On efficiently generating the sequences of an, where, $a \in \{2, 3, \cdots, 9, 10\}$ and $n \in \mathbb{N} \cup \{0\}$, up to infinite terms without being subject to any arithmetic operations and its derivatives

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Abstract.

This brief paper shall address a general assumption and its refutation by showing a result that the exact sequences of an, where, $a \in \{2, 3, \cdots, 9, 10\}$ and $n \in \mathbb{N} \cup \{0\}$, can indeed be generated up to as many terms as one pleases without relying on any arithmetic operations and others derived from them. It is done by the formation of a special table that leverages numerical patterns.

1. Introduction

$$0, 3, 6, 9, 12, 15, 18, 21, 24, 27, \cdots$$

The sequence above is obtained by multiplying 3 with the natural numbers[1]. This process involves the arithmetic operation of multiplication. Should one be tasked to achieve this without being subject to any arithmetic operations, the solution may seem impossible. Because the sequence of $3n$ extends endlessly, beyond what can be generated by the memory of a human or a computer. Thus, it may generally be agreed upon that the sequence of $3n$, where $n \in \mathbb{N}$, cannot be obtained without the use of any arithmetic operations. This assumption is reasonable. For each input n in the function $f(n) = 3n$, the corresponding output is three times that of the input n. The function may be broken as $f(n) = n + n + n$, to convert the problem of multiplication into one of addition, indicating the method of counting by an increment[2] along the number line, yet fundamentally, the problem remains unchanged.

Therefore, the general assumption is left to be declared true that *the sequence of $3n$ cannot be generated without being subject to any arithmetic operations.*

[1] including 0

[2] also known informally as "skip counting" [2]

Should such a claim be put forward, then it is false—the paper now opposes the general view.

From this point onwards, this paper shall show the method to generate the sequence of $3n$ up to endless terms without resorting to any arithmetic operations. Moreover, it shall also show for sequences of $2n, 4n, 5n, 6n, 7n, 8n, 9n$, and $10n$, under the same conditions. By doing so, the efficiency of sequence generation may be optimized, potentially inspiring new algorithms to embrace this approach. Furthermore, it holds firm promise for educational settings. From antiquity, the decimal multiplication table[3] had a great concern [5, 8, 1]. Nowadays, it's often advised—if not forced—to memorize it. In fact, it is believed that having fluency in recalling multiplication tables proves beneficial for handling other mathematical tasks, and thus learning it is an essential part of thriving in higher education [4]. However, some do believe the opposite [7]. Nevertheless, by the approach of this paper, the pain of memorizing the multiplication tables may potentially be abandoned, while still maintaining efficient recall.

Overall, this paper invites the practice of skepticism toward commonly held beliefs and assumptions. Therefore, the explanation of the method shall commence shortly. Prior to that, it is necessary to clearly define the issue being addressed and outline a few considerations regarding it. As a result, a few definitions and considerations are outlined in the following section.

2. Definition and Considerations

Definition 1 (Arithmetic Operation). An arithmetic operation is a mathematical operation that manipulates number(s) once performed [3].

Remark 1. The following are the fundamental arithmetic operations [9]:

1. Addition
2. Subtraction
3. Multiplication
4. Division

Definition 2 (Natural numbers). The set of numbers $\{0, 1, 2, 3, 4, 5, \cdots\}$ is the set of natural numbers and is often denoted by $\mathbb{N}$ [10].

In this paper, the symbol $\mathbb{N} \cup \{0\}$ shall be used to denote the set of natural numbers to avoid any potential confusion about whether to include 0.

Remark 2. Later in this paper, the sequence of natural numbers shall be used and shall be taken from a predetermined complete set of natural numbers that is independent of any arithmetic operations involved in its generation such as beginning with 0 and incrementing by 1. Therefore, had there been any arithmetic operations used to derive the set of natural numbers itself, they would've been trivially ignored.

