What is Muller’s Method?

Muller's Method is a numerical method used to **find the roots of a polynomial or a nonlinear equation**, especially when:

- The function is not easily solvable using other methods like bisection or Newton-Raphson.
- The root is complex (imaginary).

It **uses a quadratic interpolation** technique to approximate the function and then find the root of that quadratic equation.

Given **three initial guesses**:

$$x_0,\ x_1,\ x_2$$

We:

1. Fit a **quadratic polynomial** through those points:

$$f(x_0),\ f(x_1),\ f(x_2)$$

2. Find the root of that quadratic polynomial (just like solving $ax^2 + bx + c = 0$).
3. Use the root as a new approximation x_3, and repeat the process until convergence.

Mathematical Formulas

Step 1: Define three initial points

$$x_0,\ x_1,\ x_2 \quad \text{(initial guesses)}$$

Step 2: Compute differences

Let:

$$h_1 = x_1 - x_0, \quad h_2 = x_2 - x_1$$

$$\delta_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad \delta_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$d = \frac{\delta_2 - \delta_1}{h_2 + h_1}$$

Step 3: Form the quadratic polynomial

The quadratic is of the form:

$$f(x) \approx a(x - x_2)^2 + b(x - x_2) + c$$

Step 4: Solve the quadratic for root

$$x = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

Choose the denominator with **larger magnitude** (to avoid division by a small number):

$$\text{Use sign of } b: \quad \text{Denominator } = b + \operatorname{sign}(b) \cdot \sqrt{b^2 - 4ac}$$

Step 5: Update the guess

$$x_3 = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

Step 6: Repeat the steps

Continue the iteration until:

$$|x_{n+1} - x_n| < \epsilon \quad \text{(some small tolerance)}$$

Problem-01:

Find a root of an equation f(x)=2x^3-2x-5 using Muller Method

We need **3 initial guesses**, say:

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 3$$

Let's calculate their function values:

$$f(x) = 2x^3 - 2x - 5$$

$$f(x_0) = f(1) = 2(1)^3 - 2(1) - 5 = 2 - 2 - 5 = -5$$

$$f(x_1) = f(2) = 2(8) - 4 - 5 = 16 - 4 - 5 = 7$$

$$f(x_2) = f(3) = 2(27) - 6 - 5 = 54 - 6 - 5 = 43$$

Let:

$$h_1 = x_1 - x_0 = 2 - 1 = 1$$

$$h_2 = x_2 - x_1 = 3 - 2 = 1$$

$$\delta_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{7 - (-5)}{1} = 12$$

$$\delta_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{43 - 7}{1} = 36$$

$$d = \frac{\delta_2 - \delta_1}{h_2 + h_1} = \frac{36 - 12}{2} = \frac{24}{2} = 12$$

We now use point $x_2 = 3$ as our base point.

$$a = d = 12$$

$$b = \delta_2 + h_2 \cdot d = 36 + 1 \cdot 12 = 48$$

$$c = f(x_2) = f(3) = 43$$

$$x = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

First calculate:

$$\sqrt{b^2 - 4ac} = \sqrt{48^2 - 4(12)(43)} = \sqrt{2304 - 2064} = \sqrt{240} \approx 15.49$$

We now calculate two possible denominators:

- $b + \sqrt{\dots} = 48 + 15.49 = 63.49$
- $b - \sqrt{\dots} = 48 - 15.49 = 32.51$

We **choose the one with the larger magnitude** (to avoid division by small number):

So, denominator = **63.49**

Now compute:

$$x_3 = 3 + \frac{-2(43)}{63.49} = 3 - \frac{86}{63.49} \approx 3 - 1.3545 = 1.6455$$

Now use new three points:

- $x_0 = 2$
- $x_1 = 3$
- $x_2 = 1.6455$

Then repeat Steps 2–4 with updated values.

After repeating the steps, the **root converges to** approximately:

$x \approx 1.6006$

(Accurate up to 4 decimal places)

Definition:

Muller's Method is an **iterative root-finding technique** that uses a **quadratic interpolant** through three points to approximate a root of the equation:

f(x)=0

At each iteration, a quadratic polynomial is constructed using three previous approximations, and the root of this polynomial (closest to the last approximation) is taken as the next estimate.

