Math 4543: Numerical Methods

Lecture 6 — Direct Method of Interpolation

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Lecture Plan

The agenda for today

- Understand the concept of interpolation
- Why polynomials are used as interpolating function?
- Theorem about uniqueness of interpolating polynomials
- Find interpolant using the Direct method
- Use the interpolant to find the derivatives and integrals

Interpolation

What is it?

Interpolation is a type of estimation, a method of ***constructing (finding) new data points*** based on the range of a discrete set of known data points.

As given in Figure 1, data is given at discrete points such as $(x_0, y_0)\,, (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$.

A continuous function $f(x)$ may be used to represent the $n+1$ data values with $f(x)$ passing through the $n+1$ points.

Then one can find the value of *y* at any other value *x*.

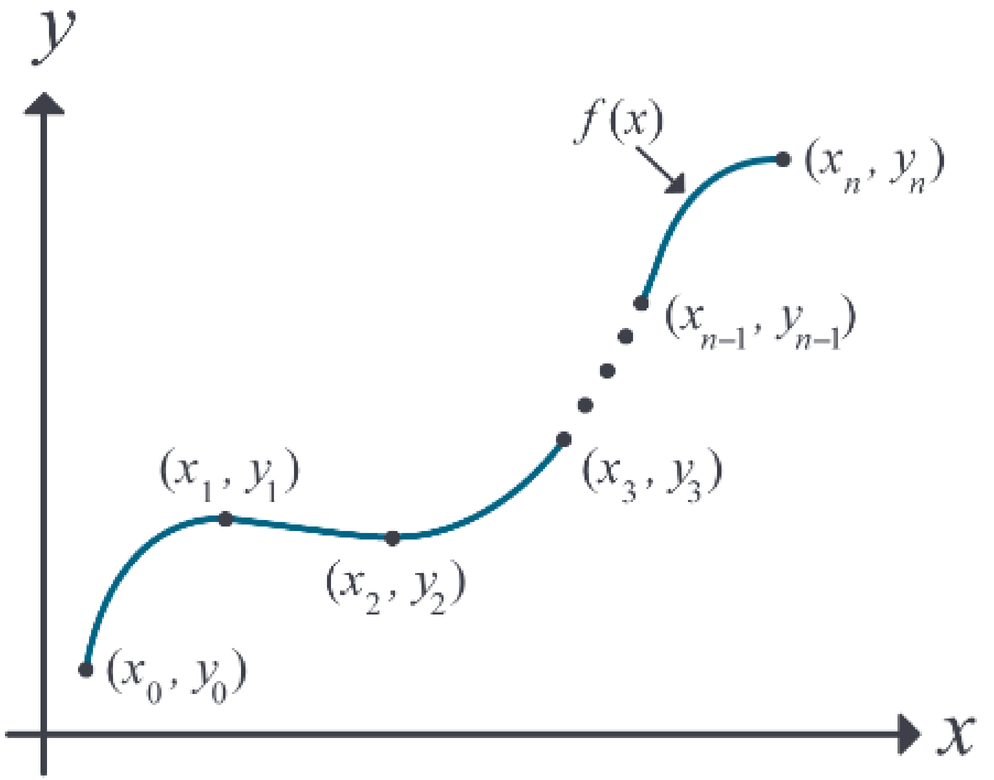


Figure 1. Interpolation of a function given at discrete points

Interpolation

Why polynomial interpolants?

The function $f(x)$ chosen for interpolation is called the ***interpolant***.

A polynomial is a common choice for an interpolating function because polynomials are ***easy*** to —

- Evaluate
- Differentiate
- Integrate

relative to other choices such as a trigonometric and exponential series.

Uniqueness of Polynomials

Theorem

A polynomial of degree n or less that passes through $n+1$ data points is unique.

Let us use proof by contradiction. If the polynomial is not unique, then at least two polynomials of order n or less pass through the $n+1$ data points.

