Date 10/8/2022

Properties of eigen values and eigenvectors

Imp

1. Eigen values of lower triangular matrix ,upper triangular matrix and diagonal matrix are just the diagonal elements only.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

eigenvalues are 1, 5, 7

2. If $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ are Eigen values of matrix A of order n, then

   (a) $\lambda_1 + \lambda_2 + \ldots + \lambda_n =$ trace of A ie, sum of eigenvalues $= Tr(A)$

   (b) $\lambda_1 \cdot \lambda_2 \ldots\ldots \cdot \lambda_n = |A|$ product of eigenvalues $= \det A$

3. 0 is an Eigen value of matrix A if and only if A is singular.

4. If all the Eigen values of A are non-zero then A is non-singular

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eigenvalues of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ are

(a) 0, 0, 3 ✓

(b) 1, 1, 1 ✗

(c) 1, 2, 1 ✗

(d) 1, -1, 4 ✗

10 marks

Detailed method

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

4 marks

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

trace = 3 ✓

det = 0 ✓

1 mark

trace ✓

det ✓

Properties of eigen values and eigenvectors

5. If λ is an eigen value of a matrix A and k is a scalar then

(a) λ^m is eigen value of matrix A^m $(\because m \in N)$

Note: Eigen vectors remains same only

(b). $k\lambda$ is an eigen value of matrix kA.

(c). $\lambda+k$ is an eigen value of matrix $A+kI$

(d). $\lambda-k$ is an eigen value of matrix $A-kI$

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(e) $a_0+a_1\lambda+a_2\lambda^2+\ldots\ldots+a_n\lambda^n$ is an eigen value of matrix

$a_0I+a_1A+a_2A^2+\ldots\ldots\ldots+a_nA^n$ ← polynomial for A

Properties of eigen values and eigenvectors

Imp

6. If λ is an Eigen value of a non-singular matrix A then

(i) $\frac{1}{\lambda}$ is an Eigen value of A^{-1} and

(ii) $\frac{|A|}{\lambda}$ is an Eigen value of adj (A) (here $\lambda \neq 0$)

Imp

7. The Eigen values of A and A^T are same

.(but The Eigen vectors of A and A^T are not same)

8. Corresponding to each Eigen value of a matrix there exist infinitely many eigen vectors.

Properties of eigen values and eigenvectors

09. If $\lambda_1, \lambda_2, \ldots \lambda_n$ are distinct eigen values of a square matrix A of order 'n' then the corresponding eigen vectors $X_1, X_2, \ldots, X_n$ of matrix A are linearly independent.

10. The Eigen vectors of A and A^{-1} same only

Imp

11. Eigen vectors corresponding to distinct Eigen values of symmetric matrix are orthogonal.

$A_{3\times3}$

eigenvalues	eigenvectors
1	x_1
2	x_2
8	x_3

$\det A = 1\times2\times8$
$= 16$

$\operatorname{adj} A - \frac{16}{1}, \frac{16}{2}, \frac{16}{8}$

$A^2 - 1^2, 2^2, 8^2$ ✓

$A^3 - 1^3, 2^3, 8^3$ ✓

$KA - K, 2K, 8K$ ✓

$A \pm KI, \; 1\pm K, 2\pm K, 8\pm K$ ✓

$A^{-1} \quad \frac{1}{1}, \frac{1}{2}, \frac{1}{8}$

eigenvectors are same

$A^2 + 3A - 4I, \; (1)^2 + 3(1) - 4(1)$
$(2)^2 + 3(2) - 4(1)$
$(8)^2 + 3(8) - 4(1)$

Matrix	Eigen values
Hermitian matrix(symmetric matrix)	Always real
Skew-Hermitian matrix (Skew-Symmetric matrix)	Purely imaginary or zero
Unitary matrix(Orthogonal matrix)	having absolute value one
Idempotent matrix	Zero , one only
Involutary matrix	1 , -1 only
Nilpotent matrix	zero only

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$\rightarrow i, 1, -1, -i, 1, \frac{1}{2} + i\frac{\sqrt{3}}{2} \ldots$

$|a+ib| = \sqrt{a^2+b^2}$

$|\lambda| = 1$

$A^2 = A$

$A^2 - A = 0$

$\lambda^2 - \lambda = 0$

$\lambda = 0, 1$

5.4 Cayley-Hamilton Theorem:

Statement:

Every square matrix of order n(>1) satisfies its own characteristic equation

Example $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$

$\lambda^2 - 5\lambda + 6 = 0 \leftarrow$ zero

$A^2 - 5A + 6I = O \leftarrow$ 2×2 zero matrix

Applications of Cayley-Hamilton theorem:

The important applications of Cayley-Hamilton theorem are

(i) To find higher powers of matrix A ✓

(ii) To find the inverse of matrix A.

(without calculating adj A)

$A_{2\times 2}$ then A^k $(k \geq 2)$ can be expressed in terms of A and I

$A_{3\times 3}$ then A^k $(k \geq 3)$ can be expressed in terms of A^2, A and I.

