A pr∞f of the infinitude of prime numbers

by Md. Shouvik Iqbal

Abstract.

This paper presents a new proof of the infinite nature of prime numbers. Grounded in fundamental Set theory and Analysis, the proof uses the method of contradiction to demonstrate the impossibility of the set of prime numbers being finite.

1 Introduction

"*Consider a scenario with five kittens. Is it possible to group them in a way that each group has the same number of kittens*[1] *other than one group with 5 kittens or 5 groups with one kitten?"*—the author.

Among the natural numbers, there exists a prime category of numbers that are so prime that they may best be described as "prime numbers". The defining property of such numbers is that they cannot be broken down into smaller whole parts other than 1 and themselves (as hinted in the question above). It was Euclid[2], who appears to have been provided the earliest *documented* definition of prime numbers in his work as a collection of books called "*Elements*", dating back to circa 300 BCE. In Book IX of *Elements*, Proposition 20, Euclid proved that there are an infinite number of such prime numbers[3] [2, p.210], [3, p.412], [4]. It was quite the proof that stands for such a high consequence that it is still valued in modernity (see [5, p.92]). From that point onward, the nonexistence of the largest prime number became evident, as evidenced by Euclid's proof. Post-Euclid, many mathematicians have supported this proposition through their own respective proofs. Several such proofs can be found in [6, ch.XVIII], [7, v.04], [8, ch.01], [9, ch.01]. In this paper, we shall also give another proof of this proposition. For that, the motivation is visible when the following is assumed,

$$\mathbb{P} = \{\text{the set of } \textit{all} \text{ prime numbers}\}$$

[1] without sacrificing any of them to the real number system

[2] an ancient Greek mathematician, who lived in Alexandria, Egypt, circa 300 BCE.

[3] it was done by showing that prime numbers are more than any assigned multitude of prime numbers. Note that, the proof was specifically for only three prime numbers. Euclid did not consider any arbitrary finite set of prime numbers, as is commonly done nowadays [1, p.01].

$$\mathbb{C} = \{\text{the set of } \textit{all} \text{ composite numbers}\}$$

This implies, $\mathbb{N}^* = \mathbb{P} \cup \mathbb{C}$, where $\mathbb{N}^* = \{n \in \mathbb{N} \mid n \geq 2\}$. However, if $|\mathbb{P}|$ is assumed finite, then an interesting point may have been ignored by the masses that $\mathbb{N}^* = \mathbb{P} \cup \mathbb{C} \cup \mathbb{S}$, where $\mathbb{S}$ is the set of all natural numbers that are neither prime nor composite, which is contradictorily impossible as every element in that set would be greater than 1. Thus, by contradiction, it is possible to prove that the assumption of a finite set containing all prime numbers is inherently false.

If the core idea of the motivation were to be explicitly stated without any formalism, it would assume that there exists a finite number of prime numbers—say only $2, 3, 5$ and 7, then consider the set of all natural numbers greater than or equal to 2 as follows,

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, \cdots$$

Given the assumption that $2, 3, 5$, and 7 are *only* prime numbers, they are thereupon removed from the set as follows,

$$4, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, \cdots$$

Now, given that prime numbers are those numbers greater than 1 that are divisible only by 1 and themselves, it follows that every composite number is a multiple of a prime number. Therefore, every multiple of $2, 3, 5$, and 7 are removed from the set as follows,

$$11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, \cdots$$

Now, this is a set of numbers that are neither prime (as the only prime numbers are assumed to be 2, 3, 5, and 7) nor composite (as they are not multiples of any assumed prime number), and all of them are greater than 1. This is a contradiction!

Having outlined the basic idea in this section, we shall give the complete proof in the next section. Following that, the paper concludes by discussing the formalized proof through a series of remarks in Section 3.