Assume two polynomials $P_n(x)$ and $Q_n(x)$ go through $n+1$ data points,

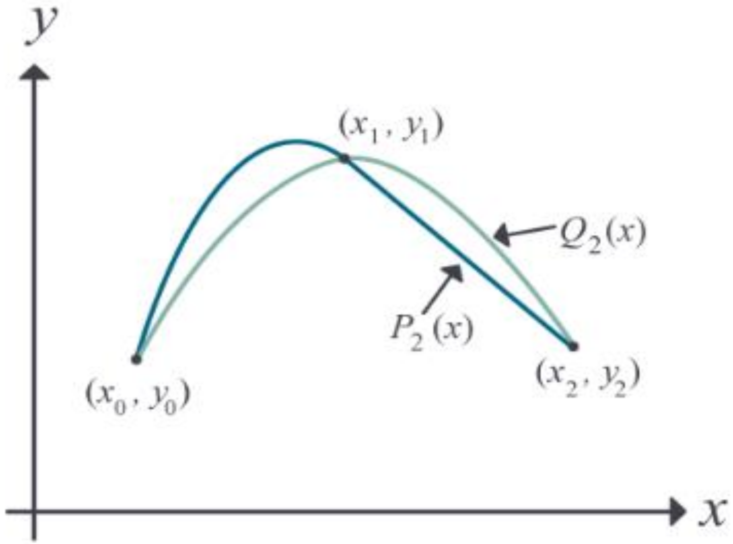
$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$$

Then

$$R_n(x) = P_n(x) - Q_n(x) \qquad (1)$$

Since $P_n(x)$and $Q_n(x)$ pass through all the $n+1$ data points,

$$P_n(x_i) = Q_n(x_i), i = 0, \ldots, n \qquad (2)$$



Hence

$$R_n(x_i) = P_n(x_i) - Q_n(x_i) = 0, i = 0, \ldots, n \qquad (3)$$

The n^{th} order polynomial $R_n(x)$ has $n+1$ zeros. A polynomial of order n can have $n+1$ zeros only if it is identical to a zero polynomial, that is,

$$R_n(x) \equiv 0 \qquad (4)$$

Hence from Equation (1)

$$P_n(x) \equiv Q_n(x)$$

Direct Method of Interpolation

How does it work?

The direct method (also called the Vandermonde polynomial method) of interpolation is based on the following premise. Given $n+1$ data points, fit a polynomial of order n as given below

$$y = a_0 + a_1 x + \ldots\ldots\ldots\ldots\ldots + a_n x^n \qquad (1)$$

through the data, where $a_0, a_1, \ldots\ldots\ldots, a_n$ are $n+1$ real constants. Since $n+1$ values of y are given at $n+1$ values of x, one can write $n+1$ equations. Then the $n+1$ constants, $a_0, a_1, \ldots\ldots\ldots, a_n$ can be found by solving the $n+1$ simultaneous linear equations. To find the value of y at a given value of x, simply substitute the value of x in Equation 1.

We ***don't need*** all the data points!

Instead we choose the ***nearest*** ones.

Direct Method of Interpolation

A first-order polynomial example

The upward velocity of a rocket is given as a function of time in Table 1.

