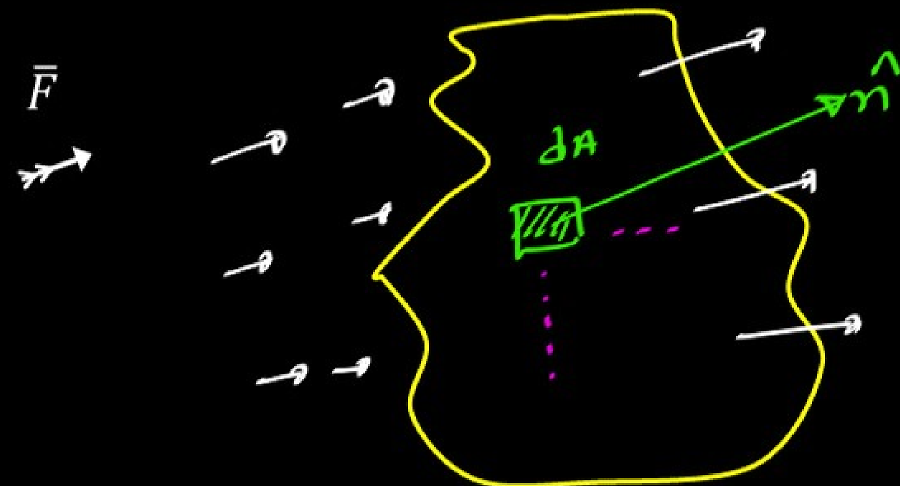
Basic (3): AREA DIRECTION

09-08-22

- The area direction is normal to it's own surface.

- For open surface normal direction is not uniquely defined but for the closed surface it is considered always out wards.

SURFACE INTEGRAL



open surface

(1) $\overline{dA} = dA\hat{n}$

(2) $\bar{F} = \overline{dA}$

(3) $\iint \bar{F} = \overline{dA}$ → Surface Integral

→ Flow

→ Flux

(4) $\oiint \bar{F} = \overline{dA}$ → Closed Surface Integral

→ Net Flow

→ Net Flux

- Closed surface encloses volume.

Ex:

(1) Vector field $\xrightarrow{\iint(\ ).\overline{dA}}$ Flux

(2) Electric Fiel $\xrightarrow{\iint(\ ).\overline{dA}}$ Electric flux (ψ_e c)

(3) Magnetic Field $\xrightarrow{\iint(\ ).\overline{dA}}$ Magnetic flux(ψ_m wb)

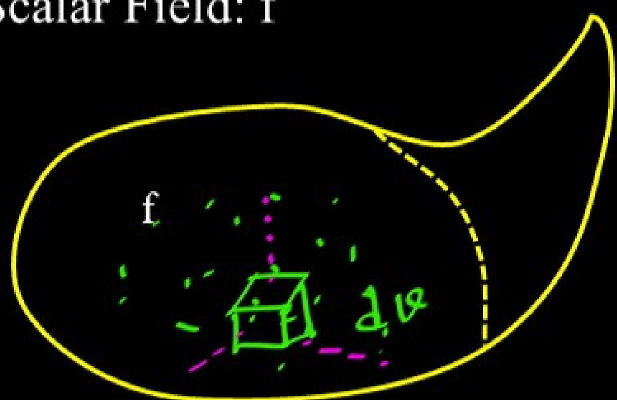
- The amount of vector field crossing the given surface ($\hat{n}$) can be evaluated by using surface integral.

Application:

"Evaluation of Flux"

VOLUME INTEGRAL

Scalar Field: f



Closed surface

(1) fdv

(2) $\iiint f dv \rightarrow$ Volume integral

- The total scalar field filled inside the volume can be evaluated by using volume integral.

Ex:

(1) Scalar field $\xrightarrow{\iiint (\ \ ) dv}$ Total Scalar Field

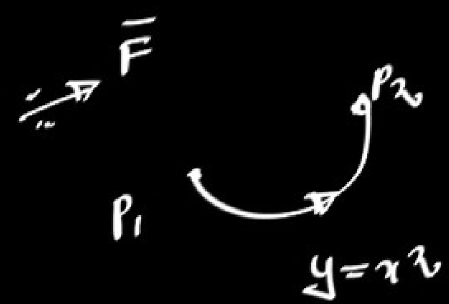
(2) Volume charge (ρ_v c/m^3) $\xrightarrow{\iiint (\ \ ) dv}$ Total charge

Application:

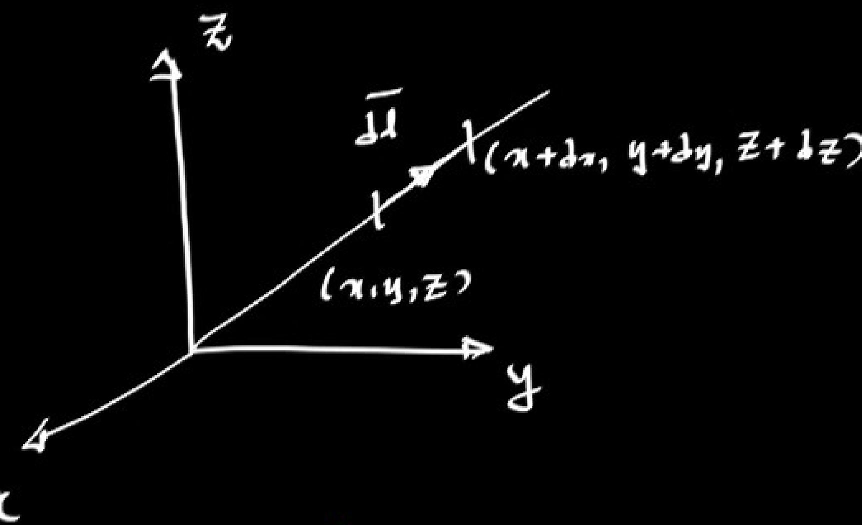
"Evaluation of Total Scalar Field"

Q. Given a vector field is defined by $\bar{F} = y\hat{x} + x\hat{y}$, evaluate line integral from point P_1 (1, 1, -1) to P_2 (2, 4, -1) along the parabola $y = x^2$

Soln:



$$I = \int_{P_1}^{P_2} \bar{F} \cdot d\bar{l}$$



$$\boxed{d\bar{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}}$$

DIFFERENTIAL LENGTH.

$$I = \int (x\hat{y} + y\hat{x}) \cdot (dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z})$$

$$= \int (x\,dy + y\,dx)$$

$$y = x^2, \quad dy = 2x\,dx$$

$$= \int x\;2x\,dx + x^2\,dx$$

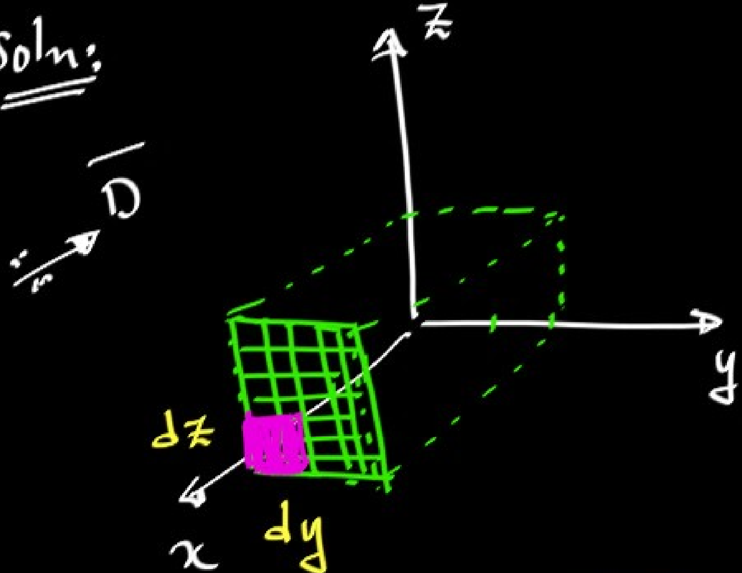
$$= \int 2x^2\,dx + x^2\,dx$$

$$= \int 3x^2\,dx = 3\,\frac{x^3}{3}$$

$$= \left[x^3\right]_1^2 = 8 - 1 = 7.$$

Q. A vector field in certain region is described as $\bar{D} = xy\hat{x} + yz\hat{y} + zx\hat{z}$. Find amount of vector field crossing the surface x = 4 m and bounded by the limits $0 \le y \le 2m$; $0 \le z \le 1m$.

Soln:



$$\boxed{\bar{dA} = dy\,dz\,(\pm\hat{x})}$$

$$\text{Flux}(\psi) = \iint \bar{D} \cdot \bar{dA}$$

$$\boxed{\bar{dA} = dx\,dy\,\hat{z} + dy\,dz\,\hat{x} + dz\,dx\,\hat{y}}$$

DIFFERENTIAL SURFACE.

$x = 4m,\ dx = 0,\quad \bar{dA} = dy\,dz\,\hat{x}$

$$\psi = \iint (xy\hat{x} + yz\hat{y} + zx\hat{z}) \cdot dy\,dz\,\hat{x}$$

$$\psi = \iint xy\,dy\,dz$$

$$\psi = x \iint y \, dy \, dz$$

$$\psi = 4 \int y \, dy \int dz$$

$$\psi = 4 \left[\frac{y^2}{2}\right]_0^2 \left[z\right]_0^1$$

$$= 2 \left[2^2\right]\left[1\right]$$

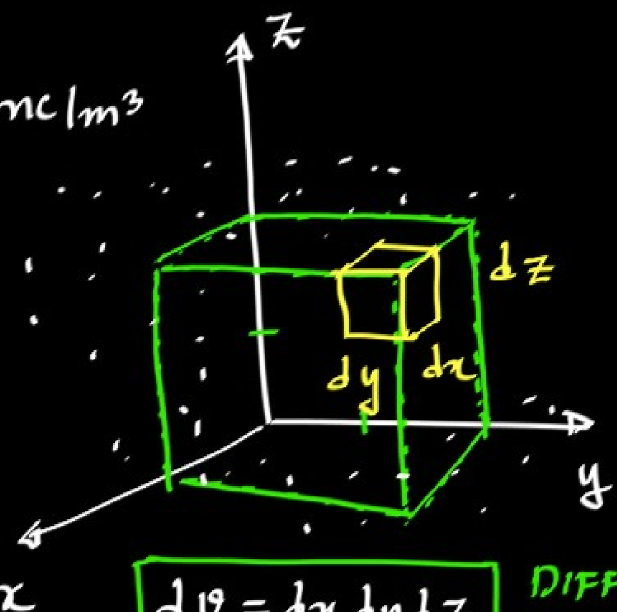
$$\psi = 8$$

Q. The volume charge density in certain region is defined as $2xy^2z$ nc/m^3. Find total charge enclosed by the box defined by the following limits.

