day / date:

Operations Research

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OR-M Lecture 1

day / date: 23 /1 Tuesday

Pg 45

2.1 Linear Programming Model:

2 variable LP model : 2D

decision variable ⇒ jis pe profit/loss depend kra ho

Maximization Problem

Example 2.1.1:

raw material	exterior paint	interior paint	max daily availability
M1	6	4	24
	1	2	6
	5	4	

① decision variables:

x_1: exterior paint x_2 : interior paint

② objective function:

max $Z = 5x_1 + 4x_2$

③ constraints restrictions:

$6x_1 + 4x_2 \le 24$ — (i)

$x_1 + 2x_2 \le 6$ — (ii)

$x_2 \le x_1 + 1 \Rightarrow x_2 - x_1 \le 1$ — (iii)

$x_2 \le 2$ — (iv)

maximizing $z = 5x_1 + 4x_2$ subject to

$6x_1 + 4x_2 \le 24$ — ①

$x_1 + 2x_2 \le 6$ — ②

$-x_1 + x_2 \le 1$ — ③

$x_2 \le 2$ — ④

$x_1, x_2 \ge 0$ — ⑤ (non -ve constraints)

④ graphical representation:

x,y intercepts by putting them = 0 one by one

i) $6x_1 + 4x_2 = 24$

putting x_1 and $x_2 = 0$

$x_1 = 0$, $x_2 = 6$ (0,6)

$x_2 = 0$, $x_1 = 4$ (4,0)

* this will result in two points in graph (0,6) and (4,0), we'll join these points and get line (i)

ii) $x_1 + 2x_2 = 6$

(0,3) , (6,0)

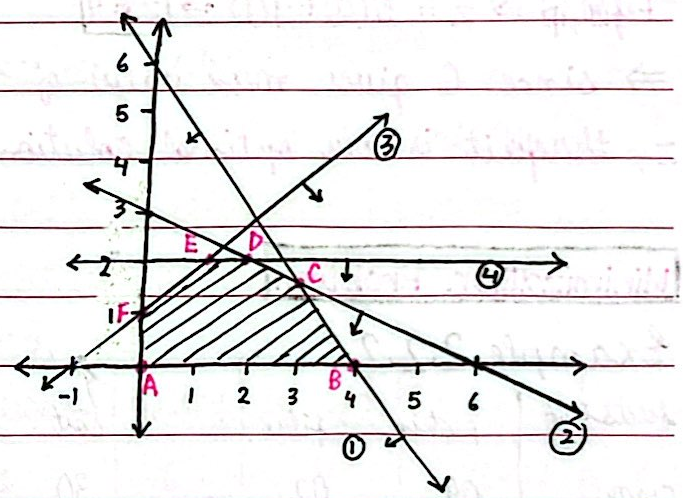
iii) $-x_1 + x_2 = 1$

(0,1) , (-1,0)

iv) $x_2 = 2$

v) $x_1 = x_2 = 0$

feasible solution space/feasible region ⇒ all points represent solution space but we take corner points for convenience



⑤ determining optimal solution:

we need to find a corner point which gives max value of z

- A(0,0) ⇒ $z = 5(0) + 4(0)$ ⇒ $\boxed{z = 0}$
- B(4,0) ⇒ $z = 5(4) + 4(0)$ ⇒ $\boxed{z = 20}$
- for point C, we solve eq.(i) and eq.(ii) since it is at intersection of line (i) and line (ii)

eq(i): $6x_1 + 4x_2 = 24$

eq(ii): $-2x_1 - 4x_2 = -12$ ~~(multiplying eq (ii) by -2)~~

$4x_1 = 12 \Rightarrow x_1 = 3$

putting this in eq (i)

$\Rightarrow x_2 = 1.5$

$C(3, 1.5) \Rightarrow z = 5(3) + 4(1.5) \Rightarrow \boxed{z = 21}$

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- for point D, solving eq (ii) and eq (iv)