2 THE THEOREM AND ITS PROOF

THEOREM 2.1. *There are infinitely many prime numbers.*

Proof. Assume—for the sake of contradiction—that the elements in the set of all prime numbers are finite, and thus define the following sets,

$$\mathbb{N}^* = \{N \in \mathbb{N} \mid N \geq 2\}$$

$$\mathbb{P} = \{p_1, p_2, p_3, \cdots, p_k\}$$

$$\mathbb{N}^* \setminus \mathbb{P} = \{N \in \mathbb{N}^* \mid N \notin \mathbb{P}\}$$

$$\mathbb{C} = \{N \in (\mathbb{N}^* \setminus \mathbb{P}) \mid N = np, \text{ where, } p \in \mathbb{P}, n \in \mathbb{N}^*\}$$

Here, $\mathbb{N}^* \setminus \mathbb{P} \subset \mathbb{N}^*$ and $\mathbb{C} \subseteq \mathbb{N}^* \setminus \mathbb{P}$. To show that $\mathbb{N}^* \setminus \mathbb{P} \neq \mathbb{C}$, notice that $\mathbb{C}$ is the set of all natural numbers of the form np, where $n \in \mathbb{N}^*$ and $p \in \{p_1, p_2, p_3, \cdots, p_k\}$; whereas $\mathbb{N}^* \setminus \mathbb{P}$ includes all natural numbers of the form nk, where $n, k \in \mathbb{N}^*$. It will now be shown that for any sufficiently large range $\varepsilon \in \mathbb{N}^*$, the total amount of natural numbers of the form $nk \leq \varepsilon$ is greater than the total amount of natural numbers of the form $np \leq \varepsilon$. By doing so, it would be concludable that $|\mathbb{N}^* \setminus \mathbb{P}| \neq |\mathbb{C}|$, for any sufficiently large range $\varepsilon \in \mathbb{N}^*$ and thus the set $\mathbb{N}^* \setminus \mathbb{P}$ always contains elements that are not in $\mathbb{C}$, up to that range $\varepsilon \in \mathbb{N}^*$. Which would then be used to form a contradiction.

Now, since $\mathbb{C}$ is the set of all natural numbers of the form np, where $n \in \mathbb{N}^*$ and $p \in \{p_1, p_2, p_3, \cdots, p_k\}$. Therefore, for the ith prime $p_i \in \{p_1, p_2, p_3, \cdots, p_k\}$, the natural numbers of the form np_i, where $n \in \mathbb{N}^*$, are as follows,

$$2p_i, 3p_i, 4p_i, 5p_i \cdots$$

Now, the number of natural numbers of the form np_i, where $n \in \mathbb{N}^*$, less than or equal to a large range $\varepsilon \in \mathbb{N}^*$ yields,

$$2p_i, 3p_i, 4p_i, \cdots, c_i p_i$$

where, $c_i p_i \leq \varepsilon$ or $c_i \leq \dfrac{\varepsilon}{p_i}$. Since $n \geq 2$ in the expression np_i, therefore $(c_i - 1)$ in the expression $c_i p_i$ represents a count of natural numbers of the form np_i less than or equal to the given range ε. Now, to count the number of natural numbers of the general form of np less than or equal to the given range ε, the following is done.

$$\begin{array}{llll}
\text{for } p_1: & 2p_1, 3p_1, 4p_1, \cdots, c_1p_1 \leq \varepsilon \implies & c_1 \leq \dfrac{\varepsilon}{p_1} & \therefore (c_1 - 1) < c_1 \leq \dfrac{\varepsilon}{p_1} \\
\text{for } p_2: & 2p_2, 3p_2, 4p_2, \cdots, c_2p_2 \leq \varepsilon \implies & c_2 \leq \dfrac{\varepsilon}{p_2} & \therefore (c_2 - 1) < c_2 \leq \dfrac{\varepsilon}{p_2} \\
\text{for } p_3: & 2p_3, 3p_3, 4p_3, \cdots, c_3p_3 \leq \varepsilon \implies & c_3 \leq \dfrac{\varepsilon}{p_3} & \therefore (c_3 - 1) < c_3 \leq \dfrac{\varepsilon}{p_3}
\end{array}$$

$$\begin{array}{llll} \vdots & & \vdots & \\ \text{for } p_n: & 2p_n, 3p_n, 4p_n, \cdots, c_n p_n \leq \varepsilon \implies & c_n \leq \dfrac{\varepsilon}{p_n} & \therefore (c_n - 1) < c_n \leq \dfrac{\varepsilon}{p_n} \end{array}$$

