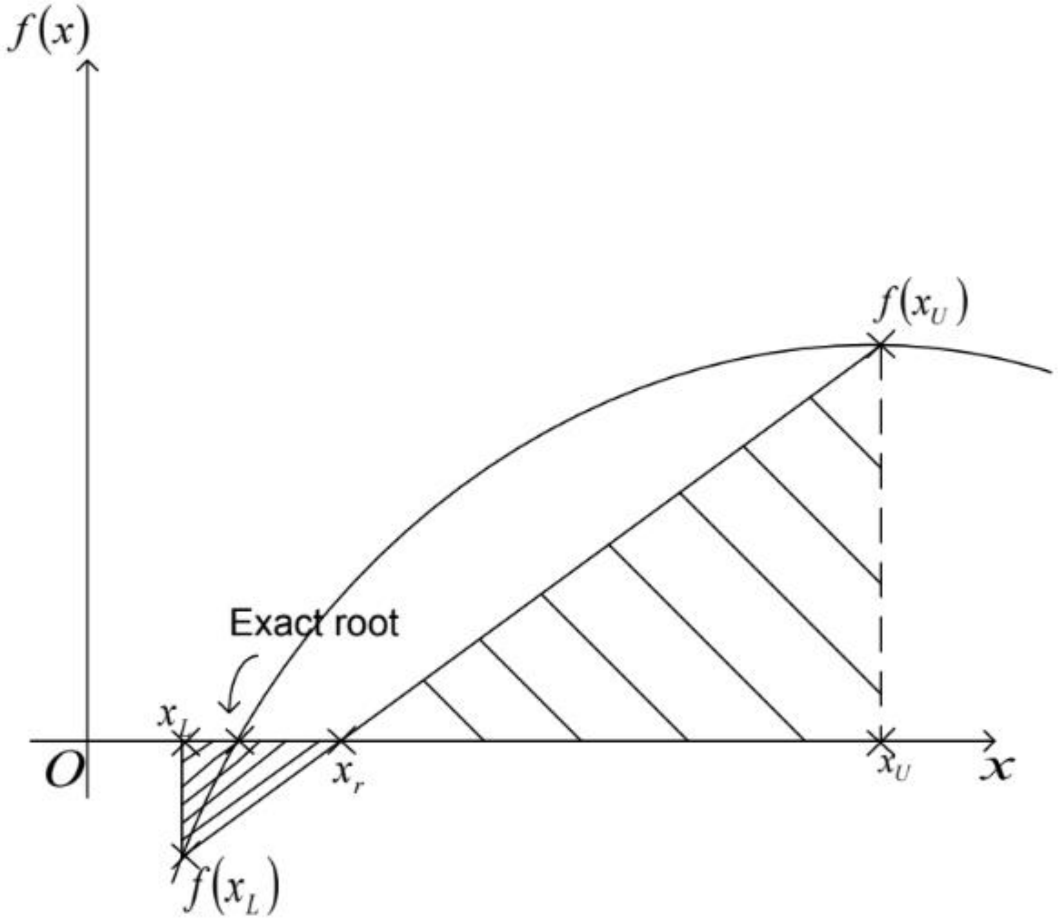
Convergence in bracketed methods is **guaranteed.**

False Position Method

Why do we need it?

