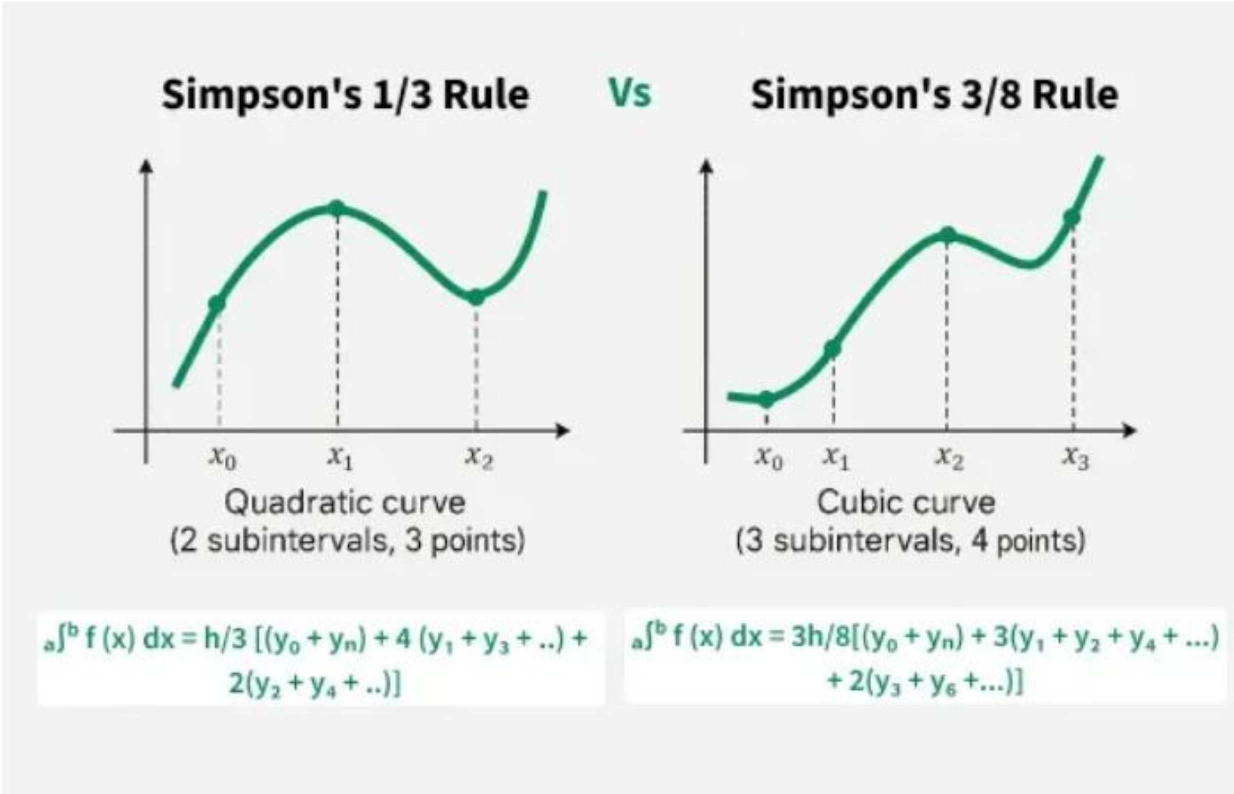

Simpson's 1/3 Rule (One-Third Rule)

Simpson's 1/3 Rule is a numerical integration method used to approximate the definite integral of a function. It is based on fitting a second-degree polynomial (a parabola) through three equally spaced points on the function and integrating the polynomial over the interval. This rule provides an accurate approximation when the function is smooth and continuous.

Mathematically, for a function $f(x)$ defined on $[a, b]$, with an even number n of subintervals, the Simpson's 1/3 Rule is given by:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

where $h = \frac{b-a}{n}$ and n is even.

Simpson's 3/8 Rule (Three-Eighth Rule)

Simpson's 3/8 Rule is another method for numerical integration, which uses cubic interpolation by fitting a third-degree polynomial through four equally spaced points. It is particularly useful when the number of subintervals is a multiple of three.

