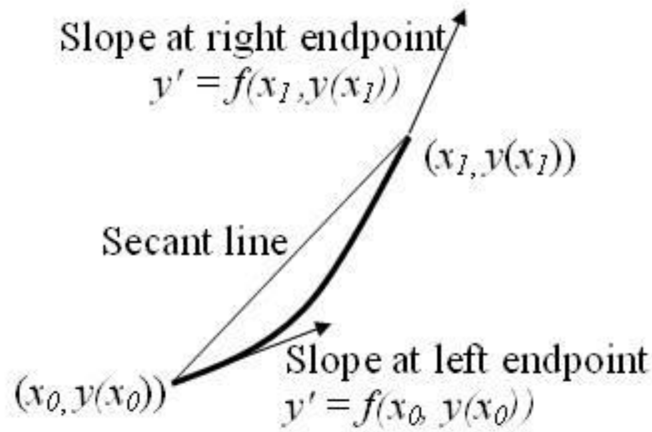
# Runge-Kutta Method




The Runge-Kutta Method is a family of iterative techniques used to obtain approximate solutions to ordinary differential equations (ODEs). It is designed to improve the accuracy of numerical integration compared to simpler methods such as Euler's method by evaluating the slope (derivative) at multiple points within each step interval and then combining these estimates in a weighted average.

Among the various Runge-Kutta methods, the classical fourth-order Runge-Kutta method (RK4) is the most commonly used due to its balance of computational efficiency and accuracy. It calculates intermediate slopes at carefully chosen points within a single step and combines them to provide a high-order approximation to the solution of the initial value problem.

Mathematically, for a first-order ODE:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

the RK4 method updates the solution from $x_n$ to $x_{n+1} = x_n + h$ by:

$$\begin{aligned}
k_1 &= hf(x_n, y_n) \\
k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\
k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\
k_4 &= hf(x_n + h, y_n + k_3) \\
y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$

# Derivation of the Classical Runge-Kutta Fourth-Order Method (RK4)

Consider the initial value problem (IVP):

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

We want to find an approximate value of $y$ at $x_{n+1} = x_n + h$, given $y_n \approx y(x_n)$.

### Goal:

Find $y_{n+1}$ such that:

$$y_{n+1} = y_n + \Phi(x_n, y_n, h)$$

where $\Phi$ is an increment function approximating the integral of the derivative over the interval $[x_n, x_{n+1}]$.

### Step 1: Taylor Series Expansion

The exact solution satisfies:

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{6}y^{(3)}(x_n) + \frac{h^4}{24}y^{(4)}(x_n) + \cdots$$

Since $y' = f(x, y)$, the higher derivatives involve derivatives of $f$ with respect to $x$ and $y$. Direct computation is often complex, so numerical methods approximate $y(x_{n+1})$ by estimating weighted slopes.

### Step 2: General Runge-Kutta Approach

The Runge-Kutta methods estimate the increment $\Phi$ as a weighted average of slopes evaluated at multiple points within the interval $[x_n, x_{n+1}]$.

The classical RK4 method uses four slope evaluations:

$$\begin{aligned}
k_1 &= hf(x_n, y_n) \quad \text{(slope at the beginning)} \\
k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \quad \text{(slope at midpoint, using } k_1\text{)} \\
k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \quad \text{(slope at midpoint, using } k_2\text{)} \\
k_4 &= hf(x_n + h, y_n + k_3) \quad \text{(slope at end)}
\end{aligned}$$

The next value is then estimated as:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

### Step 3: Matching Taylor Series Terms

The weights $1, 2, 2, 1$ and the locations where slopes are evaluated are chosen such that the local truncation error of this method is of order $O(h^5)$, thus providing fourth-order accuracy.

This is confirmed by expanding each $k_i$ term in Taylor series about $(x_n, y_n)$, substituting into the weighted average, and showing that the resulting approximation matches the Taylor expansion of $y(x_{n+1})$ up to terms in $h^4$.

This involves:

- Expanding $f$ in partial derivatives,
- Using the chain rule for $y'$, $y''$, etc.,
- Equating the coefficients of powers of $h$ in the Taylor series with those in the Runge-Kutta weighted sum,
- Ensuring that all terms up to $h^4$ are matched.

That's Why,

The RK4 method effectively uses a weighted average of slopes at four points:

- $k_1$: initial slope,
- $k_2, k_3$: slopes at midpoints (with updated $y$),
- $k_4$: slope at the endpoint (with updated $y$),

combined as:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

to achieve a high-accuracy approximation to the solution of the ODE without requiring higher derivatives explicitly.

**Example 1:**

Consider an ordinary differential equation $dy/dx = x^2 + y^2$, $y(1) = 1.2$. Find $y(1.05)$ using the fourth order Runge-Kutta method.

**Solution:**

Given,

$dy/dx = x^2 + y^2$, $y(1) = 1.2$

So, $f(x, y) = x^2 + y^2$

$x0 = 1$ and $y0 = 1.2$

Also, $h = 0.05$

Let us calculate the values of $k_1$, $k_2$, $k_3$ and $k_4$.

