NEWTON RAPHSON METHOD

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Newton-Raphson Method

Let x_0 be an approximate root of the equation $f(x) = 0$
and
Let $x = x_0 + h$ be an exact root so that $f(x_0 + h) = 0$, where h being a small quantity.
Now expanding $f(x_0 + h)$ by Taylor's series, we get –

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}hf''(x_0) + \ldots\ldots.$$

$$\therefore\ f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) = 0 \quad \because\ x_0 + h = 0$$

Since h is very small, neglecting the second and higher order terms of h, we obtain an approximate value of h say h from the above equation.

$$\therefore\ f(x_0) + h_1 f'(x_0) = 0 \Rightarrow h_1 = -\frac{f(x_0)}{f'(x_0)} \quad \ldots\ldots\ldots\ \ldots.\ (1)$$

A better approximation than x_0 is therefore may lie at x_1 where

$$x_1 = x_0 + h_1$$

$$\therefore\ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \ldots\ldots\ldots\ldots\text{ from } (1)$$

Now using x_1 in the place of x_0 and x_2 in the place of x_1 we get

Now replace x_1 for x_0
and x_2 for x_1

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, which is little better than before e.g. x_1.

Similarly

$X_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$, which is little better than x_2.

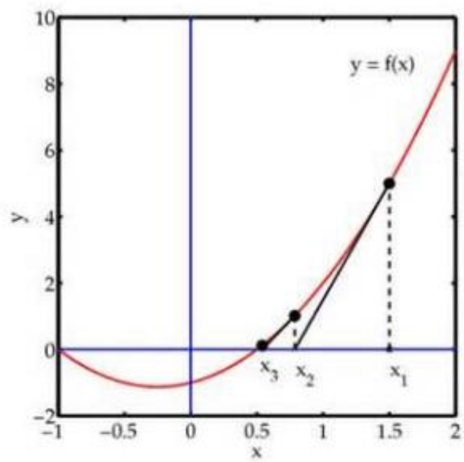
....
....

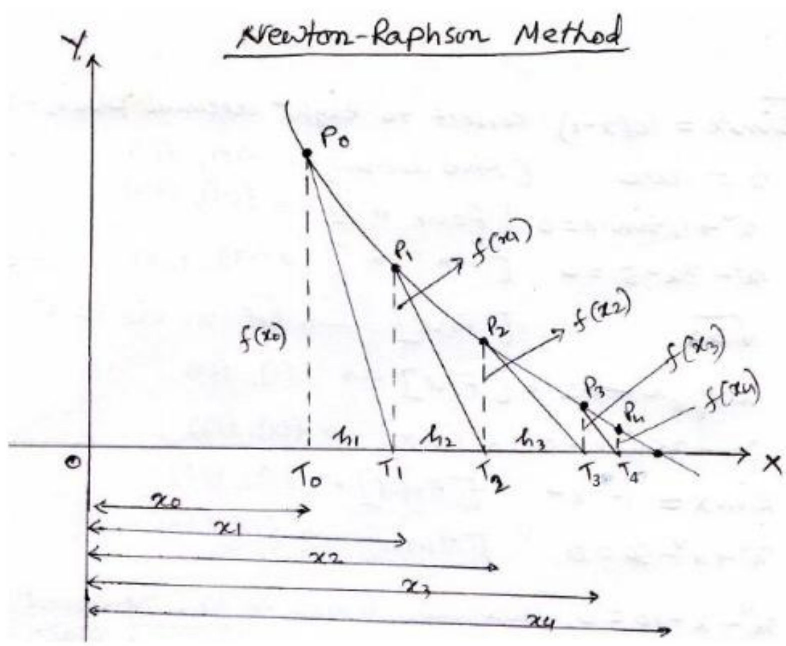
Continuing like this, we iterate this process until

$X_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, where $n = 0,1,2,3,\ldots\ldots\ldots$

This method of successive approximation is called the Newton-Raphson Method.

The method can be used for both algebraic and transcendental equations and works for complex equation with complex coefficients.





Ex: Find the root of $2x^3 - 3x - 6 = 0$ by NR method correct to five decimal places.

Sol:
Let $f(x) = 2x^3 - 3x - 6$
$f'(x) = 6x^2 - 3$

here, $f(1) = -7 < 0$
$f(2) = 4 > 0$

So, at least one root lies between 1 and 2, let $x_0 = 1.5$

By NR method, we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{2x_n^3 - 3x_n - 6}{6x_n^2 - 3}$$

$$= \frac{x_n(6x_n^2 - 3) - (2x_n^3 - 3x_n - 6)}{6x_n^2 - 3}$$

$$= \frac{6x_n^3 - 3x_n - 2x_n^3 + 3x_n + 6}{6x_n^2 - 3}$$

$$= \frac{4x_n^3 + 6}{6x_n^2 - 3}$$

From (i) putting $x_0 = 1.5$ we get

$$x_1 = \frac{4(1.5)^3 + 6}{6(1.5)^2 - 3} = 1.8571$$

$$x_2 = \frac{4(1.8571)^3 + 6}{6(1.8571)^2 - 3} = 1.7871$$

$$x_3 = \frac{4(1.7871)^3 + 6}{6(1.7871)^2 - 3} = 1.7838$$

$$x_4 = \frac{4(1.7838)^3 + 6}{6(1.7838)^2 - 3} = 1.7838$$

Hence the required root is 1.7838.

Ex: $x^3 - 6x + 4 = 0$ Correct to five decimal places by NR method.