The rule is expressed as:

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3f(x_4) + 3f(x_5) + \cdots + f(x_n)\right]$$

where $h = \frac{b-a}{n}$ and n is a multiple of 3.

For Full Derivation[https://www.cuemath.com/simpsons-rule-formula/]

Derivation of Simpson's 1/3 Rule

Let $f(x)$ be a continuous function defined over the interval $[a, b]$, and suppose the interval is divided into n **even** subintervals of equal width:

$$h = \frac{b-a}{n}$$

Consider three equally spaced points: $x_0 = a$, $x_1 = a + h$, and $x_2 = a + 2h$. Over this interval, approximate $f(x)$ using a second-degree polynomial (a parabola) passing through these points.

We assume:

$$f(x) \approx P_2(x) = Ax^2 + Bx + C$$

Using Lagrange interpolation or solving for coefficients from $f(x_0), f(x_1), f(x_2)$, one can derive the following result:

$$\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} P_2(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

If the interval $[a, b]$ is split into **n even parts**, we apply this formula to each consecutive pair of intervals and sum them:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

This is known as **Simpson's 1/3 Rule**.

Derivation of Simpson's 3/8 Rule

Simpson's 3/8 Rule is based on fitting a **cubic polynomial** through **four equally spaced points**: x_0, x_1, x_2, x_3 where:

$$x_i = a + ih, \quad \text{for } i = 0, 1, 2, 3 \quad \text{and} \quad h = \frac{b-a}{3}$$

Using a cubic polynomial interpolation:

$$f(x) \approx P_3(x)$$

The integral of $f(x)$ over $[x_0, x_3]$ using this cubic polynomial gives:

$$\int_{x_0}^{x_3} f(x)\,dx \approx \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

This formula can be extended to the entire interval $[a, b]$, provided that the number of subintervals n is a multiple of 3.

Thus, the general form is:

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3f(x_4) + \cdots + 3f(x_{n-1}) + f(x_n)\right]$$

Problem Statement

Approximate the integral of the function $f(x)$ over the interval $[0.0, 0.4]$ using **Simpson's 1/3 Rule** with the following data points:

x	$f(x)$
0.0	1.0000
0.1	0.9975
0.2	0.9900
0.3	0.9776
0.4	0.8604

Solution:

The formula for Simpson's 1/3 Rule is:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + y_n + 4\sum_{\text{odd } i} y_i + 2\sum_{\text{even } i} y_i\right]$$

where

- $h =$ step size between x-values,
- $y_i = f(x_i)$,
- n is the number of subintervals (which must be even).

Step 1: Calculate h

$$h = x_1 - x_0 = 0.1$$

Step 2: Identify function values

- $y_0 = f(0.0) = 1.0000$
- $y_1 = f(0.1) = 0.9975$
- $y_2 = f(0.2) = 0.9900$
- $y_3 = f(0.3) = 0.9776$
- $y_4 = f(0.4) = 0.8604$

Step 3: Apply Simpson's 1/3 Rule formula

$$\int_0^{0.4} f(x)\,dx \approx \frac{0.1}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)\right]$$

Substitute the values:

$$= \frac{0.1}{3}\left[(1.0000 + 0.8604) + 4(0.9975 + 0.9776) + 2(0.9900)\right]$$

Calculate the sums inside the bracket:

$$= \frac{0.1}{3}\left[1.8604 + 4(1.9751) + 1.9800\right]$$

$$= \frac{0.1}{3}\left[1.8604 + 7.9004 + 1.9800\right] = \frac{0.1}{3} \times 11.7408$$

$$= \frac{0.1}{3} \times 11.7408 = 0.39136$$

Problem Statement: Approximation of the integral using Simpson's 3/8 Rule for the Same Value.

