Computational Mathematics

Linear Interpolation

Linear interpolation:

The simplest form of interpolation is to approximate two data points by a straight line. Suppose we are given two points $(x_1, f(x_1))$ *and* $(x_2, f(x_2))$. These two points can be connected linearly as shown fig. below: using the concept of similar triangles, we show that:

$$\frac{f(x) - f(x_1)}{x - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \dots\dots\dots\dots\dots\dots\dots (1)$$

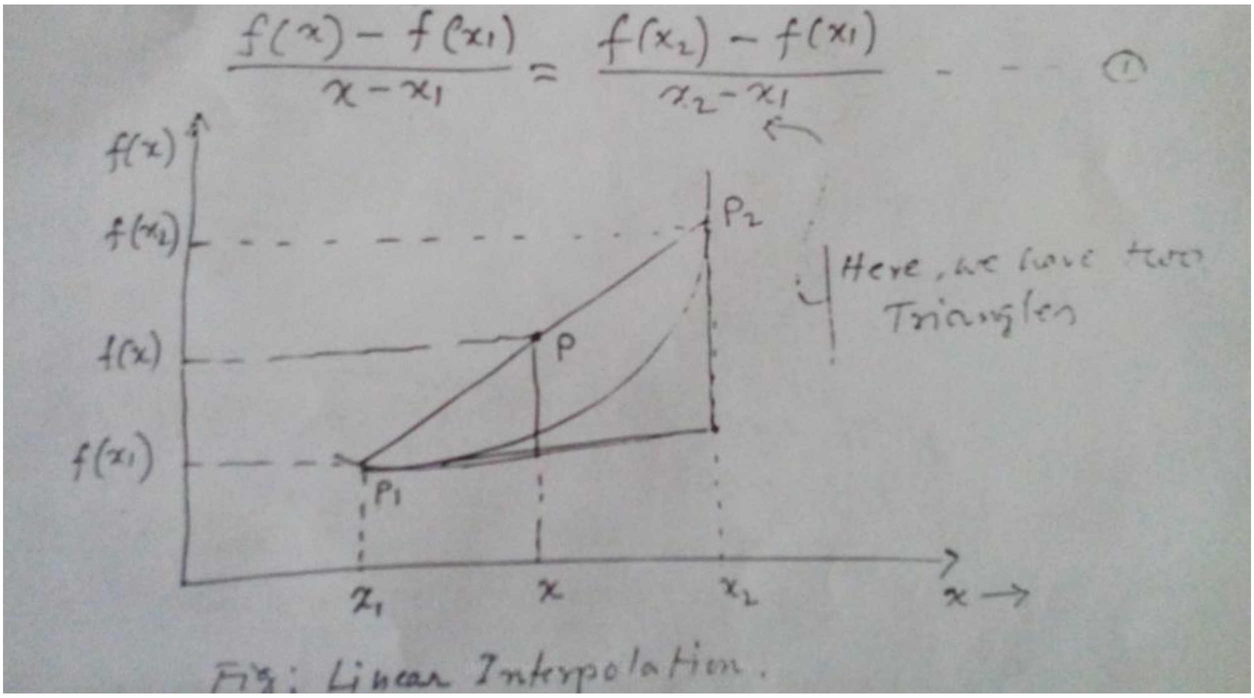

Fig: Linear Interpolation

From (1) we get: $f(x) = f(x_1) + \frac{(x - x_1)(f(x_2) - f(x_1))}{x_2 - x_1} \dots\dots\dots\dots (2)$

Which is known as Linear Interpolation formula.

Ex-1: The table below gives square roots for integers.

x	1	2	2.5	3	4	5
f(x)	1	1.4142	1.5811	1.7321	2	2.2361

Solution:

The given value of 2.5 lies between the points 2 and 3. Therefore:

$x_1 = 2: f(x_1) = 1.4142$

$x_2 = 3: f(x_2) = 1.7321$

Then by the help of linear interpolation formula (2) get:

$$f(2.5) = 1.4142 + \frac{(2.5 - 2.0)(1.7321 - 1.4142)}{3.0 - 2.0} = 1.5732$$

$$[Error = 1.5811 - 1.5732 = 0.0079]$$

The correct answer is 1.5811 $[\sqrt{2.5} = 1.5811]$.

The difference is due to the use of a linear model to an interpolation.

Cheek the results to in different interval:

Say $x_1 = 2$ and $x_2 = 4$ though we know $\sqrt{2.5}$ must lies between 2 and 3.

$$\therefore f(x_1) = 1.4142; \qquad f(x_2) = 2$$

Then by (2) we can write:

$$f(2.5) = 1.4142 + (2.5 - 2.0)\frac{(2.0 - 1.4142)}{4.0 - 2.0} = 1.5607$$

Error: 1.5811-1.5607=0.0204

Notice that the error has increased from before i.e. 0.0079 to 0.0204

Integral, the smaller the interval between the interpolating data points. The better will be the approximation.