$x_2 = 2$ (by eq (iv))

putting this in eq (ii)

$x_1 + 2(2) = 6 \Rightarrow x_1 = 2$

$D(2,2) \Rightarrow z = 5(2) + 4(2) \Rightarrow \boxed{z = 18}$

- for point E, solving eq (iii) and eq (iv)

$x_2 = 2$ (by eq (iv))

putting this in eq (iii)

$-x_1 + 2 = 1 \Rightarrow x_1 = 1$

$E(1,2) \Rightarrow z = 5(1) + 4(2) \Rightarrow z = 13$

$F(0,1) \Rightarrow z = 5(0) + 4(1) \Rightarrow \boxed{z = 4}$

$\Rightarrow$ since C gives max value of z, therefore it is our optimal solution

Minimization Problem

Example 2.2.2

feedstuff	protein	fibre	cost
corn	.09	.02	.30
soybean meal	.60	.06	.90

① decision variables:

x_1 : corn , x_2 : soybean

② objective function:

minimize $z = 0.3x_1 + 0.9x_2$

③ constraints restrictions:

$x_1 + x_2 \geq 800$ —— ①

$0.09x_1 + 0.6x_2 \geq 0.3(x_1 + x_2)$

simplifying

$-0.21x_1 + 0.3x_2 \geq 0$

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$0.21x_1 - 0.3x_2 \leq 0$ —— ②

$0.2x_1 + 0.06x_2 \leq 0.05(x_1 + x_2)$

simplifying

$0.03x_1 - 0.01x_2 \geq 0$ —— ③

$x_1, x_2 \geq 0$

④ graphical representation:

i) $x_1 + x_2 = 800$

$(0,800)$, $(800,0)$

ii) $0.21x_1 - 0.3x_2 = 0$

$0.21x_1 = 0.3x_2 \Rightarrow x_1 = \frac{0.3}{0.21}x_2$

the eq is of form $y = mx + b$

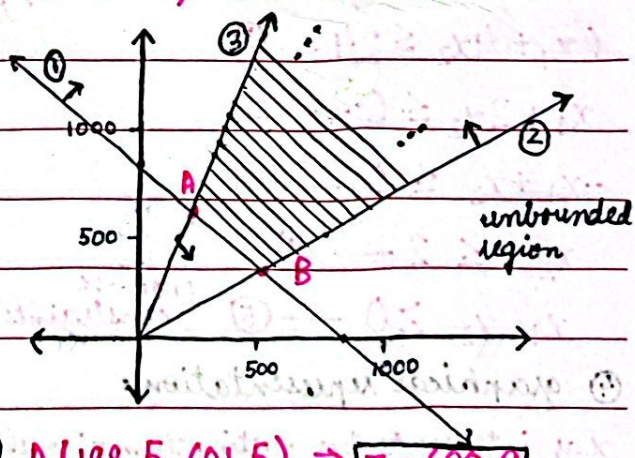
as $(0,0)$ is already a point satisfying our equation, to find second point take random value of x_2 lets say 500

$x_1 = \frac{0.3}{0.21}(500) \Rightarrow x_1 = 714$

$(714, 500)$, $(0,0)$

iii) $0.03x_1 - 0.01x_2 \geq 0$

$(0,0)$, $(167, 500)$



⑤ $A(198.5, 601.5) \Rightarrow \boxed{z = 600.9}$

$B(470.6, 329.4) \Rightarrow \boxed{z = 437.64}$

so, B is our optimum answer.

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* maximization can only be found for bounded region while minimization can be found on both bounded & unbounded regions.