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$$

$$\lambda^2 - 5\lambda + 6 = 0$$

By cayley-Hamilton Theorem

$$A^2 - 5A + 6I = 0$$

(1) multiply with A^{-1}

$$A^{-1}A^2 - 5A^{-1}A + 6A^{-1} = 0$$

$$\Rightarrow A - 5I + 6A^{-1} = 0$$

$$A^{-1} = \frac{1}{6}(5I - A)$$

(2) $A^2 = 5A - 6I$

$$A^3 = 5A^2 - 6A$$

$$= 5(5A - 6I) - 6A$$

$$= 19A - 30I$$

Similarly $A^4, A^5, \ldots$

Standard Models

1) finding eigen values for given matrix A
2) finding eigen vector X, for given matrix A and eigen value λ
3) finding eigen value λ, for given matrix A and eigen vector X
4) finding matrix, if eigen values and eigen vectors are given
5) problems related to Cayley-Hamilton Theorem
6) problems on A.M and G.M

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If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ then the eigen values of A are

(a) 8, 7, 3 ✗
(b) 0, 3, 15 ✓
(c) 1, 2, 3 ✗
(d) 1, –1, 2 ✗

trace $= 8+7+3 = 18$

det $= 8(5)+6(-10)+2(10)$
$= 0$

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24. An eigen vector corresponding to the smallest eigen value of the matrix

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is}$$

upper M

(a) $[1\ 0\ 0\ 0]^T$ ✗

(b) $[1\ 1\ 0\ 0]^T$ ✗

(c) $[1\ 0\ 2\ 0]^T$ ✓

(d) $[-1\ -1\ 2\ 2]^T$

eigen values are 1, 2, −1, 0

smallest eigen value is -1

Shortcut $\boxed{AX = \lambda X}$

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

✓ Detailed method

$(A - \lambda I)X = 0$

Infact corresponding to $\lambda = -1$

eigen vector $X = k\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$

An eigen vector of the matrix $\begin{bmatrix} 10 & -4 \\ 18 & -12 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ X

(b) $\begin{bmatrix} 2 \\ 9 \end{bmatrix}$ ✓

(c) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

trace $= 10-12$
$= -2$

det $= -120+72$
$= -48$

$\lambda^2 - (-2)\lambda - 48 = 0$

$\lambda^2 + 2\lambda - 48 = 0$

$(\lambda+8)(\lambda-6) = 0$

$\lambda = -8, 6$

$AX = \lambda X$

$\begin{pmatrix} 10 & -4 \\ 18 & -12 \end{pmatrix}\begin{pmatrix} 2 \\ 9 \end{pmatrix} = (-8)\begin{pmatrix} 2 \\ 9 \end{pmatrix}$

$\begin{pmatrix} -16 \\ -72 \end{pmatrix} = \begin{pmatrix} -16 \\ -72 \end{pmatrix}$

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Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. If – 3 and –3 are two eigen values of A then the eigen vector corresponding to the third eigen value is

(a) $[1 \;\; 2 \;\; 1]^T$ ✗ (b) $[1 \;\; 2 \;\; -1]^T$ ✓
(c) $[1 \;\; -2 \;\; 1]^T$ ✗ (d) $[-1 \;\; 2 \;\; 1]^T$ ✗

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$$AX = \lambda X$$

$$\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 10 \\ -5 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ -5 \end{pmatrix}$$

let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}$$

$$-3 - 3 + \lambda_3 = -2 + 1 + 0$$

$$\lambda_3 = 5$$

Problems Eigen Values and Eigen Vectors

Q. If $[2 \quad -2 \quad 1]^T$ is an eigen vector of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & p & -4 \\ 2 & -4 & 3 \end{bmatrix}, \text{ then } p = \_\_\_\_ .$$

$$AX = \lambda X$$

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & p & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = (\lambda) \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 30 \\ -16-2p \\ 15 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ -2\lambda \\ \lambda \end{pmatrix}$$

$$\Rightarrow \lambda = 15$$

$$-16-2p = -30$$

$$\Rightarrow -2p = -14$$

$$\Rightarrow p = 7$$

Problems Eigen Values and Eigen Vectors

Q. The eigen values and the corresponding eigen vectors of a 2×2 matrix are given by

[GATE – 06(EC)]

trace = sum = 8+4 = 12

det = product = 8×4 = 32

Eigen value	**Eigen vector**
$\lambda_1 = 8$	$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda 2 = 4$	$V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The matrix is

(a) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ ✓ (b) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$ ✗ (c) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ ✗ (d) $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$ ✗

Problems Eigen Values and Eigen Vectors

Q. A matrix has eigen values –1 and –2. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively. The matrix is

[GATE-13[EE]]

(a) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ ✗

Trace = sum = −1−2 = −3

det = product = −1×−2 = 2

(b) $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ ✗

let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix

(c) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$\lambda_1 = -1 \quad x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$A x_1 = \lambda_1 x_1$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} a-b \\ c-d \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$a - b = -1$

$c - d = 1$

$\lambda_2 = -2 \quad x_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$A x_2 = \lambda_2 x_2$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = (-2) \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} a-2b \\ c-2d \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

$a - 2b = -2$

$c - 2d = 4$

$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$

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5.5Diagonalization:

Diagonalizable matrix:

If for a given square matrix A of order n, there exists a non-singular matrix P such that $P^{-1}AP = D$ (or) $AP = PD$ where D is a diagonal matrix then A is said to be diagonalizable matrix.