$0 \le x \le 1\text{m};\ 0 \le y \le 2\text{m};\ 0 \le z \le 2\text{m}.$

Soln:

ρ_v nc/m^3



$$\boxed{dv = dx\,dy\,dz}$$

DIFFERENTIAL VOLUME.

$$Q = \iiint \rho_v \, dv$$

$$Q = \iiint 2xy^2z \, dx\,dy\,dz$$

$$= 2\iiint xy^2 z \, dx\,dy\,dz$$

$$= 2\int x\,dx\int y^2\,dy\int z\,dz$$

$$= 2\left[\frac{x^2}{2}\right]_0^1\left[\frac{y^3}{3}\right]_0^2\left[\frac{z^2}{2}\right]_0^2$$

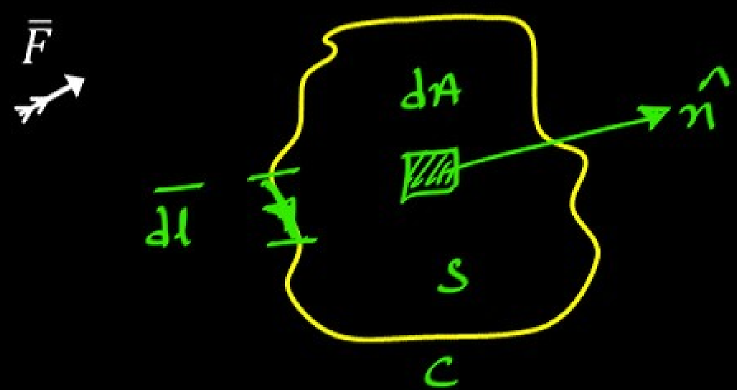
$$= 1^2 \times \frac{2^3}{3} \times \frac{2^2}{2}$$

$$= \frac{16}{3}$$

$$Q = 5.33\ nC$$

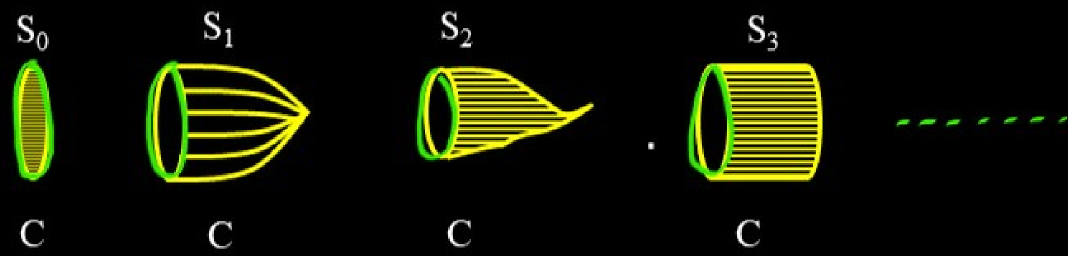
STOKE'S THEOREM

Closed Line Integral $\leftrightarrow$ Open Surface Integral



$$\oint_{C} \bar{F}.\overline{d\ell} = \iint_{S} \nabla \times \bar{F}.\overline{dA}$$

EXTENSION



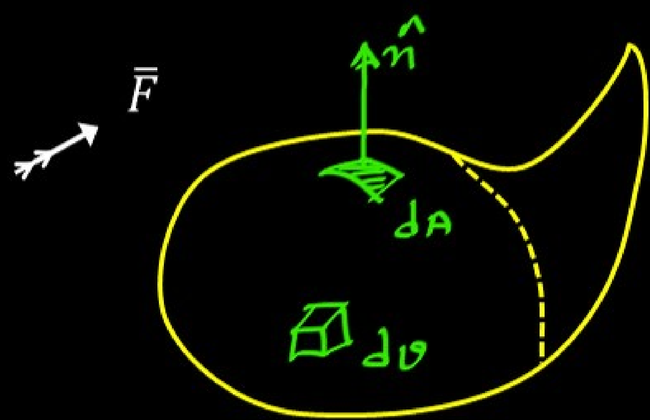
$$\oint_C \bar{F}\cdot d\bar{l} = \iint_{S_0} \nabla\times\bar{F}\cdot d\bar{A} = \iint_{S_1} \nabla\times\bar{F}\cdot d\bar{A} = \iint_{S_2} \nabla\times\bar{F}\cdot d\bar{A} \text{ ------}$$

NOTE: All the enclosed surfaces (S_0,S_1,S_2,S_3……) must have same boundary (c).

NOTE: Integral normal direction for open surface is not uniquely defined, but while relating closed line integral with open surface integral using stoke′s theorem. It can be defined uniquely with the help of right hand thumb rule.

DIVERGENCE THEOREM

Closed Surface Integral $\leftrightarrow$ Volume Integral

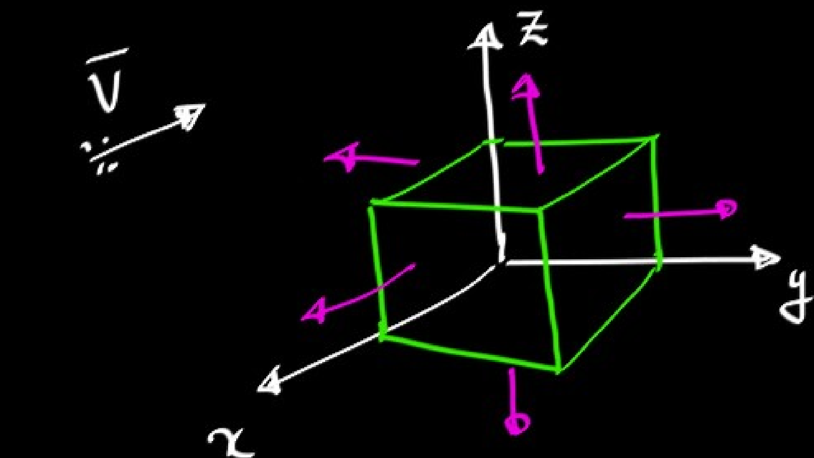


$$\oiint \bar{F}.\overline{dA} = \iiint \nabla.\bar{F}\,dv$$

Q. Given $\bar{V} = x\cos^2 y\hat{i} + x^2 e^z \hat{j} + z\sin^2 y\hat{k}$ and S. The surface of a unit cube with one corner at the origin and edges parallel to co-ordinate axes. The value of interval $\oiint \bar{V}.\underbrace{\hat{n}ds}_{\bar{ds}}.$ Is ________

Soln:

$$I = \oiint \bar{V}\cdot\bar{ds} = \iiint \nabla\cdot\bar{V}\, dv$$



$$\nabla\cdot\bar{V} = \frac{\partial}{\partial x} x\cos^2 y + \frac{\partial}{\partial y} x^2 e^z + \frac{\partial}{\partial z} z\sin^2 y$$

$$\nabla\cdot\bar{V} = \cos^2 y + \sin^2 y = 1$$

$$I = \iiint (1)\, dv = \underbrace{\iiint dv}_{\text{VOLUME}} = 1\times 1\times 1 = 1$$

(or)

$$I = \iiint dv = \iiint dx\,dy\,dz$$

$$I = \int dx \int dy \int dz = [x]_0^1 [y]_0^1 [z]_0^1$$

$$I = 1 \times 1 \times 1 = 1.$$

Q. If a vector field $\bar{V}$ is related to another vector field $\bar{A}$ through $\bar{V} = \nabla \times \bar{A}$. Which of the following is true?

(a) $\oint \overline{d\ell} = \iint \bar{A}.\overline{ds}$

(b) $\oint \nabla \times \bar{V}\overline{d\ell} = \iint \nabla \times \bar{A}.\overline{ds}$

(c) $\oint \bar{A}.\overline{d\ell} = \iint \bar{V}.\overline{ds}$

(d) $\oint \nabla \times \bar{A}.\overline{d\ell} = \iint \bar{V}.\overline{d\ell}$

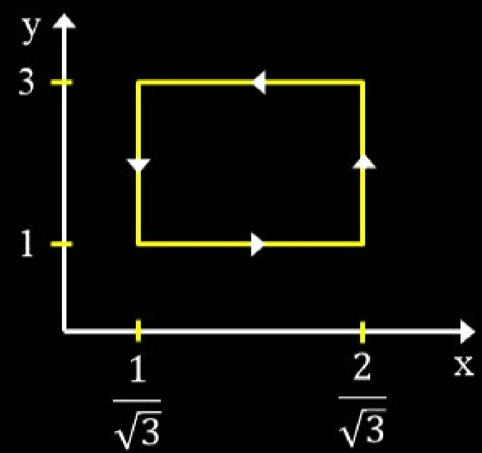
$\bar{V} = \nabla \times \bar{A}$

$\iint \bar{V}\cdot\overline{ds} = \iint \nabla \times \bar{A}\cdot\overline{ds}$

$\iint \bar{V}\cdot\overline{ds} = \oint \bar{A}\cdot\overline{dl}$

Q. If $\bar{A} = xy\hat{x} + x^2\hat{y}$, then $\oint \bar{A}.\overline{d\ell}$ over the path shown in figure is

(GATE-10)



