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TO

Ace Engineering Academy online

Engineering Mathematics

Linear Algebra (For GATE/ESE/other exams)

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Weightage Analysis and Questions Pattern

- Weightage up to 3 marks
- One mark questions and two marks questions
- Problems on theory and simplification type are frequently asked
- Multi choice questions
- Multi select questions
- Numerical answer type questions

Weightage Analysis and Questions Pattern

The following are various types of problems

- ❖ Problems on basics of matrices
- ❖ Problems on rank of matrix
- ❖ Problems on system of linear equations
- ❖ Problems on eigen values and Eigen vectors
- ❖ Problems on A.M and G.M
- ❖ Problems on Cayley- Hamilton theorem
- ❖ Problems on LU decomposition , basis and dimension

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1. Basics of matrices ✓
2. Rank of a matrix
3. System of linear equations
4. eigenvalues and eigenvectors
5. LU decomposition ← only for CSE
6. Basis and dimension ← only for ECE

Recommended books

- Ace Engineering Academy Material
- Higher Engineering Mathematics - B. S. Grewal
- Advanced Engineering Mathematics – Erwin Kreyszing
- Matrices –A. R. Vasista
- Linear Algebra – Gilbert strang

Basics of matrices

2.1 Different types of matrices

Real matrix:

If all the elements of a matrix A are real numbers, then the matrix A is called a real matrix.

For example, the matrices

$$[2], \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

are real matrices of order 1×1, 2×3 and 2×2 respectively.

2.1 Different types of matrices

Complex Matrix:

If at least one of the elements of a matrix A is purely imaginary (or) complex then the matrix A is called complex matrix.

For example, the matrices.

$$[2i], \begin{bmatrix} 0 & 2 \\ 3i & 4+3i \end{bmatrix}, \begin{bmatrix} 2 & 3i & 0 \\ 0 & 9 & 3 \end{bmatrix}$$

2.1 Different types of matrices

Diagonal or principal diagonal elements:

If $A = (a_{ij})_{n\times n}$ then the elements a_{ij} of a square matrix for which $i = j$ i.e., the elements $a_{11}, a_{22}, \ldots a_{nn}$ are called diagonal elements (or) leading diagonal elements.

For example, in the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 0 & 3 \\ 7 & 3 & -4 \end{bmatrix}$$

$a_{11}=1$, $a_{22} = 0$, $a_{33} = -4$ are diagonal elements of A.

2.1 Different types of matrices

Principal diagonal:

The line along which the diagonal elements lie is called principal diagonal of the matrix.

$$A = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$$

$$tr(A) = 2 + 6 = 8$$

Trace of a matrix:

The sum of the diagonal elements of a square matrix A is called trace of A and it is denoted by trace (A) or tr(A).

Thus, if $A = (a_{ij})_{n\times n}$

Then $tr(A_{n\times n}) = a_{11} + a_{22} + \ldots + a_{nn} = \sum_{i=1}^{n} a_{ii}$

2.1 Different types of matrices

Properties:

If A & B are square matrices of order n, then

(i) $tr(A+B) = tr(A)+tr(B)$

vi) $Tr(A) = tr(A^T)$

(ii) $tr(A-B) = tr(A) - tr(B)$

(iii) $tr(AB) \neq tr(A)\, tr(B)$

(iv) $tr(AB) = tr(BA)$

(v) $tr(kA) = k\, tr(A)$ where k is a scalar.

2.1 Different types of matrices

$$\begin{bmatrix} & \text{zero} \\ \text{zero} & \end{bmatrix}_{n\times n}$$

Diagonal matrix:

If all the non-diagonal elements in a square matrix are zero, then the matrix is called diagonal matrix.

For example, the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

are diagonal matrices of order 2, 3 respectively.

2.1 Different types of matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Note:

- A matrix $A = [a_{ij}]_{n\times n}$ is diagonal matrix if $a_{ij} = 0 \ \forall \ i \neq j$.
- A diagonal matrix of order $n\times n$ having $d_1, d_2, \ldots.. d_n$ as

diagonal elements is denoted by $\text{diag}\,[d_1, d_2, \ldots d_n]$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow D = \text{diag}\,(1, -2, 5)$$

2.1 Different types of matrices

Scalar matrix:

If all the diagonal elements of a diagonal matrix are same or equal then the matrix is called a scalar matrix.

Thus a matrix A = $[a_{ij}]_{n\times n}$ is a scalar matrix if

$$a_{ij} = \begin{cases} 0, \forall i \neq j \\ k, \forall i = j \end{cases}$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

are scalar matrices of order 2 and 3 respectively

2.1 Different types of matrices

Unit matrix or identity matrix:

If all the diagonal elements of a diagonal matrix are one then the matrix is called an identity matrix.

Thus a matrix $A = [a_{ij}]_{n\times n}$ is an identity matrix if

$$a_{ij} = \begin{cases} 0, \forall i \neq j \\ 1, \forall i = j \end{cases}$$

A unit matrix of order n is denoted by I_n.

For example, the matrices

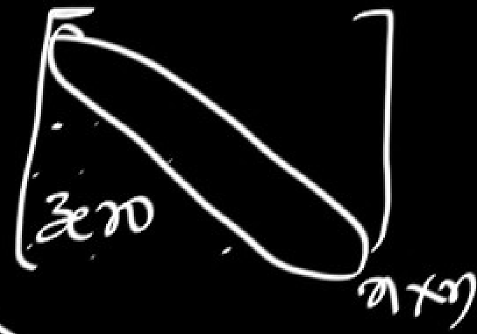
$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are identity matrices of order ,2,3 respectively.

2.1 Different types of matrices

Upper triangular matrix:

If all the elements below the principal diagonal are zero in a square matrix then the matrix is called an upper triangular matrix.



Thus, $A = [a_{ij}]_{n\times n}$ is an upper triangular matrix if $a_{ij} = 0 \ \forall \ i > j$.