Solution:

Simpson's 3/8 Rule formula:

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + \cdots) + 2(y_3 + y_6 + \cdots)\right]$$

Applying values:

$$\int_0^{0.4} f(x)\,dx \approx \frac{3 \times 0.1}{8}\left[(y_0 + y_4) + 2y_3 + 3(y_1 + y_2)\right]$$

Substitute values:

$$= \frac{0.3}{8}\left[(1.0000 + 0.8604) + 2 \times 0.9776 + 3 \times (0.9975 + 0.9900)\right]$$

Calculate inside the bracket:

$$= \frac{0.3}{8}\left[1.8604 + 1.9552 + 3 \times 1.9875\right]$$

$$= \frac{0.3}{8}\left[1.8604 + 1.9552 + 5.9625\right] = \frac{0.3}{8} \times 9.7781$$

$$= 0.0375 \times 9.7781 = 0.36668$$

SIMPSON'S $\frac{1}{3}$ RULE(USE WHEN N=EVEN NUMBER)

Example 2 a=10 ,b=16 ,h=1 ,n=6 . PAGE:8.12

X	10	11	12	13	14	15	16
Y	1.02	0.94	0.89	0.79	0.71	0.62	0.55
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\int_{10}^{16} y\,dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1.02 + 0.55) + 4(0.94 + 0.79 + 0.62) + 2(0.89 + 0.71)]$$

=0.3333[1.57+9.4+3.2]

=4.7233

$$\int_{10}^{16} y\,dx = 4.7233$$

SIMPSON'S $\frac{3}{8}$ RULE(USE WHEN N=MULTIPLE OF THREE)

EXAMPLE 3:EVALUATE $\int_0^3 \frac{1}{1+X}dx$ WITH N=6 BY USING SIMPSON'S $\frac{3}{8}$ RULE AND HENCE CALCULATE LOG 2.

- a=0 , b=3 , n=6 , h=?

$h = \frac{b-a}{n} = \frac{3-0}{6} = \mathbf{0.5}$

Y= F(X) = $\frac{1}{1+X}$

X	0	0.5	1	1.5	2	2.5	3
F(X)	1	0.6667	0.5	0.4	0.3333	0.2857	0.25
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

BY USING SIMPSON'S $\frac{3}{8}$ RULE,

$\int_0^3 \frac{1}{1+X}dx = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$

$= \frac{3(0.5)}{8}[(1 + 0.25) + 2(0.4) + 3(0.6667 + 0.5 + 0.3333 + 0.2857)]$

=1.3888

$\int_0^3 \frac{1}{1+X}dx$ =1.3888...........(1)

BY DIRRECT INTIGRATION ,

$\int_0^3 \frac{1}{1+X}dx = \left|\log(1+x)\right|_0^3$

= log 4

=log 2^2

$\int_0^3 \frac{1}{1+X}dx$ = 2log2.........(2)

FROM EQUATION 1 & 2 ,.......2log2=1.3888

=Log 2=0.6944

ALL THE FORMULA'S FOR N=6,

simpson's $\frac{3}{8}$ rule= $\frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$

simpson's $\frac{1}{3}$ rule = $\frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

Trapezoidal rule = = $\frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$

Example: Evaluate $\int_0^1 e^x dx$, by Simpson's ⅓ rule.

Solution:

Let us divide the range [0, 1] into six equal parts by taking h = 1/6.

If $x_0 = 0$ then $y_0 = e^0 = 1$.

If $x_1 = x_0 + h = ⅙$, then $y_1 = e^{1/6} = 1.1813$

If $x_2 = x_0 + 2h = 2/6 = 1/3$ then, $y_2 = e^{1/3} = 1.3956$

If $x_3 = x_0 + 3h = 3/6 = ½$ then $y_3 = e^{1/2} = 1.6487$

If $x_4 = x_0 + 4h = 4/6$ ⅔ then $y_4 = e^{2/3} = 1.9477$

If $x_5 = x_0 + 5h = ⅚$ then $y_5 = e^{5/6} = 2.3009$

If $x_6 = x_0 + 6h = 6/6 = 1$ then $y_6 = e^1 = 2.7182$

We know by Simpson's ⅓ rule;

$\int_a^b f(x)\, dx = h/3\, [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + y_6 + ... + y_{n-2})]$

Therefore,

$\int_0^1 e^x dx = (1/18)\ [(1 + 2.7182) + 4(1.1813 + 1.6487 + 2.3009) + 2(1.39561 + 1.9477)]$

$= (1/18)[3.7182 + 20.5236 + 6.68662]$

$= 1.7182$ (approx.)