Discrete Probability Distributions

The Poisson Probability Distribution

The distribution was derived by the French mathematician Siméon Poisson in 1837,

The first application was the description of the number of deaths by horse kicking in the Prussian army.

Poisson distribution

- In binomial when n is large and p is small.
- Simeon Denis Poisson(1781-1840) gave a theorem: Parameter λ = np = expected value

$$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad where\ x = 0,1,2,.....$$

here e =2.71828

Mean=E(x)=λ　　　var(x)= λ

Poisson distribution: Assumptions

An experiment is called Poisson experiment if it has the following properties:

1. The probability of observing 1 success is directly proportional to the length of the time interval Δt.

i.e. Pr(1 success) ≈ $\lambda\Delta t$ for some constant λ

2. The probability of two or more successes in any sufficiently small subinterval is 0.

i.e. Pr(>=2 success) ≈ 0 *Pr(0 success) ≈ 1- $\lambda\Delta t$*

Poisson distribution: Assumptions

3. Stationarity: The probability of success is the same for any two intervals of equal length

i.e. probability equal to the size of the region.

Time : t

p_1	p_2		---				P_n
Probability :			p_1=p_2=…….. =p_n				

Poisson distribution: Assumptions

4. Independence: The number of successes in any interval is independent of the number of successes in any other interval provided the intervals are not overlapping.

Time : t

t_1	t_1						t_1
N_1	n_2						n_n
$n_1, n_2, \ldots\ldots\ldots, n_n$ are independent							

Poisson distribution: Formal Definition

Poisson distribution is the probability distribution that results from a poisson experiment.

Let x be the number of events occurring within a given time interval in a poisson process, then this is a random variable and its probability distribution is

$$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad where\ x = 0,1,2,.....$$

λ is the shape parameter which indicates the average number of events over a time period, not per unit time.

Poisson distribution: Advantage

- No need to know n and p;
- Estimate the parameter λ from data.
- Death reports by horse-kick from 10 cavalry corps over a period of 200 years by Prussian officials.

X= Number of deaths	**Frequencies (year)**
0	109
1	65
2	22
3	3
4	1
Total	200

Poisson distribution: calculation

BINOMDIST =(A3*B3+A4*B4+A5*B5+A6*B6+A7*B7)/B8

	A	B	C	D
1	x	frequencies	Poisson probability	Expected frequencies
2				
3	0	109	0.5435	108.7
4	1	65	0.3315	66.3
5	2	22	0.101	20.2
6	3	3	0.0205	4.1
7	4	1	0.003	0.6
8		200		
9				
10				
11	mean=	=(A3*B3+A4*B4+A5*B5+A6*B6+A7*B7)/B8		
12				

Excel calculation: Calculate mean to find λ (lamda)

Poisson distribution: Calculation

x	frequencies	Poisson probability	Expected frequencies
0	109	=POISSON(A3,B11,0)	108.7
1	65	0.33144403	66.3
2	22	0.101090429	20.2
3	3	0.020555054	4.1
4	1	0.003134646	0.6
	200		

mean= 0.61

Excel calculation: Calculate Poisson probability for each cell.

Poisson distribution: Calculation

x	frequencies	Poisson probability	Expected frequencies
0	109	.5435	108.7
1	65	.3315	66.3
2	22	.101	20.2
3	3	.0205	4.1
4	1	.003	0.6
	200		

Conduct a chi-square test to see if Poisson model is compatible with data or not.

Degree of freedom is 4-1-1 = 2.
Pearson's statistic = .304;
P-value is .859

Accept null hypothesis, data compatible with model

Example: *A Poisson Process*

Suppose the number of deaths from typhoid fever over a 1-year period is Poisson distributed with parameter $\mu = \lambda t = 4.6$.

1. What is the probability distribution of the number of deaths over a 6-month period?
2. A 3-month period?

Sol.