For example, the matrix

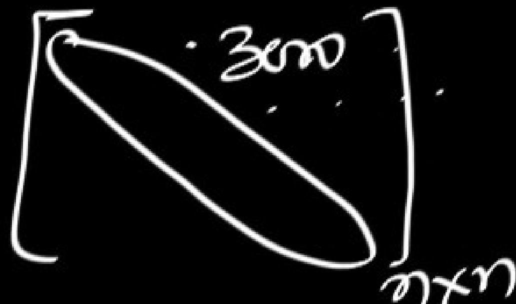
$$A = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 4 & 8 \\ 0 & 0 & 9 \end{bmatrix}$$

is an upper triangular matrix of order 3.

2.1 Different types of matrices

Lower triangular matrix:

If all the elements above the principal diagonal are zero in a square matrix then the matrix is called lower triangular matrix. Thus, $A = [a_{ij}]_{n\times n}$ is a lower triangular matrix if $a_{ij} = 0 \; \forall \; i < j$.



For example, the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 4 & 0 \\ 1 & 3 & 9 \end{bmatrix}$$ is a lower triangular matrix.

Note:

A diagonal matrix is both upper and lower triangular matrix

A matrix which is either upper triangular or lower triangular is called a triangular matrix.

2.1 Different types of matrices

Transpose of a matrix:

If a matrix $B_{n\times m}$ is obtained from a matrix $A_{m\times n}$ by changing its rows into columns and its columns into rows then the matrix $B_{n\times m}$ is called transpose of A and is denoted by A^T

For Example,

If $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 7 & -2 \end{bmatrix}_{2\times 3}$ then $A^T = \begin{bmatrix} 1 & 0 \\ 3 & 7 \\ 4 & -2 \end{bmatrix}_{3\times 2}$

2.1 Different types of matrices

Properties of Transpose:

If A & B are two matrices and A^T & B^T are transpose of A & B respectively then

(i) $(A^T)^T = A$

(ii) $(A \pm B)^T = A^T \pm B^T$

(iii) $(kA)^T = kA^T$ where k is a scalar (real or complex)

(iv) $(AB)^T = B^T A^T$ (reversal law)

2.1 Different types of matrices

Symmetric matrix:

If $a_{ij} = a_{ji}$ for all i,j; then $A_{n\times n}$ is called a symmetric matrix.

For example, the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 6 \\ -3 & 6 & 0 \end{bmatrix}$$ is a symmetric matrix of order 3.

Note:

The necessary and sufficient condition for a square matrix A to be symmetric is that $A^T = A$.

2.1 Different types of matrices

Skew-symmetric matrix:

A square matrix $A = (a_{ij})_{n\times n}$ is said to be Skew-Symmetric matrix if $a_{ij} = -a_{ji} \quad \forall$ i,j.

For diagonal elements $i = j$

$\Rightarrow a_{ii} = -a_{ii}$

$\Rightarrow 2a_{ii} = 0$

$\Rightarrow a_{ii} = 0 \quad \forall i$

For example, the matrix

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix}$$ is a Skew symmetric matrix of order 3.

Note:

(i) The diagonal elements of a skew- symmetric matrix are all zero.

(ii) The necessary and sufficient condition for a square matrix A to be skew symmetric matrix is that $A^T = -A$.

2.1 Different types of matrices

Imp

Result: Every square matrix A can be uniquely expressed as a sum of symmetric & Skew-symmetric matrices.

verify $P^T = P$

$Q^T = -Q$

$$\text{i.e., } A = \underbrace{\left[\frac{A + A^T}{2}\right]}_{P \ (\text{symm})} + \underbrace{\left[\frac{A - A^T}{2}\right]}_{Q \ (\text{skew})}$$

Result: if A and B are symmetric matrices then $A \pm B$,

$$AB + BA, A^k, B^k \text{ here } k \in N \text{ are symmetric}$$

and

$$AB - BA \text{ is skew symmetric}$$

$$AB, BA \text{ need not be symmetric}$$

2.1 Different types of matrices

Result: if A and B are skew-symmetric matrices then

$$A \pm B \ are \ skew - symmetric$$

$$, A^2, , A^4, A^6, \dots \dots B^2, B^4, B^6 \dots \dots . \ are \ symmetric$$

$$, A^3, , A^5, A^7, \dots \dots B^3, B^5, B^7 \dots \dots . \ are \ skew - symmetric$$

$$and$$

$$AB, BA \ need \ not \ be \ skew - symmetric$$

2.1 Different types of matrices

Positive integral powers of a square matrix:

If A is a square matrix then the product A.A is defined as A^2.

Similarly $A.A^2 = A^3$

$A^2A^2 = A^4$ and so on.

$A^2 = A \cdot A$

$A^3 = A^2 \cdot A = A \cdot A \cdot A$

$A^4 = A^3 \cdot A = A \cdot A \cdot A \cdot A$ - - .

Idempotent matrix:

If $A^2 = A$ for a square matrix A of order n then $A_{n\times n}$ is called an idempotent matrix. For example, the matrices

example $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ ✓

$A^2 = A \cdot A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = A$

2.1 Different types of matrices

Involutary matrix:

If $A^2 = I_n$ for a square matrix A of order n, then matrix A is called an involutary matrix..

Example: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A^2 = A \cdot A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2.1 Different types of matrices

$A^2 = O$

Nilpotent matrix:

1.If there exists a positive integer m for a square matrix A of order n such that $A^m = O$ then the matrix A is called a nilpotent matrix.

2.If m is a least positive integer for which $A^m = O$ then 'm' is called the index of the nilpotent matrix.,

For example $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ is a nilpotent matrix of index 2,

2.1 Different types of matrices

Orthogonal matrix:

A square matrix $A_{n\times n}$ is said to be an orthogonal matrix if $AA^T = A^TA = I$ (or) $A^{-1} = A^T$.