$k_1 = hf(x_0, y_0)$

$= (0.05) [x_0^2 + y_0^2]$

$= (0.05) [(1)^2 + (1.2)^2]$

$= (0.05) (1 + 1.44)$

$= (0.05)(2.44)$

$= 0.122$

$k_2 = hf[x_0 + (½)h, y_0 + (½)k_1]$

$= (0.05)\ [f(1 + 0.025, 1.2 + 0.061)]$ {since $h/2 = 0.05/2 = 0.025$ and $k_1/2 = 0.122/2 = 0.061$}

$= (0.05)\ [f(1.025, 1.261)]$

$= (0.05)\ [(1.025)^2 + (1.261)^2]$

$= (0.05)\ (1.051 + 1.590)$

$= (0.05)(2.641)$

$= 0.1320$

$k_3 = hf[x_0 + (½)h, y_0 + (½)k_2]$

$= (0.05)\ [f(1 + 0.025, 1.2 + 0.066)]$ {since $h/2 = 0.05/2 = 0.025$ and $k_2/2 = 0.132/2 = 0.066$}

$= (0.05)\ [f(1.025, 1.266)]$

$= (0.05)\ [(1.025)^2 + (1.266)^2]$

$= (0.05)\ (1.051 + 1.602)$

$= (0.05)(2.653)$

$= 0.1326$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= (0.05)\ [f(1 + 0.05, 1.2 + 0.1326)]$

$= (0.05)\ [f(1.05, 1.3326)]$

$= (0.05)\ [(1.05)^2 + (1.3326)^2]$

$= (0.05)\ (1.1025 + 1.7758)$

$= (0.05)(2.8783)$

$= 0.1439$

By RK4 method, we have;

$y_1 = y_0 + (1/6)\ (k_1 + 2k_2 + 2k_3 + k_4)$

$y_1 = y(1.05) = y_0 + (1/6)\ (k_1 + 2k_2 + 2k_3 + k_4)$

By substituting the values of $y_0$, $k_1$, $k_2$, $k_3$ and $k_4$, we get;

$y(1.05) = 1.2 + (1/6)\ [0.122 + 2(0.1320) + 2(0.1326) + 0.1439]$

$= 1.2 + (1/6)\ (0.122 + 0.264 + 0.2652 + 0.1439)$

$= 1.2 + (1/6)\ (0.7951)$

$= 1.2 + 0.1325$

$= 1.3325$

Find the value of $k_1$ by Runge-Kutta method of fourth order if $dy/dx = 2x + 3y^2$ and $y(0.1) = 1.1165$, $h = 0.1$.

**Solution:**

Given,

$dy/dx = 2x + 3y^2$ and $y(0.1) = 1.1165$, $h = 0.1$

So, $f(x, y) = 2x + 3y^2$

$x_0 = 0.1$, $y_0 = 1.1165$

By Runge-Kutta method of fourth order , we have

$k_1 = hf(x_0, y_0)$

$= (0.1)\ f(0.1, 1.1165)$

$= (0.1)\ [2(0.1) + 3(1.1165)^2]$

$= (0.1)\ [0.2 + 3(1.2465)]$

$= (0.1)(0.2 + 3.7395)$

$= (0.1)(3.9395)$

$= 0.39395$

## Problem

Given the differential equation:

$$\frac{dy}{dx} = \frac{y-x}{y+x}, \quad y(0) = 1,$$

find $y(0.2)$ using step size $h = 0.2$.

**Solution:**

## Step 1: Define the function

$$f(x, y) = \frac{y-x}{y+x}.$$

Initial values:

$$x_0 = 0, \quad y_0 = 1, \quad h = 0.2.$$

## Step 2: Calculate $k_1$

$$k_1 = hf(x_0, y_0) = 0.2 \times \frac{1-0}{1+0} = 0.2 \times 1 = 0.2.$$

## Step 3: Calculate $k_2$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times f(0.1, 1 + 0.1).$$

Calculate inside the function:

$$f(0.1, 1.1) = \frac{1.1 - 0.1}{1.1 + 0.1} = \frac{1.0}{1.2} = 0.8333.$$

Therefore,

$$k_2 = 0.2 \times 0.8333 = 0.1667.$$

### Step 4: Calculate $k_3$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \times f(0.1, 1 + 0.08335).$$

Calculate inside the function:

$$f(0.1, 1.08335) = \frac{1.08335 - 0.1}{1.08335 + 0.1} = \frac{0.98335}{1.18335} \approx 0.8310.$$

Therefore,

$$k_3 = 0.2 \times 0.8310 = 0.1662.$$

### Step 5: Calculate $k_4$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1 + 0.1662).$$

Calculate inside the function:

$$f(0.2, 1.1662) = \frac{1.1662 - 0.2}{1.1662 + 0.2} = \frac{0.9662}{1.3662} \approx 0.7071.$$

Therefore,

$$k_4 = 0.2 \times 0.7071 = 0.1414.$$

### Step 6: Calculate $y_1 = y(0.2)$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Substitute the values:

$$y_1 = 1 + \frac{1}{6}(0.2 + 2 \times 0.1667 + 2 \times 0.1662 + 0.1414).$$

Calculate inside the parenthesis:

$$= 1 + \frac{1}{6}(0.2 + 0.3334 + 0.3324 + 0.1414) = 1 + \frac{1}{6}(1.0072) = 1 + 0.1679 = 1.1679.$$

So the answer is 1.1679