Soln:

$$I = \oint \bar{A}\cdot\overline{dl} = \iint \nabla\times\bar{A}\cdot\overline{ds}$$

$$\nabla\times\bar{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2 & 0 \end{vmatrix}$$

$$= \hat{x}[0-0] - \hat{y}[0-0] + \hat{z}[2x - x]$$

$$\nabla\times\bar{A} = x\hat{z}$$

$$\overline{ds} = dx\,dy\,\hat{z} + dy\,dz\,\hat{x} + dz\,dx\,\hat{y}$$

$$I = \iint x\hat{z}\cdot dx\,dy\,\hat{z}$$

$$I = \iint x\,dx\,dy = \int x\,dx \int dy$$

$$= \left[\frac{x^2}{2}\right]_{1/\sqrt{3}}^{2/\sqrt{3}} \left[y\right]_{2}^{3}$$

$$= \frac{1}{2}\left[\frac{4}{3} - \frac{1}{3}\right]\left[3 - 2\right]$$

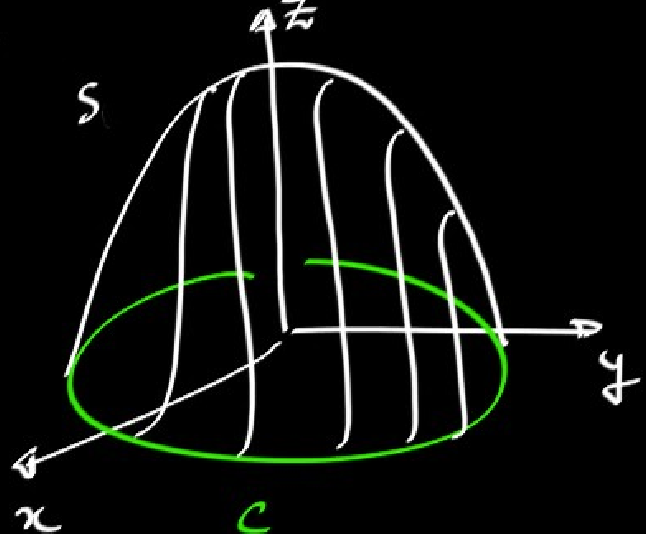
$$= \frac{1}{2}\left[2\right]\left[2\right] = 1$$

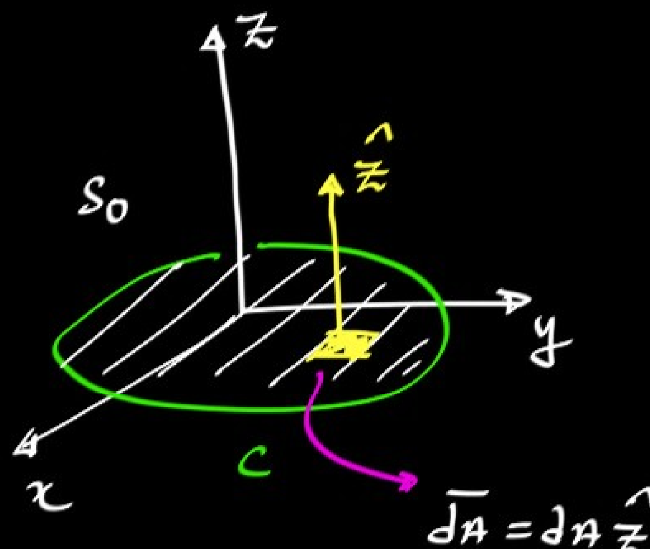
$$\underline{\underline{I = 1}}$$

Q. Given $\bar{F} = z\hat{x} + x\hat{y} + y\hat{z}$. If S represents the portion of sphere $x^2 + y^2 + z^2 = 1$ for $z \geq 0$, then $\iint_S \nabla \times \bar{F}.\overline{dA}$ is ____

(GATE-14)

Soln:





$$I = \iint_S \nabla \times \bar{F} \cdot \overline{dA} = \oint_C \bar{F} \cdot \overline{dl} = \iint_{S_0} \nabla \times \bar{F} \cdot \overline{dA}$$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$= \hat{x}[1-0] - \hat{y}[0-1] + \hat{z}[1-0]$$

$$\nabla \times \bar{F} = \hat{x} + \hat{y} + \hat{z}$$

$$I = \iint_{S_0} \nabla \times \bar{F} \cdot \overline{dA} = \iint_{S_0} (\hat{x} + \hat{y} + \hat{z}) \cdot dA\, \hat{z}$$

$$I = \underbrace{\iint_{S_0} dA}_{\text{AREA.}}$$

$$I = \pi (1)^2$$

$$I = \pi = 3.14$$

Q. The vector field is described as $\bar{F} = x\hat{x} + y\hat{y} + z\hat{z}$.

Evaluate the net flux crossing through the volume

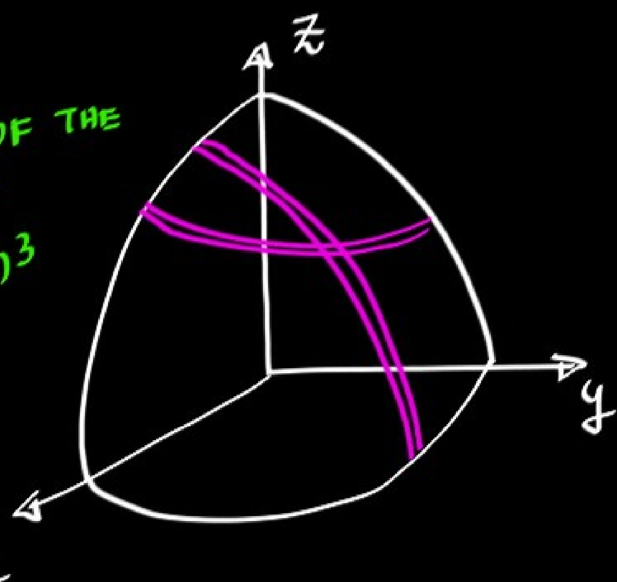
$x \geq 0, y \geq 0, z \geq 0, x^2 + y^2 + z^2 \leq 1$

Soln:

OCTANT OF THE SPHERE

$\frac{1}{8} \times \frac{4}{3} \times \pi \times (1)^3$

$= \frac{\pi}{6} \, m^3$



$$\text{NET FLUX} = \oiint \bar{F} \cdot \bar{dA} = \iiint \nabla \cdot \bar{F} \, dv$$

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 1 + 1 + 1$$

$$\nabla \cdot \bar{F} = 3$$

$$\text{NET FLUX} = \iiint (3) \, dv = 3 \underbrace{\iiint dv}_{\text{VOLUME}}$$

$$= 3 \times \frac{\pi}{6} = \frac{\pi}{2}$$

MSB

Q. The vector field $\bar{F}(x, y) = y\hat{x} + x\hat{y}$ has following characteristic(s)

(a) Divergence of curl of field $\bar{F}$ is zero

(b) Vector field $\bar{F}$ is conservative

(c) Laplacian of field $\bar{F}$ is non-zero

(d) Divergence of field $\bar{F}$ is zero

Soln:

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = \hat{x}[0-0] - \hat{y}[0-0] + \hat{z}[1-1]$$

$$\nabla \times \bar{F} = 0$$

* $\nabla^2 \bar{F} = \nabla^2 (y\hat{x} + x\hat{y}) = \nabla^2 y \, \hat{x} + \nabla^2 x \, \hat{y}$

$$= \left(\underbrace{\frac{\partial^2}{\partial x^2} y}_{0} + \underbrace{\frac{\partial^2}{\partial y^2} y}_{0} + \underbrace{\frac{\partial^2}{\partial z^2} y}_{0} \right) \hat{x} + \left(\underbrace{\frac{\partial^2}{\partial x^2} x}_{0} + \underbrace{\frac{\partial^2}{\partial y^2} x}_{0} + \underbrace{\frac{\partial^2}{\partial z^2} x}_{0} \right) \hat{y}$$

$$= 0\hat{x} + 0\hat{y} = 0$$

* $\nabla \cdot \bar{F} = \frac{\partial}{\partial x} y + \frac{\partial}{\partial y} x = 0 + 0$

$$\nabla \cdot \bar{F} = 0$$

Co-ordinate system

(i) Cartesian co-or (x, y, z)