NOTE : A matrix A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigen vectors

5.5Diagonalization:

Note:

1. If X_1, X_2, X_3 are linearly independent eigen vectors of $A_{3\times3}$ corresponding to eigen values $\lambda_1, \lambda_2, \lambda_3$ then P can be found such that $P^{-1}AP = D$ (or) $AP = PD$

$$Where\ D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$ Where $P = [X_1 \quad X_2 \quad X_3]$

$|P| \neq 0$

Note $A^2 = P D^2 P^{-1}$

$A^2 = A \cdot A = (PDP^{-1})(PDP^{-1})$
$= PDP^{-1}PDP^{-1}$
$= PD^2P^{-1}$

2. If A is diagonalizable matrix of order $n \times n$ then $A^k = P D^k P^{-1}$ or $D^k = P^{-1} A^k P$, where $k \in N$.

eigenvalues

-1

-2

eigenvectors

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$D = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$$

$$P^{-1} = \frac{1}{(-1)} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$A = P D P^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}}$$

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$P^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If 1, 2 and 3 are the eigen values of the matrix $A_{3\times3}$, then $6A^{-1} =$

(a) $A^2 + 6A - 11\,I$

(b) $A^2 - 6A - 11\,I$

(c) $A^2 - 6A + 11\,I$ ✓

(d) $A^2 + 6A + 11\,I$

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By cayley-Hamilton theorem

$A^3 - 6A^2 + 11A - 6I = 0$

multiplying with A^{-1}

$A^2 - 6A + 11I - 6A^{-1} = 0$

$\Rightarrow 6A^{-1} = A^2 - 6A + 11I$

ch equation $(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Problems Eigen Values and Eigen Vectors

Q. If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$, A^9 equals **[GATE – 07(EE)]**

(a) 511 A + 510 I

(b) 309 A + 104 I

(c) 154 A + 155 I

(d) e^{9A}

Detailed method

$\lambda^2 + 3\lambda + 2 = 0$

$\Rightarrow A^2 + 3A + 2I = 0$

$\Rightarrow A^2 = -3A - 2I$

$\Rightarrow A^3 = -3A^2 - 2A$

$= -3(-3A - 2I) - 2A$

$A^3 = 7A + 6I$

Similarly

$A^4, A^5, \ldots A^9$

Short cut method For $A_{2\times 2}$

to express $A^k = aA + bI$

Let λ_1, λ_2 be two distinct eigen values of A

solve
$$(\lambda_1)^k = a\lambda_1 + b$$
$$(\lambda_2)^k = a\lambda_2 + b$$

Note if eigen values are repeated

$$(\lambda)^k = a\lambda + b \rightarrow (1)$$

Differentiate to get second

$$k(\lambda)^{k-1} = a \rightarrow (2)$$

For above example

$$A = \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix}$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+1)(\lambda+2) = 0$$

$$\lambda = -1, -2$$

$$(-1)^9 = a(-1) + b \Rightarrow -1 = -a + b$$

$$(-2)^9 = a(-2) + b \Rightarrow -512 = -2a + b$$

$$511 = a$$

$$510 = b$$

$$\therefore A^9 = 511A + 510I$$

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Problems Eigen Values and Eigen Vectors

Q. If $A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & 2i \end{bmatrix}$ then A^4 is

(a) I

(b) 4I

(c) 16I ✓

(d) 64I

eigenvalues are $2, -2, -2i, 2i$

$\Rightarrow$ ch equation is

$$(\lambda-2)(\lambda+2)(\lambda+2i)(\lambda-2i)=0$$

$$(\lambda^2-4)(\lambda^2+4)=0$$

$$\lambda^4-16=0$$

$\Rightarrow$ By Cayley-Hamilton theorem

$$A^4-16I=0$$

$$\boxed{A^4=16I}$$

Problems Eigen Values and Eigen Vectors

Imp

Q. The matrix $A = \begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{bmatrix}$ has three distinct eigen values and one of its eigen vectors is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ Which one of the following can be another eigen vector of **(GATE-17-EE)**

symmetric

(a) $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (1\times1) + (0\times0) + (1\times-1)$

$= 0$

Problems Eigen Values and Eigen Vectors

Q. The value of x for which the matrix

$$\begin{bmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{bmatrix}$$

has zero as an eigen value is ___1___.

[GATE – 16 – EC – Set 1]

i.e., For what value of x det A is zero

clearly $x = 1 \Rightarrow |A| = 0$ ($R_1 \cong R_3$)

?

Sol: If 0 is an eigenvalue

$\Rightarrow |A| =$ product of eigenvalues

$= 0$

Problems Eigen Values and Eigen Vectors

Q.Two eigenvalues of a 3x3 real matrix P are $\left(2+\sqrt{-1}\right)$ and 3.