For example,

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

2.1 Different types of matrices

Conjugate of a complex number

$\bar{A}$ = It is a matrix obtained from A by taking conjugate of the elements of A

conjugate of $z = x + iy$ is $\bar{z} = x - iy$

$\overline{2+3i} = 2-3i$

$\overline{-4+5i} = -4-5i$

$\overline{-2i} = 2i$

$\overline{3i} = -3i$

$\bar{i} = -i$

$\bar{2} = 2$

$\overline{-3} = -3$

$\bar{4} = 4$

Note (1) $\bar{z} = z \iff z$ is real number

(2) $\bar{z} = -z \iff z$ is purely imaginary or zero

2.1 Different types of matrices

Transposed conjugate of a matrix:

If $\overline{A}$ is a conjugate matrix of a complex matrix A then $\left(\overline{A}\right)^T$ is called transposed conjugate of A and it is denoted by A^θ.

$$\therefore A^\theta = \left(\overline{A}\right)^T = \left(\overline{A^T}\right)$$

Example $A = \begin{bmatrix} 1+i & 2 \\ -3-i & 2i \end{bmatrix}$

$$\overline{A} = \begin{bmatrix} 1-i & 2 \\ -3+i & -2i \end{bmatrix}$$

$$A^\theta = \left(\overline{A}\right)^T = \begin{bmatrix} 1-i & -3+i \\ 2 & -2i \end{bmatrix}$$

2.1 Different types of matrices

Properties:

(i) $(A^{\theta})^{\theta} = A$

(ii) $(A+B)^{\theta} = A^{\theta}+B^{\theta}$

(ii) $(kA)^{\theta} = \bar{k}A^{\theta}$

(iii) $(AB)^{\theta} = B^{\theta} A^{\theta}$

2.1 Different types of matrices

Hermitian matrix:

A square matrix $A = (a_{ij})_{n\times n}$ is said to be Hermitian if $a_{ij} = \bar{a}_{ji}$ for all i, j

For diagonal elements

$i = j$

$\Rightarrow a_{ii} = \overline{a_{ii}}$

$\Rightarrow$ all a_{ii}'s are real

Note:

(i) The necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^{\theta}$.

(ii) The diagonal element of a Hermitian matrix are purely real.

$$A = \begin{bmatrix} 2 & 2+3i \\ 2-3i & 3 \end{bmatrix}$$

2.1 Different types of matrices

Skew-Hermitian matrix:

A square matrix $A = (a_{ij})$ is said to be skew- hermitian

if $a_{ij} = -\bar{a}_{ji}$ for all i, j

ie $A^{\theta} = -A$

For diagonal elements

$i = j$

$\Rightarrow a_{ii} = -\bar{a}_{ii}$

$$A = \begin{bmatrix} 2i & 4+3i \\ -(4-3i) & 0 \end{bmatrix}$$

2.1 Different types of matrices

1. The necessary & sufficient condition for a matrix A to be Skew-Hermitian is that $A^{\theta} = -A$.

2. The diagonal elements of a skew-Hermitian matrix are either zero (or) purely imaginary.

2. Every complex square matrix A can be uniquely written as a sum of Hermitian and Skew-Hermitian matrices.

$$\boxed{A = \underbrace{\left(\frac{A + A^{\theta}}{2}\right)}_{\text{Her}} + \underbrace{\left(\frac{A - A^{\theta}}{2}\right)}_{\text{skew}}}$$

2.1 Different types of matrices

Unitary Matrix:

A complex matrix $A = (a_{ij})$ is said to be unitary if $AA^{\theta} = A^{\theta}A = I$ (or) $A^{\theta} = A^{-1}$.

$$Ex: \quad A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

Note If all elements are real
then $\bar{A} = A \Rightarrow A^{\theta} = (\bar{A})^T = A^T$

→ Hermitian → symm

Skew Herm → Skew symm

unitary → orthogonal

Matrix	Definition
Symmetric matrix	$A^T = A$
Skew-Symmetric matrix	$A^T = -A$
Orthogonal matrix	$A^T A = AA^T = I$
Hermitian matrix	$A^\theta = A$
Skew- hermitian matrix	$A^\theta = -A$
Unitary matrix	$A^\theta A = AA^\theta = I$
Idempotent matrix	$A^2 = A$
Involutary matrix	$A^2 = I$
Nilpotent matrix	$A^K = O$

Matrix	Diagonal elements
Hermitian matrix	Always real
Skew-Symmetric matrix	Always zero
Skew-Hermitian matrix	Purely imaginary or zero

Algebra of matrices

Matrix multiplication

(1) let $A_{m\times n}$ $B_{p\times q}$

$(AB)_{m\times q}$ exists $\iff n = p$

$(BA)_{p\times n}$ exists $\iff q = m$

2.2. Algebra of Matrices

$xy = 0$

Note:

1.A is mxn , B is nxp are two matrices then the order of AB is mxp

2.In the product AB, A is called pre-factor & B is called post-factor.

3.If the product AB exists for two matrices A & B then the product BA may or may not exist.

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (0\times 2)+(-1\times 0) & (0\times 0)+(-2\times 0) \\ (0\times 2)+(0\times 0) & (0\times 0)+(0\times 0) \end{pmatrix}$$

4.AB = 0 does not imply either A = O or B = O

2.2. Algebra of Matrices

7.Matrix multiplication is distributive over matrix addition:

If A is a matrix of order $m\times n$ & B,C are matrices of order $n\times p$ then

$A(B+C)=AB+AC$ (left distributive law).

If A, B are matrices of order $m\times n$ & C is a matrix of order $n\times p$ then

$(A+B)C = AC+BC$ (right distributive law).

Imp 8.Result : if A is mxn , B is nxp are two matrices then the order of AB is mxp and then the number of multiplications involved in the computation of the product AB is ***mnp***

and number of additions is ***mp(n-1)***

2.3.Determinant & Inverse

Determinant of a square matrix of order two:

1. If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2 then the

determinant of 2^{nd} order and it is denoted by

$|A|$ (or) $\det(A)$ or Δ

$$\therefore \det(A)=|A|=\Delta=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$$

2.3.Determinant & Inverse

Minor of an element:

If $A = (a_{ij})$ is a square matrix of order 'n' then the minor of an element a_{ij} in A is the determinant of a square matrix that remains after deleting corresponding the i^{th} row and j^{th} column of A. It is denoted by M_{ij}.

