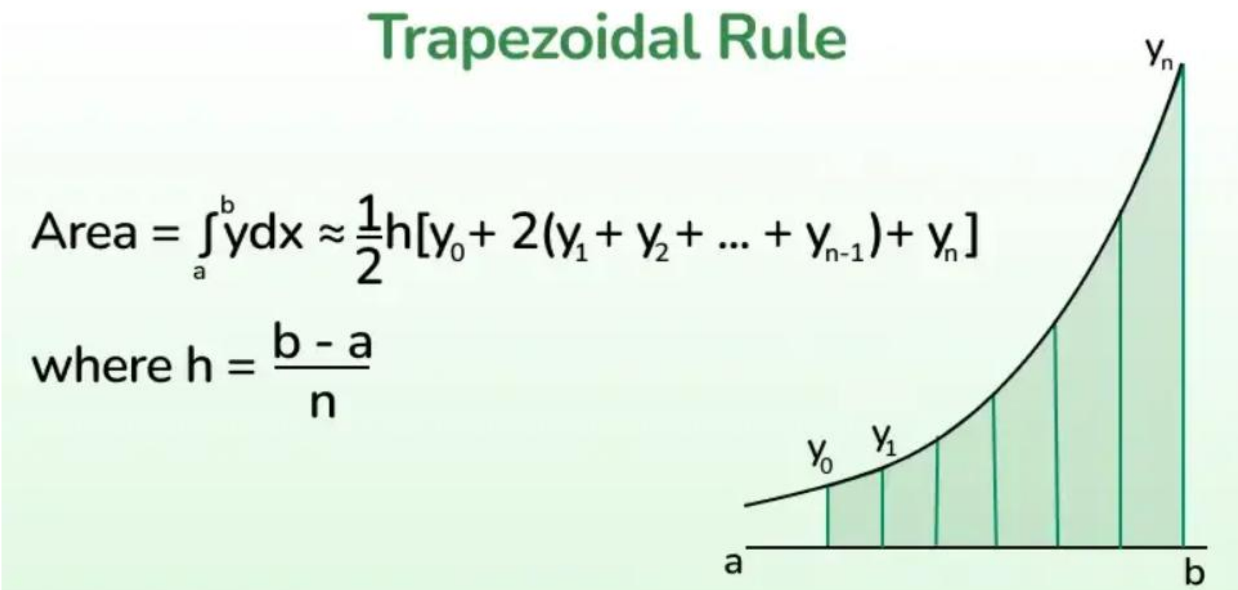

The Trapezoidal Rule is a numerical method used to approximate the definite integral of a real-valued function over a closed interval. It is based on the idea of dividing the total area under a curve into a series of adjacent trapezoids rather than rectangles, as done in Riemann sums. This approach provides a linear approximation of the function over each subinterval.

Mathematically, if a function $f(x)$ is continuous on the interval $[a, b]$, and the interval is divided into n equal subintervals of width $h = \frac{b-a}{n}$, then the Trapezoidal Rule approximates the integral $\int_a^b f(x)\,dx$ as follows:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

where $x_0 = a$, $x_n = b$, and $x_i = a + ih$ for $i = 1, 2, ..., n - 1$.

This method is particularly useful in numerical analysis when an analytical solution of the integral is difficult or impossible to obtain. The accuracy of the approximation depends on the number of subintervals and the nature of the function being integrated.

Derivation of the Trapezoidal Rule

Let $f(x)$ be a real-valued, continuous function defined on the closed interval $[a, b]$. The objective is to approximate the definite integral

$$\int_a^b f(x)\,dx$$

by a numerical method. The Trapezoidal Rule achieves this by dividing the interval $[a, b]$ into n equal subintervals, then approximating the area under the curve by the sum of areas of trapezoids formed on each subinterval.

Step 1: Partition the interval

Divide $[a, b]$ into n subintervals, each of width

$$h = \frac{b-a}{n}$$

Define the partition points as

$$x_0 = a, \quad x_1 = a + h, \quad x_2 = a + 2h, \quad \ldots, \quad x_n = a + nh = b$$

Step 2: Approximate each subinterval

On each subinterval $[x_{i-1}, x_i]$, approximate $f(x)$ using a straight line (linear interpolation), which results in a trapezoid.

