Lecture 6

Types of Iteration Methods:

Based upon the number of **initial approximation** values iteration methods can be divided into two categories:

1. Bracketing iteration methods
2. Open end iteration methods

1. Bracketing iteration method:
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These methods are also known as **Interpolation methods.** Under these methods we start with two initial guesses that 'bracket' the root and then systematically reduce the width of the bracket until the desired solution is arrived at.

There are two popular methods under this category:

1.1 Bisection method
1.2 Regular_Falsi method

2. Open end iteration method:

These methods are known as **Extrapolation methods**. Under these methods we start with one or two initial roots that do not need the bracket the root.

These methods are various types:

2.1 Netwon_raphson method
2.2 Secant method
2.3 Muller's method
2.4 Fixed-point method
2.5 Bairstow's method

Bisection, Regular_Falsi and Netwon_Raphson methods are under most common and popular root finding algorithms.

1.1 Bisection Method

It is one of the simplest and most reliable of iterative methods for the solution of nonlinear equations. This method is based on the intermediate valued theorem. That is, if $f(x)$ is a real and continuous in the interval $a < x < b$, and $f(a)$ and $f(b)$ are of opposite signs, that is,

$$f(a)\, f(b) < 0$$

then there is at least one real root in the interval between $\boldsymbol{a}$ and $\boldsymbol{b}$. (There may be more than one root in the interval).

Let $x_1 = a$ and $x_2 = b$. Let us also define another point x_0 to be the midpoint between a and b. That is

$$x_0 = \frac{x_1 + x_2}{2}$$

Now, there exists the following three conditions:

1. If $f(x_0) = 0$, we have a root at x_0.
2. If $f(x_0)\, f(x_1) < 0$, there is a root between x_0 and x_1
3. If $f(x_0)\, f(x_2) < 0$, there is a root between x_0 and x_2

It follows that by testing the sign of the function at midpoint, we can deduce which part of the interval contains the root. This is illustrated in the figure. Tt shows that, since $f(x_0)$ and $f(x_2)$ are of opposite sign, a root lies between x_0 and x_2. We can further divide this subinterval into two halves to locate a new subinterval containing the root. This process can be repeated until the interval containing the root is as small as we desire.

Procedure: There are following steps are to be taken to find the root of a function under the bisection methods:

1. Get the two initial values of x which falls on the opposite sides of roots.
2. Carry on the iteration cycle by bisecting the interval and by locating the root in one of the halves. Bisect further, the half of the interval in which the root lies.
3. See that each iteration takes us closer to the root by one binary digit.
4. Stop the iteration cycle when the interval size appears to be smaller than the specified precision required in the value of the root.

Largest Possible Roots

For a polynomial equation : $a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x + a_0 = 0$

Let $f(x_0) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x + a_0$

1. The largest possible root is given by

$$x_1 = \frac{a_{n-1}}{a_n}$$

This value is taken as the initial approximation when no other value is suggested by the knowledge of the problem at hand.

2. Search Bracket: Another relationship that might be useful for determining the search intervals that contain the real roots of a polynomial is :

$$|x| \leq \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

where x is the root the polynomial.

Then, the maximum absolute value of the root is

$$|x_{max}| = \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

Example -1: Estimate the possible initial guess value of the polynomial

$$2x^3 - 8x^2 + 2x + 12 = 0$$

Ans. The largest possible root is: $x_1 = \frac{a_{n-1}}{a_n} = \frac{-8}{2} = 4$

This implies that, no root can be larger than the value 4.

But all roots satisfy the relation

$$|x| \le \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)} = \sqrt{\left(\frac{-8}{2}\right)^2 - 2\left(\frac{2}{2}\right)} = \sqrt{14}$$

Therefore, all real roots lie in the interval ($-\sqrt{14}$, $\sqrt{14}$)

Example - 2:
Find the root of the equation $\boldsymbol{x^2 - 4x - 10 = 0}$ using **Bisection Method**.

Ans.
$$|x_{max}| = \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)} = \sqrt{\left(\frac{-4}{1}\right)^2 - 2\left(\frac{-10}{1}\right)} = 6$$

Therefore, we have both the roots in the interval (-6, 6). The table below gives the values of *f(x)* between -6 and 6 and that there is a root in the interval (- 2, - 1) and another (5, 6).