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

The minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$= a_{22}a_{32} - a_{32}a_{23}$

$A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

minor of 6 $= \begin{vmatrix} 3 & 5 \\ 0 & 0 \end{vmatrix} = 0$

minor of 3 $= \begin{vmatrix} 4 & 6 \\ 0 & 4 \end{vmatrix} = 16$

2.3.Determinant & Inverse

Cofactor of an element: If $A = (a_{ij})$ is a square matrix of order 'n' then the cofactor of an element a_{ij} is denoted by A_{ij} and defined as $(-1)^{i+j} M_{ij}$ where M_{ij} is a minor of a_{ij}.

$$\therefore A_{ij} = (-1)^{i+j} M_{ij}$$

$$\text{Ex : If } A = \begin{bmatrix} 1 & 2 & -4 \\ 3 & -5 & -2 \\ 0 & 8 & 9 \end{bmatrix}$$

$$\text{cofactor of } 1 = \underbrace{(-1)^{1+1} \underbrace{\begin{vmatrix} -5 & -2 \\ 8 & 9 \end{vmatrix}}_{\text{minor}}}_{\text{cofactor}} = -29$$

$$\text{cofactor of } -2 = \underbrace{(-1)^{2+3} \underbrace{\begin{vmatrix} 1 & 2 \\ 0 & 8 \end{vmatrix}}_{\text{minor}}}_{\text{cofactor}} = -8$$

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cofactor matrix = matrix of cofactors of elements of A.

adj A = (cofactor matrix)^T

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\text{cofactor matrix} = \begin{bmatrix} +10 & -0 & +0 \\ -(-5) & +10 & -0 \\ +(-3) & -(-6) & +4 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 10 & 5 & -3 \\ 0 & 10 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

Note: (i, j) element of $(\text{adj}A)$ = cofactor of (j, i) element of A

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2.3.Determinant & Inverse

Determinant of a square matrix of order three:

1.If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order. 3×3 then the expression

$$|A| \ or\ \Delta \ or\ Det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

is called expansion or value of the determinant of 3rd order matrix.

2.3.Determinant & Inverse

Finally, If $A = (a_{ij})$ is a square matrix of order 'n' then the sum of the products of elements of any row (or column) with the cofactors of the corresponding elements of row (or column) is called determinant

2.3.Determinant & Inverse

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Properties of determinants

1. If A and B are two square matrices of same order then $|AB| = |A|\ |B|$

$NOTE\ \ |A \pm B| \neq |A| \pm |B|$

2. $|A^m| = |A|^m (m = 2,3,4....)$

3. $If\ |A_{n\times n}| \neq 0\ then\ |A^{-1}| = \frac{1}{|A|}$

2.3.Determinant & Inverse

Properties of determinants

4.All zero property

If every element of a row (column) of a determinant of A is zero then $|A| = 0$

$$\begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & 4 \\ -1 & 3 & 7 \end{vmatrix} = 0$$

5.Reflexion Property: If A is square matrix of order n ,then $|A| = |A^T|$

2.3.Determinant & Inverse

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6.Proportionality(repetition)property:

If the corresponding elements of any two rows (or columns) of a determinant are proportional (or in the same ratio) then the value of the determinant is zero.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & 3 & 7 \end{vmatrix} = 0$$

$R_2 = 2R_1$

2.3.Determinant & Inverse

7.Switching property: If any two rows (or columns) of a determinant are interchanged then the sign of determinant changes. i.e., the determinant value is to be multiplied with -1

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = 27 \qquad \begin{vmatrix} 7 & 8 & 0 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = (-1)(27) \quad = -27 \; (\because R_1 \leftrightarrow R_3)$$

2.3.Determinant & Inverse

ACE

8 Scalar multiple property: If each element of a row (or a column) of a determinant is multiplied by a constant 'k' then the value of the determinant will be multiplied by 'k'. (or the value of the new determinant is 'k' times the value of the original determinant).

$$\begin{bmatrix} 1 & 2 & 3 \\ 2k & 3k & 5k \\ 4 & 6 & 8 \end{bmatrix}$$

For example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 4k & 5k & 6k \\ 7 & 8 & 0 \end{vmatrix} = k\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = 27k$$

Note: $|k \ A_{n\times n}| = k^n \ |A_{n\times n}|$

2.3.Determinant & Inverse

9.Sum property If The each element of a row (or column) of a determinant is expressed as a sum of two or more terms then the determinant can be expressed as the sum of two or more determinants. For example,

$$\begin{vmatrix} a+\lambda_1 & b+\lambda_2 & c+\lambda_3 \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ d & e & f \\ g & h & i \end{vmatrix}$$

Imp

10.Triangular property: The determinant of upper triangular, lower triangular, det is equal to product of its diagonal elements.

2.3.Determinant & Inverse

11. Property of invariance: If each element of a row (or column) of a determinant is multiplied by the same constant k and added to the corresponding elements of some other row (column) then the value of the determinant remains same. For example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = \begin{vmatrix} 1+4k & 2+5k & 3+6k \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix}$$

$R_1 \rightarrow R_1 + kR_2$

Imp

(note :under the elementary row /column operation the value of determinant is unchanged. $R_i \rightarrow R_i + kR_j$, $C_i \rightarrow C_i + k\,C_j$

2.3.Determinant & Inverse

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Tips to find determinant of higher order matrices

1.Try to perform the operation in a manner so we can take some thing common and create all 1

2. Once we get all 1's in a row /column then convert them in to zero as many as possible.

3. Then expand along that row/column.

Why do we get wrong answer for determinant?