The determinant of P is _______.(GATE-16-CSE-SET1)

$\rightarrow 2+i$

Sol: $2+\sqrt{-1}$ and 3 are eigenvalue

$2+i,\ 2-i,\ 3$ are eigenvalues

$|P| = (2+i)(2-i)3 = 15$

Problems Eigen Values and Eigen Vectors

$1 \le G.M \le A.M$

Q. The number of linearly independent eigen vectors of matrix corresponding to eigen value 3 is

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

eigenvalues $-2, 3, 3$

(a) 0

(b) 1 ✓

(c) 2

(d) infinite

$G.M$ of $3 = n - r = 3 - r$, n size of A

here $r = \rho(A - 3I)$

$(A - 3I) = \begin{pmatrix} -5 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ✓

$r = \rho(A - 3I) = 2$

$\therefore G.M$ of $(3) = 3 - 2 = 1$

eigen values	G.M	A.M
-2	1	1
3	1	2
	2	

Problems Eigen Values and Eigen Vectors

n - size of A

Q. The number of linearly independent eigen vectors of

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

[GATE – 07(ME)]

(a) 0

(b) 1 ✓

(c) 2

(d) infinite

eigenvalues are 2, 2, 2

A.M of eigenvalue 2 is 3

consider $(A-2I) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\rho(A)=2$

$G.M\ of\ (\lambda=2) = n - r$
$= 3 - 2$
$= 1$

eigen value	G.M	A.M
2	1	3

1

i.e. $(A-2I)X=0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$y=0$

$z=0$

clearly $x=k$

eigenvector $X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$=\begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}$

$=k\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

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Problems Eigen Values and Eigen Vectors

Q. If the characteristic polynomial of 3x3 matrix M over R (the set of real numbers) is $\lambda^3-4\lambda^2+a\lambda+30$. $a \in R$, and one eigenvalue of M is 2, then the largest among the absolute values of the eigenvalues of M is__________.

(GATE-17-CSIT)

let $P(\lambda) = \lambda^3 - 4\lambda^2 + a\lambda + 30$

given that 2 is an eigenvalue

$\Rightarrow P(2) = 0$

$(2)^3 - 4(2)^2 + a(2) + 30 = 0$

$22 + 2a = 0$

$a = -11$

$P(\lambda) = \lambda^3 - 4\lambda^2 - 11\lambda + 30$

$$\begin{array}{c|cccc} & 1 & -4 & -11 & 30 \\ 2 & 0 & 2 & -4 & -30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$(\lambda-2)(\lambda^2-2\lambda-15)=0$

$$(\lambda-2)(\lambda^2-2\lambda-15)=0$$

$$(\lambda-2)(\lambda-5)(\lambda+3)=0$$

$$\lambda=2,5-3$$

largest eigen value is 5

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Problems Eigen Values and Eigen Vectors

Q. Let the eigenvalues of a 2 × 2 matrix A be 1, – 2 with eigenvectors x_1 and x_2 respectively. Then the eigenvalues and eigenvectors of the matrix $A^2 - 3A + 4I$ would respectively, be

[GATE – 16 – EC – Set 1]

(a) 2, 14; x_1, x_2

(b) 2, 14; $x_1 + x_2$: $x_1 + x_2$

(c) 2, 0; x_1, x_2

(d) 2, 0; $x_1 + x_2$, $x_1 - x_2$

eigenvalues of A

1

−2

eigenvalues of $A^2 - 3A + 4I$

$(1)^2 - 3(1) + 4(1) = 2$

$(-2)^2 - 3(-2) + 4(1) = 14$

Problems Eigen Values and Eigen Vectors

Q. Consider the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ whose eigen values are 1, –1 and 3.

Then trace of $(A^3 - 3A^2)$ is __________.

(GATE-16-IN)

$\cancel{tr(A^3 - 3A^2) = tr(A^3) - 3\,tr(A^2)}$

$tr(KA) = K\,tr(A)$

$Tr(A^K) \neq (trA)^K$

A	$A^3 - 3A^2$
1	$(1)^3 - 3(1)^2 = -2$
-1	$(-1)^3 - 3(-1)^2 = -4$
3	$(3)^3 - 3(3)^2 = 0$

$Tr(A^3 - 3A^2) = -2 - 4 + 0 = -6$

$det(A^3 - 3A^2) = -2 \times -4 \times 0 = 0$

Problems Eigen Values and Eigen Vectors

Q. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ *and* → get the eigen values of A

→ get the eigen values of B

→ find det of B

$B = A^3 - A^2 - 4A + 5I$, where I is the 3 x 3 identity matrix. The determinant of B is ______ (up to 1 decimal place).**(GATE-18-EC)**

Problems Eigen Values and Eigen Vectors

Q. If $A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}$ then the trace of A^{1000}=

(a) $2^{1000}+1$

(b) 1

(c) 2^{1000}

(d) $2^{1000}-1$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda = -1, 2$

eigenvalues of A

-1

2

A^{1000}

$(-1)^{1000} = 1$

$(2)^{1000} = 2^{1000}$

$\text{Trace}(A^{1000}) = 2^{1000} + 1$

$\det(A^{100}) = 2^{1000} \times 1$

$= 2^{1000}$

LU decomposition (Gate CS E only For ESE also)

What is LU decomposition /factorization / Triangularization ?