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
f(x)	50	35	22	11	2	-5	-10	-13	-14	-13	-10	-5	2

Find initial root:

$$f(-2) = 4 + 8 - 10 = 2 > 0$$

$$f(-1) = 1 + 4 - 10 = -5 \ > 0$$

$$f(-2)\ and\ f(-1)\ are\ opposite\ signs. So\ root\ lies\ between\ -2\ and -1.$$

$$\therefore Initial\ root, x_0 = \frac{a+b}{2} = \frac{-2-1}{2} = \ -1.5$$

Root x_n	Signs of $f(x) = x^2 - 4x - 10$	Comments
$x_0 = -1.5$	$f(-1.5) = -1.75 < 0$	Since f(-2)>0 and f(-1.5)<0, the root must in the interval (-2,-1.5)

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$x_1 = \frac{-2 - 1.5}{2} = -1.75$	$f(-1.75) = 0.0625 > 0$	Since f(-1.5)<0 and f(-1.75)>0, the root must in the interval (-1.75, -1.5)
$x_2 = \frac{-1.75 - 1.5}{2} = -1.625$	$f(-1.625) = -0.859 < 0$	The root must in the interval (-1.75, -1.625)
$x_3 = \frac{-1.75 - 1.625}{2} = -1.6875$	$f(-1.6875) = -0.4 < 0$	The root must in the interval (-1.75,-1.6875)
$x_4 = \frac{-1.75 - 1.6875}{2} = -1.7188$	$f(-1.7188) = -0.1705 < 0$	The root must in the interval (-1.75, -1.7188)
$x_5 = \frac{-1.75 - 1.7188}{2} = -1.7344$	$f(-1.7344) = -0.054 < 0$	The root must in the interval (-1.75, -1.7344)
$x_6 = \frac{-1.75 - 1.7344}{2} = -1.7422$	$f(-1.7422) = 0.004 > 0$	The root must in the interval (-1.7344, -1.7422)
$x_7 = \frac{-1.7344 - 1.7422}{2} = -1.7383$	$f(-1.7383) = -0.025 < 0$	The root must in the interval (-1.7422, -1.7383)
$x_8 = \frac{-1.7422 - 1.7383}{2} = -1.7402$	$f(-1.7402) = -0.0109 < 0$	The root must in the interval (-1.7422, -1.7402)
$x_9 = \frac{-1.7422 - 1.7402}{2} = -1.7412$	$f(-1.7412) = -0.003 \approx 0$	$\therefore$ *The appropriate root is* -1.7412

Example – 3:
Find the root of the equation $\boldsymbol{x^3 - x - 1 = 0}$ using **Bisection Method** correct to two decimal places.
Solution:

$$let, f(x) = x^3 - x - 1$$

$$\therefore f(1) = 1 - 1 - 1 = -1 < 0$$

$$f(2) = 8 - 2 - 1 = 5 > 0$$

So, f(1) and f(2) are of opposite sign. So at least one root lies between 1 and 2.

The largest possible root is

$$|x_{max}| = \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)} = \sqrt{\left(\frac{-1}{1}\right)^2 - 2\left(\frac{-1}{1}\right)} = \sqrt{3} = 1.732$$

<u>Iteration 1:</u>

Let $x_0 = \frac{1+2}{2} = 1.5 \;\; so, f(x_0) = f(1.5) = 0.875 > 0$

So, the root lies between 1 and 1.5

Iteration 2:

Let $x_1 = \frac{1+1.5}{2} = 1.25 \;\; so, f(x_1) = f(1.25) = -0.297 < 0$

So, the root lies between 1.25 and 1.5

Iteration 3:

Let $x_2 = \frac{1.5+1.25}{2} = 1.375 \;\; so, f(x_2) = f(1.375) = 0.2246 > 0$

So, the root lies between 1.25 and 1.375

Iteration 4:

Let $x_3 = \frac{1.25+1.375}{2} = 1.3125 \;\; so, f(x_3) = f(1.3125) = -0.0515 < 0$

So, the root lies between 1.3125 and 1.375

Iteration 5:

Let $x_4 = \frac{1.3125+1.375}{2} = 1.34375 \;\; so, f(x_4) = f(1.34375) = 0.0826 > 0$

So, the root lies between 1.3125 and 1.34375

Iteration 6:

Let $x_5 = \frac{1.3125+1.34375}{2} = 1.3281 \;\; so, f(x_5) = f(1.3281) = 0.018447 > 0$

So, the root lies between 1.3125 and 1.3281

Iteration 7:

Let $x_6 = \frac{1.3125+1.3281}{2} = 1.3208 \;\; so, f(x_6) = f(1.3208) = -0.019 < 0$

So, the root lies between 1.3208 and 1.3281

Iteration 8:

Let $x_7 = \frac{1.3208+1.3281}{2} = 1.3242 \;\; so, f(x_7) = f(1.3242) = -0.002 < 0$

So, the root lies between 1.3242 and 1.3281

Iteration 9:

Let $x_8 = \frac{1.3242+1.3281}{2} = 1.3261 \;\; so, f(x_8) = f(1.3261) = 0.005970 > 0$

So, the root lies between 1.3242 and 1.3261.