Here, p_n is such a prime number for which $np_n \leq \varepsilon$ and $np_{n+1} \nleq \varepsilon$, for all $n \in \mathbb{N}^*$. If ε is large enough, then $np_n = np_k \leq \varepsilon$, where $p_k = \max(\mathbb{P})$. This implies that the amount of natural numbers of the form np_1, np_2, np_3, $\cdots$, np_n less than or equal to ε are $(c_1 - 1), (c_2 - 1), (c_3 - 1), \cdots, (c_n - 1)$, respectively. Now, summing these quantities results in the total amount of natural numbers of the general form np, less than or equal to ε. Let such a total amount be α. Now, note that c_i in the expression $c_i p_i$ for all $i \in \{1, 2, 3, \cdots, k\}$ is greater than 1, as each n in the expression np_i is an element of $\mathbb{N}^*$ (i.e., greater than 1). As a result, $\alpha = (c_1 - 1) + (c_2 - 1) + (c_3 - 1) + \cdots + (c_n - 1) > 1$, as shown below.

$$\begin{aligned} 1 &< (c_1 - 1) + (c_2 - 1) + (c_3 - 1) + \cdots + (c_n - 1) = \alpha \\ &< c_1 + c_2 + c_3 + \cdots + c_n \\ &\leq \frac{\varepsilon}{p_1} + \frac{\varepsilon}{p_2} + \frac{\varepsilon}{p_3} + \cdots + \frac{\varepsilon}{p_k} \\ &= \varepsilon \left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \cdots + \frac{1}{p_n} \right) \\ \therefore 1 &< \alpha \leq \varepsilon \left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \cdots + \frac{1}{p_n} \right) \end{aligned} \tag{1}$$

where, α represents the total amount of natural numbers of the form np, where $n \in \mathbb{N}^*$ and $p \in \{p_1, p_2, p_3, \cdots, p_k\}$, less than or equal to $\varepsilon \in \mathbb{N}^*$.

Now, on the other hand, since the set $\mathbb{N}^* \setminus \mathbb{P}$ contains all natural numbers of the form nk, where $n, k \in \mathbb{N}^*$. Therefore, for the ith natural number $k_i \in \mathbb{N}^*$, the natural numbers of the form nk_i, where $n \in \mathbb{N}^*$, are as follows,

$$2k_i, 3k_i, 4k_i, 5k_i \cdots$$

Now, the number of natural numbers of the form nk_i less than or equal to a large range $\varepsilon \in \mathbb{N}^*$ yields,

$$2k_i, 3k_i, 4k_i, \cdots, c_i k_i$$

where, $c_i k_i \leq \varepsilon$ or $c_i \leq \dfrac{\varepsilon}{k_i}$. Since $n \geq 2$ in the expression nk_i, therefore $(c_i - 1)$ in the expression $c_i k_i$ represents a count of natural numbers of the form nk_i less than or equal to the given range ε. Now, to count the amount of natural numbers of the general form nk less than or equal to the given range ε, the following is done.

$$\begin{array}{llll} \text{for } k_1: & 2k_1, 3k_1, 4k_1, \cdots, c_1 k_1 \leq \varepsilon \implies & c_1 \leq \dfrac{\varepsilon}{k_1} & \therefore (c_1 - 1) < c_1 \leq \dfrac{\varepsilon}{k_1} \\ \text{for } k_2: & 2k_2, 3k_2, 4k_2, \cdots, c_2 k_2 \leq \varepsilon \implies & c_2 \leq \dfrac{\varepsilon}{k_2} & \therefore (c_2 - 1) < c_2 \leq \dfrac{\varepsilon}{k_2} \end{array}$$

$$\text{for } k_3: \quad 2k_3, 3k_3, 4k_3, \cdots, c_3k_3 \leq \varepsilon \implies \quad c_3 \leq \frac{\varepsilon}{k_3} \quad \therefore (c_3-1) < c_3 \leq \frac{\varepsilon}{k_3}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\text{for } k_n: \quad 2k_n, 3k_n, 4k_n, \cdots, c_nk_n \leq \varepsilon \implies \quad c_n \leq \frac{\varepsilon}{k_n} \quad \therefore (c_n-1) < c_n \leq \frac{\varepsilon}{k_n}$$