__Ex-2:__ Table below gives values of square of integers:

x	1	2	3	3.25	4	5
x^2	1	4	9	10.5625	16	25

Using the linear interpolation formula estimate the square root of 3.25

(a) Using the points 3 and 4
(b) Using the points 2 and 4

[Compare and comment on the results.]

__Ex-3:__ Find the linear interpolation polynomial for each of the following pairs of points:

(a) (0,1) and (1,3)
(b) (-2,3) and (7,12)

[Matlab]

Other polynomial Form/Interpolation

(1) Power Form:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots \ldots \ldots . + a_n x^n \Rightarrow p(x) = a_0 + a_1 x[power = 1]$$

(2) Shifted Form:

$$p(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots \ldots \ldots . + a_n(x - c)^n [known\ as\ taylor\ expansion/series]$$

(3) Newton Form:

$$p(x) = a_0 + a_1(x - c_1) + a_2(x - c_1)(x - c_1) + \cdots \ldots . + a_n(x - c_1)(x - c_2) \ldots \ldots (x - c_n)$$

(4) Linear Interpolation:

$$f(x) = f(x_1) + f(x - x_1)\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

(5) Newton-Gregory:

(5.1) Forward Interpolation with equal intervals:

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots \ldots \ldots + \frac{u(u-1)(u-2) \ldots . . (u-n+1)}{n!}\Delta^n y_n$$

Useful for interpolating the values of f(x) near the beginning of the set of given values.

(5.2) Backward Interpolation with equal interval:

$$y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots \ldots \ldots \ldots + \frac{u(u+1)(u+2) \ldots \ldots \ldots (u+n-1)}{n!}\nabla^n y_n$$

Where $y_0, y_1, y_2, \ldots \ldots \ldots \ldots \ldots y_n$ are the values of y for the (n+1) equidistant values of $x_0, x_1, x_2, \ldots \ldots \ldots \ldots x_n$.

(6) Divided Difference:

Let the arguments/points $x_0, x_1, x_2, \ldots \ldots . . x_n$ to the correspond values of $f(x_0), f(x_1), f(x_2), \ldots \ldots \ldots \ldots f(x_n)$ of the polynomial f(x); not necessarily equal spaced. Then the generalized formula for:

(6.1) First divided difference:

$$f(x_i, x_{i+1}) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \ ; where\ i = 0,1,2, \ldots \ldots 4$$

Then $f(x_0,\ x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_3)}{x_2 - x_1}$$

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$$f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

(6.2) Second divided difference:

$$f(x_i, x_{i+1}, x_{i+2}) = \frac{f(x_{i+1}, x_{i+2}) - f(x_i\ , x_{i+1})}{x_{i+2} - x_i}\ [i = 0,1,2, \ldots \ldots \ldots .]$$

Then, $f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$

(6.3) Nth Divided Difference:

$$f(x_0, x_1, \ldots \ldots \ldots .. x_n) = \frac{f(x_1, x_2, \ldots \ldots \ldots x_n) - f(x_0, x_1, \ldots \ldots .. x_{n-1})}{x_n - x_0}$$

(7) Newton Divided Difference Formula for unequal interval:

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) \\ + \cdots \ldots \ldots \ldots . +(x - x_0)(x - x_1) \ldots .. (x - x_{n-1})f(x_0, x_1, x_2 \ldots \ldots . x_n)$$

(8) Lagrange's Interpolation formula for unequal Intervals:

$$f(x) = \frac{(x - x_1)(x - x_2) \ldots \ldots \ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \ldots \ldots \ldots (x_0 - x_n)} f(x_0) \\ + \frac{(x - x_0)(x - x_2) \ldots .. (x - \mathrm{x_n})}{(x_1 - x_0)(x_1 - x_2) \ldots \ldots \ldots (x_1 - x_n)} f(x_1) \\ + \frac{(x - x_0)(x - x_1)(x - x_3) \ldots .. (x - \mathrm{x_n})}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3) \ldots \ldots \ldots (x_2 - x_n)} f(x_2) \\ + \cdots \ldots \ldots \ldots \ldots .. + \frac{(x - x_0)(x - x_1)(x - x_2) \ldots .. (x - \mathrm{x_{n-1}})}{(x_n - x_0)(x_n - x_1)(x_n - x_2) \ldots \ldots \ldots (x_2 - x_{n-1})} f(x_n)$$

Ex-1: Newton's Forward and Backward (both):

Find the value of y at x=21 and x=28 from the following data:

X=	20	23	26	29
Y=	0.342	0.3907	0.4384	0.4848

Solution: First we construct the difference table for the given data as follows

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.342			
		0.0487		
23	0.3907		-0.0010	
		0.0477		-0.0003
26	0.4384		-0.0013	
		0.0464		
29	0.4848			

