Computational Mathematics

Iterative/ Iteration method:

In computational mathematics, an **iterative method** is a mathematical procedure that generates a sequence of improving approximate solutions for a class of problems. A specific implementation of an iterative method, including the termination criteria, is an algorithm of the iterative method. An iterative method is called **convergent** if the corresponding sequence converges for given initial approximations.

An iterative method is a powerful device of solving and finding the roots of the nonlinear equations. It is a process that uses successive approximations to obtain more accurate solutions to a linear system at each step. However, iterative methods are often useful even for linear problems involving a large number of variables (sometimes of the order if millions). It is a method involves a large number of iterations of arithmetic operations to arrive at a solution for which the computers are very often used in its process to make the task simple and efficient.

Iteration means the act of repeating a process usually with the aim of approaching a desired goal or target or result. Each repetition of the process is also called iteration and the result of the iteration are used as the starting point for the next iteration. In the problems of finding the root of an equation (or a solution of a system of equations), an iterative method uses an initial guess to generate successive approximations to a solutions.

Let the equation be f(x)=0, and the value of x to be determined.

By using the iteration method, for root finding, first we have to write equation like below:

X = f(x)

Let $x = x_0$ be an **initial approximation** of the required root, then the first approximation x_1 is given by

$$x_1 = f(x_0)$$

Similarly for second, third and so on. We can obtain the successive approximation as

$$x_2 = f(x_1)$$

$$x_3 = f(x_2)$$

$$x_4 = f(x_3)$$

… … … … … … … … … ..

 typed by Tajim-IIT-JU43

… … … … … … … … ….

$x_n = f(x_{n-1})$

Thus, we get a sequence of successive approximations which may converge to desired root.

Example-1:

Find the real root of the equation $x^3 + x^2 = 1$ by iteration method.

We can rewrite the above equation by $x^3 + x^2 - 1 = 0$;

Let $f(x) = x^3 + x^2 - 1$

$f(0) = -1\ (negative)$

$f(1) = 1\ (positive)$

Hence the root value lies between 0 to 1

$x^3 + x^2 - 1 = 0$

$\Rightarrow x^2(x+1) = 1$

$$\Rightarrow x^2 = \frac{1}{x+1}$$

$$\Rightarrow x = \frac{1}{\sqrt{(x+1)}}$$

$$f(x) = \frac{1}{\sqrt{(x+1)}}$$

Let the initial approximation be $x_0 = 0.5$ $\left[because, \frac{0+1}{2} = 0.5\right]$

$$x_1 = f(x_0) = \frac{1}{\sqrt{(1+0.5)}} = 0.81649$$

$$x_2 = f(x_1) = \frac{1}{\sqrt{1+0.81649}} = 0.74196$$

$$x_3 = f(x_2) = \frac{1}{\sqrt{1+0.74196}} = 0.75767$$

$$x_4 = f(x_3) = \frac{1}{\sqrt{1+0.75767}} = 0.75427$$

$$x_5 = f(x_4) = \frac{1}{\sqrt{1 + 0.75427}} = 0.75500$$

$$x_6 = f(x_5) = \frac{1}{\sqrt{1 + 0.75500}} = 0.75485$$

$$x_7 = f(x_6) = \frac{1}{\sqrt{1 + 0.75485}} = 0.75488$$

Since the difference between x_6 and x_7 are very small, so the root is 0.75488.

Example-2:

Find the root of the equation: $\cos x = 3x - 1$, correct to four decimal places by iteration method.

