Iteration Method

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Iteration Method (Iterative Method)

Definition:

- An **iterative method** is a computational technique that generates a sequence of improving approximate solutions.
- It's used to **find roots** of equations or **solve systems** of equations.
- An iterative method is **convergent** if the sequence of approximations converges to the actual solution.

- Uses **initial guess** to start.
- **Successive approximations** improve the solution.
- Especially useful for **nonlinear equations** and **large linear systems**.
- Computers are often used due to the **repetitive arithmetic operations**.

Concept of Iteration:

- **Iteration** means repeating a process to get closer to a desired result.
- Each repetition = one **iteration**.
- Output of each step becomes the **input** for the next.

Root Finding Using Iteration:

Given:
$\mathbf{f(x) = 0}$

Convert to:
$\mathbf{x = \varphi(x)}$

Let:
Initial guess $\mathbf{x_0}$

Then:

$$\mathbf{x_1 = \varphi(x_0)}$$
$$\mathbf{x_2 = \varphi(x_1)}$$
$$\mathbf{x_3 = \varphi(x_2)}$$
$$\ldots$$
$$\mathbf{x_n = \varphi(x_{n-1})}$$

This sequence continues until desired accuracy is achieved.

Ex: Find the real root of the equation $x^3 + x^2 - 1 = 0$ by iteration method.

Sol:

We can rewrite the above equation by $x^3 + x^2 = 1$.

Let, $f(x) = x^3 + x^2 - 1$

$f(0) = -1 < 0$

$f(1) = 1 > 0$

Hence the root value lies between 0 and 1.

$x^3 + x^2 = 1$

$\implies x^2(x+1) = 1$

$\implies x^2 = \frac{1}{1+x}$

$\implies x = \frac{1}{\sqrt{1+x}}$

$f(x) = \frac{1}{\sqrt{1+x}}$

Let the initial approximation be $x_0 = 0.5$

$$x_1 = f(x_0) = \frac{1}{\sqrt{1+0.5}} = \frac{1}{\sqrt{1.5}} \approx 0.81649$$

$$x_2 = f(x_1) = \frac{1}{\sqrt{1+0.81649}} = \frac{1}{\sqrt{1.81649}} \approx 0.74196$$

$$x_3 = f(x_2) = \frac{1}{\sqrt{1+0.74196}} \approx 0.75787$$

$$x_4 = f(x_3) = \frac{1}{\sqrt{1+0.75787}} \approx 0.75427$$

$$x_5 = f(x_4) = \frac{1}{\sqrt{1+0.75427}} \approx 0.75501$$

$$x_6 = f(x_5) = \frac{1}{\sqrt{1+0.75501}} \approx 0.75485$$

$$x_7 = f(x_6) = \frac{1}{\sqrt{1+0.75485}} \approx 0.75488$$

$$x_8 = f(x_7) = \frac{1}{\sqrt{1+0.75488}} \approx 0.75488$$

Since the difference between x_7 and x_8 are very small.

So the root is 0.75488.

Ex: Find the root of the equation $\cos x = 3x - 1$, correct to four decimal places.

Sol:

Let, $f(x) = \cos x - 3x + 1$

If, $f(0) = 2 > 0$

$f(1) = -1.4596 < 0$

Hence the at least one root lies between 0 and 1.

Let the initial value $x_0 = \frac{0+1}{2} = 0.5$

The given equation may be written as:

$3x = 1 + \cos x$

$\implies x = \frac{1}{3}(1 + \cos x) = f(x)$

$x_1 = \frac{1}{3}(1 + \cos x_0) = \frac{1}{3}(1 + \cos 0.5) = 0.6258$

$x_2 = \frac{1}{3}(1 + \cos x_1) = \frac{1}{3}(1 + \cos 0.6258) = 0.6035$

$x_3 = \frac{1}{3}(1 + \cos x_2) = \frac{1}{3}(1 + \cos 0.6035) = 0.6078$

$x_4 = \frac{1}{3}(1 + \cos x_3) = \frac{1}{3}(1 + \cos 0.6078) = 0.6070$

$x_5 = \frac{1}{3}(1 + \cos x_4) = \frac{1}{3}(1 + \cos 0.6070) = 0.6071$

$x_6 = \frac{1}{3}(1 + \cos x_5) = \frac{1}{3}(1 + \cos 0.6071) = 0.6071$

Due to repetition of x_5 and x_6 we stopped. Hence root is 0.6071.

Ex: $2x = \cos x + 3$ [Correct to three decimal places]

Sol:

Let, $f(x) = \cos x - 2x + 3$

$f(1) = 1.54030$

$f(2) = -1.4161 < 0$

One root lies between 1 and 2.

Initial root $x_0 = \frac{1+2}{2} = 1.5$

The given equation may be written as:

$2x = \cos x + 3$

$x = \frac{1}{2}(\cos x + 3)$

$$
\begin{aligned}
x_1 &= \tfrac{1}{2}(\cos 1.5 + 3) = 1.5354 \\
x_2 &= \tfrac{1}{2}(\cos 1.5354 + 3) = 1.5177 \\
x_3 &= \tfrac{1}{2}(\cos 1.5177 + 3) = 1.5265 \\
x_4 &= \tfrac{1}{2}(\cos 1.5265 + 3) = 1.5221 \\
x_5 &= \tfrac{1}{2}(\cos 1.5221 + 3) = 1.5243 \\
x_6 &= \tfrac{1}{2}(\cos 1.5243 + 3) = 1.5232 \\
x_7 &= \tfrac{1}{2}(\cos 1.5232 + 3) = 1.5238 \\
x_8 &= \tfrac{1}{2}(\cos 1.5238 + 3) = 1.5235
\end{aligned}
$$

$x_9 = \frac{1}{2}(3 + \cos 1.5235) = 1.5236$

$x_{10} = \frac{1}{2}(3 + \cos 1.5236) = 1.5236$

Due to repetition of x9 and x10, we stop calculation. Hence, the root is 1.523 correct to three decimal places.

