BISECTION METHOD

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One of the simplest and most reliable iterative methods.

Based on Intermediate Value Theorem.

If $f(x)$ is continuous on (a, b) and $f(a)f(b) < 0$, then at least one real root lies in (a, b).

Basic Concept

Let $x_1 = a$, $x_2 = b$.

Midpoint $x_0 = \frac{x_1+x_2}{2}$.

Three possible cases:

1. $f(x_0) = 0 \rightarrow$ root at x_0
2. $f(x_0)f(x_1) < 0 \rightarrow$ root in (x_1, x_0)
3. $f(x_0)f(x_2) < 0 \rightarrow$ root in (x_0, x_2)

Iterative Approach

Test sign of $f(x_0)$ to determine which half contains the root.

Repeat the process to narrow down the interval.

Continue until the interval is sufficiently small.

Procedure

1. Choose two initial values of x such that $f(x_1)f(x_2) < 0$.
2. Bisect the interval and choose the half with the sign change.
3. Each iteration improves accuracy by one binary digit.
4. Stop when interval size is less than required precision.

Largest Possible Roots

For polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$:

Largest possible root: $x_1 = \frac{a_{n-1}}{a_n}$.

Used as initial guess if no better estimate is available.

Search Bracket

Useful inequality:

$$|x| \leq \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

Maximum absolute value of root:

$$|x_{\max}| = \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

Ex-1: Estimate the possible initial guess value of the Polynomial.

$$2x^3 - 8x^2 + 2x + 12 = 0$$

Answer:

The largest possible root is:

$$x_1 = \frac{a_{n-1}}{a_n} = \frac{-8}{2} = +4$$

This implies that, no root can be larger than the value 9.

But all roots satisfy the relation.

$$|x| \leq \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

$$= \sqrt{\left(\frac{-8}{2}\right)^2 - 2\left(\frac{2}{2}\right)}$$

$$= \sqrt{16 - 2}$$

$$= \sqrt{14}$$

Therefore all real roots lie in the interval $(-\sqrt{14}, \sqrt{14})$.

Ex-2:

Find the real of the equation $x^2 - 4x - 10 = 0$ using Bisection Method.

Sol:

$$|x|_{max} = \sqrt{\left(\frac{-4}{1}\right)^2 - 2\left(\frac{-10}{1}\right)}$$

$$= \sqrt{16+20}$$

$$= \sqrt{36} = 6$$

Therefore, we have both the roots in the interval $[-6, 6]$.

The table gives the values of $f(x)$ between -6 and 6.

x	**-6**	**-5**	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	**5**	**6**
f(x)	50	35	22	11	2	-5	-10	-13	-14	-13	-10	-5	2

Find initial root:

$f(-2) = 2 > 0$

$f(-1) = -5 < 0$

$f(-2)$ and $f(-1)$ are opposite signs, so root lies between -2 and -1.

Initial root, $x_0 = \frac{a+b}{2} = \frac{-2+(-1)}{2} = \frac{-3}{2} = -1.5$

Root x_n	Signs of $f(x) = x^2 - 4x - 10$	Comments
$x_0 = -1.5$	$f(-1.5) = -1.75 < 0$	Since f(-2)>0 and f(-1.5)<0, the root must in the interval (-2,-1.5)

$x_1 = \frac{-2-1.5}{2} = -1.75$	$f(-1.75) = 0.0625 > 0$	Since f(-1.5)<0 and f(-1.75)>0, the root must in the interval (-1.75, -1.5)
$x_2 = \frac{-1.75-1.5}{2} = -1.625$	$f(-1.625) = -0.859 < 0$	The root must in the interval (-1.75, -1.625)
$x_3 = \frac{-1.75-1.625}{2} = -1.6875$	$f(-1.6875) = -0.4 < 0$	The root must in the interval (-1.75,-1.6875)
$x_4 = \frac{-1.75-1.6875}{2} = -1.7188$	$f(-1.7188) = -0.1705 < 0$	The root must in the interval (-1.75, -1.7188)
$x_5 = \frac{-1.75-1.7188}{2} = -1.7344$	$f(-1.7344) = -0.054 < 0$	The root must in the interval (-1.75, -1.7344)
$x_6 = \frac{-1.75-1.7344}{2} = -1.7422$	$f(-1.7422) = 0.004 > 0$	The root must in the interval (-1.7344, -1.7422)
$x_7 = \frac{-1.7344-1.7422}{2} = -1.7383$	$f(-1.7383) = -0.025 < 0$	The root must in the interval (-1.7422, -1.7383)
$x_8 = \frac{-1.7422-1.7383}{2} = -1.7402$	$f(-1.7402) = -0.0109 < 0$	The root must in the interval (-1.7422, -1.7402)
$x_9 = \frac{-1.7422-1.7402}{2} = -1.7412$	$f(-1.7412) = -0.003 \approx 0$	$\therefore$ *The appropriate root is* -1.7412

