elements are all real, so that $I_{ik} = I_{ki}^*$. Therefore, subtracting the second of these equations from the first, we find

$$(I_m - I_n^*)\sum_l \omega_{lm}\omega_{ln}^* = 0 \tag{11.88}$$

For the case $m = n$, we have

$$(I_m - I_m^*)\sum_l \omega_{lm}\omega_{lm}^* = 0 \tag{11.89}$$

The sum is just the definition of the scalar product of $\boldsymbol{\omega}_m$ and $\boldsymbol{\omega}_m^*$:

$$\boldsymbol{\omega}_m \cdot \boldsymbol{\omega}_m^* = |\boldsymbol{\omega}_m|^2 \geq 0 \tag{11.90}$$

Therefore, because the squared magnitude of $\boldsymbol{\omega}_m$ is in general positive, it must be true that $I_m - I_m^*$ for Equation 11.89 to be satisfied. If a quantity and its complex conjugate are equal, then the imaginary parts must vanish identically. Thus, the principal moments of inertia are all real. Because $\{\mathbf{I}\}$ is real, the vectors $\boldsymbol{\omega}_m$ must also be real.

If $m \neq n$ in Equation 11.88 and if $I_m \neq I_n$, then the equation can be satisfied only if $\boldsymbol{\omega}_m \cdot \boldsymbol{\omega}_n = 0$; that is, these vectors are orthogonal, as before.

In all the proofs carried out in this section, we have referred to the inertia tensor. But examining these proofs reveals that the only properties of the inertia tensor that have actually been used are the facts that the tensor is symmetrical and that the elements are real. We may therefore conclude that *any* real, symmetric tensor* has the following properties:

1. Diagonalization may be accomplished by an appropriate rotation of axes, that is, a similarity transformation.
2. The eigenvalues† are obtained as roots of the secular determinant and are real.
3. The eigenvectors† are real and orthogonal.

11.8 Eulerian Angles

The transformation from one coordinate system to another can be represented by a matrix equation of the form

$$\mathbf{x} = \boldsymbol{\lambda}\mathbf{x}'$$

If we identify the fixed system with $\mathbf{x}'$ and the body system with $\mathbf{x}$, then the rotation matrix $\boldsymbol{\lambda}$ completely describes the relative orientation of the two systems. The rotation matrix $\boldsymbol{\lambda}$ contains three independent angles. There are many possible

*To be more precise, we require only that the elements of the tensor obey the relation $I_{ik} = I_{ki}^*$; thus we allow the possibility of complex quantities. Tensors (and matrices) with this property are said to be **Hermitean.**

†The terms *eigenvalues* and *eigenvectors* are the generic names of the quantities, which, in the case of the inertia tensor, are the principal moments and the principal axes, respectively. We shall encounter these terms again in the discussion of small oscillations in Chapter 12.

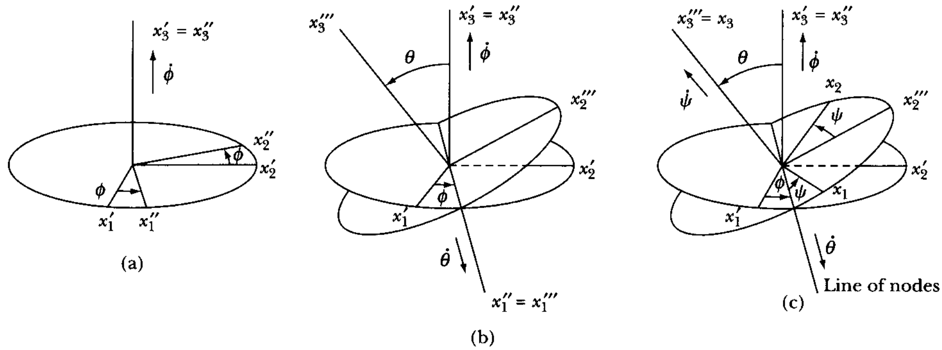

FIGURE 11-9 The Eulerian angles are used to rotate from the x_i' system to the x_i system. (a) First rotation is counterclockwise through an angle ϕ about the x_3'-axis. (b) Second rotation is counterclockwise through an angle θ about the x_1''-axis. (c) Third rotation is counterclockwise through an angle ψ about the x_3'''-axis.

choices for these angles; we find it convenient to use the **Eulerian angles*** ϕ, θ, and ψ.