Correct

$R_i \to R_i + kR_j$

$C_i \to C_i + kC_j$

$R_i \leftrightarrow R_j$

$C_i \leftrightarrow C_j$

But in this case final value is to be multiplied with (−1)

Incorrect

$R_i \to kR_i + R_j$

$C_i \to k \cdot C_i + C_j$

$R_i \to \alpha R_i + \beta R_j$

$C_i \to \alpha C_i + \beta C_j$

$R_i \to 1/R_i$, $C_i \to \frac{1}{C_i}$

$R_i \to R_i \circ R_j$

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 8 \end{vmatrix} = 11$$

$R_2 \to R_2 + kR_1$

$$\begin{vmatrix} 2 & 5 \\ 1+2k & 8+5k \end{vmatrix}$$

$= 16 + \cancel{10k} - 5 - \cancel{10k}$

$= 11$

$C_1 \to C_1 + kC_2$

$$\begin{vmatrix} 2+5k & 5 \\ 1+8k & 8 \end{vmatrix} = 16 + \cancel{40k} - 5 - \cancel{40k}$$

$= 11$

But $R_2 \to 3R_2 + R_1$

$$\begin{vmatrix} 2 & 5 \\ 5 & 29 \end{vmatrix} = 58 - 25 = \underline{33} \times$$

2.3.Determinant & Inverse

12. If A is an orthogonal matrix of order 'n' then $|A| = \pm 1$

$$AA^T = A^TA = I$$

13. If A is an unitary matrix of order 'n' then $|A| = \pm 1$

$$AA^T = I$$

$$|AA^T| = |I|$$

$$|A|\,|A^T| = 1$$

$$|A|\,|A| = 1 \Rightarrow |A|^2 = 1$$

$$|A| = \pm 1$$

14. $|\overline{A}| = \overline{|A|}$

15. $|A^\theta| = \overline{|A|}$

Imp

16. The determinant of a Hermitian matrix is always a real number.

2.3.Determinant & Inverse

17. If A is an idempotent matrix then $|A| = 0\ (or)\ 1$

18. $|I_n| = 1$

Imp

19. The determinant of a skew symmetric matrix of odd order is zero.

20. The determinant of a skew symmetric matrix of even order is perfect square

21. If A and B are two nonzero square matrices and AB=O then detA=0 and detB=0.

2.3. Determinant & Inverse

Properties of adjoint matrix:

1 If A is a square matrix of order ‘n’ then

$A\,adj(A) = adj(A)\,A = |A|I_n$.

2. If O is a zero matrix of order ‘n’ then $adj(O) = O$.

3. If A is a square matrix of order ‘n’

then $adj\,(A^T) = (adj(A))^T$.

4. If A and B are non-singular matrices

then $adj(AB) = adj(B).adj(A)$.

2.3.Determinant & Inverse

5. $Adj(KA)=K^{n-1}Adj(A)$

6. If $|A| = 0$ then $|adj(A)| = 0$

Imp

7. If A is a square matrix of order 'n' then $|adj(A)| = |A|^{n-1}$.

Imp

8. If A is a square matrix of order 'n' then

$|adj(adj(A)| = |A|^{(n-1)^2}$

$\underbrace{|adj\ adj \ldots adj\, A|}_{k\ times} = |A|^{(n-1)^k}$

n is order of A.

9..If A is a non-singular matrix of order 'n' then $adj(adj(A)) = |A|^{n-2} A$.

2.3.Determinant & Inverse

Singular matrix:

A square matrix A of order 'n' is said to be singular matrix if $|A_{n\times n}| = 0$

Non-singular matrix:

A square matrix A of order 'n' is said to be non-singular matrix if $|A_{n\times n}| \neq 0$

2.3.Determinant & Inverse

Inverse (or) reciprocal of a square matrix:

If for a non-singular matrix A of order 'n' there exists another non-singular matrix B of order 'n' such that $AB = BA = I_n$ then B is called the inverse of A. It is denoted by A^{-1}.

$$\therefore B = A^{-1} \text{ and } AA^{-1} = A^{-1}A = I_n.$$

2.3.Determinant & Inverse

Note:

1. If the inverse of a square matrices A exists then the matrix is called invertible matrix.
2. If A is a non-singular matrix of order n then

$$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$$

Note

(i, j) element of A^{-1}

$= \frac{1}{|A|}\left(\begin{array}{c}\text{cofactor of } (j, i) \\ \text{element of } A\end{array}\right)$

2.3.Determinant & Inverse

Properties of inverse of a square matrix:

01.The necessary and sufficient condition for a square matrix A to possess (have) the inverse is that $|A| \neq 0$ (i.e., nonsingular matrix).

2. If the inverse of a square matrix A exists then it is unique.

03. If A, B and C are non-singular matrices then

(i) $(AB)^{-1} = B^{-1}A^{-1}$ (reversal law)

(ii) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

04. If A is non-singular matrix of order 'n'

then $(A^T)^{-1} = (A^{-1})^T$ and $(A^\theta)^{-1} = (A^{-1})^\theta$.

2.3.Determinant & Inverse

06. If AB = BA for two non-singular matrices then $A^{-1}B^{-1} = B^{-1}A^{-1}$.

07. Cancellation law: If A is a non-singular matrix of order 'n' and B, C are square matrices of same order as A then

(i) $AB = AC \Rightarrow B = C$ (left cancellation law).

(ii) $BA = CA \Rightarrow B = C$ (right cancellation law).

2.4. Vectors

$$X = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad Y = \begin{pmatrix}2\\4\\5\end{pmatrix}$$

$$X^T \cdot Y = (1\times 2) + (2\times 4) + (3\times 5)$$
$$= 2 + 8 + 15$$
$$= 25$$

Inner product:

The inner product of two vectors

$$X = \begin{bmatrix} x_1 \\ x_2 \\ | \\ | \\ x_n \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 \\ y_2 \\ | \\ | \\ y_n \end{bmatrix}$$ is denoted by $X \cdot Y$ and defined as

$$X.Y = X^T Y = [x_1 x_2 \ldots x_n] \cdot \begin{bmatrix} y_1 \\ y_2 \\ | \\ | \\ y_n \end{bmatrix} = \underbrace{x_1 y_1 + x_2 y_2 + \ldots + x_n y_n}$$

(Which is a scalar quantity)

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orthogonal vectors $\quad x^T \cdot y = 0$

$$\begin{pmatrix}1\\2\\-1\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix} = (1\times 0)+(2\times 1)+(-1\times 2)$$
$$= 0$$

