Computational Mathematics

Numerical Integration

General Quadrature Formula for Equidistance:

Let $I = \int_a^b ydx \, , where \; y = f(x)$.

Let f(x) be given for certain equidistance values of

$x = x_0, \quad x_0 + h, \quad x_0 + 2h, \dots \dots \dots . . , x_0 + nh.$

Suppose $y_0, y_1, y_2, \dots \dots \dots \dots \dots y_n$ are the points corresponding to the arguments: $x_0 = a, \quad x_1 = a + h, \quad x_2 = a + 2h, \dots \dots \dots . . , x_n = a + nh = b$

$$\therefore I = \int_a^b ydx = \int_{x_0}^{x_0+nh} ydx \quad \left[we\ know: u = \frac{x - x_0}{h} \Rightarrow x = x_0 + uh \quad \therefore dx = hdu\right]$$

Limit: when $x = x_0$ then $u = 0$ when $x = x_0 + nh$ then $u = n$

$$\therefore I = \int_{x_0}^{x_0+nh} ydx = \int_0^n yhdu$$

$$= h \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \dots \dots \dots \right.$$
$$\left. + \frac{u(u-1)(u-2) \dots \dots \dots (u-n+1)}{n!}\Delta^n y_0\right] du$$

After completing the integration we get:

$$I = \int_{x_0}^{x_0+nh} ydx$$
$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{1}{2!}\Delta^2 y_0 + \left(\frac{n^4}{4} - n^3\right)\frac{1}{3!}\Delta^3 y_0 + \cdots \dots \dots \dots \right.$$
$$+ \left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - 3n^2\right)\frac{1}{4!}\Delta^4 y_0$$
$$\left. + \cdots \dots . + \left(\frac{n^7}{7} - \frac{15n^6}{6} + 17n^5 - \frac{225n^4}{4} + \frac{274n^3}{3} - 60n^2\right)\frac{1}{6!}\Delta^6 y_0\right] \dots \dots \dots (1)$$

This is the general quadrature formula.

If we put n=1 in (1) we get: Trapezoidal rule

If we put n=2 in (1) we get: Simpson's One-Third rule

If we put n=3 in (1) we get: Simpson's Three-Eight's rule

If we put n=4 in (1) we get: Weddle's rule

Trapezoidal rule:

Putting n=1 in the formula (1) and neglecting the **second and higher order** differences, we get:

$$\int_{x_0}^{x_0+h} ydx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{1}{2}h(y_0 + y_1)$$

Similarly for n=2,3,…………..

$$\int_{x_0}^{x_0+2h} ydx = h\frac{1}{2}(y_1 + y_2)$$

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$$\int_{x_0}^{x_0+nh} ydx = h\frac{1}{2}(y_{n-1} + y_n)$$

Adding these n integrals, we get:

$$\int_{x_0}^{x_0+nh} ydx = h\left[\frac{1}{2}(y_0 + y_n) + (y_1 + y_2 + \cdots \dots \dots .. + y_{n-1})\right]$$

Simpson's One-Third rule:

Putting n=2 in the equation (1) and neglecting the **third and higher order** differences, we get:

$$\int_{x_0}^{x_0+2h} ydx = h\left[2y_0 + 2\Delta y_1 + \frac{\frac{8}{3} - 2}{2}\,\Delta^2 y_2\right] = \frac{1}{3}h(y_0 + 4y_1 + y_2)$$

Similarly for n=4,6,……………..

$$\int_{x_0}^{x_0+4h} ydx = \frac{1}{3}h(y_2 + 4y_3 + y_4)$$

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$$\int_{x_0}^{x_0+nh} ydx = \frac{1}{3}h(y_{n-2} + 4y_{n-1} + y_n)$$

Adding all these integrals, we get:

$$\int_{x_0}^{x_0+nh} ydx = \frac{1}{3}h[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots .. + y_{n-1}) + 2(y_2 + y_4 + \cdots \ldots \ldots + y_{n-2})]$$

This formula can be used only when the number of subdivisions of the interval is even.