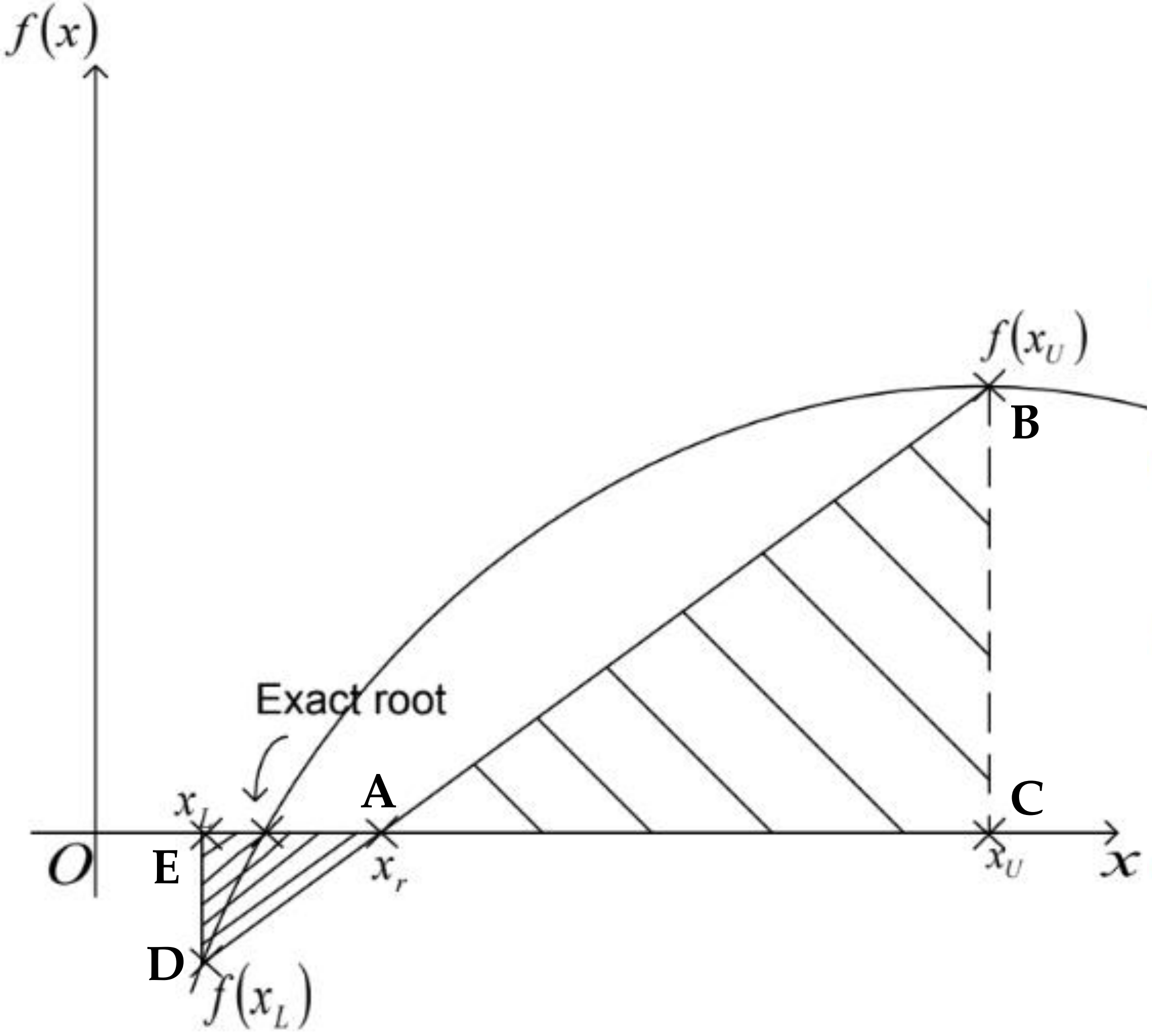
Remember what may happen when one of the initial guesses is very close to the root in the Bisection Method? (a lot of wasteful iterations!)

False Position method takes advantage of the fact that the intersection point between the secant and the X-axis is ***<u>closer to the actual root</u>***, compared to the mid-point between the two guesses.



False Position Method

Derivation of the formula



$$\triangle ABC \sim \triangle DAE$$

So,

$$\frac{DE}{AE} = \frac{BC}{AC}$$

$$\frac{0 - f(x_L)}{x_r - x_L} = \frac{0 - f(x_U)}{x_r - x_U} \qquad (4)$$

From Equation (4), one obtains

$$(x_r - x_L) f(x_U) = (x_r - x_U) f(x_L)$$

$$x_U f(x_L) - x_L f(x_U) = x_r \{f(x_L) - f(x_U)\}$$

The above equation can be solved to obtain the next predicted root x_r as

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)} \qquad (5)$$

The above equation, through simple algebraic manipulations, can also be expressed as

$$x_r = x_U - \frac{f(x_U)}{\left\{\dfrac{f(x_L) - f(x_U)}{x_L - x_U}\right\}} \qquad (6)$$

or

$$x_r = x_L - \frac{f(x_L)}{\left\{\dfrac{f(x_U) - f(x_L)}{x_U - x_L}\right\}} \qquad (7)$$

Observe the resemblance of Equations (6) and (7) to the secant method.

False Position Method

The Algorithm

The steps to apply the false-position method to find the root of the equation $f(x)=0$ are as follows.

1. Choose x_L and x_U as two guesses for the root such that $f(x_L)f(x_U)<0$, or in other words, $f(x)$ changes sign between x_L and x_U.

2. Estimate the root, x_r of the equation $f(x)=0$ as

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$

3. Now check the following

If $f(x_L)f(x_r)<0$, then the root lies between x_L and x_r; then $x_L = x_L$ and $x_U = x_r$.