[3] also known informally as "times table" or sometimes as the "table of Pythagoras" [6, p. 17]

DEFINITION 3 (Counting by an increment). Let a_0 be the initial value and let d be the fixed value added to each term in the sequence. Then the sequence obtained by

$$a_n = a_0 + n \cdot d$$

is defined as counting by an increment. Where, $a_0, n \in \mathbb{N} \cup \{0\}$ and $d \in \mathbb{N}$.

In order to address the issue, it is essential to clearly define it. Therefore, the issue to be addressed is defined below by the term "Problem".

PROBLEM. Generate the sequences of an, where $a \in \{2, 3, \cdots, 9, 10\}$ and $n \in \mathbb{N} \cup \{0\}$, up to endless terms, ensuring adherence to the provided guidelines, which may include but are not limited to:

1. No arithmetic operations noted in Remark 1 can be used.
2. The terms generated must directly correspond to the original terms in the sequence supposed to be generated, one by one, in the same order. That is, strict correspondence between the terms generated and the original terms in the sequence must be met. *For example, the sequence* 0*,* 6*,* 3*,* 9*,* 12*,* 18*,* 20*,* 21*,* 15*,* 24*,* $\cdots$ *would be an invalid generation of the sequence* 0*,* 3*,* 6*,* 9*,* 12*,* 15*,* 18*,* 21*,* 24*,* $\cdots$
3. The method used must be efficient, implying that the generation of each term in the sequence must be done with no significantly more steps than the general arithmetic approach. *For example, in the general arithmetic approach, each term of the sequence is generated by a single step of multiplication. Therefore, any method requiring significantly more steps than this to generate the same term shall be treated as inefficient.*
4. The sequence must be generated without any prior influence or dependency on the sequence supposed to be generated. *For example, no method can be used similar to that of running through the sequence of natural numbers and matching the terms corresponding to the sequence supposed to be generated.*
5. The use of number incrementation other than 1 (i.e., skip counting) to generate the sequence is prohibited, as it involves the use of arithmetic operations, whether directly or indirectly.
6. Besides the arithmetic operations noted in Remark 1, the use of any other arithmetic operations that can be derived from or are similar to those four, are prohibited (such as exponents, logarithms, roots, etc.).

In general, the problem above is concerned with generating the sequences of an, where $a \in \{2, 3, \cdots, 9, 10\}$ and $n \in \mathbb{N} \cup \{0\}$, up to endless terms, without the traditional use of any arithmetic operation and others derived from it.

Having the definition and considerations set forth, the following section now delves into an explanation of the method by which the problem can be addressed.

3. The Methodology for Generating the Sequence of $3n$, $n \in \mathbb{N} \cup \{0\}$

In order to generate the sequence of $3n$, where, $n \in \mathbb{N} \cup \{0\}$, a special table is used. The process of construction is outlined below.

First, begin with the following,

0	3	6	9
2	5	8	
1	4	7	

Now, write the natural numbers starting from 0 in the following order,

$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

Then, write the natural numbers once again, starting from 0, above the horizontal line and to the left of the vertical line. Note that, this row is for counting the number of *repetition*, which is used when division is performed and is *NOT* the focus of this paper at all[4].

$\cdots\leftarrow$	$3\leftarrow$	$2\leftarrow$	$1\leftarrow$	0				
$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

Accordingly, the following table is created.

[4] as such, readers are welcome to disregard it completely.