1. $\lambda = 4.6 * .5 = 2.3$

If random variable x follows poisson distribution

$$p(x=0) = \frac{e^{-2.3} 2^{0}}{0!} = 0.1$$

Example: *A Poisson Process*

$$p(x=1)=\frac{e^{-2.3}2^{1}}{1!}=0.231$$

$$p(x=2)=\frac{e^{-2.3}2^{2}}{2!}=0.265$$

$$p(x=3)=\frac{e^{-2.3}2^{3}}{3!}=0.203$$

$$p(x=4)=\frac{e^{-2.3}2^{4}}{4!}=0.117$$

$$p(x=5)=\frac{e^{-2.3}2^{5}}{5!}=0.054$$

$$p(x>=6)=1-\sum=0.030$$

Example: *A Poisson Process*

Suppose the number of deaths from typhoid fever over a 1-year period is Poisson distributed with parameter $\mu = \lambda t = 4.6$.

1. What is the probability distribution of the number of deaths over a 6-month period? Ans 2.3
2. A 3-month period?

Sol.

2. $\lambda = 4.6 * 0.25 = 1.15$

If random variable x follows poisson distribution

$$p(x=0) = \frac{e^{-1.15} 1.15^{0}}{0!} = 0.317$$

Example: *A Poisson Process*

Average number of homes sold by a company is 2 homes per day. What is the probability that exactly 3 homes will be sold tomorrow?

Sol. $\lambda=2.0 \quad x=3$

If random variable x follows poisson distribution

$$p(x=3)=\frac{e^{-2}2^{3}}{3!}=0.18$$

Example: *Computing Poisson Probabilities*

The Food and Drug Administration sets a Food Defect Action Level (FDAL) for various foreign substances that inevitably end up in the food we eat and liquids we drink.

For example, the FDAL level for insect filth in chocolate is 0.6 insect fragments (larvae, eggs, body parts, and so on) per 1 gram.

(a) Determine the mean number of insect fragments in a 5 gram sample of chocolate.

(b) What is the standard deviation?

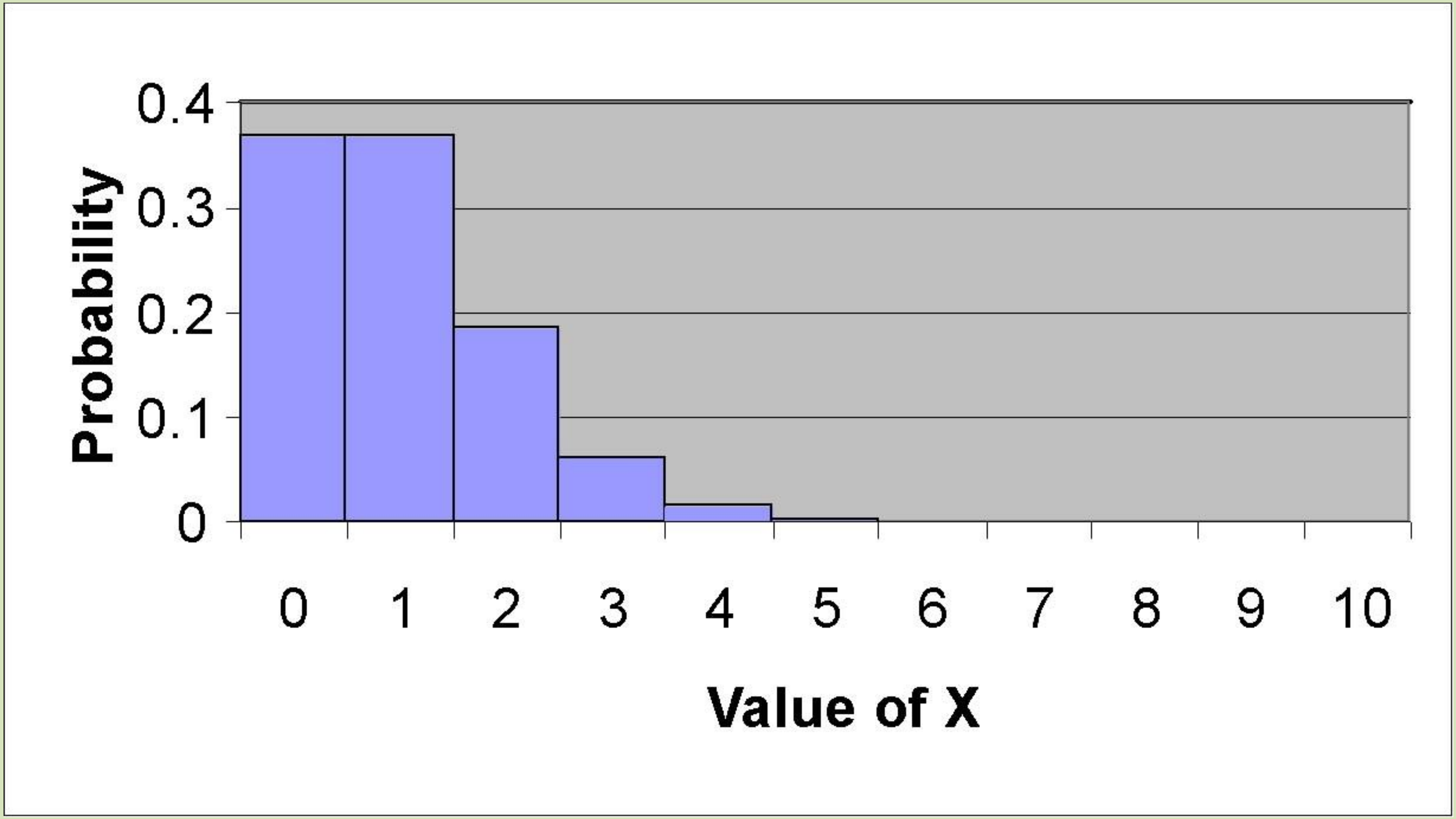
Assumptions: Why poisson?