Here, k_n is such a natural number for which $nk_n \leq \varepsilon$ and $nk_{n+1} \not\leq \varepsilon$, for all $n \in \mathbb{N}^*$. This implies that the amount of natural numbers of the form $nk_1, nk_2, nk_3, \cdots, nk_n$ less than or equal to ε are $(c_1-1), (c_2-1), (c_3-1), \cdots, (c_n-1)$, respectively. Now, summing these quantities results in the total amount of natural numbers of the general form nk, less than or equal to ε. Let such a total quantity be β. Now, note that c_i in the expression c_ik_i for all $i \in \{1, 2, 3, \cdots, n\}$ is greater than 1, as each n in the expression nk is an element of $\mathbb{N}^*$ (i.e., greater than 1). As a result, $\beta = (c_1-1)+(c_2-1)+(c_3-1)+\cdots+(c_n-1) > 1$, as shown below.

$$\begin{aligned}
1 &< (c_1-1)+(c_2-1)+(c_3-1)+\cdots+(c_n-1) = \beta \\
&< c_1+c_2+c_3+\cdots+c_n \\
&\leq \frac{\varepsilon}{k_1}+\frac{\varepsilon}{k_2}+\frac{\varepsilon}{k_3}+\cdots+\frac{\varepsilon}{k_n} \\
&= \varepsilon\left(\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}+\cdots+\frac{1}{k_n}\right) \\
\therefore 1 &< \beta \leq \varepsilon\left(\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}+\cdots+\frac{1}{k_n}\right) \qquad (2)
\end{aligned}$$

where, β represents the total amount of natural numbers of the form nk, where $n, k \in \mathbb{N}^*$, less than or equal to $\varepsilon \in \mathbb{N}^*$

Now, comparing equation (1) and (2), we get,

$$\frac{1}{1} < \frac{\beta}{\alpha} \leq \frac{\varepsilon\left(\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}+\cdots+\frac{1}{k_n}\right)}{\varepsilon\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\cdots+\frac{1}{p_n}\right)}$$

$$1 < \frac{\beta}{\alpha} \leq \frac{\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}+\cdots+\frac{1}{k_n}}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\cdots+\frac{1}{p_n}}$$

Since, in the expression nk, $k \in \mathbb{N}^*$, therefore $\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}+\cdots+\frac{1}{k_n}$ represents $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}$. Which is the Harmonic number H_n minus 1. Therefore, $\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}+\cdots+\frac{1}{k_n} = \sum_{k=1}^{n}\frac{1}{k} - 1$. Similarly, $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\cdots+\frac{1}{p_n}$

represents the sum of the reciprocals of the prime numbers. Therefore, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \cdots + \frac{1}{p_n} = \sum_{i=1}^{n} \frac{1}{p_i}$. As a result, we have the following.

$$1 < \frac{\beta}{\alpha} \leq \frac{\sum_{k=1}^{n} \frac{1}{k} - 1}{\sum_{i=1}^{n} \frac{1}{p_i}}$$

Now, since $\sum_{k=1}^{n} \frac{1}{k} - 1 \sim \ln(n)$ and $\sum_{i=1}^{n} \frac{1}{p_i} \sim \ln(\ln(n))$ [10] for sufficiently large values of the terms n obtained by choosing a sufficiently large range ε, and $\ln(n) > \ln(\ln(n))$; therefore, $\sum_{k=1}^{n} \frac{1}{k} - 1 > \sum_{i=1}^{n} \frac{1}{p_i}$ for sufficiently large values of n. As a result, $\frac{\sum_{k=1}^{n} \frac{1}{k} - 1}{\sum_{i=1}^{n} \frac{1}{p_i}} = L > 1$, for sufficiently large n. This implies that

$$\begin{aligned} 1 < \frac{\beta}{\alpha} &\leq L \\ \alpha < \beta &\leq \alpha L \\ \therefore \alpha &< \beta \end{aligned}$$