Simpson's Three-Eight's rule (3/8 rule):

Putting n=3 in the formula (1) and neglecting the **four and higher order** differences, we get:

$$\int_{x_0}^{x_0+3h} ydx = h\left[3y_0 + \frac{9}{2}\Delta y_1 + \frac{\frac{27}{3} - \frac{9}{2}}{2!}\Delta^2 y_2 + \frac{\left(\frac{81}{4} - 27 + 9\right)(\Delta^3 y_3)}{3!}\right]$$

$$= \frac{3}{8}h(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly for n=6,9,………………

$$\int_{x_0}^{x_0+6h} ydx = \frac{3}{8}h(y_3 + 3y_4 + 3y_5 + y_6)$$

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$$\int_{x_0}^{x_0+nh} ydx = \frac{3}{8}h(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

Adding all these integrals, we get:

$$\int_{x_0}^{x_0+nh} ydx = \frac{3}{8}h[(y_0 + y_n) + 3(y_1 + y_3 + y_5 + \cdots .. + y_{n-1}) + 2(y_3 + y_6 + \cdots \ldots \ldots + y_{n-3})]$$

This formula is used when the number of subdivisions of the interval is a multiple of 3.

Weddle's rule:

Putting n=6 in (1) and neglecting the difference of orders **higher than six**, we get:

$$\int_{x_0}^{x_0+nh} y dx = \frac{3}{10} h[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \cdots \ldots.]$$

This formula requires at least **seven** consecutive values of the function.

Ex-1: Find $\int_0^1 \frac{1}{1+x^2} dx$ by using (i) $\frac{1}{3}$ rule, (ii) $\frac{3}{8}$ rule, (iii) Trapezoidal rule, (iv) Weddle's rule.

Solution: we have here: a=0; b=1;

We shall divide the interval into six equal parts. So, n=6.

Here, $h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

Now the value of $y = \frac{1}{1+x^2}$ for each point of sub division are given below:

x	$y = \frac{1}{1+x^2}$
$x_0 = 0$	$y_0 = 1$
$x_1 = x_0 + h = \frac{1}{6}$	$y_1 = 0.97297897$
$x_2 = x_0 + 2h = \frac{1}{3}$	$y_2 = 0.90000000$
$x_3 = x_0 + 3h = \frac{1}{2}$	$y_3 = 0.80000000$
$x_4 = x_0 + 4h = \frac{2}{3}$	$y_4 = 0.69230769$
$x_5 = x_0 + 5h = \frac{5}{6}$	$y_5 = 0.59016393$
$x_6 = x_0 + 6h = 1$	$y_6 = 0.50000000$

By Simpson's 1/3 rule, we get:

$$\int_0^1 \frac{1}{1+x^2} dx = \int_{x_0}^{x_6} \frac{1}{1+x^2} dx = \frac{1}{3} h[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{18}[1.50000000 + 9.4525760 + 3.18461538]$$

$$\Rightarrow \frac{\pi}{4} = 0.78539952$$

$$\therefore \pi = 3.141598 \quad (approx)$$

By Simpson's 3/8 rule:

$$\int_0^1 \frac{1}{1+x^2}dx = \int_{x_0}^{x_6} \frac{1}{1+x^2}dx \ = \frac{3}{8}h[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3]$$

$$\Rightarrow \frac{\pi}{4} = 0.78539586 \quad \therefore \pi = 3.14158344$$

By Trapezoidal rule:

$$\int_0^1 \frac{1}{1+x^2}dx = \int_{x_0}^{x_6} \frac{1}{1+x^2}dx \ = \frac{1}{2}h[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$$

$$\Rightarrow \frac{\pi}{4} = 0.78424077 \quad \therefore \pi = 3.13696306$$

By Weddle's rule:

$$\int_0^1 \frac{1}{1+x^2}dx = \int_{x_0}^{x_6} \frac{1}{1+x^2}dx \ = \frac{3}{10}h[y_0+y_2+y_4+5(y_1+y_5)+6y_3+2y_6]$$

$$\Rightarrow \frac{\pi}{4} = 0.81040565538 \quad \therefore \pi = 3.241622621$$

- $\frac{1}{3}$ Rule and $\frac{3}{8}$ rule give more accurate result than Trapezoidal and Weddle's rules.

Ex-2: Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin x}dx$ by using Simpson's (i) $\frac{1}{3}$ rule, (ii) $\frac{3}{8}$ rule.

Solution: we have here: a=0; b=$\frac{\pi}{2}$;

We shall divide the interval into six equal parts. So, n=6.