Solution:

Let $f(x) = \cos x - 3x + 1$

For x=0 $\Rightarrow f(0) = \cos 0 - 0 + 1 = 2 > 0 \; (positive)$

For $x = \frac{\pi}{2} \; \Rightarrow f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - 3\frac{\pi}{2} + 1 = \; -3.7124 < 0 \; (negative)$

Hence the at least one root lies between 0 and $\frac{\pi}{2}$

Let the initial value (root): $x_0 = \frac{0+\frac{\pi}{2}}{2} = \frac{3.14}{2} * \frac{1}{2} = 0.785$

The given equation may be written as:

$3x = 1 + \cos x$

$\Rightarrow x = \frac{1}{3}(1 + \cos x) = f(x)$

$$\therefore x_1 = \frac{1}{3}[1 + \cos x_0] = \frac{1}{3}[1 + \cos(0.785)] = 0.5691$$

$$x_2 = \frac{1}{3}[1 + \cos x_1] = \frac{1}{3}[1 + \cos(0.5691)] = 0.6141$$

$$x_3 = \frac{1}{3}[1 + \cos x_2] = \frac{1}{3}[1 + \cos(0.6141)] = 0.6057$$

$$x_4 = \frac{1}{3}[1 + \cos x_3] = \frac{1}{3}[1 + \cos(0.6057)] = 0.6073$$

$$x_5 = \frac{1}{3}[1 + \cos x_4] = \frac{1}{3}[1 + \cos(0.6073)] = 0.6070$$

$$x_6 = \frac{1}{3}[1 + \cos x_5] = \frac{1}{3}[1 + \cos(0.6070)] = 0.6071$$

$$x_7 = \frac{1}{3}[1 + \cos x_6] = \frac{1}{3}[1 + \cos(0.6071)] = 0.6071$$

Due to repetition of x_6 *and* x_7 we stop our work.

So, the required root is: 0.6071 correct to four decimal places.

Example-3: Find the root of the equation $2x = \cos x + 3$ correct to three decimal places by using iteration method.

Solution:

Let $f(x) = \cos x - 2x + 3$

For $x = 1 \Rightarrow f(1) = \cos(1) - 2 + 3 = 1.5403 > 0$ $(positive)$

For $x = \frac{\pi}{2} \Rightarrow f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) - 2\frac{\pi}{2} + 3 = -0.14159 < 0$ $(negative)$

So, the root lies between 1 and $\frac{\pi}{2}$

The initial root $x_0 = \frac{1+\frac{\pi}{2}}{2} = 1.2853$

The given equation can be written as:

$$2x = \cos x + 3$$

$$\Rightarrow x = \frac{1}{2}(\cos x + 3)$$

So, $x_1 = \frac{1}{2}[\cos x_0 + 3] = \frac{1}{2}[\cos(1.2853) + 3] = 1.6408$

$$x_2 = \frac{1}{2}[\cos x_1 + 3] = \frac{1}{2}[\cos(1.6408) + 3] = 1.4650$$

$$x_3 = \frac{1}{2}[\cos x_2 + 3] = \frac{1}{2}[\cos(1.4650) + 3] = 1.5527$$

$$x_4 = \frac{1}{2}[\cos x_3 + 3] = \frac{1}{2}[\cos(1.5527) + 3] = 1.5090$$

$$x_5 = \frac{1}{2}[\cos x_4 + 3] = \frac{1}{2}[\cos(1.5090) + 3] = 1.5308$$

$$x_6 = \frac{1}{2}[\cos x_5 + 3] = \frac{1}{2}[\cos(1.5308) + 3] = 1.5199$$

$$x_7 = \frac{1}{2}[\cos x_6 + 3] = \frac{1}{2}[\cos(1.5199) + 3] = 1.5254$$

$$x_8 = \frac{1}{2}[\cos x_7 + 3] = \frac{1}{2}[\cos(1.5254) + 3] = 1.5226$$

$$x_9 = \frac{1}{2}[\cos x_8 + 3] = \frac{1}{2}[\cos(1.5226) + 3] = 1.5240$$

$$x_{10} = \frac{1}{2}[\cos x_9 + 3] = \frac{1}{2}[\cos(1.5240) + 3] = 1.5233$$

$$x_{11} = \frac{1}{2}[\cos x_{10} + 3] = \frac{1}{2}[\cos(1.5233) + 3] = 1.5237$$

Due to repetition of x_{10} and x_{11}, we stop calculation. Hence, the root is: 1.523 correct to three decimal places.

Example-4: Find the root of the equation $e^x - 3x = 0$, using iteration method correct to four decimal places.