Norm of a vector $\quad x = \begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} \quad \|x\| = \sqrt{(x_1)^2+(x_2)^2+\cdots(x_n)^2}$

Normalized vector $\quad x = \dfrac{x}{\|x\|}$

example $\quad x = \begin{pmatrix}1\\2\\-1\end{pmatrix} \quad \|x\| = \sqrt{(1)^2+(2)^2+(-1)^2} = \sqrt{6}$

$$\frac{x}{\|x\|} = \begin{pmatrix}1/\sqrt{6}\\2/\sqrt{6}\\-1/\sqrt{6}\end{pmatrix}$$

Orthonormal vectors Let X, Y be two vectors these are said to be orthonormal if

1) X, Y are orthogonal

2) $\|X\| = 1$ and $\|Y\| = 1$

Result If A is orthogonal matrix then its rows (columns) are orthonormal

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Orthonormal vectors let X, Y be two vectors

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Result If A is orthogonal matrix then its rows (columns) are orthonormal

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Problems on Basics of Matrices

Q. What is the minimum number of multiplications involved in computing the matrix product PQR? Matrix P has 4 rows and 2 columns, matrix Q has 2 rows and 4 columns and matrix R has 4 rows and 1 column _______.

[GATE-2013[CE]]

PQR

P(QR)

(PQ)R

(1) $P(QR)$ — (1) $\underset{2\times4}{Q}\ \underset{4\times1}{R}$ — * $2\times4\times1 = 8$

(2) $\underset{4\times2}{P}\ \underset{2\times1}{(QR)}$ — * $4\times2\times1 = 8$

$\underline{\underline{16}}$

$A_{m\times n}\ B_{n\times p}$

* mnp

(2) $(PQ)R$ — (1) $\underset{4\times2}{P}\ \underset{2\times4}{Q}$ — * $4\times2\times4 = 32$

(2) $\underset{4\times4}{(PQ)}\ \underset{4\times1}{R}$ — * $4\times4\times1 = 16$

$\underline{48}$

minimum is 16

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Problem If $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = A^{-1}$ then the element in the second row third column of B is ____

(a) 0 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1

Soly $(2,3)$ element of $A^{-1} = \frac{1}{|A|}$ cofactor of $(3,2)$ element of A

$$= \frac{1}{2}(-1)^{3+2}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -\frac{1}{2}$$

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problem: Suppose that A is of order $n-1$. The elements of A are defined by

$$a_{ij} = \begin{cases} n-1 & i = j \\ -1 & i \neq j \end{cases}$$

Then det of A is ___

(a) n^{n-1} (c) $(n-1)^{n-2}$

(b) n^{n-2} (d) $(n-1)^{n-3}$

$n=3 \rightarrow$ A is of order 2×2

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$\Rightarrow |A| = 3$

clearly option (b)

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problem If $\operatorname{adj} A = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$ then absolute value of determinant of A is ___

soln we know that

$$|\operatorname{adj} A| = |A|^{n-1}$$
$$= |A|^{3-1}$$
$$= |A|^2$$

$$|\operatorname{adj} A| = \begin{vmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{vmatrix} = 2116$$

$$\Rightarrow |A|^2 = 2116$$
$$|A| = \pm 46$$

absolute value of $|A|$ is 46

Problems on Basics of Matrices

Q. The number of terms in the expansion of general determinant of order n is

[GATE – 99(CE)]

(a) n^2

(b) $n!$

(c) n

(d) $(n+1)^2$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

two terms

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} | \; | + a_{13} | \; |$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - (\;) + (\;)$$

$= 6$ terms $\rightarrow 3!$

Problems on Basics of Matrices

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Q. Perform the following operations on the matrix

$$\begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{bmatrix}$$

(i) Add the third row to the second row ✓ $R_2 \to R_2 + R_3$

(ii) Subtract the third column from the first column. ✓ $C_1 \to C_1 - C_3$

The determinant of the resultant matrix is ✓ 0.

[GATE – 15 – CS – Set 2]

we know that det Remains same under elementary operations

|Resultant matrix|

= |given matrix|

= 0

$(C_3 = 15 C_1)$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

Problems on Basics of Matrices

Q. The determinant of the matrix M shown below is___.

(GATE-21-IN)

$$M = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

Sol:

$$\begin{array}{cccc} + & - & + & - \end{array}$$
$$\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 2 & 1 \end{vmatrix}$$

$$1\begin{vmatrix} \overset{+}{4} & \overset{-}{0} & \overset{+}{0} \\ 0 & 4 & 3 \\ 0 & 2 & 1 \end{vmatrix} - 2\begin{vmatrix} \overset{+}{3} & \overset{-}{0} & \overset{+}{0} \\ 0 & 4 & 3 \\ 0 & 2 & 1 \end{vmatrix} + 0 - 0$$

$$1\{4(4-6) - 0 + 0\} - 2\{3(4-6) - 0 + 0\}$$

$$= -8 + 12$$

$$= 4$$

Problems on Basics of Matrices

Q. The determinant of matrix A= $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ is

[GATE-14-CE-Set2]

Sol:

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$$

$R_3 \to R_3 - 2R_2$

$R_4 \to R_4 - 3R_2$

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & -6 & 1 \\ 0 & 0 & -8 & 2 \end{vmatrix}$$

$$+0 - 1\begin{vmatrix} 1 & 2 & 3 \\ 3 & -6 & 1 \\ 0 & -8 & 2 \end{vmatrix} + 0 - 0$$

$$= -1\{1(-4) - 3(28) + 0\}$$

$$= -\{-88\}$$

$$= 88$$

Problem: The maximum value of the determinant of all 2×2 real symmetric matrices with trace 10 is ____

Sol: $A = \begin{bmatrix} x & c \\ c & y \end{bmatrix}$

$|A| = xy - c^2$

$|A|$ is maximum if $c=0$

$|A| = xy$

given that $x+y=10 \Rightarrow y=10-x$

let $f(x) = |A| = x(10-x) = 10x - x^2$

$f'(x) = 10-2x$

$f''(x) = -2$

$f'(x) = 0 \Rightarrow 10-2x=0 \Rightarrow x=5$

$f''(5) = -2<0$ maximum exists

$\therefore |A|$ is maximum for $x=5 \Rightarrow y=5$

$\therefore |A| = xy = 25$

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Problems on Basics of Matrices

