ICT 2105: Numerical Analysis

Newton-Raphson Method

Let x_0 be an approximate root of the equation $f(x) = 0$
and
Let $x = x_0 + h$ be an exact root so that $f(x_0 + h) = 0$, where h being a small quantity.
Now expanding $f(x_0 + h)$ by Taylor's series, we get –

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} hf''(x_0) + \ldots\ldots.$$

$$\therefore\ f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) = 0 \qquad \because\ x_0 + h = 0$$

Since h is very small, neglecting the second and higher order terms of h, we obtain an approximate value of h say h from the above equation.

$$\therefore\ f(x_0) + h_1 f'(x_0) = 0 \Rightarrow h_1 = -\frac{f(x_0)}{f'(x_0)} \quad \ldots\ldots\ldots\ \ldots.\ (1)$$

A better approximation than x_0 is therefore may lie at x_1 where

$$x_1 = x_0 + h_1$$

$$\therefore\ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \ldots\ldots\ldots\ldots.\ \text{from (1)}$$

Now using x_1 in the place of x_0 and x_2 in the place of x_1 we get

Now replace x_1 for x_0
and x_2 for x_1

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, which is little better than before e.g. x_1.

Similarly

$X_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$, which is little better than x_2.

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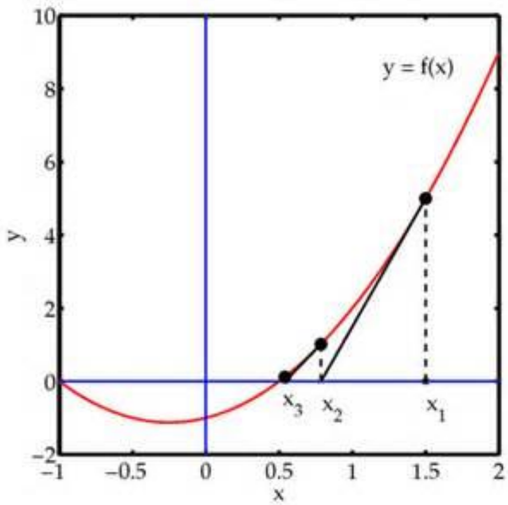
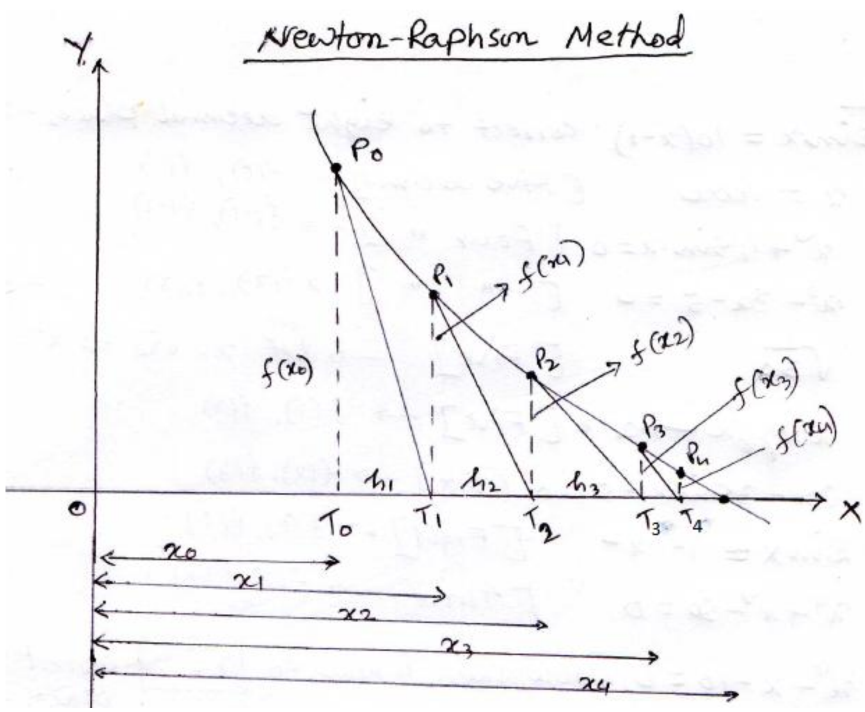
Continuing like this, we iterate this process until

$X_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, where $n = 0,1,2,3,\ldots\ldots\ldots$

This method of successive approximation is called the Newton-Raphson Method.