- For a sufficiently small interval, the probability of two successes is 0.

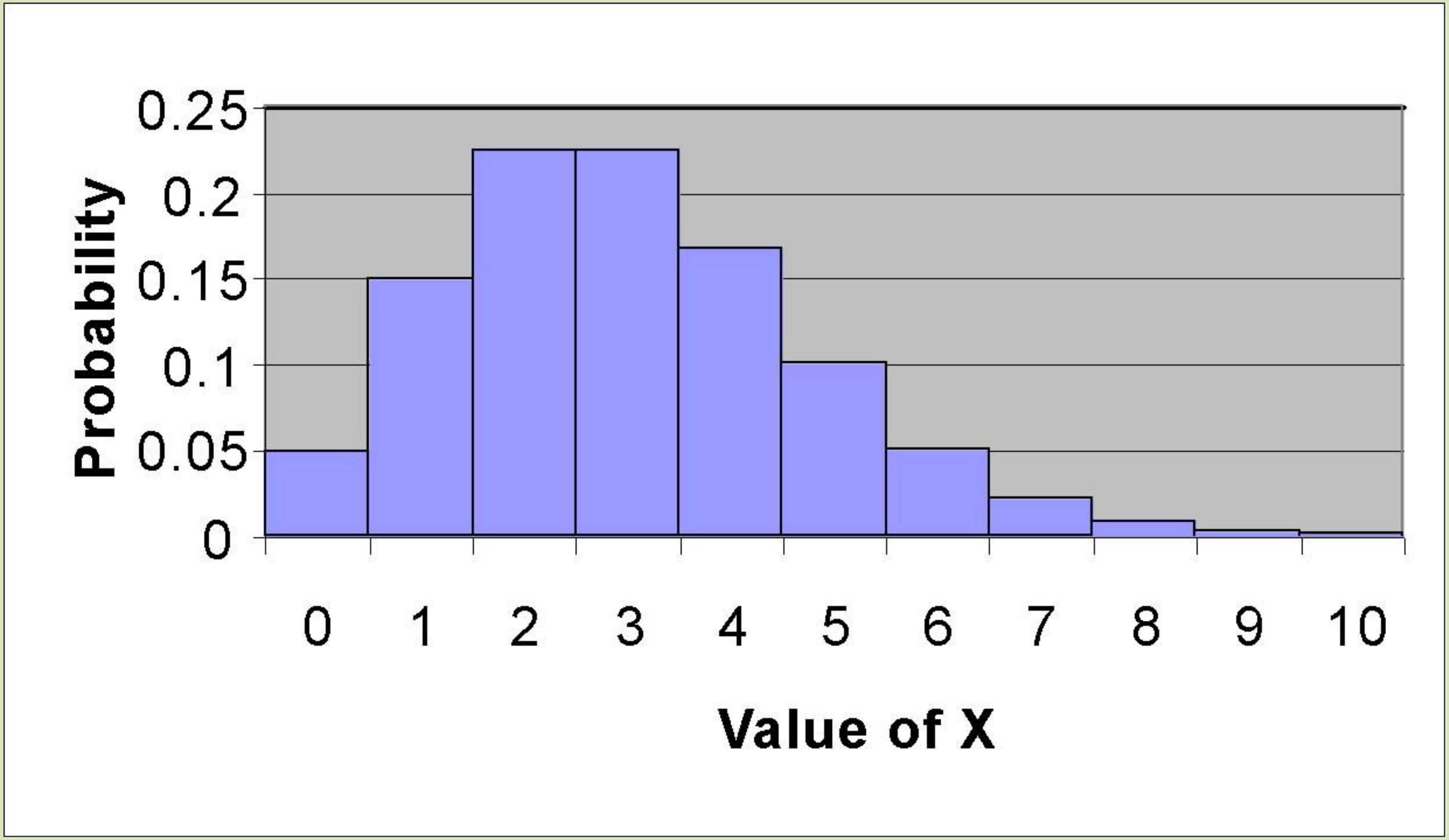
- The probability of insect filth in one region of a candy bar is equal to the probability of insect filth in some other region of the candy bar.