$\cdots$	3	2	1	0				
$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

Now that the table has been created, careful observation is made. In the table, it is apparent that the number in the first column of the first row to the left of the vertical line and under the horizontal line is 0. When it is distributed to concatenate by preceding the digits in the first row to the right of the vertical line (that is, with the digits 0, 3, 6, and 9), we get 00, 03, 06, and 09, which are the first four terms of the sequence $3n$ or the terms of the sequence $3n$ for $n = 0, 1, 2, 3$, respectively, as shown below,

$\cdots$	3	2	1	0				
$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

00

$\cdots$	3	2	1	0				
$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

03

$\cdots$	3	2	1	0				
$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

06

$\cdots$	3	2	1	0				
$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

09

Next, the number in the first column of the second row to the left of the vertical line is distributed to concatenate by preceding the digits in the second row to the right of the vertical line. That is, 1 is distributed to precede the digits 2, 5, and 8 to form 12, 15, and 18 by concatenation. These are the terms of the sequence $3n$, for $n = 4, 5, 6$, respectively, as shown below,

$\cdots$	3	2	1	0				
$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

12

$\cdots$	3	2	1	0				
$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

15

$\cdots$	3	2	1	0				
$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

18

Next, the number in the first column of the third row to the left of the vertical line is distributed to concatenate by preceding the digits of the third row to the right of the vertical line. That is, 2 is distributed to precede the digits 1, 4, and 7 to form 21, 24, and 27 by concatenation. These are the terms of the sequence $3n$, for $n = 7, 8, 9$, respectively, as shown below,

$\cdots$	3	2	1	0				
$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

21

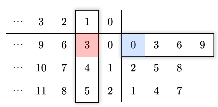
$\cdots$	3	2	1	0				
$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

24

$\cdots$	3	2	1	0				
$\cdots$	9	6	3	0	0	3	6	9
$\cdots$	10	7	4	1	2	5	8	
$\cdots$	11	8	5	2	1	4	7	

27

When all the numbers in the first column are distributed, we move to the second column to the left of the vertical line. That is, the number in the second column of the *first row* to the left of the vertical line and below the horizontal line is distributed to concatenate by preceding the digits of the *first row* to the right of the vertical line. That is, 3 is distributed to precede the digits 0, 3, 6, and 9 to form 30, 33, 36, and 39 by concatenation. These are the terms of the sequence $3n$, for $n = 10, 11, 12$, respectively, as shown below,



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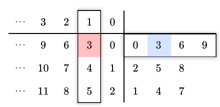

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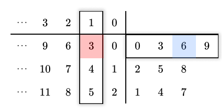

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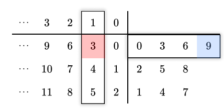

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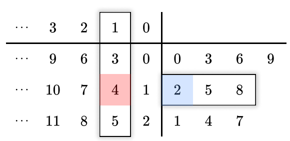
The subsequent steps follow the same process as before as shown in the following figures (recommended to be viewed as an animation).



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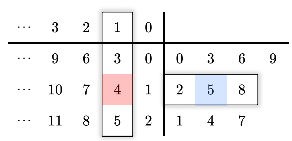

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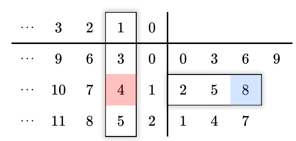

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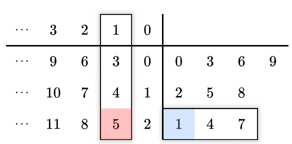

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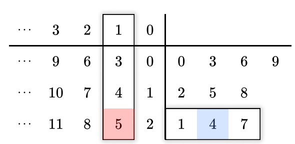

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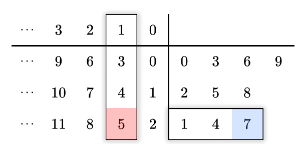

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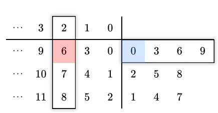

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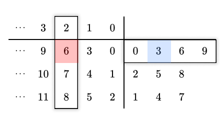

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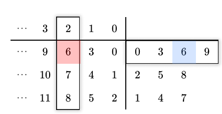

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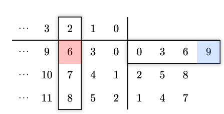