7. LU decomposition

LU decomposition /factorization / triangulariazation

1.The matrix A is decomposed or factorized as the product of a lower triangular matrix L and upper triangular matrix U

2.The method is to solve the system of equations or

For $A_{3\times 3}$

3..To find the inverse of matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$A = L \quad U$$

LU decomposition

Doolittle Method

$$l_{ii} = 1 \ in \ L$$

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}}_{U}$$

Crout's method

$$u_{ii} = 1 \ in \ U$$

$$A = \underbrace{\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}}_{U}$$

Cholesky method

$$A = LL^T$$

A is symmetric

$$A = \underbrace{\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}}_{L^T}$$

7.LU decomposition

Note:

1.The method fails when any of the diagonal elements l_{ii} in L and u_{ii} in U becomes zero

2. A has a unique LU decomposition $\Leftrightarrow$ all leading principal minors of A are non singular

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|a_{11}| \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

leading principal minors
(naturally ordered minors)

positive definite A symmetric matrix A is said to be positive definite if all leading principal minors are positive

(or)

all eigen values are positive

Note leading principal minors and principal minors are not same

7.LU decomposition

Result : suppose an $n \times n$ matrix A can be reduced to its row echelon form U without any row interchange , that is by using only the elementary row operations $R_i \longrightarrow R_i - m_{ij}R_j$ for j=1….n-1 and i = j+1….n .define an $n \times n$ matrix L = $\{l_{ij}\}$as follows

$R_i \rightarrow R_i - m_{ij}R_j$

$l_{ij} = m_{ij}$ *if* $i > j$.

=1 if i= j

= 0 if i < j

$R_2 \rightarrow R_2 + 3R_1$

(2,1) element of L is (-3)

Then A = LU

Method

$$A = LU$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Step1 find first row of first colomn elements of A

Step2 second Row of second colum elements of A

Step3 third Row of third columnts of A

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Problems on LU decomposition

Detailed method

Q. Consider the matrix $A = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 8 & 13 \\ 6 & 27 & 31 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 & 5 \\ 4 & 8 & 13 \\ 6 & 27 & 31 \end{pmatrix} = L \; U$

Doolittle

If $A = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$ $\begin{pmatrix} \; \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & - \\ - & - & - \end{pmatrix}$

then $l_{32} =$ ________

$$A = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 8 & 13 \\ 6 & 27 & 31 \end{pmatrix}$$

$$\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \begin{pmatrix} 2 & 1 & 5 \\ 0 & 6 & 3 \\ 0 & 24 & 16 \end{pmatrix}$$

$$R_3 \to R_3 - 4R_2 \quad \begin{pmatrix} 2 & 1 & 5 \\ 0 & 6 & 3 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 5 \\ 4 & 8 & 13 \\ 6 & 27 & 31 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 5 \\ 0 & 6 & 3 \\ 0 & 0 & 4 \end{pmatrix}$$

?

$$\boxed{R_i \to R_i - m_{ij} R_j}$$

Problems on LU decomposition

Q.The matrix $A=\begin{bmatrix}2 & 1\\4 & -1\end{bmatrix}$

is decomposed into a product of lower triangular matrix [L] and an upper triangular [U]. The properly decomposed [L] and [U] matrices respectively are

(a) $\begin{bmatrix}1 & 0\\4 & -2\end{bmatrix}$ and $\begin{bmatrix}1 & 1\\0 & -2\end{bmatrix}$, (b) $\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$ and $\begin{bmatrix}2 & 1\\0 & -3\end{bmatrix}$,

(c) $\begin{bmatrix}1 & 0\\4 & 1\end{bmatrix}$ and $\begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}$, (d) $\begin{bmatrix}2 & 0\\4 & -3\end{bmatrix}$ and $\begin{bmatrix}1 & 0.5\\0 & 1\end{bmatrix}$,

$$A=\begin{pmatrix}1 & 0\\l_{21} & 1\end{pmatrix}\begin{pmatrix}u_{11} & u_{12}\\0 & u_{22}\end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{pmatrix}2 & 1\\0 & -3\end{pmatrix}$$

$$A=\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\0 & -3\end{pmatrix}$$

Problems on LU decomposition

In the LU decomposition of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$ if the diagonal elements of U are both 1, then the lower diagonal entry l_{22} of L is [Gate 2015 CS]

Crout's method

(a) 4

(b) 5 ✓

(c) 6

(d) 7

$$\begin{pmatrix} 2 & 2 \\ 4 & 9 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} 1 & u_{12} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 4 & 9 \end{pmatrix} = \begin{bmatrix} l_{11} & l_{11} u_{12} \\ l_{21} & l_{21} u_{12} + l_{22} \end{bmatrix}$$

$l_{11} = 2$, $l_{11} u_{12} = 2 \Rightarrow u_{12} = 1$

$l_{21} = 4$ $\quad l_{21} u_{12} + l_{22} = 9$

$(4)(1) + l_{22} = 9$

$(l_{22}) = 9 - 4 = 5$

Suppose $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & -4 & 13 \\ 2 & 1 & -5 \end{bmatrix}$. If $A = L.U$ where L is lower triangular matrix with diagonal elements as unity, then U = ______ and L = ______.