- The number of successes in any random sample is independent of the number of successes in any other random sample.

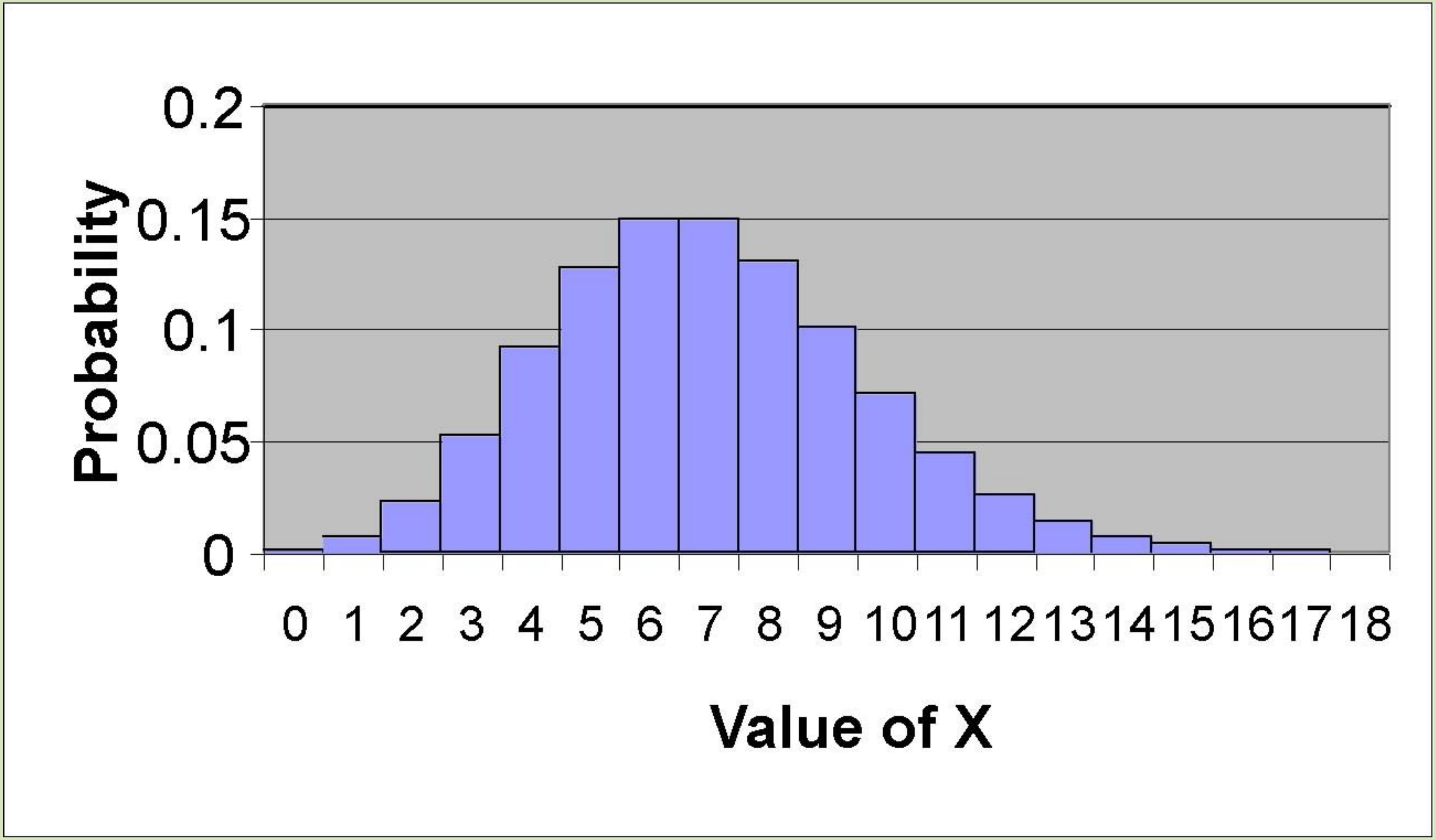
Probability Histogram of a Poisson Distribution with $\lambda = 1$