(i) We have to find y(21):

Here, $u = \frac{x-x_0}{h} = \frac{21-20}{3} = 0.3333$

Since x=21 is nearer to the beginning of the table. So, we use Newton's forward interpolation.

i.e. $y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$

$$\therefore y(21) = 0.342 + 0.3333(0.0487) + \frac{0.3333(0.3333-1)}{2!}(0.0010) + \frac{0.3333(0.3333-1)(0.3333-2)}{3!}(-0.0003) = 0.3583$$

(ii) Again we have to find y(28):

Since, x=28 is nearer to end value. So, we use Newton's Backward Interpolation formula.
Here $u = \frac{x-x_n}{h} = \frac{28-29}{3} = -0.3333$

$$\therefore y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$\Rightarrow y(28) = 0.4848 + (-0.3333)(0.0464) + \frac{(-0.3333)(-0.3333+1)}{2!}(-0.0013) + \frac{(-0.3333)(-0.3333+1)(-0.3333+2)}{3!}(-0.0003) = 0.46946$$

Therefore the value of y at x=21 and x=28 are 0.3583 and 0.46946 respectively. (**Ans.**)

Ex-2: The population of a town in the decimal census was as given below. Estimate the population for the year 1895.

Year (x)	1891	1901	1911	1921	1931
Population (y)	46	66	81	93	101

Solution: Construct the difference table. We have to find y(1895). Since x=1895 is near to the beginning of the table/data. So, we use Newton's forward interpolation formula.

Here, $u = \frac{x-x_0}{h} = \frac{1995-1891}{10} = 0.4$

Ex-3: The following table gives the population of a town during the last six censuses. Estimate using any suitable interpolation formula, the increase in the population during the period from year 1996 to 1998.

Year (x)	1961	1971	1981	1991	2001	2011
Population (th)	12	15	20	27	39	52

Solution: Construct the difference table. Use Newton's backward interpolation formula.

Here, $(i)\ u = \frac{x-x_n}{h} = \frac{1996-2011}{10} = -1.5 \rightarrow for\ 1996$

Then, $(ii)\ u = \frac{x-x_n}{h} = \frac{1998-2011}{10} = -1.3 \rightarrow for\ 1998$

Ex-4: Find the annual premium at the age of 46 and 63 from the following table:

Age (x)	45	50	55	60	65
Premium (y)	114.84	96.16	83.32	74.48	68.48

(i) Forward $\rightarrow u = \frac{x-x_0}{h} = \frac{46-45}{5} = 0.2$

(ii) Backward $\rightarrow u = \frac{x-x_n}{h} = \frac{63-65}{5} = -0.4$

Ex-5: Marks obtained by the students in an examination are given below:

Marks (x)	45	50	55	60	65
No of students (y)	114.84	96.16	83.32	74.48	68.48

Estimate the number of students who obtained less than

(i) 42 marks

(ii) 70 marks

Solution: First we prepare the cumulative frequency table as below:

Marks Obtained	No of students
<20	41
<40	41+62=103
<60	103+65=168
<80	168+50=228
<100	218+17=235

Difference table:

Marks (x)	y	Δ y	Δ^2 y	Δ^3 y	Δ^4y
20	41				
		62			
40	103		3		
		65		-18	
60	168		-15		0
		50		-18	
80	218		-33		
		17			
100	235				

(i) For 42:

Here, $u = \frac{x-x_0}{h} = \frac{42-20}{20} = 1.1$

We use Newton's forward interpolation.

i.e. $y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 = 110$

The number of students who obtained less than 42 marks=110 (**Ans.**)

(ii) For 70:

We use Newton's Backward Interpolation formula. Here $u = \frac{x-x_n}{h} = \frac{70-100}{20} = -1.5$

$$\therefore y(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n = 196$$

The number of students who obtained less than 70 marks =196 (**Ans.**)

Ex-6: Marks obtained by the students in an examination are given below:

Marks (x)	30-40	40-50	50-60	60-70	70-80
No of students (y)	31	42	51	35	31

Estimate the number of students who obtained:

(i) Less than 45 marks
(ii) Greater than 45 marks
(iii) Between 40 and 45 marks

Ex-7: Find the polynomial f(x) passes through the points: $(0,1), (1,3), (2,7)$ *and* $(3,13)$.

Solution: we construct the difference table for the given data is as below:

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

Here, $u = \frac{x-x_0}{h} = \frac{x-0}{1} = x$

So, by Newton's forward interpolation formula: we get:

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 = 1 + x(2) + \frac{x(x-1)}{2}.2 = 1 + 2x + x^2 - x$$

$$\Rightarrow f(x) = x^2 + x + 1 \rightarrow \textit{which is the required polynomial of second degree.}$$

Ex-8: Using the following table find the Function f(x):

X:	0	1	2	3	4
f(x):	3	6	11	18	27