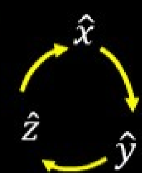
Limits

$-\infty < x < +\infty$

$-\infty < y < +\infty$

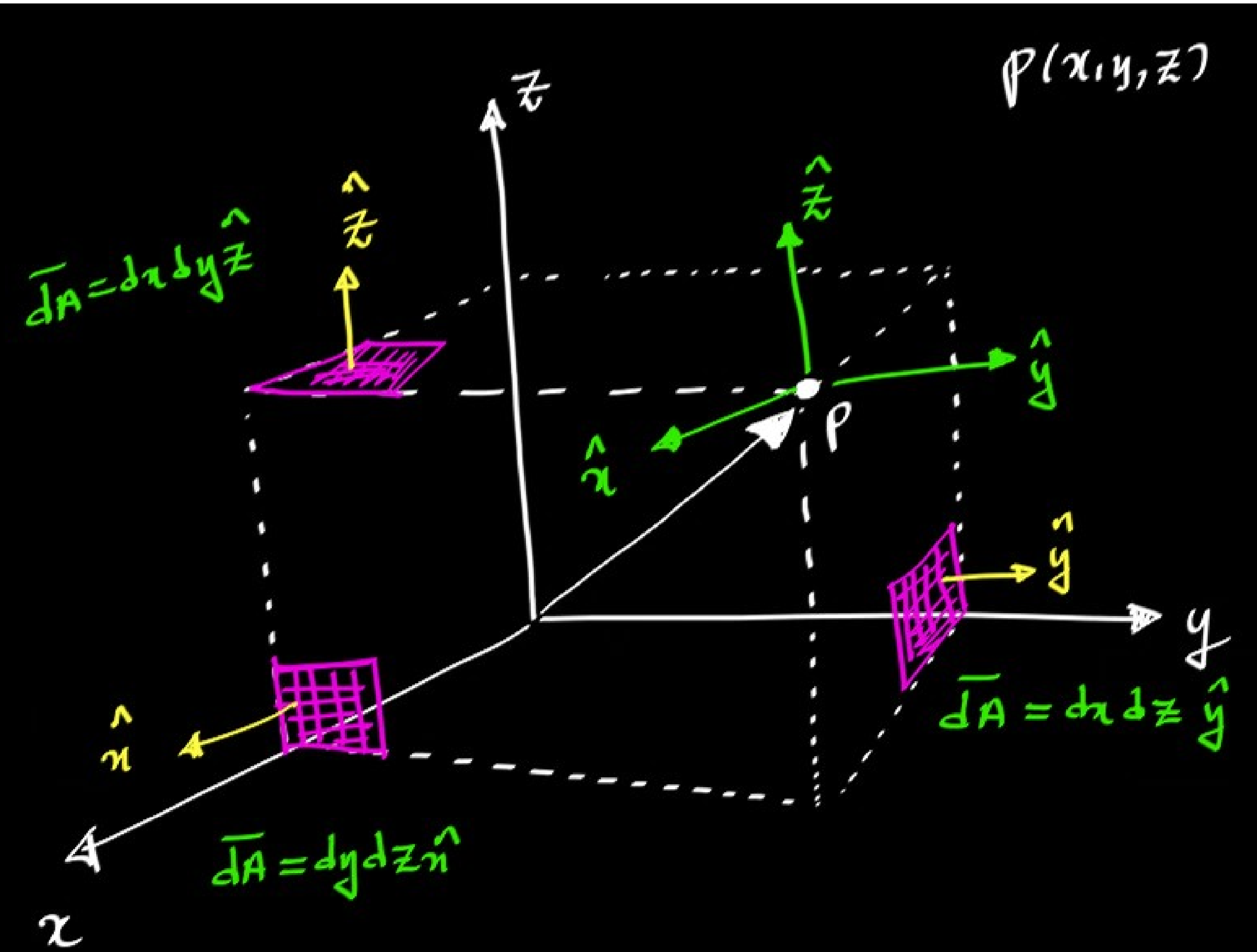
$-\infty < z < +\infty$

- Right handed and orthogonal co-ordinate system.
- $\hat{x} \perp \hat{y} \perp \hat{z}$



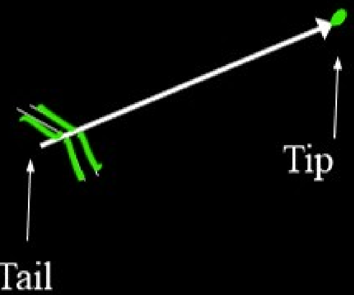
- $\hat{x} \times \hat{y} = \hat{z}$
- $\hat{y} \times \hat{x} = -\hat{z}$
- $\hat{y} \times \hat{z} = \hat{x}$
- $\hat{z} \times \hat{y} = -\hat{x}$

⋮



p(x,y,z)
z
$\hat{z}$
$\bar{dA} = dx\,dy\,\hat{z}$
$\hat{z}$
$\hat{y}$
P
$\hat{x}$
$\hat{y}$
y
$\bar{dA} = dx\,dz\,\hat{y}$
$\hat{x}$
$\bar{dA} = dy\,dz\,\hat{x}$
x

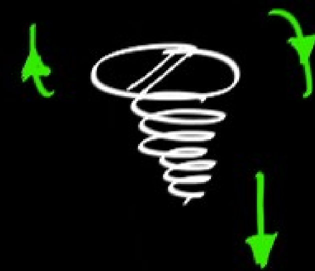
Ex:

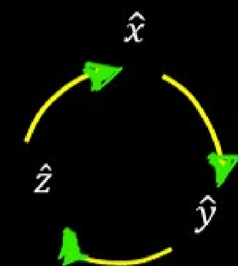


$\odot$ “Towards us”

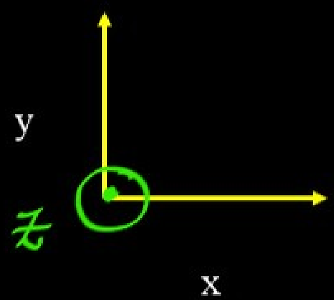
$\otimes$ “Away from us”

Mechanical Screw

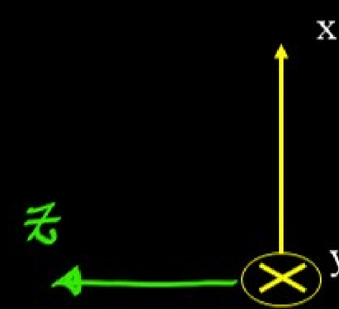




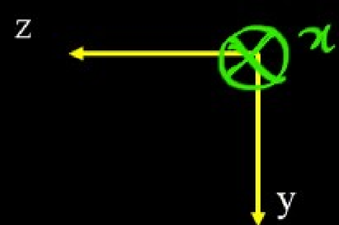
Ex: (1)



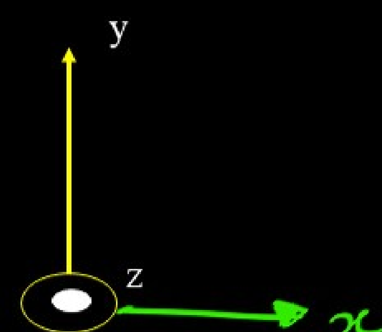
Ex: (2)



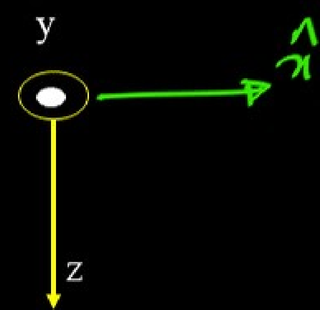
Ex: (3)



Ex: (4)

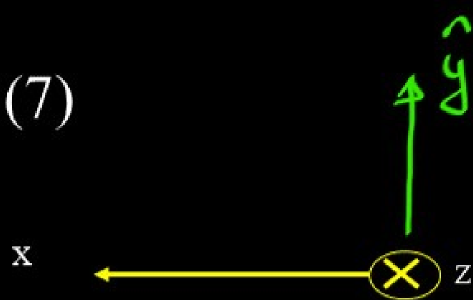


Ex: (5)

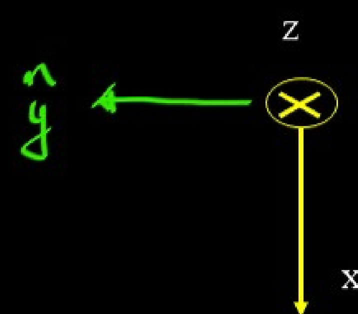


Ex: (6)

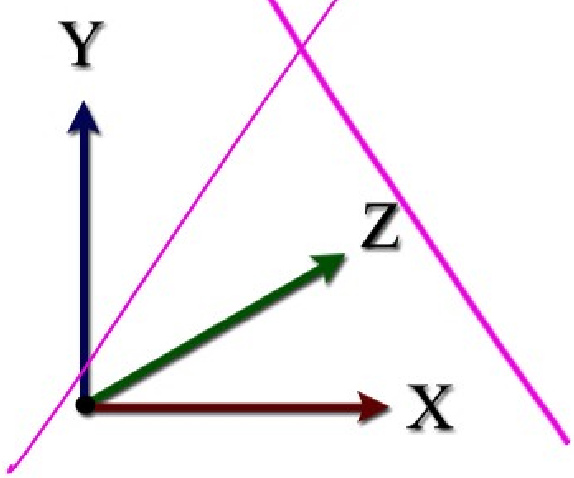
Ex: (7)



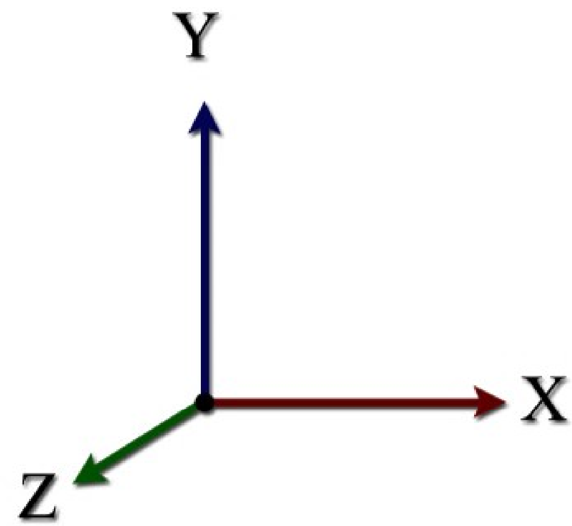
Ex: (8)