This implies that for any sufficiently large range $\varepsilon \in \mathbb{N}^*$, the total amount of natural numbers of the form $np \leq \varepsilon$ is always less than that of the form $nk \leq \varepsilon$. In other words, the number of elements in $\mathbb{N}^* \setminus \mathbb{P}$ is always greater than the number of elements in $\mathbb{C}$, up to any sufficiently large range $\varepsilon \in \mathbb{N}^*$. As a result, for any sufficiently large range ε, since $\alpha < \beta$ for ε, therefore $\exists \delta = nk \in \mathbb{N}^* \setminus \mathbb{P}$ such that $\delta \notin \mathbb{C}$. Since $\delta \notin \mathbb{C}$, therefore, $\delta \neq np$, $\forall p \in \mathbb{P}$, $\forall n \in \mathbb{N}^*$, where $p, n < \delta$. However, if $\delta \neq np$, then, $\forall p \in \mathbb{P}$, $\frac{\delta}{p} \neq n$, where $n \in \mathbb{N}^*$. This implies that $\forall p \in \mathbb{P}$, $p \nmid \delta$, therefore, by the definition of a prime number, $\delta \in \mathbb{P}$. However, this contradicts the premise that $\delta \in \mathbb{N}^* \setminus \mathbb{P}$.

Should δ be in $\mathbb{C}$ for even large range $\varepsilon_1 > \varepsilon$, then, since $\alpha < \beta$ also for ε_1, therefore $\exists \delta_1 = nk \in \mathbb{N}^* \setminus \mathbb{P}$ such that $\delta_1 \notin \mathbb{C}$ and the rest follows a similar contradiction! Thus, the argument can be repeated endlessly with every new chosen large range $\varepsilon \in \mathbb{N}^*$, proving the infinitude of the prime numbers each time by contradiction. This completes the proof. ■

3 Discussion and remarks

This section is dedicated to gaining a comprehensive understanding of the proof of Theorem 2.1.

The proof of Theorem 2.1, makes use of 'proof by contradiction' method. Rooted in the principles of reductio ad absurdum, this method of mathematical proof involves assuming the negation of the proposition and then deriving a logical contradiction from that assumption. By showing that the assumption of the negation leads to a contradiction, one concludes that the original proposition must be true and cannot be denied otherwise (see [11, ch.06]). In this case,

Proposition: *There exists infinitely many prime numbers.*

A few examples of the proof using the method of contradiction are provided below as remarks, where,

$$\mathbb{N}^* = \{N \in \mathbb{N} \mid N \geq 2\}$$

$$\mathbb{P} = \{\text{the set of } \textit{all} \text{ prime numbers}\}$$

$$\mathbb{N}^* \setminus \mathbb{P} = \{N \in \mathbb{N}^* \mid N \notin \mathbb{P}\}$$

$$\mathbb{C} = \{N \in (\mathbb{N}^* \setminus \mathbb{P}) \mid N = np, \text{ where, } p \in \mathbb{P}, n \in \mathbb{N}^*\}$$

are considered.

Remark 01. (Numerical example of the proof): Assume, on the contrary, that the proposition is false and the only prime numbers are 2, 3, and 5. Then,

$$\mathbb{N}^* = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, \cdots\}$$

$$\mathbb{P} = \{2, 3, 5\}$$

$$\mathbb{N}^* \setminus \mathbb{P} = \{4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, \cdots\}$$

$$\mathbb{C} = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, \cdots\}$$

Now, since $\mathbb{C} \subset \mathbb{N}^* \setminus \mathbb{P}$, therefore, there exists an element $7 \in \mathbb{N}^* \setminus \mathbb{P}$ such that $7 \notin \mathbb{C}$. Since $7 \notin \mathbb{C}$, therefore, $7 \neq np$, $\forall p \in \{2, 3, 5\}$, $\forall n \in \mathbb{N}^*$, where $n, p < 7$. This implies that $\forall p \in \{2, 3, 5\}$, $p \nmid 7$, and thus, by definition, $7 \in \mathbb{P}$. However, this contradicts the premise that $7 \in \mathbb{N}^* \setminus \mathbb{P}$, and the assumption that the only prime numbers are 2, 3, and 5.

Remark 02. (Randomized example of the proof): Assume, on the contrary, that the proposition is false and the only prime numbers are 11, 13, 17, 5623, 7901, and 53982894593057. Then,

$$\mathbb{N}^* = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, \cdots\}$$

$$\mathbb{P} = \{11, 13, 17, 5623, 7901, 53982894593057\}$$

$$\mathbb{N}^* \setminus \mathbb{P} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 19, 20, 21, \cdots\}$$

$$\mathbb{C} = \{22, 26, 33, 34, 39, 44, 51, 52, 55, 65, 66, 68, 77, 78, 85, \cdots\}$$