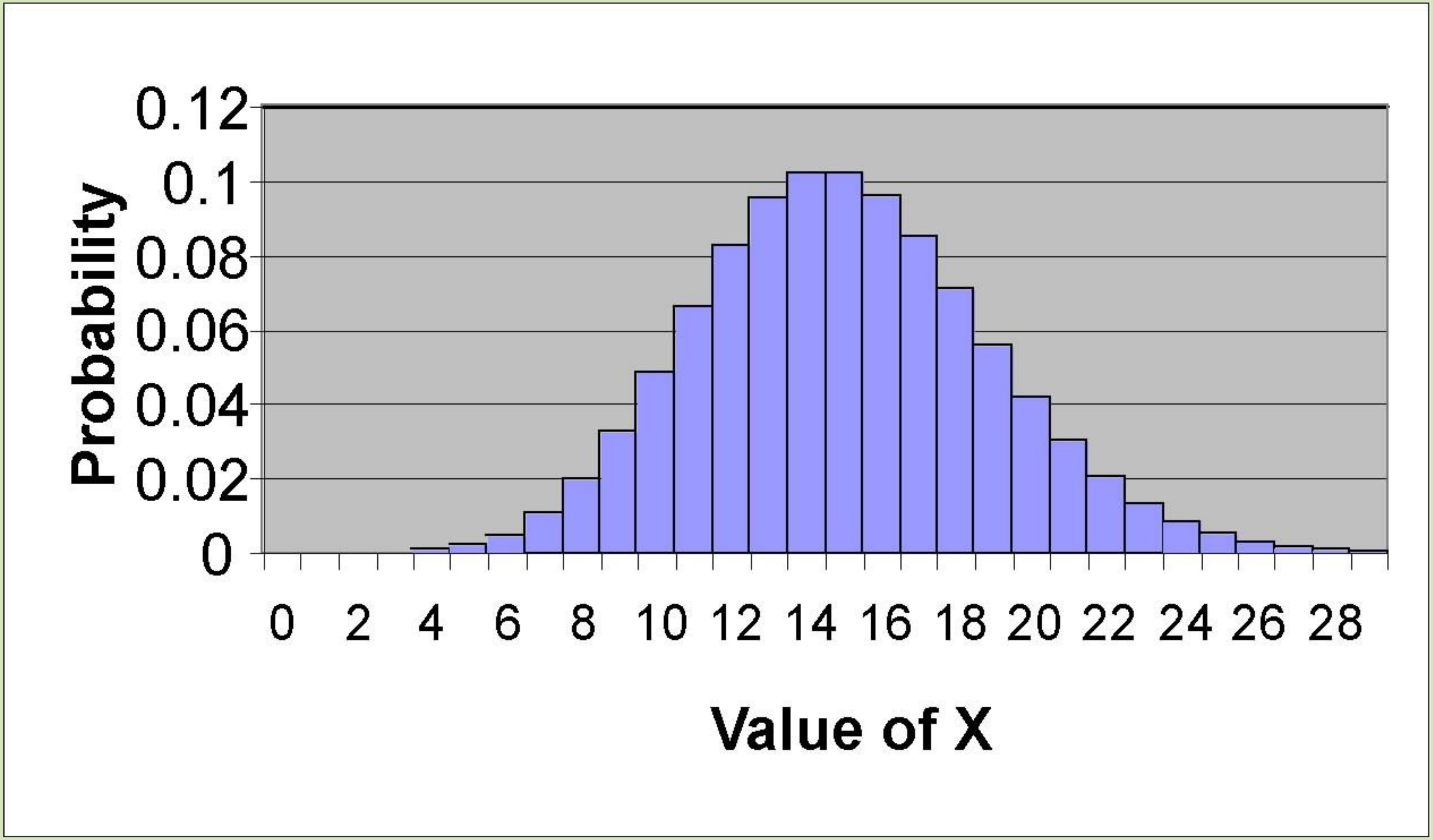
Probability Histogram of a Poisson Distribution with λ = 3