The method can be used for both algebraic and transcendental equations and works for complex equation with complex coefficients.





Examples of NR(Newton-Rapson Method):

Ex-1: Find the root of $2x^3 - 3x - 6 = 0$ by NR method correct to **five** decimal places.

Solution: let, $f(x) = 2x^3 - 3x - 6 \;\; so, f'(x) = 6x^2 - 3$

$here, f(1) = -7 < 0 \quad ; \quad f(2) = 4 > 0$

So, at least one root lies between 1 and 2. $\Rightarrow x_0 = 1.5$

Now by NR method, we get:

$$x_{n+1} = x_n - \left(\frac{f(x_n)}{f'(x_n)}\right) = x_n - \frac{2x_n^3 - 3x_n - 6}{6x_n^2 - 3} = \frac{6x_n^3 - 3x_n - 2x_n^3 + 3x_n + 6}{6x_n^2 - 3}$$

$$\Rightarrow x_{n+1} = \frac{4x_n^3 + 6}{6x_n^2 - 3} \dots\dots\dots\dots\dots\dots\dots\dots.(1)$$

From (1) putting $x_0 = 1.5$ we get,

$$x_1 = \frac{4x_0^3 + 6}{6x_0^2 - 3} = \frac{4(1.5)^3 + 6}{6(1.5)^2 - 3} = 1.85714$$

$$x_2 = \frac{4x_1^3 + 6}{6x_1^2 - 3} = \frac{4(1.85714)^3 + 6}{6(1.85714)^2 - 3} = 1.78711$$

$$x_3 = \frac{4x_2^3 + 6}{6x_2^2 - 3} = \frac{4(1.78711)^3 + 6}{6(1.78711)^2 - 3} = 1.78373$$

$$x_4 = \frac{4x_3^3 + 6}{6x_3^2 - 3} = \frac{4(1.78373)^3 + 6}{6(1.78373)^2 - 3} = \mathbf{1.78377}$$

$$x_5 = \frac{4x_4^3 + 6}{6x_4^2 - 3} = \frac{4(1.78377)^3 + 6}{6(1.78377)^2 - 3} = \mathbf{1.78377}$$

Hence, the required root is 1.78377.

Ex-2: Find the root of $x^3 = 6x - 4$ correct to **Five** decimal places by NR method.

Solution: let, $f(x) = x^3 - 6x + 4 \;\; so, \;\; f'(x) = 3x^2 - 6$

Here, f(0)=4>0 and f(1)=-1<0 .

So, at least one root lies between o and 1. $\Rightarrow x_0 = 0.75$

Now, by NR method we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 6x_n + 4}{3x_n^2 - 6} = \frac{2x_n^3 - 4}{3x_n^2 - 6} \dots\dots\dots\dots.(1)$$

Taking $x_0 = 0.75$ the relation (1) gives,

$$x_1 = \frac{2(x_0)^3 - 4}{3(x_0)^2 - 6} = \frac{2(0.75)^3 - 4}{3(0.75)^2 - 6} = 0.72188$$

$$x_2 = \frac{2(x_1)^3 - 4}{3(x_1)^2 - 6} = \frac{2(0.72188)^3 - 4}{3(0.72188)^2 - 6} = \mathbf{0.73205}$$

$$x_3 = \frac{2(x_2)^3 - 4}{3(x_2)^2 - 6} = \frac{2(0.73205)^3 - 4}{3(0.73205)^2 - 6} = \mathbf{0.73205}$$

so, the required root is $\mathbf{0.73205}$ **.**

Ex-3 Find the root of 3x-cosx-1=0 by Newton-Raphson method correct to **six** decimal places.

