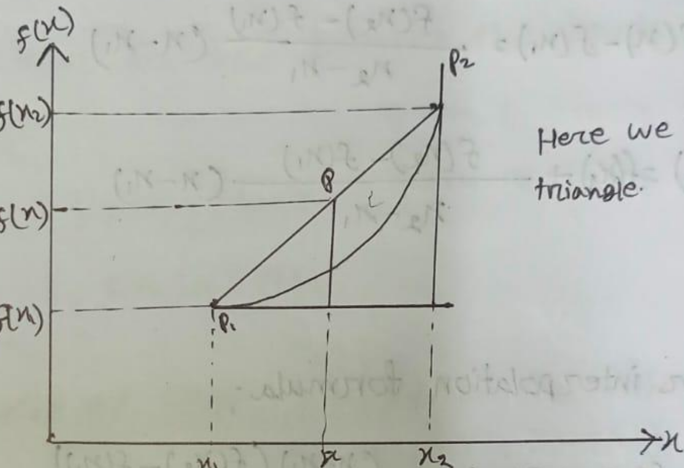
LINEAR INTERPOLATION

Hand Note

Linear Interpolation

Linear interpolation is the process of estimating the value of a function between two known values by assuming the function behaves linearly between those points.



Here we have two triangle.

Two known data points:

$(x_1, f(x_1))$ and $(x_2, f(x_2))$

We are to estimate $f(x)$ for a value x such that $x_1 < x < x_2$, assuming a straight line between the two points.

From the geometry of the straight line and concept of similar triangles:

$$\frac{f(x)-f(x_1)}{x-x_1} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$\Rightarrow f(x)-f(x_1) = \frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1)$$

$$\therefore f(x) = f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1)$$

The linear interpolation formula:

$$f(x) = f(x_1) + \frac{(x-x_1)(f(x_2)-f(x_1))}{x_2-x_1}$$

Ex:

x	1	2	2.5	3	4	5
$f(x)$	1	1.4142	1.5811	1.7321	2	2.2361

Sol:

The given value 2.5 lies between the points 2 and 3.

Therefore

$$x_1 = 2 ;\ f(x_1) = 1.4142$$

$$x_2 = 3 ;\ f(x_2) = 1.7321$$

By the help of linear interpolation formula,

$$f(2.5) = 1.4142 + \frac{(2.5-2)(1.7321-1.4142)}{3-2}$$

$$= 1.5732$$

[Error : $1.5811 - 1.5732 = 0.0079$]

Ex:

x	1	2	3	3.25	4	5	
x^2	1	4	9	10.5625	16	25	

using the linear interpolation formular estimate the square root of 3.25.

(a) using the points 3 and 4.

(b) using the points 2 and 4.

Sol:

a) Let,

$x_1 = 3, \; f(x_1) = 9$

$x_2 = 4, \; f(x_2) = 16$

Given that,

$f(x) = 10.5625$

Now apply the linear interpolation formula:

$$x = x_1 + \frac{f(x) - f(x_1)}{f(x_2) - f(x_1)} (x_2 - x_1)$$

$$\therefore x = 3 + \frac{10.5625 - 9}{16 - 9} (4-3)$$

$$= 3.2232$$

$x = 3.2232$

(b)

Let,

$x_1 = 2,\ f(x_1) = 4$

$x_2 = 4,\ f(x_2) = 16$

Given, $f(x) = 10.5625$

Apply the interpolation formula,

$$x = x_1 + \frac{f(x) - f(x_1)}{f(x_2) - f(x_1)} (x_2 - x_1)$$

$$= 2 + \frac{10.5625 - 4}{16 - 4} (4-2)$$

$$= 3.09375$$

so x is 3.09375

Ex:

X	20	23	26	29
Y	0.342	0.3907	0.4384	0.4848

Find the value of y at $x=21$ and $x=28$.

Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.342			
		0.0487		
			-0.001	
23	0.3907			-0.0003
		0.0477		
26	0.4384		-0.0013	
		0.0464		
29	0.4848			

(i) we have to find $y(21)$

Here, $u = \frac{x - x_0}{h} = \frac{21-20}{3} = 0.3333$

using newtons forward interpolation,

$$y(x) = x_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= 0.342 + 0.3333 \times 0.0487 + \frac{0.3333(0.3333-1)}{2}(-0.001) +$$

$$\frac{0.3333(0.3333-1)(0.3333-2)}{6}(-0.0003)$$

$$= 0.342 + 0.01623 + 0.00011 - 0.0000185$$

$$= 0.3583$$

(ii) we have to find $y(28)$

$$u = \frac{x - x_n}{h} = \frac{28-29}{3} = -0.3333$$

$$y(x) = x_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$= 0.4848 + (-0.3333 \times 0.0464) + \frac{-0.3333(-0.3333+1)}{2}(-0.0013) +$$

$$\frac{-0.3333(-0.3333+1)(-0.3333+2)}{6}(-0.0003)$$

$$= 0.4848 - 0.01546 + 0.00014 + 0.0000185$$

$$= 0.4695$$

Ex:

Year (x)	1891	1901	1911	1921	1931
population (y)	96	66	81	93	101

Estimate the population for the year 1895.

Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	96				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

We have to find $y(1895)$

$$u = \frac{x - x_0}{h} = \frac{\cancel{1995}\,1895 - 1891}{10}$$

$$= 0.4$$

$$f(y) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$= 46 + 0.4\times 20 + \frac{0.4(-0.6)}{2}(-5) + \frac{0.4(-0.6)(-1.6)}{6}(3) + \frac{0.4(-0.6)(-1.6)(-2.6)}{24}(-3)$$

$$= 46 + 8 - \cancel{0.12}\ 0.6 + 0.128 + 0.1248$$

$$= \cancel{54.1328}\ 54.8528$$

Ex:

X	1961	1971	1981	1991	2001	2011
Population	12	15	20	27	39	52

The increase in the population during the period from year 1996 to 1998.

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1961	12					
		3				
			2			
1971	15					
		5		0		
			2		3	
1981	20					
		7		3		-10
			5		-7	
1991	27					
		12		-4		
			1			
2001	39					
		13				
2011	52					

for, 1996

$$u = \frac{1996-2011}{10} = -1.5$$

using newton's backward interpolation formula:

$$f(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n$$

$$+ \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_n$$

$$= 52 + (-1.5)13 + \frac{-1.5(-0.5)}{2}(1) + \frac{-1.5(-0.5)(0.5)}{6}(-4) + \frac{-1.5(-0.5)(0.5)(1.5)}{24}(-7) + \frac{-1.5(-0.5)(0.5)(1.5)(2.5)}{120}(-10)$$

$$= 52 - 19.5 + 0.375 - 0.25 - 0.1640 - 0.011718$$

$$= 32.3438$$

for, 1998,

$$u = \frac{1998-2011}{10} = -1.3$$

$$f(1998) = 52 + (-1.3)\times 13 + \frac{-1.3(-0.3)}{2}(1) + \frac{-1.3(-0.3)(0.7)}{6}(-4) + \frac{-1.5\times(-0.3)(0.7)(1.7)}{24}(-7) + \frac{-1.3(-0.3)(0.7)(1.7)(2.7)}{120}(-10)$$

$$= 52 - 16.9 + 0.195 - 0.182 - 0.02231 - 0.1044225$$

$$= 34.9862$$

Inencease the population,

$= 34.9862 - 32.3438$

$= 2.6424$

Marks(x)	30-40	40-50	50-60	60-70	70-80
No of students (Y)	31	42	51	35	31

Estimate the number of students who obtained:

i) Less than 45 marks

ii) Greater than 45 marks

iii) Between 40 and 45 marks.

(i)

Marks	No. of Students	Cumulative Frequency
40	31	31
50	42	73
60	51	124
70	35	159
80	31	190

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

to get $f(45)$

$$u = \frac{45-40}{10} = 0.5$$

$$f(45) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2}9 + \frac{0.5(-0.5)(-1.5)}{6}(-25) + \frac{0.5 \times (-0.5)(-1.5)(-2.5)}{24}37$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453$$

$$= 47.86 \approx 48$$

Student get less than 45 is 48.

ii) so students gets mone than 45 is

$= 190 - 48$

$= 142$

iii) Students between 40 and 45 marks

is $= 48 - 31 = 17$