Exercise Problems : 1-27 Pg 77

Q2.3. Let U_1 be unused amount for M_1

$U_1 = 24-(6x_1+4x_2)$

likewise

$U_2 = 6-(x_1+2x_2)$

given $x_1=1$ and $x_2=2$

which gives $\boxed{M_1=10}$ $\boxed{M_2=1}$

H.W 2.7, 2.13
2.16 ; 2.22 (tricky)

OR-M Lecture 2

25/1 Thursday

Example : 2.4.1

Type	Interest Rate	Bad-debt ratio	Remaining max (1-bad-debt)
Personal x_1	0.140	0.10	0.9
Car x_2	0.130	0.07	0.93
Home x_3	0.120	0.03	0.97
Farm x_4	0.125	0.05	0.95
Commercial x_5	0.100	0.02	0.98

objective function:

Z = total interest - bad debt

total interest $= 0.14(0.9x_1)+0.130(0.93x_2)$
$+0.120(0.97x_3)+0.125(0.95x_4)+0.1(0.98x_5)$

bad debt $= 0.1x_1+0.07x_2+0.03x_3$
$+0.05x_4+0.02x_5$

$Z=(0.126x_1+0.1209x_2+0.1164x_3+0.1187x_4$
$+0.098x_5)-(0.1x_1+0.7x_2+0.03x_3+0.05x_4+0.02x_5)$

$Z=0.026x_1+0.0509x_2+0.0864x_3+0.6875x_4+0.78x_5$

... baki from book

Exercise : 2.4.2 from book

S ⇒ items not delivering on time

Exercise : 2.4.3 from book

I ⇒ extra inventory

* cost is always minimized and profit is always maximized

Q22.

Workstation	Hi-Fi1	Hi-Fi2	time used	time unused
1	6	4	10% = 0.1	0.9
2	5	5	14% = 0.14	0.86
3	4	6	12% = 0.12	0.88

$6x_1+4x_2 \le 480\times 0.9 = 432$

$5x_1+5x_2 \le 480\times 0.86 = 412.8$

$4x_1+6x_2 \le 480\times 0.88 = 422.4$

To develop objective function:

$6x_1+4x_2+Z_1 = 432$

$5x_1+5x_2+Z_2 = 412.8$

$4x_1+6x_2+Z_3 = 422.4$

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$15x_1 + 15x_2 + \overbrace{Z_1 + Z_2 + Z_3}^{Z} = 1267.2$

minimizing

$Z = \underbrace{1267.2}_{\text{ignore}} - 15x_1 - 15x_2$

$Z = 15x_1 + 15x_2$

subject to

$6x_1 + 4x_2 \leq 432$

$5x_1 + 5x_2 \leq 412.8$

$4x_1 + 6x_2 \leq 422.4$

$x_1, x_2 \geq 0$

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$x_1 = 50.88$

$x_2 = 31.68$

$Z = 1238.4$ → maximum value

$\boxed{Z = 31.68}$ → idle min.

OR-E Lecture 3

30/1 Tuesday

For **Simplex method**, following restrictions are imposed on LP model:

1. All constraints shud be eq with +ve right hand side.
2. All variables are +ve.

- to make R.H.S +ve if it is -ve, multiply whole eq by -ve.

converting inequalities into equalities:

- if there is $\leq$, add slack variable S
- if there is Z, subtract surplus variable S

transition from graphical to algebraic solution:

- $m \Rightarrow$ no of eq, $n \Rightarrow$ no of variables
- $m < n$
- no of basic solutions : ${}^nC_m = \frac{n!}{m!(n-m)!}$
- all variables − basic = non-basic
- for basic solution ⇒ put non basic variables = 0