(a) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 4 \\ 0 & 0 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & \frac{-3}{2} & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & \frac{-3}{2} & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & -7 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & \frac{3}{2} & 1 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & \frac{3}{2} & 1 \end{bmatrix}$

Doolittle method

$R_2 \to R_2 + 3R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & -3 & 1 \end{pmatrix}$$

$R_3 \to R_3 + \frac{3}{2} R_2$

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -\frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{pmatrix}$$

$R_i \to R_i - m_{ij} R_j$

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Break - 20 mins

Vector space , basis and dimension

(ECE Branch Gate and For ESE exam)

what is a vector space?

Vector space , basis and dimension

Let V be a non empty set of certain objects , which may be vectors, matrices, functions or some other objects. Each object is an element of V and is called vector. The elements in V are denoted by u, v *or* $v_1, v_2, \ldots..$

Let F be a field of scalars.

V is said to be vector filed over F if the following conditions are satisfied .

Vector space , basis and dimension

1.(V,+) is an abelian group

Properties with respect to vector addition

(a) $u + v \in V$ (closure under addition)

(b) $u + v = v + u$ (commutative)

(c) $(u + v) + w = u + (v + w)$ (associative)

(d) $u + 0 = 0 + u = u$ (existence of identity)

(e) $u + (-u) = 0$ (existence of inverse)

Vector space , basis and dimension

2. Properties with respect Scalar multiplication

α, β are scalars

is denoted by . For $\alpha, \beta \in F, u \in V$

(f) $\alpha . u \in V$ *such that* , $\alpha \in F$, *and* $u \in V$ (closure under scalar multiplication)

(g) $\alpha(u + v) = \alpha u + \alpha v$ (left distributive)

(h) $(\alpha + \beta)u = \alpha u + \beta u$ (right distributive)

(i) $\alpha(\beta u) = (\alpha\beta)u = \beta(\alpha u)$

(j) $1. u = u$

Properties (a) and (f)are called closure properties

when these two properties are satisfied, we say that the vector space is closed under the vector addition and scalar multiplication.

Vector space , basis and dimension

Note :If even the one of the above properties is not satisfied then V is not a vector space . We usually check the closure properties before checking other Properties

Examples :

1. the set V of real numbers or complex numbers is a vector space Under usual addition and multiplication

2. the set V of all $n\times n$ matrices under usual matrix multiplication and addition is a vector space

3. Set of all polynomials of degree $\leq$n

Vector space , basis and dimension

Linearly independent and dependent vectors

1. Two vectors x_1, and x_2 are said to be linearly dependent if $X_1 = \alpha X_2$

otherwise independent

2.Linear combination of vectors:

If $X_1, X_2, \ldots. X_r$ are 'r' vectors of order 'n' and $\alpha_1, \alpha_2, \ldots \ldots \alpha_r$ are 'r' scalars then the expression of the form $\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 \ldots\ldots + \alpha_r X_r$ is also a vector and it is called linear combination of the vectors $X_1, X_2, \ldots\ldots X_r$.

Linearly independent and dependent vectors

Linearly dependent vectors:

The vectors X_1, X_2, …. X_r of same order 'n' are said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, \ldots \ldots \alpha_r$ not all zero such that $\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 \ldots\ldots + \alpha_r X_r = O$ where O denotes the zero vector of order n.

Linearly independent vectors:

The vectors X_1, X_2, …. X_r of same order 'n' are said to be linearly independent if there exist scalars $\alpha_1, \alpha_2, \ldots \ldots \alpha_r$ such that

$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 \ldots\ldots + \alpha_r X_r = O \Rightarrow \alpha_1 = 0, \alpha_2 = 0, \ldots \ldots \alpha_r = 0$

where O denotes the zero vector of order n.

Vector space , basis and dimension

Linearly independent and dependent vectors

Define Matrix A by considering each vector as a row

$A_{m \times n}$

$m = n$	$m \neq n$

$m = n$

1. $|A| \neq 0$ (or) $e(A) = n$

Then the rows and columns of A are L.I

2. $|A| = 0$ (or) $e(A) < n$ then the rows and columns of A are L.D

$m \neq n$

The vectors are said to be L.I if no row is zero row in row echelon form

Linearly independent and dependent vectors

Dr.A

Important points

1.If X_1, X_2, …….., X_r are linearly dependent vectors then at least one of the vectors can be expressed as a linear combination of other vectors uniquely

2.Any subset of a linearly independent set is itself linearly independent set.

3. Any super set of L. D set is L.D

4. If a set of vectors includes a zero vector then the set of vectors is linearly dependent set.

5.A set of (n+1) vectors in R^n is linearly dependent

3 vectors in R^2 are L.D

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} \begin{pmatrix} -1 \\ 7 \end{pmatrix} \right\} \rightarrow \text{L-D}$$

4 vectors in R^3 are L-D

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \right\} \rightarrow \text{L-D}$$

Vector space , basis and dimension

what is a subspace of a vector space ?