Probability Histogram of a Poisson Distribution with $\lambda = 7$



Probability Histogram of a Poisson Distribution with λ = 15



Example: Poisson Particles

In 1910, Ernest Rutherford and Hans Geiger recorded the number of α-particles emitted from a polonium source in eighth-minute (7.5 second) intervals. The results are reported in the table on the next slide. Does a Poisson probability function accurately describe the number of α-particles emitted?

Source: Rutherford, Sir Ernest; Chadwick, James; and Ellis, C.D.. Radiations from Radioactive Substances. London, Cambridge University Press, 1951, p. 172.

Poisson distribution: Calculation

Number Detected	Frequency
0	57
1	203
2	383
3	525
4	532
5	408
6	273
7	139
8	45
9	27
10	10
11	6

Poisson distribution: Calculation

x	Frequency, *f*	$x \cdot f$	Expected Proportion, if Poisson	Expected Frequency, if Poisson
0	57	0	0.020866	54.4
1	203	203	0.080744	210.6
2	383	766	0.156225	407.4
3	525	1575	0.201511	525.5
4	532	2128	0.194943	508.4
5	408	2040	0.150872	393.5
6	273	1638	0.097303	253.8
7	139	973	0.053789	140.3
8	45	360	0.026018	67.9
9	27	243	0.011187	29.2
10	10	100	0.004329	11.3
11	6	66	0.001523	4.0
	Sum: 2608	10092		

$\mu = 3.869632$

Results from calculation

Pearson's chi-squared statistics = 12.955;

d.f.=12-1-1=10

Poisson parameter = 3.87,

P-value between .95 and .975.

Accept null hypothesis : data are compatible with Poisson model

Poisson Approximation to the Binomial Distribution

The binomial distribution with large n and small p can be accurately approximated by

a Poisson distribution with parameter $\mu = np$.

Example

According to the U.S. National Center for Health Statistics, 7.6% of male children under the age of 15 years have been diagnosed with Attention Deficit Disorder (ADD). In a random sample of 120 male children under the age of 15 years, what is the probability that at least 4 of the children have ADD?

Example

Suppose 6 of 15 students in a grade-school class develop influenza, whereas 20% of grade-school students nationwide develop influenza. Is there evidence of an excessive number of cases in the class? That is, what is the probability of obtaining at least 6 cases in this class if the nationwide rate holds true?

1. What is the expected number of students in the class who will develop influenza?
2. What is the probability of obtaining exactly 6 events for a Poisson distribution with parameter $\mu = 4.0$?
3. What is the probability of obtaining at least 6 events for a Poisson distribution with parameter $\mu = 4.0$?
4. What is the expected value and variance for a Poisson distribution with parameter $\mu = 4.0$?