Table 1. Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

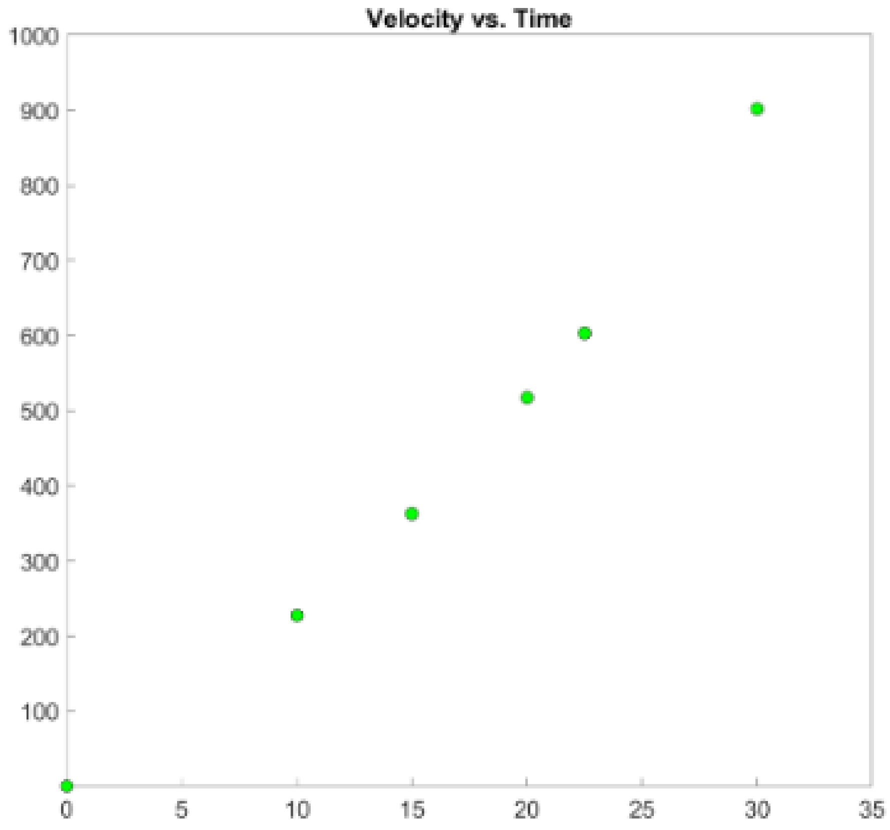


Figure 1. Graph of velocity vs. time data for the rocket example.

Estimate the velocity at $t = 16$ seconds using the direct method of interpolation with a first-order polynomial.

Direct Method of Interpolation

A first-order polynomial example

Solution

For first-order polynomial interpolation (also called linear interpolation), the velocity given by

$$v(t) = a_0 + a_1 t \qquad (E1.1)$$

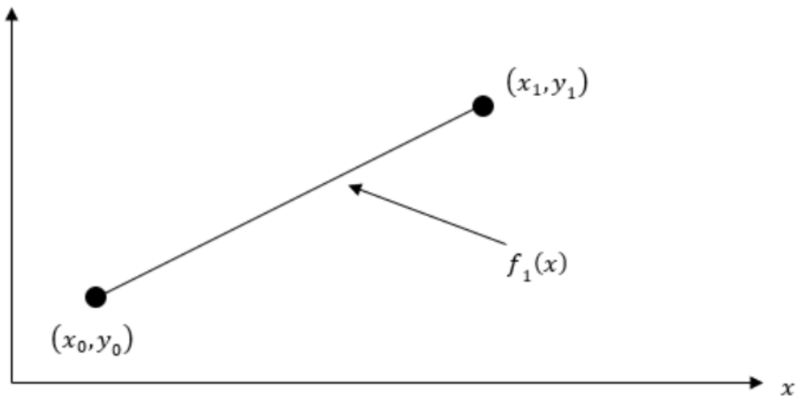


Figure 2. Linear interpolation.

Since we want to find the velocity at $t = 16$, and we are using a first-order polynomial, we need to choose the two data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The two points are $t_0 = 15$ and $t_1 = 20$.

$$t_0 = 15,\ v(t_0) = 362.78$$
$$t_1 = 20,\ v(t_1) = 517.35$$

Equation (E1.1) gives

$$v(15) = a_0 + a_1(15) = 362.78$$
$$v(20) = a_0 + a_1(20) = 517.35$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 15 \\ 1 & 20 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 362.78 \\ 517.35 \end{bmatrix}$$

Solving the above two equations gives

$$a_0 = -100.93$$
$$a_1 = 30.914$$

Hence from Equation (E1.1)

$$v(t) = a_0 + a_1 t$$
$$= -100.93 + 30.914t,\ 15 \le t \le 20$$

$$\boxed{\begin{aligned} v(16) &= -100.92 + 30.914 \times 16 \\ &= 393.70 \text{ m/s} \end{aligned}}$$

Direct Method of Interpolation

A second-order polynomial example

The upward velocity of a rocket is given as a function of time in Table 1.