Vector space , basis and dimension

Subspace: let v be vector space , A non empty set W of V is said to be subspace of if w is itself a vector space under same operations

Note: To show that W is a subspace of V , it is not necessary to verify all the properties just verify closure properties and the existence of zero element and the additive inverse in W .

$\omega_1, \omega_2 \in \omega, \alpha \in F$

1. $\omega_1 + \omega_2 \in \omega$
2. $\alpha \cdot \omega \in \omega$
3. $0 \in \omega$

Vector space , basis and dimension

Example :

1.If V is set of polynomials of degree$\leq$ n , then W is set of polynomials of degree$\leq$ n is a subspace

2.If V is the set of all $n \times n$ matrices is a vector space

Then W= set of all symmetric / skew symmetric matrices of order n is subspace

And W = set of all upper/lower triangular matrices of order n is also subspace

Vector space , basis and dimension

Imp

What is span or spanning set ?

Vector space , basis and dimension

Spanning set (linear span): let v be vector space and A non empty set S (need not be subspace)of V

If every element of v is linear combination of elements of S then S is said to be spanning set or s S spans V .

Note: I f s is a spanning set then it is denoted by L[s]

2.L[S]= set of all possible linear combination of elements of S

3. L[s] is a subspace of V

Vector space , basis and dimension

Example: let v be vector space of all 2×2 matrices

$V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in R \right\}$

$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a spanning set

example $\begin{pmatrix} 2 & 4 \\ -5 & 6 \end{pmatrix} \in V$

clearly $\begin{pmatrix} 2 & 4 \\ -5 & 6 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (-5)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 6\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

ie, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{l.c of elements of } S}$

Vector space , basis and dimension

What is basis and dimension of vector space ?

Vector space , basis and dimension

Basis : let v be vector space and A non empty set B of V is said to be Basis if

1. B is linearly independent set

Basis - spanning set + L.I

2.L[B]= V ie…, B generates V

ie., in other words a linearly independent set which spans the vector space V is a basis

Dimension : number of elements in Basis is called dimension

Note: The vector space is said to be finite dimensional if it has a finite basis

Vector space , basis and dimension

Important results:

standard basis in R^2 $e_1 = (1,0)$ and $e_2 = (0,1)$

$$(a,b) = a(1,0) + b(0,1)$$

Standard basis in R^3 is $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$

$$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

Vector space , basis and dimension

Important results:

In a vector space V(F) the following hold good

1.B=$\{u_1, u_2, \ldots\ldots u_n\}$ is a basis for V and if for $u \in V$

$u = \alpha_1 v_1 + \alpha_2 v_2 \ldots\ldots + \alpha_n$ Then $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ are unique

Imp 2. If B=$\{u_1, u_2, \ldots\ldots u_n\}$ is a set of n dimensional vectors in R^n is independent set then its basis for R^n

R^2 $B=\{u_1, u_2\}$ is L.I then it is a Basis

3. For a vector space V(F) more than one basis can exists but dimension of all the basis is same

4.If W is a sub space of V then dimW≤ dim V and dimW= dimV if and only if W=V

Bases for $\mathbb{R}^n$ Imp

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$.

Theorem 1 If $k < n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ do not span $\mathbb{R}^n$.

Theorem 2 If $k > n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent.

Theorem 3 If $k = n$ then the following conditions are equivalent:

(i) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;
(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.

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$R^3 - \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix};$ $\longrightarrow$ not a span set

$R^3 - \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} \begin{pmatrix} a_4 \\ b_4 \\ c_4 \end{pmatrix}$ $\longrightarrow$ L.D

$R^3 - \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix}.$

Important results:

5.Let W_1 *and* W_2 be two subspaces of finite dimensional vector space V

$$\dim(W_1 + W_2) = dimW_1 + dimW_2 - \dim(W_1 \cap W_2)$$

6. dim(V/W) = dimV- dimW

Vector space, basis and dimension

How do we find basis and dimension for given set S of vectors?

?

Vector space, basis and dimension

How do we find basis and dimension :

1.Form a matrix by considering each vector as a row

2.Reduce the matrix in to row echelon form

3.The non zero rows are the elements in basis

4.The number of elements in basis is called dimension of the given set

Vector space, basis and dimension

Q. If $v_1, v_2, \ldots\ldots\ldots v_6$, are six vectors in R^4 , which one of the following statements is FALSE ? [GATE 2020-EC]

(a) These vectors are not linearly independent correct

(b) Any four of these vectors form a basis for R^4 Not correct

(c) It is not necessary that these vectors span R^4 correct

(d) If $\{v_1, v_3, v_5, v_6\}$ spans R^4 , then it forms a basis R^4 correct

4 vectors

Basis — spanning + L.I set

Vector space, basis and dimension

Q. The vectors $e_1, = (1,0,2)$ $e_2 = (0,1,0)$ and $e_3 = (-2,0,1)$, form an orthogonal basis the three dimensional space R^3 ,then the vector u=(4,3,-3) can be expressed as [GATE 2016-EC]