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Q. The determinant of the matrix

given below is $\begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$ [GATE – 05]

(a) -1

(b) 0

(c) 1

(d) 2

$$\begin{vmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{vmatrix} = 0 - 0 + 0 - 1\begin{vmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -2 & 0 \end{vmatrix}$$

$$= -1\left\{0 - 1\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} + 0\right\}$$

$$= -1\{-1(-1)\} = -1$$

Problems on Basics of Matrices

Q. The determinant of

the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 100 & 1 & 0 & 0 \\ 100 & 200 & 1 & 0 \\ 100 & 200 & 300 & 1 \end{bmatrix}$ is **[GATE – 02(EE)]**

(a) 100

(b) 200

(c) 1 ✓

(d) 300

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 100 & 1 & 0 & 0 \\ 100 & 200 & 1 & 0 \\ 100 & 200 & 300 & 1 \end{vmatrix} = 1\times1\times1\times1 = 1$$

→ lower triangular

Problems on Basics of Matrices

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Q. For $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, the determinant of $A^T A^{-1}$ is

[GATE – 15 – EC – SET3]

(a) $\sec^2 x$

(b) $\cos 4x$

(c) 1 ✓

(d) 0

$$|A^T A^{-1}| = |A^T|\,|A^{-1}|$$
$$= \cancel{|A|}\frac{1}{\cancel{|A|}}$$
$$= 1$$

Problems on Basics of Matrices

Q. Which one of the following equations is a correct identify for arbitrary 3 × 3 real matrices P, Q and R ? **[GATE-14-ME-Set4]**

(a) $P(Q + R) = PQ + RP$

(b) $(P - Q)^2 = P^2 - 2PQ - Q^2$

(c) $\text{Det}\,(P + Q) = \det P + \det Q$

(d) $(P + Q)^2 = P^2 + PQ + QP + Q^2$

$PR \neq RP$

$(P-Q)^2 = P^2 - PQ - QP + Q^2$

Problems on Basics of Matrices

Q. If $A = \begin{bmatrix} 6 & 7 \\ 2 & 2 \end{bmatrix}$ then the determinant of $A^{2004} - 2A^{2003}$ is____

(a) 2^{2004}

(b) (7) (2^{2004})

(c) (7) (2^{2005})

(d) (-7) (2^{2005})

$$|A^{2004} - 2A^{2003}| \neq |A^{2004}| - |2A^{2003}|$$

$$|A^{2004} - 2A^{2003}| = |A^{2003}(A - 2I)|$$

$$= |A^{2003}|\,|A - 2I|$$

$$= |A|^{2003}\,|A - 2I|$$

$$|A| = \begin{vmatrix} 6 & 7 \\ 2 & 2 \end{vmatrix} = (-2)$$

$$(A-2I) = \begin{pmatrix} 6 & 7 \\ 2 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 7 \\ 2 & 0 \end{pmatrix}$$

$$|A-2I| = \begin{vmatrix} 4 & 7 \\ 2 & 0 \end{vmatrix}$$

$$= -14$$

$$|A^{2004} - 2A^{2003}|$$

$$= |A|^{2003} |A-2I|$$

$$= (-2)^{2003}(-14)$$

$$= (-2)^{2003}(-2)(7)$$

$$= (-2)^{2004}(7)$$

$$= \underline{\underline{2^{2004}\,7}}$$

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Problems on Basics of Matrices

Q. If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$ then the top row of R^{-1} is **[GATE – 05(EE)]**

(a) [5 6 4] ✗

(b) [5 -3 1] ✓

(c) [2 0 -1]

(d) [2 -1 0]

$$R^{-1} R = I$$

$$\begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problems on Basics of Matrices

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Q. If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$ then the top row of R^{-1} is **[GATE – 05(EE)]**

(a) [5 6 4] ✗

(b) [5 -3 1] ✓

(c) [2 0 -1]

(d) [2 -1 0]

$$R^{-1} R = I$$

$$\begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Break – 20 mni

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Rank of a matrix

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3.1.Elementary Matrices

Definition : If a matrix E is obtained from a unit matrix I by applying a single elementary row (or column) operation on I then the matrix E is called an elementary matrix .

Example : $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$R_2 \to R_2 - R_1$ $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \leftarrow$ elementary matrix

$C_1 \to C_1 + C_2$

$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \leftarrow$ elementary matrix

3.1Elementary Matrices

Result : let A and B two matrices of order $k \times l$. We say that A is row equivalent to B if and only if there exists elementary matrices$E_1, E_2, \dots \dots E_n$ such that $E_1, E_2, \dots \dots E_n A = B$

Result : If two matrices A and B are row (column) equivalent then rank of A= rank B

$$A = \begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$$

Finally $$\boxed{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}}$$

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3.2 Definition of rank of a Matrix

Sub-Matrix: A matrix obtained by deleting some rows or columns or both of a given matrix is called its sub matrix

Example: $A = \begin{bmatrix} 2 & 4 & 5 & 7 \\ 3 & 2 & 4 & 3 \\ 1 & -1 & 2 & 0 \end{bmatrix}_{3\times 4}$

2×3 submatrix

$\begin{bmatrix} 2 & 4 & 5 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 5 & 7 \\ 2 & 4 & 3 \end{bmatrix}$ - - - - - .

3×3 submatrices $\begin{bmatrix} 2 & 4 & 5 \\ 3 & 2 & 4 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 5 & 7 \\ 2 & 4 & 3 \\ -1 & 2 & 0 \end{bmatrix}$ - - . . . .

2×2 Submatrices $\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 2 & 0 \end{bmatrix}$ - - - .