Table 1. Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

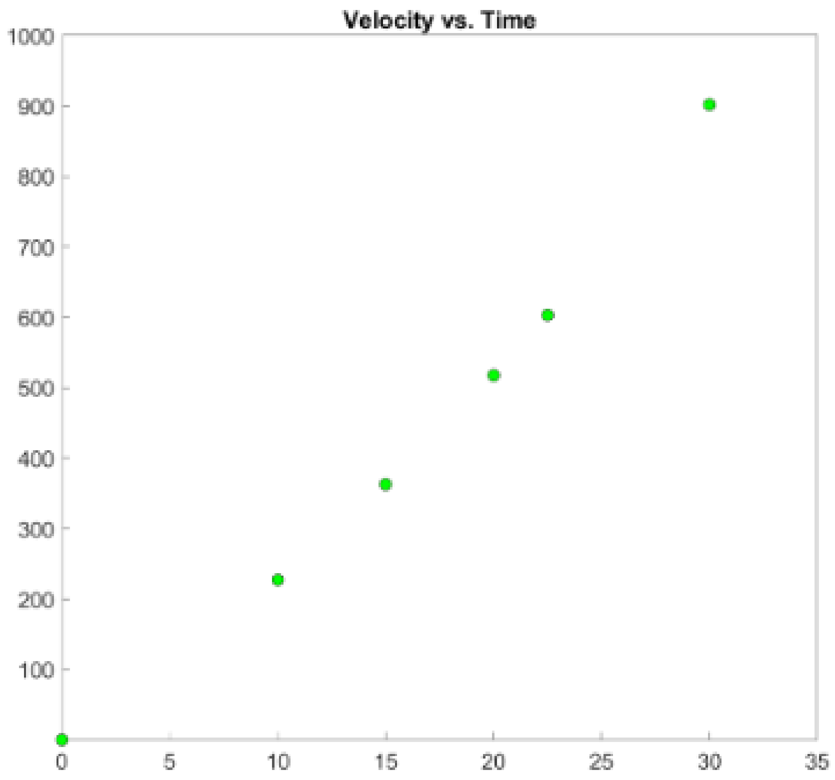


Figure 1. Graph of velocity vs. time data for the rocket example.

a) Estimate the velocity at $t = 16$ seconds using the direct method of interpolation with a second-order polynomial.

Direct Method of Interpolation

A second-order polynomial example

Solution

For second-order polynomial interpolation (also called quadratic interpolation), the velocity is given by

$$v(t) = a_0 + a_1 t + a_2 t^2 \qquad (E2.1)$$

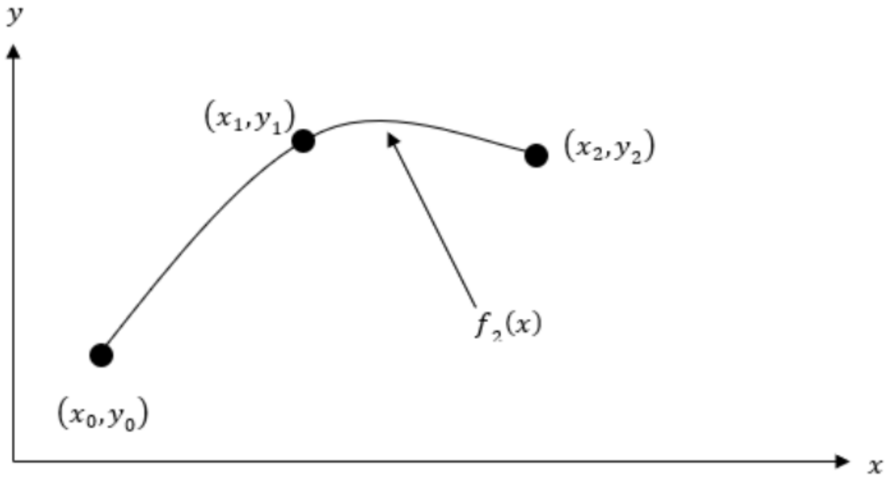


Figure 3. Quadratic interpolation.