The Eulerian angles are generated in the following series of rotations, which takes the x_i' system into the x_i system.†

1. The first rotation is counterclockwise through an angle ϕ about the x_3'-axis (Figure 11-9a) to transform the x_i' into the x_i''. Because the rotation takes place in the x_1'-x_2' plane, the transformation matrix is

$$\boldsymbol{\lambda}_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11.91}$$

and

$$\mathbf{x}'' = \boldsymbol{\lambda}_\phi \mathbf{x}' \tag{11.92}$$

2. The second rotation is counterclockwise through an angle θ about the x_1''-axis (Figure 11-9b) to transform the x_i'' into the x_i'''. Because the rotation is now in the x_2''-x_3'' plane, the transformation matrix is

$$\boldsymbol{\lambda}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{11.93}$$

and

$$\mathbf{x}''' = \boldsymbol{\lambda}_\theta \mathbf{x}'' \tag{11.94}$$

*The rotation scheme of Euler was first published in 1776.

†The designations of the Euler angles and even the manner in which they are generated are not universally agreed upon. Therefore, some care must be taken in comparing any results from different sources. The notation used here is that most commonly found in modern texts.

3. The third rotation is counterclockwise through an angle ψ about the x_3'''-axis (Figure 11-9c) to transform the x_i''' into the x_i. The transformation matrix is

$$\boldsymbol{\lambda}_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11.95}$$

and

$$\mathbf{x} = \boldsymbol{\lambda}_\psi \mathbf{x}''' \tag{11.96}$$

The line common to the planes containing the x_1- and x_2-axes and the x_1'- and x_2'-axes is called the **line of nodes.** The complete transformation from the x_i' system to the x_i system is given by

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\lambda}_\psi \mathbf{x}''' = \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \mathbf{x}'' \\ &= \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \boldsymbol{\lambda}_\phi \mathbf{x}' \end{aligned} \tag{11.97}$$

and the rotation matrix $\boldsymbol{\lambda}$ is

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \boldsymbol{\lambda}_\phi \tag{11.98}$$

The components of this matrix are

$$\left.\begin{aligned} \lambda_{11} &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi \\ \lambda_{21} &= -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi \\ \lambda_{31} &= \sin\theta\sin\phi \\ \lambda_{12} &= \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi \\ \lambda_{22} &= -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi \\ \lambda_{32} &= -\sin\theta\cos\phi \\ \lambda_{13} &= \sin\psi\sin\theta \\ \lambda_{23} &= \cos\psi\sin\theta \\ \lambda_{33} &= \cos\theta \end{aligned}\right\} \tag{11.99}$$

(The components λ_{ij} are offset in the preceding equation to assist in the visualization of the complete $\boldsymbol{\lambda}$ matrix.)

Because we can associate a vector with an infinitesimal rotation, we can associate the time derivatives of these rotation angles with the components of the angular velocity vector $\boldsymbol{\omega}$. Thus,

$$\left.\begin{aligned} \omega_\phi &= \dot{\phi} \\ \omega_\theta &= \dot{\theta} \\ \omega_\psi &= \dot{\psi} \end{aligned}\right\} \tag{11.100}$$

The rigid-body equations of motion are most conveniently expressed in the body coordinate system (i.e., the x_i system), and therefore we must express the components of $\boldsymbol{\omega}$ in this system. We note that in Figure 11-9 the angular velocities

$\dot{\boldsymbol{\phi}}$, $\dot{\boldsymbol{\theta}}$, and $\dot{\boldsymbol{\psi}}$ are directed along the following axes:

$\dot{\boldsymbol{\phi}}$ along the x_3'- (fixed) axis

$\dot{\boldsymbol{\theta}}$ along the line of nodes

$\dot{\boldsymbol{\psi}}$ along the x_3- (body) axis

The components of these angular velocities along the body coordinate axes are

$$\left.\begin{aligned} \dot{\phi}_1 &= \dot{\phi} \sin\theta \sin\psi \\ \dot{\phi}_2 &= \dot{\phi} \sin\theta \cos\psi \\ \dot{\phi}_3 &= \dot{\phi} \cos\theta \end{aligned}\right\} \tag{11.101a}$$

$$\left.\begin{aligned} \dot{\theta}_1 &= \dot{\theta} \cos\psi \\ \dot{\theta}_2 &= -\dot{\theta} \sin\psi \\ \dot{\theta}_3 &= 0 \end{aligned}\right\} \tag{11.101b}$$

$$\left.\begin{aligned} \dot{\psi}_1 &= 0 \\ \dot{\psi}_2 &= 0 \\ \dot{\psi}_3 &= \dot{\psi} \end{aligned}\right\} \tag{11.101c}$$

Collecting the individual components of $\boldsymbol{\omega}$, we have, finally,

$$\boxed{\begin{aligned} \omega_1 &= \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\ \omega_2 &= \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\ \omega_3 &= \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi} \cos\theta + \dot{\psi} \end{aligned}} \tag{11.102}$$

These relations will be of use later in expressing the components of the angular momentum in the body coordinate system.

EXAMPLE 11.9

Using the Eulerian angles, find the transformation that moves the original x_1'-axis to the x_2'-x_3' plane halfway between x_2' and x_3' and moves x_2' perpendicular to the x_2'-x_3' plane (Figure 11-10).

Solution. The key to transformations using Eulerian angles is the second rotation about the line of nodes, because this single rotation must move x_3' to x_3. From the statement of the problem, x_3 must be in the x_2'-x_3' plane, rotated 45° from x_3'. The first rotation must move x_1' to x_1'' to have the correct position to rotate $x_3' = x_3''$ to $x_3''' = x_3$.

In this case, $x_3' = x_3''$ is rotated $\theta = 45°$ about the original $x_1' = x_1''$-axis so that $\phi = 0$ and

$$\boldsymbol{\lambda}_\phi = \mathbf{1} \tag{11.103}$$

$$\boldsymbol{\lambda}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \tag{11.104}$$

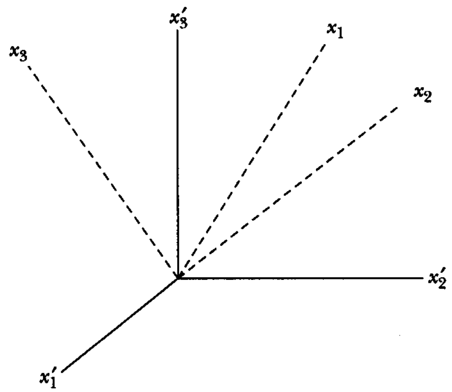

FIGURE 11-10 Example 11.9. We use Eulerian angles to rotate the x'_i system into the x_i system.

The last rotation, $\psi = 90°$, moves $x'_1 = x''_1 = x'''_1$ to x_1 to the position desired in the original x_2-x_3 plane.

$$\boldsymbol{\lambda}_\psi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11.105}$$

The transformation matrix $\boldsymbol{\lambda}$ is $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\psi\boldsymbol{\lambda}_\theta\boldsymbol{\lambda}_\phi = \boldsymbol{\lambda}_\psi\boldsymbol{\lambda}_\theta$:

$$\boldsymbol{\lambda} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\boldsymbol{\lambda} = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \tag{11.106}$$

Direction comparison between the x_i- and x'_i-axes shows that $\boldsymbol{\lambda}$ represents a single rotation describing the transformation.

11.9 Euler's Equations for a Rigid Body

Let us first consider the force-free motion of a rigid body. In such a case, the potential energy U vanishes and the Lagrangian L becomes identical with the rotational kinetic energy T.* If we choose the x_i-axes to correspond to the principal

*Because the motion is force free, the translational kinetic energy is unimportant for our purposes here. (We can always transform to a coordinate system in which the center of mass of the body is at rest.)