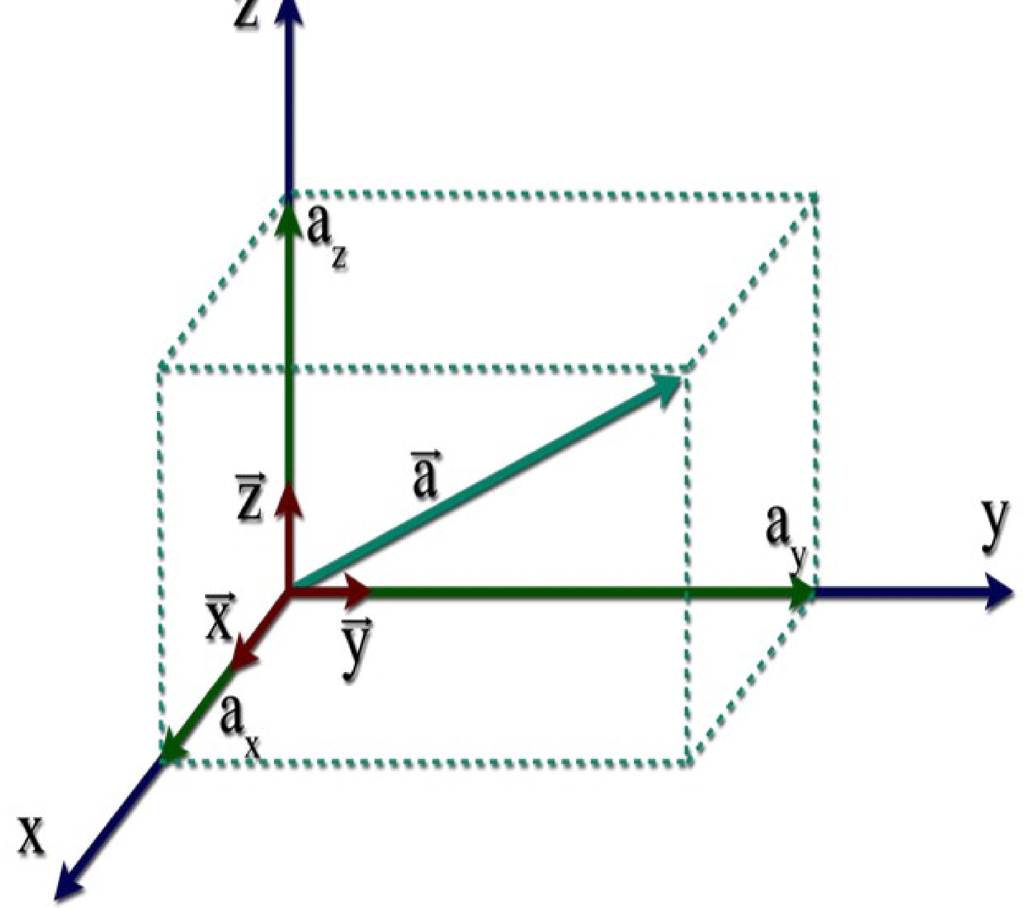


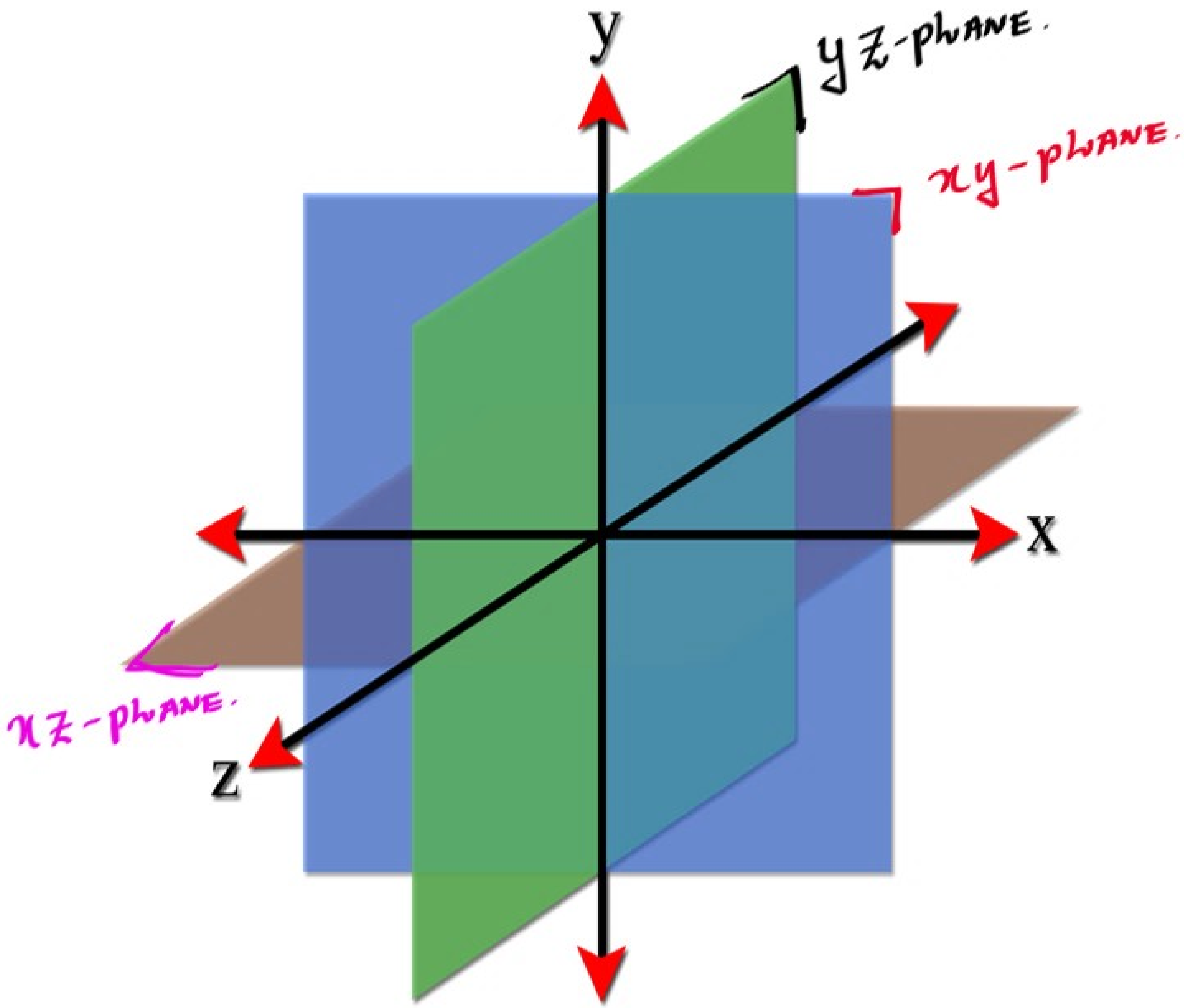
~~Left-handed~~
~~Cartesian Coordinates~~



Right-handed
Cartesian Coordinates




z
a_z
$\vec{z}$
$\vec{a}$
a_y
y
$\vec{x}$
$\vec{y}$
a_x
x


y
yz-plane.
xy-plane.
x
xz-plane.
z

(II) CYLINDRICAL CO-OR (ρ, ϕ,z)

LIMITS

$$\left\{\begin{aligned} &0 \le \rho < \infty \\ &0 \le \phi \le 2\pi \\ &-\infty < z < +\infty \end{aligned}\right\}$$

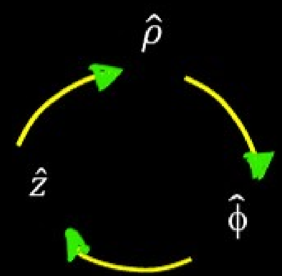
ρ: Radius of CYLINDER.

ϕ: Azimuthal Angle

z: Axial Length

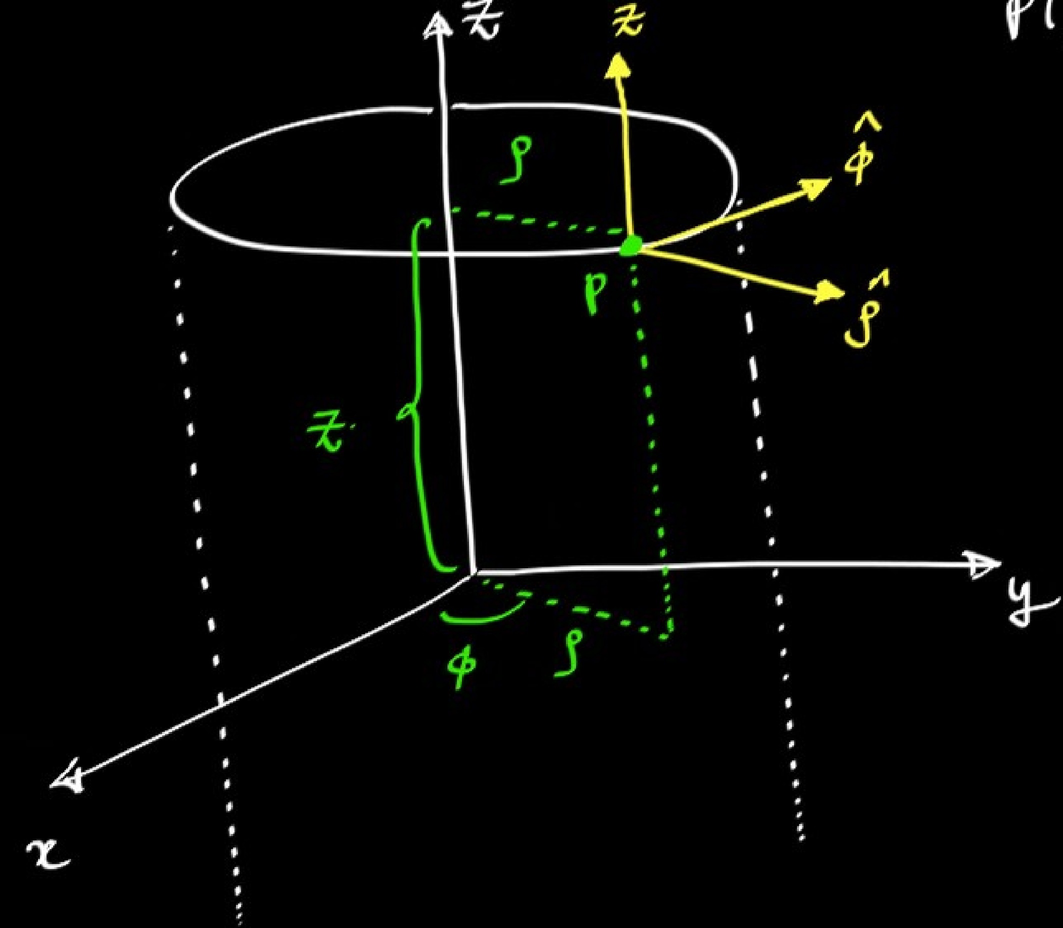
- Right handed and orthogonal co-ordinate system.
- $\hat{\rho} \perp \hat{\phi} \perp \hat{z}$

- $\hat{\rho} \times \hat{\phi} = \hat{z}$
- $\hat{\phi} \times \hat{\rho} = -\hat{z}$
- $\hat{\phi} \times \hat{z} = \hat{\rho}$
- $\hat{z} \times \hat{\phi} = -\hat{\rho}$