Here, $h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{6} = \frac{\pi}{12}$

Now the value of $y = e^{\sin x}$ for each point of sub division are given below:

x	$y = e^{\sin x}$

$x_0 = 0$	$y_0 = 1.00000$
$x_1 = x_0 + h = \frac{\pi}{12}$	$y_1 = 1.0045797$
$x_2 = x_0 + 2h = \frac{2\pi}{12}$	$y_2 = 1.64872$
$x_3 = x_0 + 3h = \frac{3\pi}{12}$	$y_3 = 2.02811$
$x_4 = x_0 + 4h = \frac{4\pi}{12}$	$y_4 = 2.37744$
$x_5 = x_0 + 5h = \frac{5\pi}{12}$	$y_5 = 2.62722$
$x_6 = x_0 + 6h = \frac{6\pi}{12}$	$y_6 = 2.71828$

<u>By Simpson's 1/3 rule, we get:</u>

$$\int_0^{\frac{\pi}{2}} e^{\sin x} dx = \int_{x_0}^{x_6} e^{\sin x} dx = \frac{1}{3} h[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] = 3.10438$$

<u>By Simpson's 3/8 rule:</u>

$$\int_0^{\frac{\pi}{2}} e^{\sin x} dx = \int_{x_0}^{x_6} e^{\sin x} dx = \frac{3}{8} h[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3] = 3.10437$$

<u>Ex-3:</u> Calculate $\int_{0.5}^{0.7} e^{-x} x^{\frac{1}{2}} dx$ by using Simpson's $\frac{1}{3}$ rule.

<u>Solution:</u> we have here: a=0.5; b=0.7;

We shall divide the interval into 4 equal parts. So, n=4.

Here, $h = \frac{b-a}{n} = \frac{0.7-0.5}{4} = 0.05$

Now the value of $y = e^{-x} x^{\frac{1}{2}}$ for each point of sub division are given below:

x	$y = e^{-x} x^{\frac{1}{2}}$
$x_0 = 0.5$	$y_0 = 0.4289$
$x_1 = x_0 + h = 0.55$	$y_1 = 0.4279$

$x_2 = x_0 + 2h = 0.60$	$y_2 = 0.4251$
$x_3 = x_0 + 3h = 0.65$	$y_3 = 0.4209$
$x_4 = x_0 + 4h = 0.70$	$y_4 = 0.4155$

By Simpson's 1/3 rule, we get:

$$\int_{0.5}^{0.7} e^{-x}x^{\frac{1}{2}}\,dx = \int_{x_0}^{x_4} e^{-x}x^{\frac{1}{2}}\,dx = \frac{1}{3}h[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] = 0.0848$$

Ex-4: Calculate the value of $\int_4^{5.2} \ln x\,dx$ by using (i) Trapezoidal rule, (ii) $\frac{1}{3}$ rule, (iii) $\frac{3}{8}$ rule, (iv) Weddle's rule.

Solution: we have here: a=4; b=5.2;

We shall divide the interval into 6 equal parts. So, n=6.

Here, $h = \frac{b-a}{n} = \frac{5.2-4}{6} = 0.2$

Now the value of $y = \ln x$ for each point of sub division are given below:

x	$y = \ln x$
$x_0 = 4$	$y_0 = 1.38629$
$x_1 = x_0 + h = 4.2$	$y_1 = 1.43508$
$x_2 = x_0 + 2h = 4.4$	$y_2 = 1.48160$
$x_3 = x_0 + 3h = 4.6$	$y_3 = 1.52606$
$x_4 = x_0 + 4h = 4.8$	$y_4 = 1.56862$
$x_5 = x_0 + 5h = 5.0$	$y_5 = 1.60944$
$x_6 = x_0 + 6h = 5.2$	$y_6 = 1.648666$

By Trapezoidal rule:

$$\int_{4}^{5.2} \ln x\,dx = \int_{x_0}^{x_6} \ln x\,dx = \frac{1}{2}h[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)] = 1.82766$$

By Simpson's 1/3 rule, we get:

$$\int_{4}^{5.2} \ln x\,dx = \int_{x_0}^{x_6} \ln x\,dx = \frac{1}{3}h[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] = 1.82785$$

By Simpson's 3/8 rule:

$$\int_{4}^{5.2} \ln x\, dx = \int_{x_0}^{x_6} \ln x\, dx = \frac{3}{8} h[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3] = 1.82785$$

By Weddle's rule:

$$\int_{4}^{5.2} \ln x\, dx = \int_{x_0}^{x_6} \ln x\, dx = \frac{3}{10} h[y_0 + y_2 + y_4 + 5(y_1 + y_5) + 6y_3 + 2y_6] = 1.82785$$

Ex-5: Compute the value $\int_{1.2}^{1.6} \left(x + \frac{1}{x}\right) dx$ by Simpson's 1/3 rule and compare with the exact value.

Solution:

we have here: a=1.2; b=1.6;

We shall divide the interval into 8 equal parts. So, n=8.

Here, $h = \frac{b-a}{n} = \frac{1.6-1.2}{8} = 0.05$

- 1/3 rule = 0.8477
- Exact value = 0.8477

So, NO ERROR.