3.2 Definition of rank of a matrix

Minor of Matrix: let A be an $m \times n$ matrix . The determinant of a square – sub matrix of A is called minor of the matrix. If the order of the square sub-matrix is t then its determinant is called a minor of order t

Example: 3×3 minors $\begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 4 \\ 1 & -1 & 2 \end{vmatrix}$ $\begin{vmatrix} 4 & 5 & 7 \\ 2 & 4 & 3 \\ -1 & 2 & 0 \end{vmatrix}$ - - - - - - -

2×2 minors $\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$ $\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix}$ - - - - - - -

3.2 Definition of rank of a matrix

Definition -1

Let A be $m \times n$ *non zero* [redacted] *matrix* A.

A non negative integer ' r ' is said to be the rank of matrix A, if

(i) there exists at least one non-zero minor of order 'r'

(ii) all minors of order (r+1) if they exist, are zeros.

ie., *Rank of A=* $\rho(A)$ *= r= the order of the highest non-zero minor*

$A_{5\times4}$

$A_{10\times7}$

$A_{3\times3}$

3.2 Definition of rank of a matrix

Important points

1.Rank of a matrix A = The order of largest non vanishing minor of A.

non zero

2.Rank of matrix is unique

3. Every matrix will have a rank

4. The Rank of a matrix is $= r$ means all minors of $(r+1)^{th}$ order are vanishes and at least one minor of $(r)^{th}$ order which is non zero exists

Note: for large size of matrices, this definition will not in general be of much use

3.2 Definition of rank of a matrix

Definition -2 Row Echelon form: (only Row operations)

A matrix A of order m×n is said to be in row echelon form if

(i) zero rows (if any occur) then they must be below the non-zero rows

(ii) the number of zeros before the first non-zero element in each row is less than the number of such zeros in the next non zero row.

(iii) Zero rows may have equal number of zeros

Finally , rank of A = number of non-zero rows in the row Echelon form of A.

NOTE :If A is n× n square matrix then row echelon form is same as upper triangular form only

Row Echelon form:

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \qquad \rho(A) = 3$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \qquad \rho(A) = 2$$

$$\begin{bmatrix} 0 & 5 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \rho(A) = 1$$

Row Echelon form:

$$\begin{bmatrix} 6 & 0 & -4 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \rho(A) = 4$$

$$\begin{bmatrix} 5 & 0 & -14 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{pmatrix} 5 & 0 & -14 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rho(A) = 3$

$$\begin{bmatrix} -10 & 0 & 11 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 2$$

$$\begin{bmatrix} 4 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 2$$

$$\begin{bmatrix} 9 & 0 & 2 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 2$$

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To Reduce the given matrix A into Row Echelon form we can apply the following operations

$$\left.\begin{array}{l} 1)\ R_i \to \alpha R_i \\ 2)\ R_i \to \alpha R_i + \beta R_j \\ 3)\ R_i \to R_i + \alpha R_j \\ 4)\ R_i \leftrightarrow R_j \end{array}\right\}$$

Important points

Elementary operations do not change the rank of a matrix.

To find rank of a matrix, we have to reduce the matrix A into its row Echelon form, using only elementary row transformations(operations)

Given matrix $\xrightarrow{\text{Elementary row operations}}$ row echelon form

Method:

Step-1 first row and first column element is to be non zero (if possible)

Step-2 using R_1 make first column elements $a_{21}, a_{31}, a_{41}, a_{51}, \ldots\ldots$ as zero

Step-3 using R_2 make second column elements $a_{32}, a_{42}, a_{52}, \ldots\ldots$ as zero

Step-4 using R_3 make third column elements $a_{43}, a_{53}, a_{63}, \ldots\ldots$ as zero

$$\begin{bmatrix} \text{nonzero} & - & - & - & - & - \\ 0 & \text{nonzero} & & & & \\ 0 & 0 & \text{nonz} & & & \\ 0 & 0 & 0 & & & \\ \vdots & 0 & 0 & & & \\ 0 & 0 & \vdots & - & - & - \end{bmatrix}$$

3.3 Properties of rank of matrix

1. Rank of a matrix is zero $\Leftrightarrow$ A is a null matrix.

$A_{5\times 4}$

2. If $A \neq O$ then $\rho(A) \geq 1$.

Imp

03. If $A_{m\times n} \neq O$ then $\rho(A_{m\times n}) \leq \min\{m, n\}$.

Imp

$A_{3\times 3}$

04. If $|A_{n\times n}| \neq 0$ then $\rho(A_{n\times n}) = n$

Imp

05. If $|A_{n\times n}| = 0$ then $\rho(A_{n\times n}) < n$

3.3 Properties of rank of matrix

6. $\rho\,(I_n) = n.$ ✓

Imp

7. $\rho\,(A) = \rho\,(A^T).$ ✓

Imp

8. $\rho\,(AB) \le \min\{\rho\,(A), \rho\,(B)\}.$

9. $\rho\,(A+B) \le \{\rho\,(A) + \rho\,(B)\}.$

10. $\rho\,(A-B) \ge \{\rho\,(A) - \rho\,(B)\}$

3.3 Properties of rank of matrix

Imp 11. $\rho\,(AA^T) = \rho\,(A^T A) = \rho\,(A)$

$$\begin{bmatrix} 2 & 4 & 5 \\ 0 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

12. The rank of a diagonal matrix is equal to the number of non-zero diagonal elements.

$\rho(A) = 2$

diagonal elements are 2, 6, 0

Nonzero diagonal elements (2, 6) — 2

13. (a). If $\rho\,(A_{n\times n}) = n \Rightarrow \rho\,(\text{adj}A) = n$

(b). If $\rho(A_{n\times n}) = n - 1 \Rightarrow \rho\,(\text{adj}A) = 1$

(c). If $\rho\,(A_{n\times n}) \leq n - 2 \Rightarrow \rho\,(\text{adj}A) = 0$

In this case adj A is a zero matrix

Problem Find Rank of the matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}_{3\times 4}$ is ___

$R_2 \to 3R_2 + R_1$
$R_3 \to 3R_3 - 7R_1$
$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ 0 & 42 & 28 & 58 \\ 0 & -42 & -28 & -58 \end{bmatrix}$$

$R_3 \to R_3 + R_2$
$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$