Ex: $e^x - 3x = 0$ [Correct to four decimal places]

Let,

$f(x) = e^x - 3x$
$f(0) = 1 > 0$
$f(1) = e - 3 \approx 2.71828 - 3 = -0.28172 < 0$

So one root lies between 0 and 1.

So therefore, $x_0 = \frac{0+1}{2} = 0.5$

The equation can be written as, $3x = e^x$, $x = \frac{1}{3}e^x$

$x_1 = \frac{1}{3}e^{0.5} \approx \frac{1}{3}(1.6487) \approx 0.5496$

$x_2 = \frac{1}{3}e^{0.5496} \approx \frac{1}{3}(1.7328) \approx 0.5775$

$x_3 = \frac{1}{3}e^{0.5775} \approx \frac{1}{3}(1.7819) \approx 0.5939$

$x_4 = \frac{1}{3}e^{0.5939} \approx \frac{1}{3}(1.8117) \approx 0.6039$

$x_5 = \frac{1}{3}e^{0.6039} \approx \frac{1}{3}(1.8299) \approx 0.6096$

$x_6 = \frac{1}{3}e^{0.6096} \approx \frac{1}{3}(1.8399) \approx 0.6133$

$x_7 = \frac{1}{3}e^{0.6133} \approx \frac{1}{3}(1.8467) \approx 0.6155$

$x_8 = \frac{1}{3}e^{0.6155} \approx \frac{1}{3}(1.8507) \approx 0.6169$

$x_9 = \frac{1}{3}e^{0.6169} \approx \frac{1}{3}(1.8533) \approx 0.6178$

$x_{10} = \frac{1}{3}e^{0.6177} \approx 0.6183$

$x_{11} = \frac{1}{3}e^{0.6183} \approx 0.6185$

$x_{12} = \frac{1}{3}e^{0.6185} \approx 0.6187$

$x_{13} = \frac{1}{3}e^{0.6187} \approx 0.6188$

$x_{14} = \frac{1}{3}e^{0.6188} \approx 0.6189$

$x_{15} = \frac{1}{3}e^{0.6189} \approx 0.6190$

$x_{16} = \frac{1}{3}e^{0.6190} \approx 0.6190$

Due to repetition of x_{15} and x_{16} we stop calculation.

So the required root is 0.6190.

Ex: $3x - \sqrt{1 + \sin x} = 0$ [Correct to five decimal places]

$f(x) = 3x - \sqrt{1 + \sin x}$
$f(0) = -1 < 0$
$f(1) = 3 - \sqrt{1 + \sin 1} \approx 3 - \sqrt{1 + 0.84147} \approx 3 - \sqrt{1.84147} \approx 3 - 1.35701 \approx 1.64299 > 0$

$x_0 = 0.5$

Equation can be written as, $3x = \sqrt{1 + \sin x}$, $x = \frac{1}{3}\sqrt{1 + \sin x}$

$x_1 = 0.28358$
$x_2 = 0.48986$
$x_3 = 0.48596$
$x_4 = 0.48775$
$x_5 = 0.48692$
$x_6 = 0.48730$
$x_7 = 0.48713$
$x_8 = 0.48721$
$x_9 = 0.48717$
$x_{10} = 0.48719$
$x_{11} = 0.48718$
$x_{12} = 0.48718$

Due to repetition of x_{11} and x_{12}, the root is 0.48718.

Ex: $\cos x - xe^x = 0$

Let $f(x) = \cos x - xe^x$

$f(0) = 1 > 0$

$f(1) = -2.1779 < 0$

So one root lies between 0 and 1.

Let $x_0 = 0.5$

Equation can be written as $x = \cos(x)/e^x$

$x_1 = 0.5323$

$x_2 = 0.5060$

$x_3 = 0.5273$

$x_4 = 0.5100$

$x_5 = 0.5241$

$x_6 = 0.5126$

$x_7 = 0.5219$

$x_8 = 0.5144$

$x_9 = 0.5205$

$x_{10} = 0.5155$

$x_{11} = 0.5195$

$x_{12} = 0.5162$

$x_{13} = 0.5189$

$x_{14} = 0.5168$

$x_{15} = 0.5186$

$x_{16} = 0.5171$

$x_{17} = 0.5183$

$x_{18} = 0.5173$

$x_{19}=0.5181$

$x_{20}=0.5175$

$x_{21}=0.5179$

$x_{22}=0.5176$

$x_{23}=0.5179$

$x_{24}=0.5176$

$x_{25}=0.5178$

$x_{26}=0.5176$

$x_{27}=0.5178$

$x_{28}=0.5177$

$x_{29}=0.5178$

$x_{30}=0.5177$

$x_{3}1=0.5177$

So root is 0.5177.