Solutions: Let, $f(x) = e^x - 3x$

For $x = 0 \Rightarrow f(0) = e^0 - 3 * 0 = 1 > 0\ (positive)$

For $x = 1 \Rightarrow f(1) = e^1 - 3 * 1 = -0.281718 < 0\ (negative)$

So, the initial root lies between o and 1.

Therefore, $x_0 = \frac{0+1}{2} = 0.5$

The equation can be written $x = \frac{e^x}{3}$

Now, $x_1 = \frac{1}{3}e^{x_0} = \frac{1}{3}e^{0.5} = 0.54957$

$$x_2 = \frac{1}{3}e^{x_1} = \frac{1}{3}e^{0.54957} = 0.57750$$

$$x_3 = \frac{1}{3}e^{x_2} = \frac{1}{3}e^{0.57750} = 0.59386$$

$$x_4 = \frac{1}{3}e^{x_3} = \frac{1}{3}e^{0.59386} = 0.603655$$

$$x_5 = \frac{1}{3}e^{x_4} = \frac{1}{3}e^{0.603655} = 0.609597$$

$$x_6 = \frac{1}{3}e^{x_5} = \frac{1}{3}e^{0.609597} = 0.61323$$

$$x_7 = \frac{1}{3}e^{x_6} = \frac{1}{3}e^{0.61323} = 0.615462$$

$$x_8 = \frac{1}{3}e^{x_7} = \frac{1}{3}e^{0.615462} = 0.616837$$

$$x_9 = \frac{1}{3}e^{x_8} = \frac{1}{3}e^{0.616837} = 0.617685$$

$$x_{10} = \frac{1}{3}e^{x_9} = \frac{1}{3}e^{0.617685} = 0.618210$$

$$x_{11} = \frac{1}{3}e^{x_{10}} = \frac{1}{3}e^{0.618210} = 0.618534$$

$$x_{12} = \frac{1}{3}e^{x_{11}} = \frac{1}{3}e^{0.618534} = 0.618735$$

$$x_{13} = \frac{1}{3}e^{x_{12}} = \frac{1}{3}e^{0.618735} = 0.618859$$

$$x_{14} = \frac{1}{3}e^{x_{13}} = \frac{1}{3}e^{0.618859} = 0.618996$$

$$x_{15} = \frac{1}{3}e^{x_{14}} = \frac{1}{3}e^{0.618996} = 0.618984$$

$$x_{16} = \frac{1}{3}e^{x_{15}} = \frac{1}{3}e^{0.618984} = 0.6190134$$

$$x_{17} = \frac{1}{3}e^{x_{16}} = \frac{1}{3}e^{0.6190134} = 0.6190316$$

Due to repetition of $x_{16} and\ x_{17}$ we stop calculation.

So, the required root is 0.6190 correct to four decimal places.

<u>Example-5:</u> Find the root of the equation $3x - \sqrt{1 + \sin x} = 0$ correct to five decimal places by using iteration method.

typed by Tajim-IIT-JU43

Hints: $f(0) = -1 < 0 \; and \; f(1) = 1.99 > 0$ $\left[x = \frac{1}{3}\sqrt{1 + \sin x}\,; x_0 = 0.7\right]$

Example-6: Solve the equation $\cos x - xe^x = 0$ by using iteration method.

Hints: $f(0) = 1 > 0 \; and \; f(1) = -1.71844 < 0$

The root lies between 0 and 1.

$\cos x - xe^x = 0 \;\Rightarrow x = \dfrac{\cos x}{e^x}$

Example-7: Solve the equation $x^3 + x^2 - 1 = 0$ by iteration method.

Hints: $f(0) = -1 < 0 \; and \; f(1) = 1 > 0$

The root lies between 0 and 1. So, $x_0 = 0.7$

$$x^3 + x^2 = 1 \;\Rightarrow x^2(1 + x) = 1 \;\Rightarrow x = \frac{1}{\sqrt{1 + x}}$$

Example-8: Starting with x=0.12, solve $x = 0.21\sin(.5 + x)$ by using iteration method.

Hints: here, $x_0 = 0.12$

Example-9: Find the root of the equation $\sin^2 x = x^2 - 1$