The area of a trapezoid on $[x_{i-1}, x_i]$ is:

$$\text{Area}_i = \frac{h}{2}\left[f(x_{i-1}) + f(x_i)\right]$$

Step 3: Sum over all subintervals

Summing the areas over all subintervals:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \frac{h}{2}\left[f(x_{i-1}) + f(x_i)\right]$$

Factor out the constant:

$$= \frac{h}{2} \sum_{i=1}^{n} \left[f(x_{i-1}) + f(x_i)\right]$$

Rewriting the summation:

$$= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

This is the **Trapezoidal Rule formula**:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

Problem Statement

Find the area enclosed by the function $f(x) = 2x$ between $x = 0$ and $x = 2$ using the **Trapezoidal Rule** with **2 intervals**.

Given:

- Function: $f(x) = 2x$
- Lower limit: $a = 0$
- Upper limit: $b = 2$
- Number of intervals: $n = 2$

Step 1: Calculate the width of each interval

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{2} = 1$$

Step 2: Determine the x-values

$$x_0 = a = 0, \quad x_1 = x_0 + \Delta x = 1, \quad x_2 = x_1 + \Delta x = 2$$

Step 3: Evaluate the function at each x-value

$$\begin{aligned} f(x_0) &= f(0) = 2 \times 0 = 0 \\ f(x_1) &= f(1) = 2 \times 1 = 2 \\ f(x_2) &= f(2) = 2 \times 2 = 4 \end{aligned}$$

Step 4: Apply the Trapezoidal Rule formula

The general formula for the Trapezoidal Rule is:

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$$

Substituting the known values:

$$T_2 = \frac{1}{2}\left[f(0) + 2f(1) + f(2)\right]$$

$$T_2 = \frac{1}{2}\left[0 + 2(2) + 4\right]$$

$$T_2 = \frac{1}{2}(0 + 4 + 4) = \frac{1}{2}(8) = 4$$

Final Answer:

$$\boxed{\text{Area} = 4}$$

Source: byjus

Problem Statement

Find the area enclosed by the function $f(x) = x^3 + 1$ between $x = 0$ and $x = 4$, using the **Trapezoidal Rule** with **4 intervals**.

Given:

- Function: $f(x) = x^3 + 1$
- Lower limit: $a = 0$
- Upper limit: $b = 4$
- Number of intervals: $n = 4$

Step 1: Calculate the width of each interval

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$$

Step 2: Determine the x-values

$$x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad x_4 = 4$$

Step 3: Evaluate the function at each x-value

$$\begin{aligned}
f(x_0) &= f(0) = 0^3 + 1 = 1 \\
f(x_1) &= f(1) = 1^3 + 1 = 2 \\
f(x_2) &= f(2) = 2^3 + 1 = 9 \\
f(x_3) &= f(3) = 3^3 + 1 = 28 \\
f(x_4) &= f(4) = 4^3 + 1 = 65
\end{aligned}$$

Step 4: Apply the Trapezoidal Rule formula

The general formula for the Trapezoidal Rule is:

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

Substituting the known values:

$$\begin{aligned} T_4 &= \frac{1}{2}\left[1 + 2(2) + 2(9) + 2(28) + 65\right] \\ &= \frac{1}{2}\left[1 + 4 + 18 + 56 + 65\right] \\ &= \frac{1}{2}(144) = 72 \end{aligned}$$

Final Answer:

$$\boxed{\text{Area} = 72}$$

Example 1: Find the area under the curve using the Trapezoidal Rule for the following points:

x	0	1	2	3
y	2	4	8	16

Solution:

The width of the interval is $h = x_{i+1} - x_i = 1$

Apply the Trapezoidal Rule Formula: $\int_a^b f(x)dx \approx \frac{h}{2}[y_0 + 2\Sigma_{i-1}^{n-1}(y_i) + y_n]$

Here:

- $y_0 = 2, y_1 = 4, y_2 = 8, y_3 = 16$
- $n = 3$

$\int_0^3 f(x)\,dx \approx ½[2 + 2(4 + 8) + 16]$
$= ½[2 + 24 + 16]$
$= ½[42]$
$= 21$

Example 2: Calculate the area under the curve passing through the following points using the Trapezoidal rule:

x	0	2	4	6
y	5	9	13	17

Solution:

The width of the interval is h = x_{i+1} - x_i = 2

Apply the Trapezoidal Rule Formula: $\int_a^b f(x)dx \approx \frac{h}{2}[y_0 + 2\Sigma_{i-1}^{n-1}(y_i) + y_n]$

Here:

- $y_0 = 5$, $y_1 = 9$, $y_2 = 13$, $y_3 = 17$
- *n = 3*

$^6\int_0 f(x)\,dx \approx 2/2[5 + 2(9 + 13) + 17]$
$= 1[5 + 2(22) + 17]$
$= 66$

Example 3: Find the area enclosed by the function f(x) between x = 0 to x = 4 with 4 intervals: f(x) = 4x

Solution:

Here a = 0, b = 4 and n = 4.

$$\Delta x = \frac{b-a}{n}$$
$$= \Delta x = \frac{4-0}{4}$$
$$= \Delta x = 1$$

$$x_i = a + i\Delta x$$
$$= x_i = 0 + i$$
$$= x_i = i$$

The trapezoidal rule for n = 4 is,

$$T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$$

Substituting the values in this equation,

$$T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$$
T_n = ½[0 + 2(4 + 8 + 12) + 16]
= ½[0 + 48 + 16]
= ½ [64]
= 32

Example 4: Find the area enclosed by the function f(x) between x = 0 to x = 3 with 3 intervals: f(x) = x

Solution:

Here a = 0, b = 3 and n = 3.

$$\Delta x = \frac{b-a}{n}$$
$$= \Delta x = \frac{3-0}{3}$$
$$= \Delta x = 1$$

$$x_i = a + i\Delta x$$
$$= x_i = 0 + i$$
$$= x_i = i$$

The trapezoidal rule for n = 3 is,

$$T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3))$$

Substituting the values in this equation,

$$T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3))$$
$$\Rightarrow T_n = \frac{1}{2}(f(0) + 2f(1) + 2f(2) + f(3))$$
$$\Rightarrow T_n = \frac{1}{2}(0 + 2 + 2(2) + 2(3))$$
$$\Rightarrow T_n = \frac{1}{2}(2 + 4 + 6) = 6$$

Example 6: Find the area enclosed by the function f(x) between x = 0 to x = 3 with 3 intervals: f(x) = x^2

Solution:

Here a = 0, b = 3 and n = 3.

$$\Delta x = \frac{b-a}{n}$$
$$= \Delta x = \frac{3-0}{3}$$
$$= \Delta x = 1$$

$$x_i = a + i\Delta x$$
$$= x_i = 0 + i$$
$$= x_i = i$$

The trapezoidal rule for n = 3 is,

$$T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3))$$

Substituting the values in this equation,

$$T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3))$$
$$\Rightarrow T_n = \frac{1}{2}(f(0) + 2f(1) + 2f(2) + f(3))$$
$$\Rightarrow T_n = \frac{1}{2}(0 + 2(1) + 2(4) + 2(9))$$
$$\Rightarrow T_n = \frac{1}{2}(2 + 8 + 18) = 14$$

Example 7: Find the area enclosed by the function f(x) between x = 0 to x = 4 with 4 intervals: f(x) = $x^3 + 1$

Solution:

Here a = 0, b = 4 and n = 4.

$$\Delta x = \frac{b-a}{n}$$
$$= \Delta x = \frac{4-0}{4}$$
$$= \Delta x = 1$$

$$x_i = a + i\Delta x$$
$$= x_i = 0 + i$$
$$= x_i = i$$

The trapezoidal rule for n = 4 is,

$$T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$$

Substituting the values in this equation,

$$T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$$
$$\Rightarrow T_n = \frac{1}{2}(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4))$$
$$\Rightarrow T_n = \frac{1}{2}(1 + 2(2) + 2(9) + 2(28) + (65))$$
$$\Rightarrow T_n = \frac{1}{2}(1 + 4 + 18 + 56 + 65)$$
$$\Rightarrow T_n = 72$$

Example 8: Find the area enclosed by the function f(x) between x = 0 to x = 4 with 4 intervals: f(x) = e^x

Solution:

Here a = 0, b = 4 and n = 4.

$$\Delta x = \frac{b-a}{n}$$
$$= \Delta x = \frac{4-0}{4}$$
$$= \Delta x = 1$$

$$x_i = a + i\Delta x$$
$$= x_i = 0 + i$$
$$= x_i = i$$

The trapezoidal rule for n = 4 is,

$$T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$$

Substituting the values in this equation,

$$T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$$
$$\Rightarrow T_n = \frac{1}{2}(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4))$$
$$\Rightarrow T_n = \frac{1}{2}(e^0 + 2e + 2e^2 + 2e^3 + e^4)$$
$$\Rightarrow T_n = \frac{1}{2} + e + e^2 + e^3 + \frac{e^4}{2}$$