Sol:

Let $f(x) = x^3 - 6x + 4$

$f'(x) = 3x^2 - 6$

here, $f(0) = 4 > 0$

$f(1) = -1 < 0$

So at least one root lies between 0 and 1, let $x_0 = 0.5$

By NR method we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 6x_n + 4}{3x_n^2 - 6}$$

$$= \frac{x_n(3x_n^2 - 6) - (x_n^3 - 6x_n + 4)}{3x_n^2 - 6}$$

$$= \frac{3x_n^3 - 6x_n - x_n^3 + 6x_n - 4}{3x_n^2 - 6}$$

$$= \frac{2x_n^3 - 4}{3x_n^2 - 6} \quad ...(i)$$

Taking $x_0 = 0.5$ the relation (i) gives

$$x_1 = \frac{2(0.5)^3 - 4}{3(0.5)^2 - 6} = \frac{2(0.125) - 4}{3(0.25) - 6} = \frac{0.25 - 4}{0.75 - 6} = \frac{-3.75}{-5.25} = 0.71428$$

$$x_2 = \frac{2(0.71428)^3 - 4}{3(0.71428)^2 - 6} = \frac{2(0.3644) - 4}{3(0.5099) - 6} = \frac{0.7288 - 4}{1.5297 - 6} = \frac{-3.2712}{-4.4703} = 0.73189$$

$$x_3 = \frac{2(0.73189)^3 - 4}{3(0.73189)^2 - 6} = \frac{2(0.3923) - 4}{3(0.5357) - 6} = \frac{0.7846 - 4}{1.6071 - 6} = \frac{-3.2154}{-4.3929} = 0.73205$$

$$x_4 = \frac{2(0.73205)^3 - 4}{3(0.73205)^2 - 6} = \frac{2(0.3926) - 4}{3(0.5360) - 6} = \frac{0.7852 - 4}{1.6080 - 6} = \frac{-3.2148}{-4.3920} = 0.73205$$

So the required root is 0.73205.

Ex: $3x - \cos(x) - 1 = 0$ by NR method correct to six decimal places.

Sol:

Let $f(x) = 3x - \cos(x) - 1$

$f'(x) = 3 + \sin(x)$

Here, $f(0) = -1 - 1 = -2 < 0$

$f(1) = 3 - \cos(1) - 1 = 2 - 0.5403 = 1.4597 > 0$

So one root lies between 0 and 1.

Iteration 1:

- $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$
- $x_1 = 0.5 - \frac{3(0.5) - \cos(0.5) - 1}{3 + \sin(0.5)}$
- $x_1 \approx 0.607097$

Iteration 2:

- $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
- $x_2 \approx 0.607102$

Iteration 3:

- $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$
- $x_3 \approx 0.607102$

So the answer is 0.607102

Ex: Find the root of $\cos x - xe^x = 0$ by NR.

Sol:

Let $f(x) = \cos x - xe^x$

$f'(x) = -\sin x - e^x - xe^x$

$f(0) = 1 > 0$

$f(1) = -2.17798 < 0$

By NR method we get,

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \frac{\cos x_n - x_n e^{x_n}}{-\sin x_n - e^{x_n} - x_n e^{x_n}}$

$= \frac{-x_n \sin x_n - x_n e^{x_n} - x_n^2 e^{x_n} - \cos x_n + x_n e^{x_n}}{-\sin x_n - e^{x_n} - x_n e^{x_n}}$

$= \frac{-x_n^2 e^{x_n} - x_n \sin x_n - \cos x_n}{-\sin x_n - e^{x_n} - x_n e^{x_n}}$

$= \frac{x_n^2 e^{x_n} + x_n \sin x_n + \cos x_n}{\sin x_n + e^{x_n} + x_n e^{x_n}}$ (1)

Taking $x_0 = 0.5$ in eqn. (1),

$x_1 = \frac{(0.5)^2 e^{0.5} + 0.5 \sin 0.5 + \cos 0.5}{\sin 0.5 + e^{0.5} + 0.5 e^{0.5}} = 0.51803$

$x_2 = \frac{(0.51803)^2 e^{0.51803} + 0.51803 \sin 0.51803 + \cos 0.51803}{\sin 0.51803 + e^{0.51803} + 0.51803 e^{0.51803}} = 0.51776$

$x_3 = \frac{(0.51776)^2 e^{0.51776} + 0.51776 \sin 0.51776 + \cos 0.51776}{\sin 0.51776 + e^{0.51776} + 0.51776 e^{0.51776}} \approx 0.51776$

So the required root is 0.51776.

Ex: $x \log_{10} x - 1.2 = 0$ [Five decimal $\implies f(2), f(3)$]

Let:

$f(x) = x \log_{10} x - 1.2$

$f'(x) = \log_{10} x + x \cdot \frac{1}{x \ln 10} = \log_{10} x + \frac{1}{\ln 10}$

$f(2) = 2 \log_{10} 2 - 1.2 = 2(0.3010) - 1.2 = 0.6020 - 1.2 = -0.5979 < 0$

$f(3) = 3 \log_{10} 3 - 1.2 = 3(0.4771) - 1.2 = 1.4313 - 1.2 = 0.2313 > 0$

By NR method:

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_{n+1} = x_n - \frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + \frac{1}{\ln 10}}$

By putting $x_0 = 2.5$

$x_1 = 2.5 - \frac{2.5 \log_{10} 2.5 - 1.2}{\log_{10} 2.5 + \frac{1}{\ln 10}} = 2.7465$

$x_2 = 2.7465 - \frac{2.7465 \log_{10} 2.7465 - 1.2}{\log_{10} 2.7465 + \frac{1}{\ln 10}} = 2.7406$

$x_3 = 2.7406 - \frac{2.7406 \log_{10} 2.7406 - 1.2}{\log_{10} 2.7406 + \frac{1}{\ln 10}} = 2.7406$

So the one root is 2.7406.