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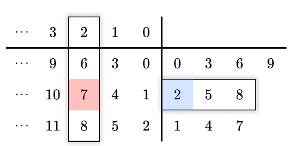

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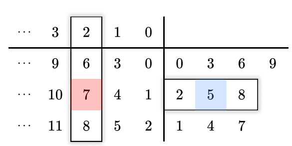

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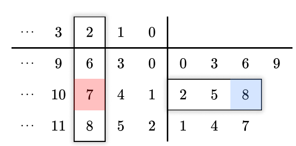

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···	3	2	1	0				
···	9	6	3	0	0	3	6	9
···	10	7	4	1	2	5	8	
···	11	8	5	2	1	4	7	

81

···	3	2	1	0				
···	9	6	3	0	0	3	6	9
···	10	7	4	1	2	5	8	
···	11	8	5	2	1	4	7	

84

···	3	2	1	0				
···	9	6	3	0	0	3	6	9
···	10	7	4	1	2	5	8	
···	11	8	5	2	1	4	7	

87

···	3	2	1	0				
···	9	6	3	0	0	3	6	9
···	10	7	4	1	2	5	8	
···	11	8	5	2	1	4	7	

90

···	3	2	1	0				
···	9	6	3	0	0	3	6	9
···	10	7	4	1	2	5	8	
···	11	8	5	2	1	4	7	

93

···	3	2	1	0				
···	9	6	3	0	0	3	6	9
···	10	7	4	1	2	5	8	
···	11	8	5	2	1	4	7	

96

···	3	2	1	0				
···	9	6	3	0	0	3	6	9
···	10	7	4	1	2	5	8	
···	11	8	5	2	1	4	7	

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Having outlined the methodology for generating the sequence of $3n$, where $n \in \mathbb{N} \cup \{0\}$, the process for generating sequences of $2n, 4n, 5n, \cdots, 9n, 10n$ follows a similar pattern. As a result, the table of these sequences is presented in the following section.

4. The Table for the Sequence of $2n, 3n, \cdots, 9n, 10n,\ n \in \mathbb{N} \cup \{0\}$

4.1. The Table for $2n$, where $n \in \mathbb{N} \cup \{0\}$

The sequence of $2n$, where $n \in \mathbb{N} \cup \{0\}$, is:

$$2n = 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, \cdots$$

The sequence above can be generated by the following table,

···	3	2	1	0					
···	3	2	1	0	0	2	4	6	8

4.2. The Table for $3n$, where $n \in \mathbb{N} \cup \{0\}$

The sequence of $3n$, where $n \in \mathbb{N} \cup \{0\}$, is:

$$3n = 0, 3, 6, 9, 12, 15, 18, 21, 24, 27, \cdots$$

The sequence above can be generated by the following table,

···	3	2	1	0				
···	9	6	3	0	0	3	6	9
···	10	7	4	1	2	5	8	
···	11	8	5	2	1	4	7	

4.3. The Table for $4n$, where $n \in \mathbb{N} \cup \{0\}$

The sequence of $4n$, where $n \in \mathbb{N} \cup \{0\}$, is:

$$4n = 0, 4, 8, 12, 16, 20, 24, 28, 32, 36, \cdots$$

The sequence above can be generated by the following table,

$$\begin{array}{ccccc|ccc}
\cdots & 3 & 2 & 1 & 0 & & & \\
\hline
\cdots & 6 & 4 & 2 & 0 & 0 & 4 & 8 \\
\cdots & 7 & 5 & 3 & 1 & 2 & 6 &
\end{array}$$

4.4. The Table for $5n$, where $n \in \mathbb{N} \cup \{0\}$

The sequence of $5n$, where $n \in \mathbb{N} \cup \{0\}$, is:

$$5n = 0, 5, 10, 15, 20, 25, 30, 35, 40, 45, \cdots$$

The sequence above can be generated by the following table,

$$\begin{array}{ccccc|cc}
\cdots & 3 & 2 & 1 & 0 & & \\
\hline
\cdots & 3 & 2 & 1 & 0 & 0 & 5
\end{array}$$