Ex: Find the root of the equation $x^3 - x - 1 = 0$, correct to two decimal places.

Solution:
Let $f(x) = x^3 - x - 1$
$f(1) = 1 - 1 - 1 = -1 < 0$
$f(2) = 8 - 2 - 1 = 5 > 0$

So the root lies between 1 and 2.

a	x	b	f(a)	f(x)	f(b)
1	1.5	2	-1	0.875	5
1	1.25	1.5	-1	-0.2968	0.875
1.25	1.375	1.5	-0.2968	0.2246	0.875
1.25	1.3125	1.375	-0.2968	-0.0515	0.2246
1.3125	1.3438	1.375	-0.0515	0.0828	0.2246
1.3125	1.3282	1.3438	-0.0515	0.0149	0.0828
1.3125	1.3204	1.3282	-0.0515	-0.0183	0.0149
1.3204	1.3243	1.3282	-0.0515	-0.0018	0.0149
1.3243	1.3262	1.3282	-0.0018	0.0063	0.0149

So up to two decimal places the root is 1.32

Ex: $f(x) = x^3 + 3x - 5$, where [a = 1, b = 2], DOA = 0.0001

a	x	b	f(a)	f(x)	f(b)
1	1.5	2	-1	2.875	9
1	1.25	1.5	-1	0.7031	2.875
1	1.125	1.25	-1	-0.2012	0.7031
1.125	1.1875	1.25	-0.2012	0.2370	0.7031
1.125	1.15625	1.1875	-0.2012	0.0145	0.2370
1.125	1.140625	1.15625	-0.2012	-0.0941	0.0145
1.140625	1.1484375	1.15625	-0.0941	-0.0400	0.0145
1.1484375	1.15234375	1.15625	-0.0400	-0.0128	0.0145
1.1484375	1.154296	1.15625	-0.0128	0.0008	0.0145

Answer is: 1.154296

Ex: Find the root $f(x)=3x+\sin(x)-e^x=0$

a	x	b	f(a)	f(x)	f(b)
1	1.5	2	0.2991	0.0444	-1.3541
1.5	1.75	2	0.0444	-0.4740	-1.3541
1.5	1.625	1.75	0.0444	-0.1750	-0.4740
1.5	1.5625	1.625	0.0444	-0.0559	-0.1750
1.5	1.53125	1.5625	0.0444	-0.0035	-0.0559
1.5	1.515625	1.53125	0.0444	0.0210	-0.0035
1.515625	1.5234375	1.53125	0.0210	0.0089	-0.0035
1.5234375	1.52734375	1.53125	0.0089	0.0027	-0.0035
1.52734375	1.529296875	1.53125	0.0027	-0.0003	-0.0035
1.52734375	1.528320313	1.529296875	0.0027	0.0012	-0.0003
1.528320313	1.528808594	1.529296875	0.0012	0.0004	-0.0003
1.528808594	1.529052734	1.529296875	0.0004	0.0000	-0.0003

The root is 1.5292966875

Ex: $X^3 - 7X^2 + 14X - 6 = 0$ on[0,1]

a	x	b	f(a)	f(x)	f(b)
0	0.5	1	-6	-0.625	2
0.5	0.75	1	-0.625	0.9843	2
0.5	0.625	0.75	-0.625	0.2597	0.9843
0.5	0.5625	0.625	-0.625	-0.1619	0.2597
0.5625	0.59375	0.625	-0.1619	0.0540	0.2597
0.5625	0.578125	0.59375	-0.1619	-0.0526	0.0540
0.578125	0.5859375	0.59375	-0.0526	0.00103	0.0540

Root is approximately: $x = 0.5859375$