Computational Mathematics

Taylor Series Representation

Let we consider an ordinary differential equation of first order and first degree like:

$\frac{dy}{dx} = f(x, y) \Rightarrow y' = f(x, y) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$

With initial condition $y(x_0) = y_0$. This represents $x = x_0 \; when \; y = y_0$

Differentiating w.r.t. x we get from (1)

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} . \frac{\partial y}{\partial x} \Rightarrow y'' = fx + fy . y' \ldots \ldots \ldots \ldots \ldots \ldots . (2)$$

Differentiating successively, we get:

$$y''', y^{iv}, \ldots \ldots \ldots \ldots \ldots$$

- Putting the initial condition at $x = x_0$ then

$y = y_0$, We get $y_0', y_0'', y_0''', \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$

Now the Taylor's series of y(x) at $x = x_0$ is [the Taylor series]

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \cdots \ldots \ldots$$

$$\Rightarrow y(x) = y_0 + (x - x_0) y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \cdots \ldots \ldots \ldots . (3)$$

Putting $x = x_1$ and hence $x_1 - x_0 = h$ $\Rightarrow$ we get from (3)

At $x = x_0 \Rightarrow y(x_1) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \cdots \ldots \ldots \ldots \ldots \ldots$

$\therefore y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots \ldots \ldots \ldots \ldots \ldots (4)$

Where, $y(x_0) = y_0 \; then \; y(x_1) = y_1$

If the values of $y_0, y_0', y_0'', y_0''', \ldots \ldots \ldots \ldots \ldots \ldots$ ...are known, then (4) gives a series for y_1 .

Once y_1 is known, we can compute y', y'', y''', ...from (1), (2) and so on.

$\therefore$ Now if y be expanded in a Taylor's series at $x = x_1$ then:

At $x = x_1 \Rightarrow y(x_2) = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \cdots \ldots \ldots \ldots \ldots \ldots$

$\therefore y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \cdots \dots\dots\dots\dots\dots$

Similarly, $\quad y_3 = y_2 + hy_2' + \frac{h^2}{2!} y_2'' + \cdots \dots\dots\dots\dots\dots\dots\dots\dots\dots at\ x = x_2$

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$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!} y_n'' \dots\dots\dots\dots\dots.\dots\dots\dots\dots.at\ x = x_n$

Which are the required Taylor series representation of Differential equation.

<u>Ex-1:</u> Using Taylor's series method, find y(0.1) and y(0.2) by solving $\frac{dy}{dx} = x^2 + y^2$ with the initial condition y(0)=1.

<u>Solution:</u>

(i) We know the Taylor series for ordinary differential equation is:

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots \dots\dots\dots\dots\dots (1)$$

$here, \quad x_0 = 0, y_0 = 1, h = 0.1; \;\; y(0) = 1\ for\ x.$

Now we have given:

$$\frac{dy}{dx} = x^2 + y^2$$

$\Rightarrow y' = x^2 + y^2$

$[\because x_0 = 0, y_0 = 1, h = 0.1 \;\; \therefore y_0' = x_0^2 + y_0^2 = 0 + 1 = 1]$

$\Rightarrow y'' = 2x + 2yy' \rightarrow \therefore y_0'' = 2x_0 + 2y_0y_0' = 2(0) + 2(1)(1) = 2$

$\Rightarrow y''' = 2 + 2(y')^2 + 2y'' \rightarrow \therefore y_0''' = 2 + 2(1)^2 + 2(2) = 8$

$\Rightarrow y^{iv} = 0 + 2[2y'y'' + 2\{y'y'' + yy'''\}] = 6y'y'' + 2yy'''$

$$\rightarrow \therefore y_0^{iv} = 6 * 1 * 2 + 2 * 1 * 8 = 28$$

Now from (1) $y_1 = 1 + \frac{0.1}{1!} * 1 + \frac{0.1^2}{2!} * 2 + \frac{0.1^3}{3!} * 8 + \frac{0.1^4}{4!} * 28 = 1.11145$

$$\therefore y(0.1) = 1.11145$$

$$[y(0.1) \rightarrow means\ initial\ value\ of\ h = 0.1; y(0.2) \rightarrow next\ initial\ value\ of\ h = 0.2 - 0.1 = 0.1\ interval\ size; y(0.1) = 1.11145 \rightarrow x = 0.1]$$

(ii) By Taylor's series method, we have

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \cdots \ldots \ldots \ldots \ldots \ldots (2)$$

Now we have given:

$$\frac{dy}{dx} = x^2 + y^2$$

$$\Rightarrow y' = x^2 + y^2$$

$$[\because x_1 = 0.1, y_1 = 1.1115, h = 0.1 \ \therefore y_1' = x_1^2 + y_1^2 = 1.24543]$$

$$\Rightarrow y'' = 2x + 2yy' \rightarrow \therefore y_1'' = 2x_1 + 2y_1y_1' = 2(0.1) + 2(1.1115)(1.24543) = 2.96859$$

$$\Rightarrow y''' = 2 + 2(y')^2 + 2y'' \rightarrow \therefore y_1''' = 11.70137$$

$$\Rightarrow y^{iv} = 0 + 2[2y'y'' + 2\{y'y'' + yy'''\}] = 6y'y'' + 2yy''' \rightarrow \therefore y_0^{iv} = 48.19517$$

Now from (2) $y_2 = 1.1115 + \frac{0.1}{1!} * 1.24543 + \frac{0.1^2}{2!} * 2.96859 + \frac{0.1^3}{3!} * 11.70137 + \frac{0.1^4}{4!} * 48.19517 = 1.25304$

$$\therefore y(0.2) = 1.25304$$

[Try for the interval (0.04) using two subintervals of size 0.2. with y(0)=0]

Ex-2: Using Taylor series method, find y(0.2) and y(0.4) correct to four decimal places by solving $\frac{dy}{dx} = 1 - 2xy$; With initial condition y(0)=0.

Hints: $x_0 = 0, y_0 = 0, h = 0.2; next\ h = 0.4 - 0.2 = 0.2$

Ex-3: Evaluate y(0.2) correct to six decimal places by Taylor's series method if y(x) satisfies: $y' = y^2 + x$ with y(0)=0.

Hints: $y_0 = 0; x_0 = 0; h = 0.2$

Ex-4: Solve: $\frac{dy}{dx} = 1 + xy$ with y(0)=2. [x_0=0; y_0=2]

Find

(i) y (0.1) →h=0.1

(ii) y (0.2) →h=0.2-0.1=0.1

(iii) y (0.3)→h=0.3-0.1=0.2