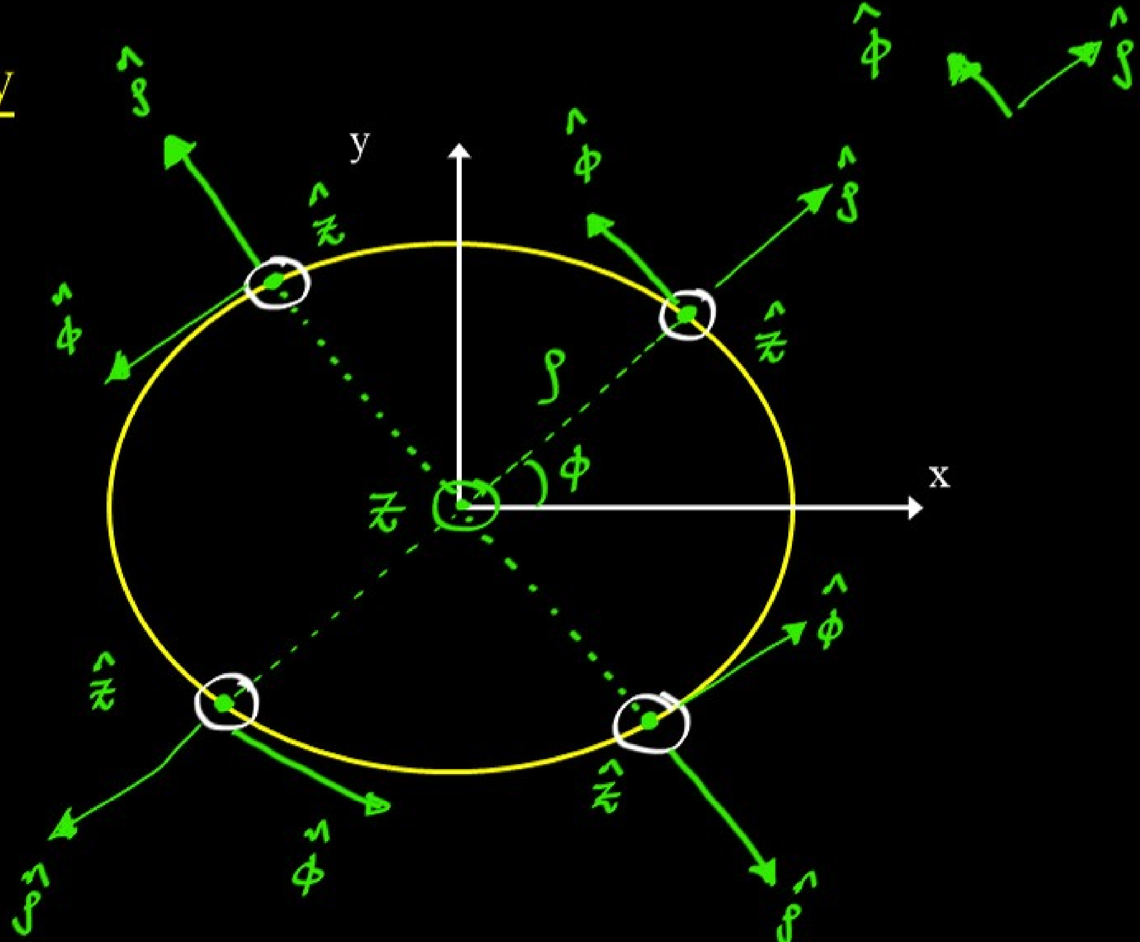
- 

NOTE: Angle direction is tangent direction at that point.

$P(\rho, \phi, z)$



TOP-VIEW



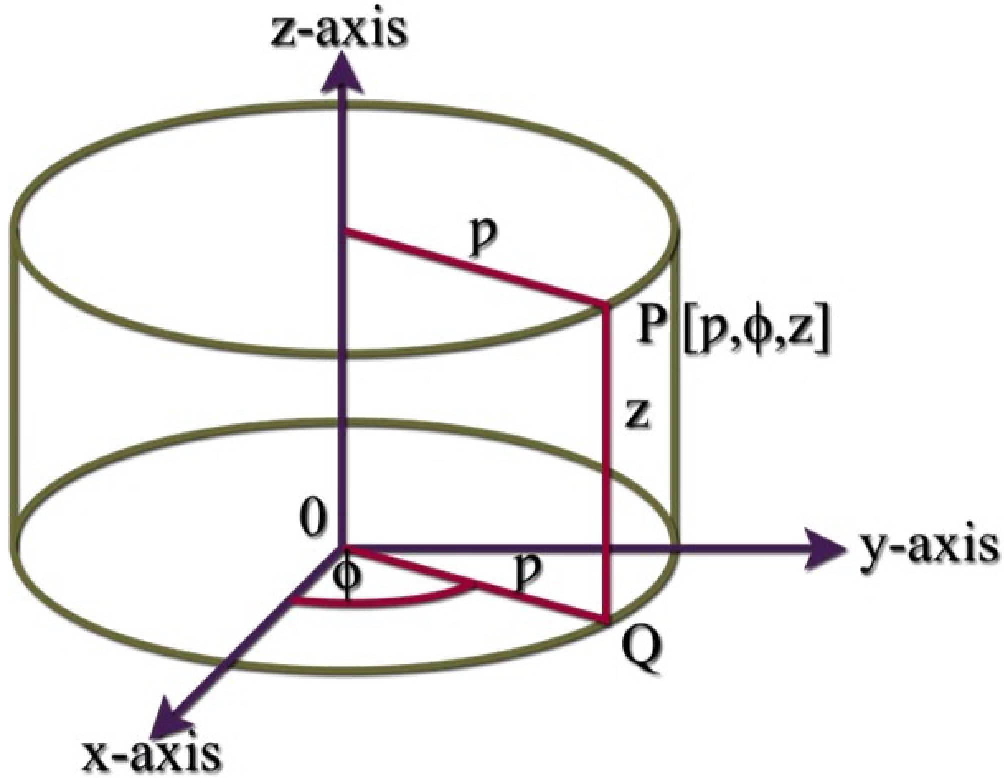
CONVERSIONS

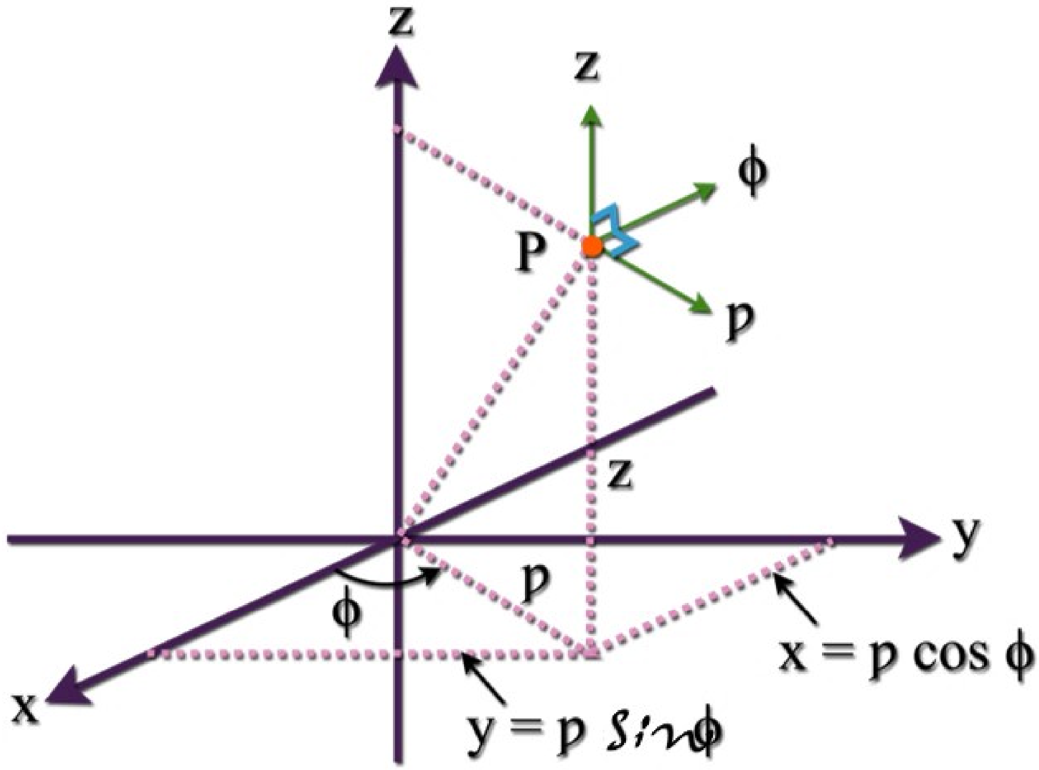
$(\rho, \phi, z) \rightarrow (x, y, z)$

- $x = \rho \cos\phi$
- $y = \rho \sin\phi$
- $z = z$

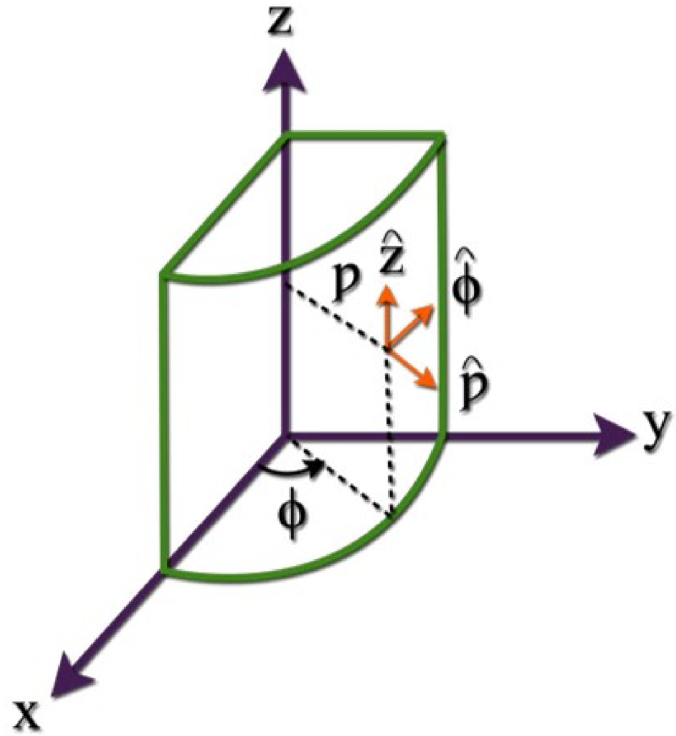
$(x, y, z) \rightarrow (\rho, \phi, z)$

- $\rho = \sqrt{x^2 + y^2}$
- $\phi = \tan^{-1}\left[\frac{y}{x}\right]$
- $z = z$


z-axis
p
P [p,ϕ,z]
z
0
y-axis
ϕ
p
Q
x-axis



z
z
ϕ
P
p
z
y
p
ϕ
x = p cos ϕ
x
y = p sinϕ


z
ρ
$\hat{z}$
$\hat{\phi}$
$\hat{\rho}$
y
φ
x

(III) SPHERICAL CO-ORD (r,θ,ϕ)