Solution: Let, f(x) =3x-cosx-1

So, f ′(x) = 3+sinx

Here, f(0) = 0-0-1 <0

And f(1) = 3-cos(1)-1 = 1.45970 >0

So, one root lies between 0 and 1.

Now, by NR method we get:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \ldots\ldots\ldots\ldots\ldots(1)$$

Taking $x_0 = 0.5$ the relation (1) gives,

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{0.5\sin(0.5) + \cos(0.5) + 1}{3 + \sin(0.5)} = 0.60852$$

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.60852\sin(0.60852) + \cos(0.60852) + 1}{3 + \sin(0.60852)} = \mathbf{0.60710}$$

$$x_3 = \frac{x_2 \sin x_2 + \cos x_2 + 1}{3 + \sin x_2} = \frac{0.60710\sin(0.60710) + \cos(0.60710) + 1}{3 + \sin(0.60710)} = \mathbf{0.60710}$$

So, The required root is: $\mathbf{0.60710}$**.**

Ex-4: Find the root of $\cos x - xe^x = 0$ by NR.

Solution:

Let, $f(x) = \cos x - xe^x \quad so, f'(x) = -\sin x - e^x - xe^x$

Here, f(0)=1>0 and f(1)=-2.17798<0 so, one root lies between 0 and 1.

By NR method we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\cos x_n - x_n e^{x_n}}{-\sin x_n - e^{x_n} - x_n e^{x_n}} = \frac{x_n \sin x_n + x_n^2 e^{x_n} + \cos x_n}{\sin x_n + e^{x_n} + x_n e^{x_n}} \ldots \ldots (1)$$

Taking $x_0 = 0.5$ we get from (1)

$$x_1 = \frac{x_0 \sin x_0 + x_0^2 e^{x_0} + \cos x_0}{\sin x_0 + e^{x_0} + x_0 e^{x_0}} = \frac{0.5 \sin 0.5 + 0.5^2 e^{0.5} + \cos 0.5}{\sin 0.5 + e^{0.5} + 0.5 e^{0.5}} = 0.51803$$

$$x_2 = \frac{x_1 \sin x_1 + x_1^2 e^{x_1} + \cos x_1}{\sin x_1 + e^{x_1} + x_1 e^{x_1}}$$
$$= \frac{0.51803 \sin 0.51803 + 0.51803^2 e^{0.51803} + \cos 0.51803}{\sin 0.51803 + e^{0.51803} + 0.51803 e^{0.51803}} = \mathbf{0.51776}$$

$$x_3 = \frac{x_2 \sin x_2 + x_2^2 e^{x_2} + \cos x_2}{\sin x_2 + e^{x_2} + x_2 e^{x_2}}$$
$$= \frac{0.51776 \sin 0.51776 + 0.51776^2 e^{0.51776} + \cos 0.51776}{\sin 0.51776 + e^{0.51776} + 0.51776 e^{0.51776}} = \mathbf{0.51776}$$

So the required root is: $\mathbf{0.51776}$.

Exercise for NR method:

1. $\sin x = 10(x-1)$ correct to eight decimal places ⇒ f(1),f(2)
2. $x = \cos x$ [five decimal] ⇒ f(0), f(1)
3. $x^2 + 4\sin x = 0$ [Four decimal] ⇒ f(-1), f(-2)
4. $x^3 - 3x - 5 = 0$ [Four decimal] ⇒ f(2), f(3)
5. $\sqrt{30}$; $(let\ x = \sqrt{30} \Rightarrow x^2 = 30 \Rightarrow x^2 - 30 = 0)$ [Five decimal] ⇒ f(5), f(6)
6. $xlog_{10}x - 1.2 = 0$ [Five decimal] ⇒ f(2), f(3)
7. $2x - 3\sin x - 5 = 0$ [Six decimal] ⇒ f(1), f(2)
8. $\sin x = 1 - x^2$ [Eight decimal] ⇒ f(0), f(1)
9. $x^4 + x^2 - 80 = 0$ [Three decimal] ⇒ f(2), f(3)
10. $x^4 - x - 10 = 0$, where/which is near to x=2 [Three decimal] ⇒ f(1), f(2)