Let the current approximations be:

$$x_1,\ x_2,\ x_3$$

With corresponding function values:

$$f_1 = f(x_1), \quad f_2 = f(x_2), \quad f_3 = f(x_3)$$

Define the differences:

$$h_1 = x_1 - x_3, \quad h_2 = x_2 - x_3$$

$$d_1 = f_1 - f_3, \quad d_2 = f_2 - f_3$$

Let the denominator for the interpolation be:

$$D = h_1 \cdot h_2 \cdot (h_1 - h_2)$$

The coefficients of the quadratic polynomial:

$$P(x) = a_0 + a_1(x - x_3) + a_2(x - x_3)^2$$

Are given by:

$$a_0 = f_3$$

$$a_1 = \frac{d_2 h_1^2 - d_1 h_2^2}{D}$$

$$a_2 = \frac{d_1 h_2 - d_2 h_1}{D}$$

Since $P(x_4) = 0$, we solve:

$$a_0 + a_1(x_4 - x_3) + a_2(x_4 - x_3)^2 = 0$$

Let $\Delta = x_4 - x_3$, then:

$$\Delta = \frac{-2a_0}{a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}$$

Choose the sign in the denominator so that the absolute value of the denominator is maximized (to improve numerical stability).

Thus, the next approximation is:

$$x_4 = x_3 + \Delta = x_3 + \frac{-2a_0}{a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}$$

Iterative Algorithm Steps:

1. Choose initial guesses:

   x_1, x_2, x_3 close to the desired root.

2. Compute f_1, f_2, f_3

3. Form h_1, h_2, d_1, d_2, D

4. Compute coefficients a_0, a_1, a_2

5. Evaluate:

$$x_4 = x_3 + \frac{-2a_0}{a_1 + \operatorname{sgn}(a_1) \cdot \sqrt{a_1^2 - 4a_0a_2}}$$

6. Update:

$$(x_1, x_2, x_3) \leftarrow (x_2, x_3, x_4)$$

7. Repeat until:

$$|f(x_4)| < \text{tolerance}$$

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0$$

Iteration 1:

Let:

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 2$$

$$f_1 = -20, \quad f_2 = -7, \quad f_3 = 16$$

$$h_1 = -2, \quad h_2 = -1$$

$$d_1 = -36, \quad d_2 = -23$$

$$D = (-2)(-1)((-2) - (-1)) = -2$$

$$a_0 = 16$$

$$a_1 = \frac{-23 \cdot (-2)^2 - (-36) \cdot (-1)^2}{-2} = \frac{-92 - 36}{-2} = \frac{-128}{-2} = 64$$

$$a_2 = \frac{-36 \cdot (-1) - (-23) \cdot (-2)}{-2} = \frac{36 - 46}{-2} = \frac{-10}{-2} = 5$$

Discriminant:

$$\sqrt{a_1^2 - 4a_0a_2} = \sqrt{64^2 - 4 \cdot 16 \cdot 5} = \sqrt{4096 - 320} = \sqrt{3776} \approx 61.43$$

Denominator choice (larger magnitude):

$$64 + 61.43 \approx 125.43$$

$$x_4 = 2 - \frac{2 \cdot 16}{125.43} \approx 2 - 0.2559 = 1.7441$$

Repeat using:

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = x_4$$

Eventually, you will converge to the root:

$$\boxed{x \approx 1.36808}$$

Feature	Version 1 (Earlier Given)	Version 2 (Later, Textbook Style)
Reference Points	x_0, x_1, x_2	x_1, x_2, x_3
Formula Style	Based on finite difference + Newton form	Based on quadratic interpolation through 3 points
Root Formula	$x_3 = x_2 + \frac{-2c}{b \pm \sqrt{b^2-4ac}}$	$x_4 = x_3 + \frac{-2a_0}{a_1 \pm \sqrt{a_1^2-4a_0a_2}}$
Coefficients	$a = d, b = \delta_2 + h_2 d, c = f(x_2)$	Interpolated a_0, a_1, a_2 from divided differences
Use of Notation	Simple polynomial-fitting	General interpolation form using D, h_1, h_2, d_1, d_2

1. Solve $x^3 - x - 2 = 0$ with $x_1 = 1,\ x_2 = 1.2,\ x_3 = 1.4.$
2. Solve $g(x) = 1 + 2x - \tan x = 0$ with $x_1 = 1.5,\ x_2 = 1.4,\ x_3 = 1.3.$