Now, since $\mathbb{C} \subset \mathbb{N}^* \setminus \mathbb{P}$, therefore, there exists an element $4 \in \mathbb{N}^* \setminus \mathbb{P}$ such that $4 \notin \mathbb{C}$. Since $4 \notin \mathbb{C}$, therefore, $4 \neq np$, $\forall p \in \mathbb{P}$, $\forall n \in \mathbb{N}^*$, where $n, p < 4$. This implies that $\forall p \in \mathbb{P}$, $p \nmid 4$, and thus, by definition, $4 \in \mathbb{P}$. However, this contradicts the premise that $4 \in \mathbb{N}^* \setminus \mathbb{P}$, and the assumption that the only prime numbers are 11, 13, 17, 5623, 7901, and 53982894593057.

Remark 03. (Generalized example of the proof): Assume, on the contrary, that the proposition is false and the only prime numbers are $2, 3, 5, 7, \cdots, n$; where $n \in \mathbb{N}^*$. Then,

$$\mathbb{N}^* = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, \cdots\}$$

$$\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 27, \cdots, n\}$$

$$\mathbb{N}^* \setminus \mathbb{P} = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, \cdots, n+1, n+2, n+3, \cdots\}$$

$$\mathbb{C} = \{N \in (\mathbb{N}^* \setminus \mathbb{P}) \mid N = bp, \text{ where, } p \in \mathbb{P}, b \in \mathbb{N}^*\}$$

Now, since $\mathbb{C} \subset \mathbb{N}^* \setminus \mathbb{P}$, therefore, there exists an element $n + k \in \mathbb{N}^* \setminus \mathbb{P}$ such that $n+k \notin \mathbb{C}$, where $n, k \in \mathbb{N}^*$. Since $n+k \notin \mathbb{C}$, therefore, $n+k \neq bp$, $\forall b \in \mathbb{N}^*$, $\forall p \in \mathbb{P}$, $b, p < n + k$, where $n, k \in \mathbb{N}^*$. This implies that $\forall p \in \mathbb{P}$, $p \nmid n + k$, and thus, by definition, $n + k \in \mathbb{P}$, where $n, k \in \mathbb{N}^*$. However, this contradicts the premise that $n + k \in \mathbb{N}^* \setminus \mathbb{P}$, where $n, k \in \mathbb{N}^*$, and the assumption that the only prime numbers are $2, 3, 5, 7, \cdots, n$, where $n \in \mathbb{N}$.

References.

[1] Narkiewicz W. The Development of Prime Number Theory: From Euclid to Hardy and Littlewood. Springer Monographs in Mathematics. Springer Berlin Heidelberg; 2000.

[2] Fitzpatrick R, Heiberg JL. Euclid's Elements. Richard Fitzpatrick; 2007.

[3] Heath TL, Heiberg JL. The Thirteen Books of Euclid's Elements. No. v. 2 in The Thirteen Books of Euclid's Elements. The University Press; 1908.

[4] Joyce DE. Euclid's Elements, Book IX, Proposition 20;. Available from: http://aleph0.clarku.edu/~djoyce/java/elements/bookIX/propIX20.html. [Online; accessed 19-Feb-2024].

[5] Hardy GH, Snow CP. A Mathematician's Apology. Canto Classics. Cambridge University Press; 2012.

[6] Dickson LE. History of the Theory of Numbers: Divisibility and Primality. v. 1. Dover Publications; 2012.

[7] Meštrović R. Euclid's theorem on the infinitude of primes: a historical survey of its proofs (300 B.C.–2022) and another new proof. 2023. Available from: https://arxiv.org/abs/1202.3670.

[8] Aigner M, Hofmann KH, Ziegler GM. Proofs from THE BOOK. Springer Berlin Heidelberg; 2018.

[9] Ribenboim P. The Little Book of Bigger Primes. Springer-Verlag; 1991.

[10] Gasarch W, Washington L. $\sum_{p\leq n} 1/p = \ln(\ln n) + O(1)$: An Exposition. 2015. Available from: https://arxiv.org/abs/1511.01823.

[11] Hammack RH. Book of Proof. Richard Hammack; 2019. Available from: https://www.people.vcu.edu/~rhammack/BookOfProof.

Md. Shouvik Iqbal has recently passed Higher Secondary Certificate (HSC) and is presently preparing for undergraduate studies.