If $f(x_L)f(x_r)>0$, then the root lies between x_r and x_U; then $x_L = x_r$ and $x_U = x_U$.

If $f(x_L)f(x_r)=0$, then the root is x_r. Stop the algorithm.

4. Find the new estimate of the root

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$

Find the absolute relative approximate error as

$$|\epsilon_a| = \left|\frac{x_r^{new} - x_r^{old}}{x_r^{new}}\right| \times 100$$

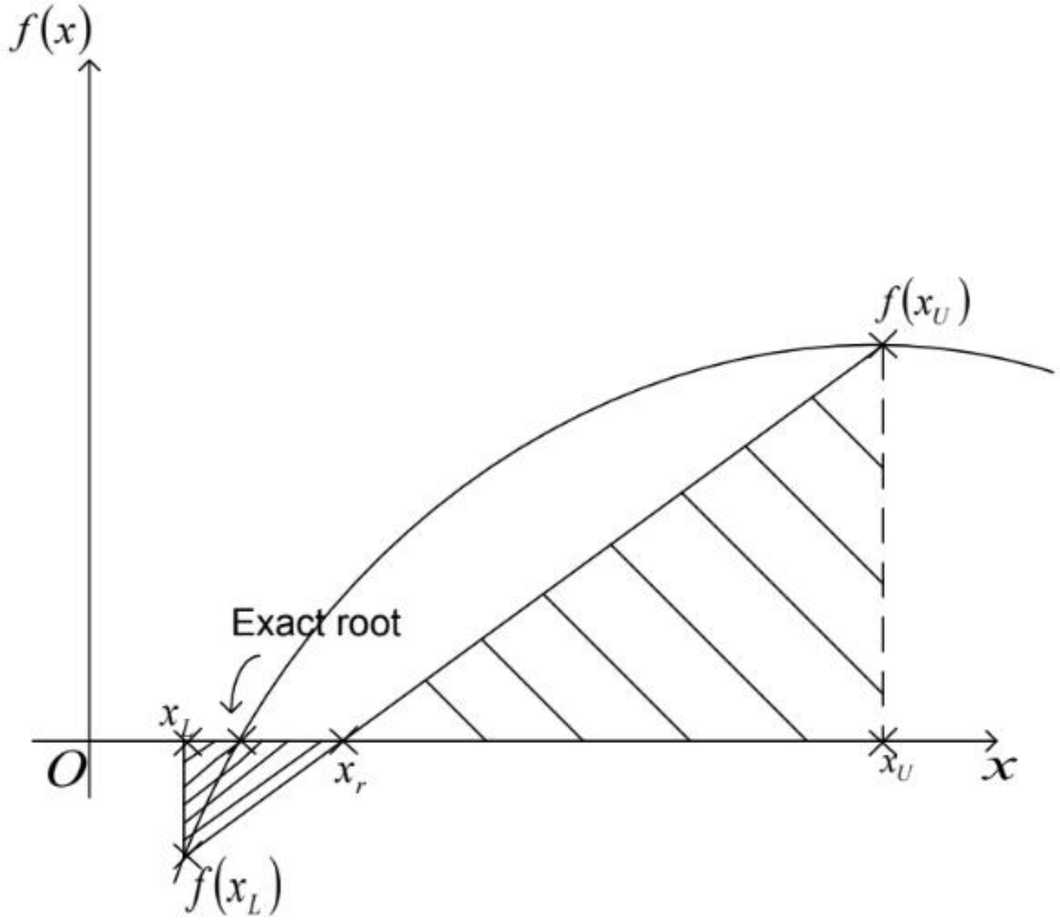
where

x_r^{new} = estimated root from present iteration

x_r^{old} = estimated root from previous iteration

5. Compare the absolute relative approximate error $|\epsilon_a|$ with the pre-specified relative error tolerance ϵ_s. If $|\epsilon_a|>\epsilon_s$, then go to step 3, else stop the algorithm. Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Note that the false-position and bisection algorithms are quite similar. The only difference is the formula used to calculate the new estimate of the root x_r as shown in steps #2 and #4!



False Position Method

Pros and Cons

Advantages

- ***May converge faster*** than Bisection Method.
- Since it is a bracketed method, the convergence is ***guaranteed***.

Disadvantages

- Almost the same as the drawbacks of the Bisection method. (think about it later…)

Mini Quiz

Which one reigns supreme?

Think intuitively — For the following graph, which method ***performs better***? Bisection or False Position?

(*i.e.* ***lesser*** number of iterations)