4.5. The Table for $6n$, where $n \in \mathbb{N} \cup \{0\}$

The sequence of $6n$, where $n \in \mathbb{N} \cup \{0\}$, is:

$$6n = 0, 6, 12, 18, 24, 30, 36, 42, 48, 54, \cdots$$

The sequence above can be generated by the following table,

$$\begin{array}{ccccc|cc}
\cdots & 3 & 2 & 1 & 0 & & \\
\hline
\cdots & 9 & 6 & 3 & 0 & 0 & 6 \\
\cdots & 10 & 7 & 4 & 1 & 2 & 8 \\
\cdots & 11 & 8 & 5 & 2 & 4 &
\end{array}$$

4.6. The Table for $7n$, where $n \in \mathbb{N} \cup \{0\}$

The sequence of $7n$, where $n \in \mathbb{N} \cup \{0\}$, is:

$$7n = 0, 7, 14, 21, 28, 35, 42, 49, 56, 63, \cdots$$

The sequence above can be generated by the following table,

⋯	3	2	1	0		
⋯	21	14	7	0	0	7
⋯	22	15	8	1	4	
⋯	23	16	9	2	1	8
⋯	24	17	10	3	5	
⋯	25	18	11	4	2	9
⋯	26	19	12	5	6	
⋯	27	20	13	6	3	

4.7. The Table for $8n$, where $n \in \mathbb{N} \cup \{0\}$

The sequence of $8n$, where $n \in \mathbb{N} \cup \{0\}$, is:

$$8n = 0, 8, 16, 24, 32, 40, 48, 56, 64, 72, \cdots$$

The sequence above can be generated by the following table,

⋯	3	2	1	0		
⋯	12	8	4	0	0	8
⋯	13	9	5	1	6	
⋯	14	10	6	2	4	
⋯	15	11	7	3	2	

4.8. The Table for $9n$, where $n \in \mathbb{N} \cup \{0\}$

The sequence of $9n$, where $n \in \mathbb{N} \cup \{0\}$, is:

$$9n = 0, 9, 18, 27, 36, 45, 54, 63, 72, 81, \cdots$$

The sequence above can be generated by the following table,

⋯	3	2	1	0		
⋯	27	18	9	0	0	9
⋯	28	19	10	1	8	
⋯	29	20	11	2	7	
⋯	30	21	12	3	6	
⋯	31	22	13	4	5	
⋯	32	23	14	5	4	
⋯	33	24	15	6	3	
⋯	34	25	16	7	2	
⋯	35	26	17	8	1	

4.9. The Table for $10n$, where $n \in \mathbb{N} \cup \{0\}$

The sequence of $10n$, where $n \in \mathbb{N} \cup \{0\}$, is:

$$10n = 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, \cdots$$

The sequence above can be generated by the following table,

$$\begin{array}{ccccc|c} \cdots & 3 & 2 & 1 & 0 & \\ \hline \cdots & 3 & 2 & 1 & 0 & 0 \end{array}$$

5. Conclusion

In essence, this paper discusses a method for generating the sequence of $2n$, $3n$, $4n$, $5n$, $6n$, $7n$, $8n$, $9n$, and $10n$[5] up to any desired number of terms. The method operates without relying on arithmetic or related operations, distinguishing its uniqueness. This is significant as it introduces an approach to a problem previously viewed as impossible. Moreover, it frees the practitioners with a tool that helps abandon what is often memorized blindly. Nevertheless, leveraging numerical patterns, this method showcases its efficacy in resolving the issue to be resolved.

[5] although, sequences exceeding $10n$ can be generated if number incrementation other than 1 is allowed. However, since this paper focuses only on incrementation of 1, it therefore restricts itself to sequences of up to $10n$.

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