$e_1 \cdot e_2 = e_2 \cdot e_3 = e_3 \cdot e_1 = 0 \leftarrow$ orthogonal

(a) u= $-2/5\, e_1 - 3e_2 - 11/5 e_3$

(b) u= $-2/5\, e_1 - 3e_2 + 11/5 e_3$

(c) u= $-2/5\, e_1 + 3e_2 + 11/5 e_3$

(d) u= $-2/5\, e_1 + 3e_2 - 11/5 e_3$

Basis $\rightarrow$ spanning set

$(4,3,-3) = c_1(1,0,2) + c_2(0,1,0) + c_3(-2,0,1)$

$(4,3,-3) = (c_1 - 2c_3,\; c_2,\; 2c_1 + c_3)$

$c_1 = \frac{-2}{5}$ $\quad c_2 = 3$ $\quad c_3 = \frac{-11}{5}$

Q. Determine whether the set of vectors $S = \{(1,5,3), (0,1,2), (0,0,6)\}$ is a basis for R^3 or not ?

$$A = \begin{pmatrix} 1 & 5 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{pmatrix}$$

$|A| \neq 0$

clearly the vectors are L.I

Basis

Problems on Basis and Dimension

For S Basis is what

Q. Consider the set of vectors

S={ (1, 2, 3, 4) ,(2, 0, 1, -2) , (3, 2, 4, 2) }

Then the maximum number of linearly independent vectors in S is/are ---------

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & -2 \\ 3 & 2 & 4 & 2 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$ $\quad$ $R_3 \to R_3 - 3R_1$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & -4 & -5 & -10 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis for set S is

$= \{(1,2,3,4)(0,-4,-5,-10)\}$

dimension — 2

common doubts asked by the students on Linear Algebra

Why do we get wrong answers for determinant?

Because of incorrect operations

$R_i \to \alpha R_i$

$R_i \to \alpha R_i + R_j$

$R_i \to \alpha R_i + \beta R_j$

$C_i \to \alpha C_i$

$C_i \to \alpha C_i + C_j$

$C_i \to \alpha C_i + \beta C_j$

What is invertible matrix ?

Inverse exists

ie $|A| \neq 0$

ie. A is nonsingular

Is det $(A \pm B) = \det(A) \pm \det(B)$?

NO

Can we find the rank for any matrix ?

Yes.

Is rank always non zero ?

Yes.

Rank is zero for only zero matrix

Can we use column operations to find rank in row echelon form method ?

NO

Can we use row interchange operations while finding the rank of matrix in row echelon method ?

Yes

Consider the system of linear equations $AX = B$ here A is $n \times n$ matrix and det A =0, does the system pose infinitely many solutions ?

No garuntee

$|A| \neq 0$ unique solutions

$|A| = 0$ either infinitely many solutions (or) no solution

i.e., No unique solutions

what are the possible solutions for the system of linear equations $AX = B$?

1. unique solution
(or)
2. infinitely many solutions
(or)
3. NO solutions

Is homogenous system of linear equations $AX = O$ always consistent ?

Yes, with solution $X = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

For homogenous system of linear equations $AX = O$ how many non trivial solutions exits(if exists) and out of these how many are linearly independent ?

Infinitely many

$n-r$ are L.I

$r = \rho(A)$, $n =$ variables

For homogenous system of linear equations $AX = O$ what is nullity?

nullity = dimension of null space

= NO of L.I solutions

$= n - r$

Can we find eigen values of given matrix by reducing into upper triangular matrix /lower triangular matrix/diagonal matrix?

$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$

$\lambda^2 - 5\lambda + 6 = 0$

$\lambda = 2, 3$ are eigenvalue

NO

Trace $= 5 = 2+3$

det $= 6 = 2 \times 3$

suppose $R_2 \to R_2 + R_1$

$\begin{pmatrix} 1 & 2 \\ 0 & 6 \end{pmatrix}$

If we consider 1, 6 as eigenvalue

Trace $= 5 \neq 1 + 6$

Is there any shortcut method to find eigen values of higher order matrices?

NO

only $|A-\lambda I|=0$ is to be simplified

Can we have zero as an eigen value for a matrix ?

Yes

Can we have complex numbers as eigen values for a matrix ?

Yes

Corresponding to an eigen value how many eigen vectors will exists ?

Infinitely many

What is A.M and G.M ? of eigenvalue λ

A.M of λ = Number of times λ repeated

G.M of λ = Number of L.I eigenvectors of λ

$= n - r$

n = order of A

$r = \rho(A - \lambda I)$

For any eigenvalue λ,

$1 \leq G.M \leq A.M$

What is the necessary and sufficient condition for diagonalization?

A is diagonalizable $\Longleftrightarrow$ n L.I eigen vectors exists

Can we have $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, as an eigen vector corresponding to an eigen value of a matrix ?

NO

eigen vector is always non zero vector

If λ is an eigen value and X is corresponding eigen vector of matrix A , then the eigen value and eigen vector of the matrix $a_0 I + a_1 A + a_2 A^2 + \ldots\ldots\ldots + a_n A^n$ are …..

$f(A) = a_0 I + a_1 A + a_2 A^2 - - - + a_n A^n$

then $f(\lambda)$ is eigen value of $f(A)$

but same X is eigenvector