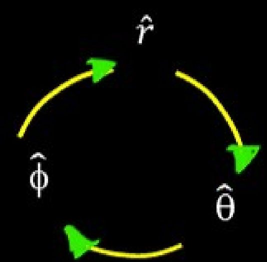
LIMITS

$$\left\{\begin{array}{l} 0 \le r < \infty \\ 0 \le \theta \le \pi \\ 0 \le \phi \le 2\pi \end{array}\right\}$$

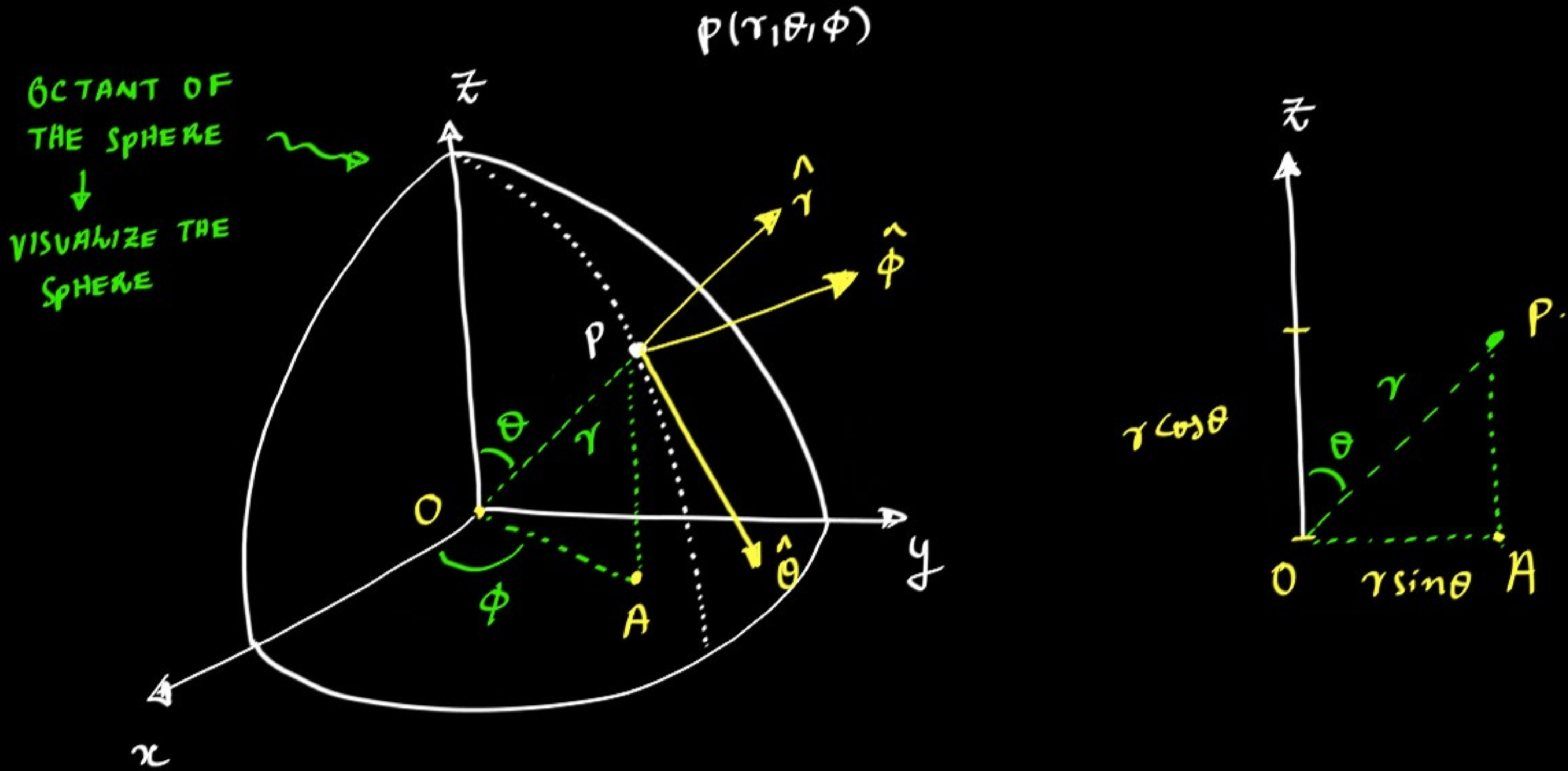
r: Radius of Sphere

θ: Elevation Angle

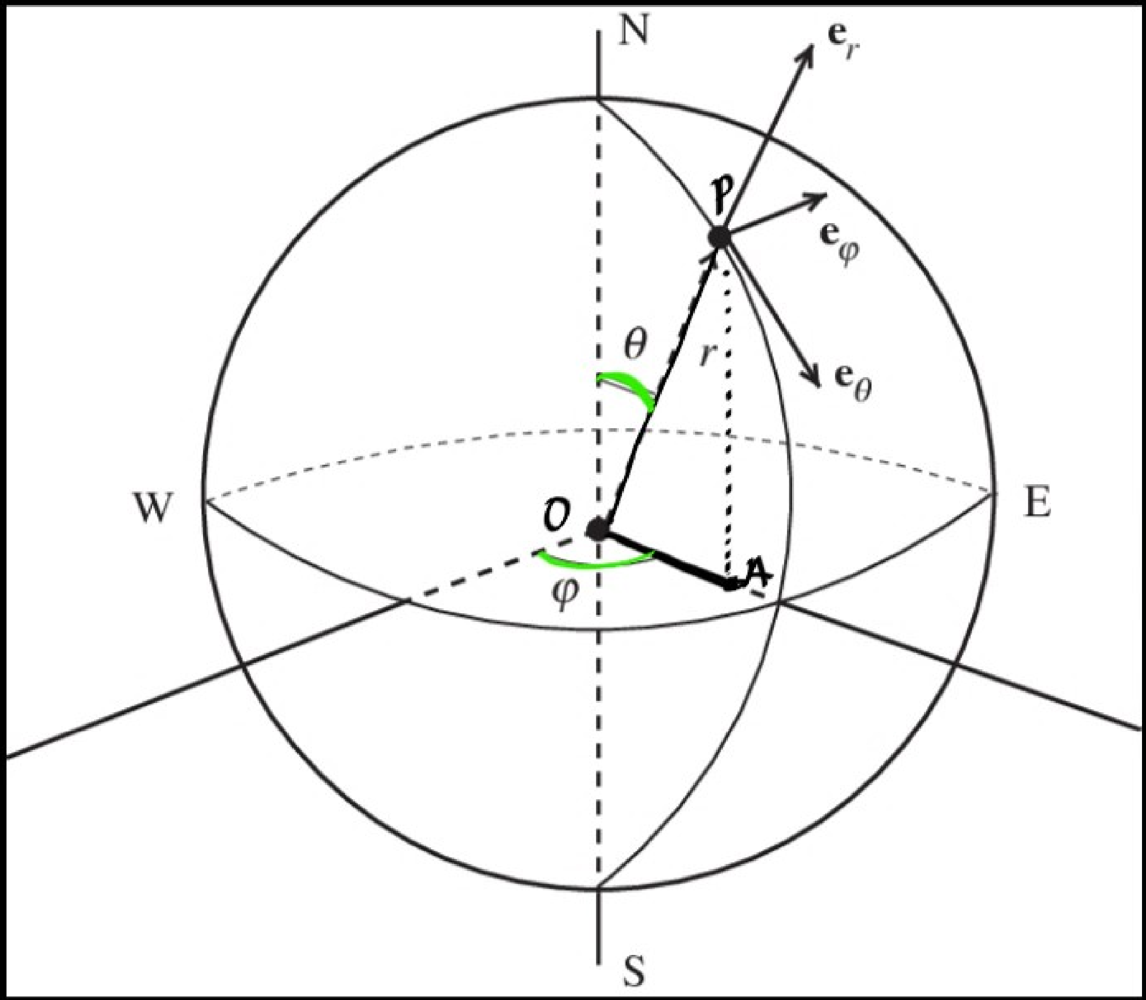
ϕ : Azimuthal Angle

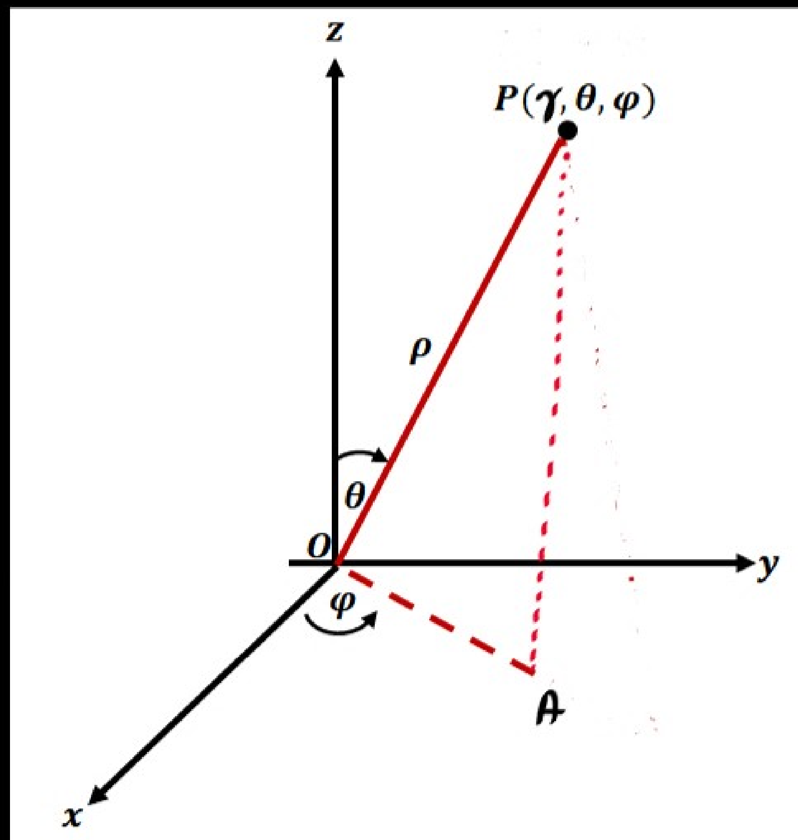
- Right handed and orthogonal co-ordinate system.
- $\hat{r} \perp \hat{\theta} \perp \hat{\phi}$
- 