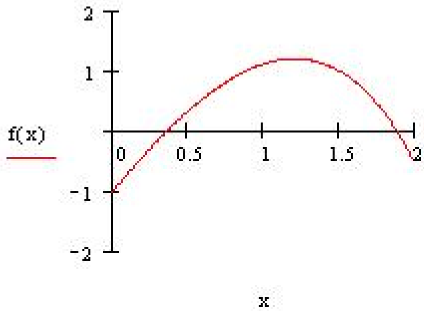
So, up to two decimal places the root is 1.32.

EXAMPLE-4: Consider $f(x) = x^3 + 3x - 5$, where [a = 1, b = 2] and DOA = 0.001.

i	A	X	b	f(a)	f(x)	f(b)
1	1	1.5	2	–1	2.875	9
2	1	1.25	1.5	–1	0.703125	2.875
3	1	1.125	1.25	–1	–0.201171875	0.703125
4	1.125	1.1875	1.25	–0.201171875	0.237060546875	0.703125
5	1.125	1.15625	1.1875	–0.201171875	0.014556884765625	0.237060546875
6	1.125	1.140625	1.15625	–0.201171875	–0.0941429138183594	0.014556884765625
7	1.140625	1.1484375	1.15625	–0.0941429138183594	–0.0400032997131348	0.014556884765625
8	1.1484375	1.15234375	1.15625	–0.0400032997131348	–0.0127759575843811	0.014556884765625
9	1.15234375	1.154296875	1.15625	–0.0127759575843811	0.000877253711223602	0.014556884765625

Numerical Example:
Find a root of $f(x) = 3x + \sin(x) - \exp(x) = 0$.

The graph of this equation is given in the figure.



Its clear from the graph that there are two roots, one lies between 0 and 0.5 and the other lies between 1.5 and 2.0.

Consider the function f (x) in the interval [0, 0.5] since f (0) * f (0.5) is less than zero.

Then the bisection iterations are given by

Iteration No.	a	B	c	f(a) * f(c)
1	0	0.5	0.25	0.287 (+ve)
2	0.25	0.5	0.393	-0.015 (-ve)

3	0.65	0.393	0.34	9.69 E-3 (+ve)
4	0.34	0.393	0.367	-7.81 E-4 (-ve)
5	0.34	0.367	0.354	8.9 E-4 (+ve)
6	0.354	0.367	0.3605	-3.1 E-6 (-ve)

So one of the roots of 3x + sin(x) - exp(x) = 0 is approximately 0.3605.

- Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on [0,1].

Solution: Let $f(x) = x^3 - 7x^2 + 14x - 6 = 0$. Note that $f(0) = -6 < 0$ and $f(1) = 2 > 0$, therefore, based on the Intermediate Value Theorem, since f is continuous, there is $p \in (0, 1)$ such that $f(p) = 0$.

Let $a_0 = 0$, $b_0 = 1$, with $f(a_0) < 0$, $f(b_0) > 0$.

Let $p_0 = a_0 + \frac{b_0 - a_0}{2} = 0.5$, and we have $f(p_0) = -0.6250 < 0$ (the same sign as $f(a_0)$), therefore $a_1 = p_0 = 0.5$, $b_1 = b_0 = 1$ and repeat: $p_1 = 0.75$, ... This yields the following results for p_n and $f(p_n)$:

n	p_n	$f(p_n)$
0	0.5	-0.6250000
1	0.75000000	+0.9843750
2	0.62500000	+0.2597656
3	0.56250000	-0.1618652
4	0.59375000	+0.0540466
5	0.57812500	-0.0526237
6	0.58593750	+0.0010313

<u>Exercise:</u>

i. $x^3 + 3x - 5 = 0$ $[where\ a = 1, b = 2]$

ii. $3x + \sin x - e^x = 0$

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