a) Since we want to find the velocity at $t = 16$, and we are using a second-order polynomial, we need to choose the three data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The three points are $t_0 = 10, \ t_1 = 15, \text{ and } t_2 = 20.$

Then

$$\begin{aligned} t_0 &= 10, \ v(t_0) = 227.04 \\ t_1 &= 15, \ v(t_1) = 362.78 \\ t_2 &= 20, \ v(t_2) = 517.35 \end{aligned}$$

Equation (E2.1) gives

$$\begin{aligned} v(10) &= a_0 + a_1(10) + a_2(10)^2 = 227.04 \\ v(15) &= a_0 + a_1(15) + a_2(15)^2 = 362.78 \\ v(20) &= a_0 + a_1(20) + a_2(20)^2 = 517.35 \end{aligned}$$

Direct Method of Interpolation

A second-order polynomial example

Writing the three equations in matrix form, we have

$$\begin{bmatrix} 1 & 10 & 100 \\ 1 & 15 & 225 \\ 1 & 20 & 400 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 227.04 \\ 362.78 \\ 517.35 \end{bmatrix}$$

Solving the above three equations gives

$$a_0 = 12.050$$
$$a_1 = 17.733$$
$$a_2 = 0.37660$$

Hence from Equation (E2.1)

$$v(t) = 12.050 + 17.733t + 0.37660t^2, \ 10 \le t \le 20$$

At $t = 16$,

$$\boxed{\begin{aligned} v(16) &= 12.050 + 17.7333(16) + 0.37660(16)^2 \\ &= 392.19 \text{ m/s} \end{aligned}}$$

Direct Method of Interpolation

A second-order polynomial example

b) Find the absolute relative approximate error for the second-order polynomial approximation.

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the first- and second-order polynomial is

$$|\epsilon_a| = \left|\frac{392.19 - 393.70}{392.19}\right| \times 100$$
$$= 0.38410\%$$

Direct Method of Interpolation

A second-order polynomial example

c) Using the second-order polynomial interpolant for velocity from part (a), find the distance covered by the rocket from $t = 11\ s$ to $t = 16\ s$.

The distance covered by the rocket between $t = 11\ s$ and $t = 16\ s$ can be calculated from the interpolating polynomial (Equation E2.2)

$$v(t) = 12.050 + 17.733t + 0.37660t^2,\ 10 \le t \le 20$$

Note that the polynomial is valid between $t = 10$ s and $t = 20$ s and hence includes the limits of integration of $t = 11$ s and $t = 16$ s can

So

$$\begin{aligned} s(16) - s(11) &= \int_{11}^{16} v(t)\, dt \\ &= \int_{11}^{16} \left(12.050 + 17.733t + 0.37660t^2\right) dt \\ &= \left[12.050t + 17.733\frac{t^2}{2} + 0.37660\frac{t^3}{3}\right]_{11}^{16} \\ &= 1604.3 \text{ m} \end{aligned}$$

Direct Method of Interpolation

A second-order polynomial example

d) Using the second-order polynomial interpolant for velocity from part (a), find the acceleration of the rocket at $t = 16$ s.

The acceleration at $t = 16$ s is given by

$$a(16) = \frac{d}{dt} v(t)\bigg|_{t=16}$$

Given that from Equation (E2.2)

$$v(t) = 12.050 + 17.733t + 0.37660t^2, \ 10 \leq t \leq 20$$

we get

$$\begin{aligned} a(t) &= \frac{d}{\mathrm{dt}} v(t) \\ &= \frac{d}{\mathrm{dt}} \left(12.050 + 17.733t + 0.37660t^2\right) \\ &= 17.733 + 0.75320t, \ 10 \leq t \leq 20 \end{aligned}$$

Hence

$$\begin{aligned} a(16) &= 17.733 + 0.75320(16) \\ &= 29.784 \ \mathrm{m/s}^2 \end{aligned}$$