- $\hat{r} \times \hat{\theta} = \hat{\phi}$
- $\hat{\theta} \times \hat{r} = -\hat{\phi}$
- $\hat{\theta} \times \hat{\phi} = \hat{r}$
- $\hat{\phi} \times \hat{\theta} = -\hat{r}$



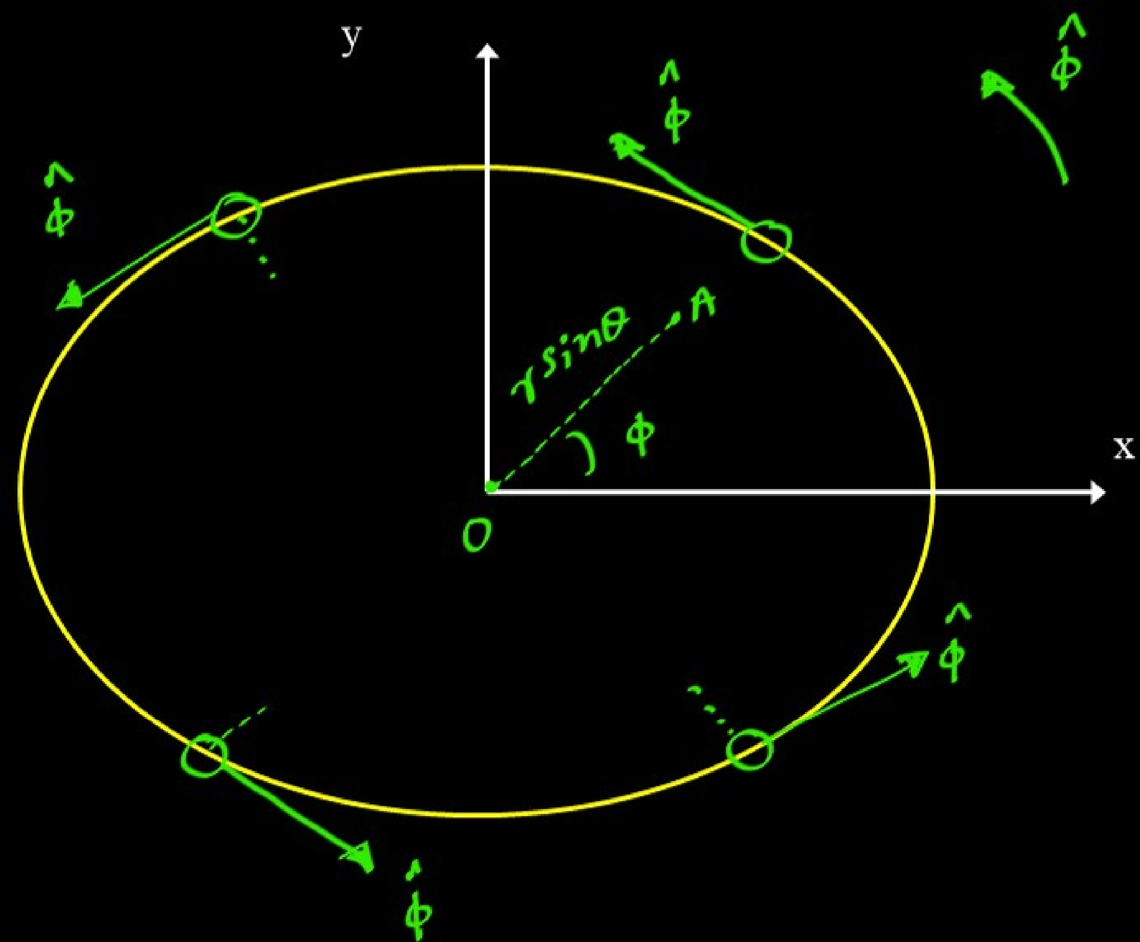
p(r, θ, φ)
OCTANT OF THE SPHERE
VISUALIZE THE SPHERE
z
P
θ
r
O
φ
A
y
x
r̂
φ̂
θ̂
z
P.
r
θ
r cosθ
O
r sinθ
A


N
$\mathbf{e}_r$
P
$\mathbf{e}_\varphi$
θ
r
$\mathbf{e}_\theta$
W
O
E
A
φ
S

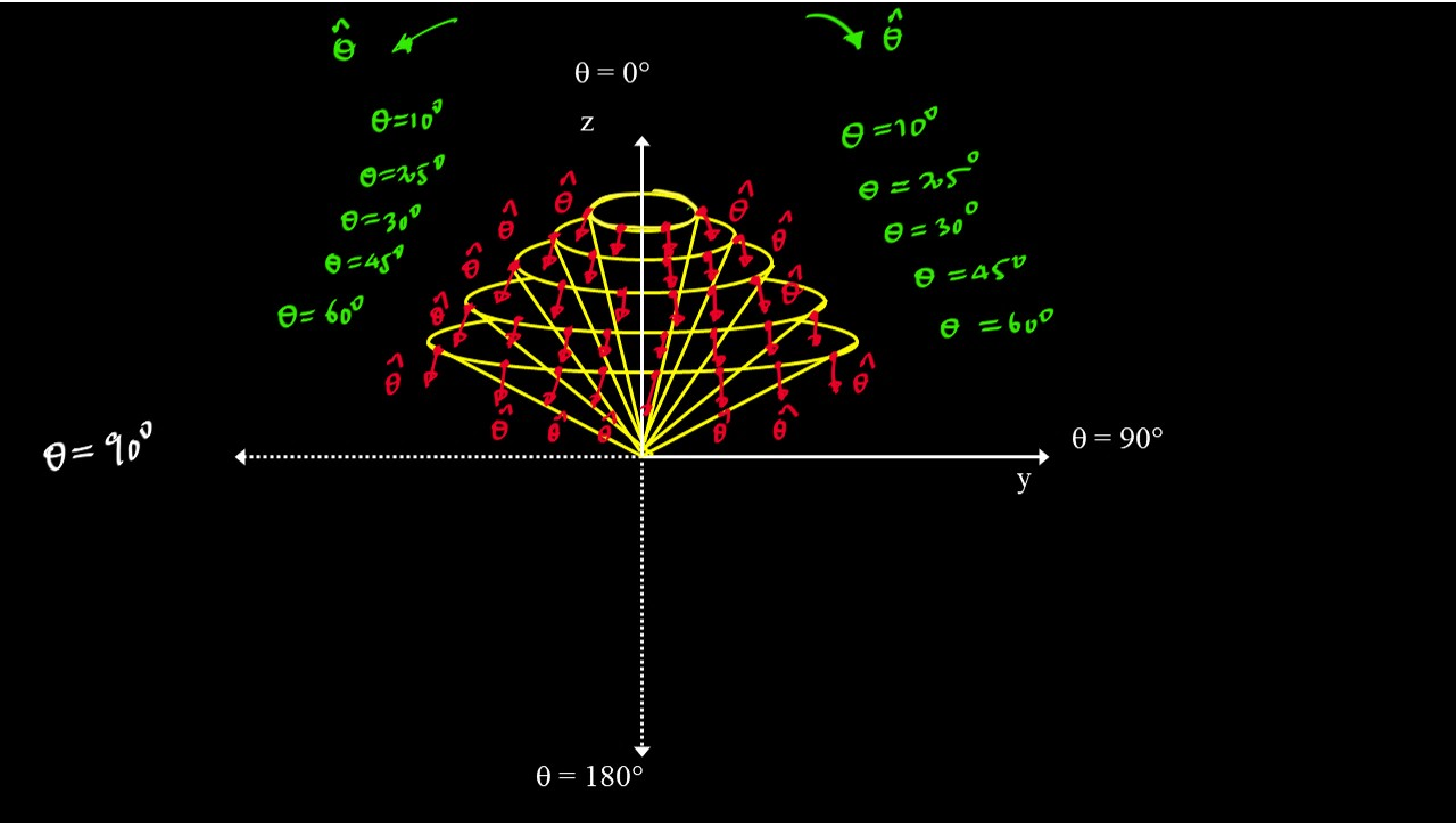


z
P(γ, θ, φ)
ρ
θ
O
y
φ
A
x

TOP-VIEW



y
x
$r\sin\theta$
A
ϕ
O
$\hat{\phi}$
$\hat{\phi}$
$\hat{\phi}$
$\hat{\phi}$
$\hat{\phi}$
$\hat{\phi}$



θ = 0°
z
y
θ = 90°
θ = 180°
θ=10°
θ=25°
θ=30°
θ=45°
θ=60°
θ=90°

CONVERSIONS

$(r, \theta, \phi) \to (x, y, z)$

- $z = r\cos\theta$
- $x = (r\sin\theta)\cos\phi$
- $y = (r\sin\theta)\sin\phi$

$(x, y, z) \to (r, \theta, \phi)$

- $r = \sqrt{x^2 + y^2 + z^2}$
- $\theta = \cos^{-1}\left[\frac{z}{r}\right]$
- $\phi = \tan^{-1}\left[\frac{y}{x}\right]$