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Example 3.2.1

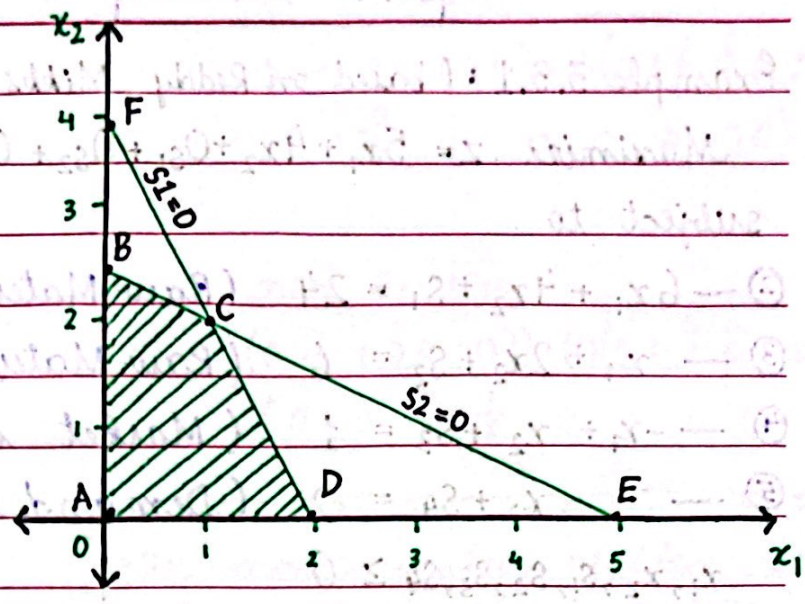
Maximize $z = 2x_1 + 3x_2$

subject to

$2x_1 + x_2 \leq 4$

$x_1 + 2x_2 \leq 5$

$x_1, x_2 \geq 0$



algebric solution :

slack variable

$2x_1 + x_2 + \underline{S_1} = 4$

$x_1 + 2x_2 + S_2 = 5$

$x_1, x_2, S_1, S_2 \geq 0$

Nonbasic (zero) variables	Basic variables	Basic solution	Associated corner points	Feasible?	Objective value (z)
(x_1, x_2)	(S_1, S_2)	$(4, 5)$	A	✓	0
(x_1, S_1)	(x_2, S_2)	$(4, -3)$	F	X	-
(x_1, S_2)	(x_2, S_1)	$(2.5, 1.5)$	B	✓	7.5
(x_2, S_1)	(x_1, S_2)	$(2, 3)$	D	✓	4
(x_2, S_2)	(x_1, S_1)	$(5, -6)$	E	X	-
(S_1, S_2)	(x_1, x_2)	$(1, 2)$	C	✓	8 optimum

- basic solutions are obtained by putting $n-m = 4-2 = 2$ variables (nonbasic) $= 0$
  $\Rightarrow$ we make all combinations of $n-m$ variables (non basic) and put them $= 0$
- e.g when we take (x_1, S_1) and put x_1 and $S_1 = 0$ in algebric solution
  $2(0) + x_2 + (0) = 4 \Rightarrow \boxed{x_2 = 4}$ and $0 + 2(4) + S_2 = 5 \Rightarrow \boxed{S_2 = -3}$
- (max) no of corner points $= {}^nmC = {}^4_2C = 6$, here 4 are feasible and 2 are infeasible
- to find associated corner points we are just taking values of x_1 and x_2 (since lines are drawn according to them) and finding corresponding corner points.
- similarly, for objective function we're considering values of x_1 and x_2 and putting them in z equation

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OR-D Lecture 4

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Example 3.3.1 : (based on Reddy Mikks model Example 2.1.1)

Maximize $z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to

② — $6x_1 + 4x_2 + s_1 = 24$ (Raw Material M1)

③ — $x_1 + 2x_2 + s_2 = 6$ (Raw Material M2)

④ — $-x_1 + x_2 + s_3 = 1$ (Market demand)

⑤ — $x_2 + s_4 = 2$ (Demand limit)

$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$

first we rewrite the objective equation:

① — $z - 5x_1 - 4x_2 = 0$

following table can be formed by these equations

Basic	z	↳ x_1 (enters)	x_2	s_1	s_2	s_3	s_4	Solution	Ratio or intercept
z row → z	1	-5	-4	0	0	0	0	0	- - -
s_1 row → s_1 (leaves)	0	6	4	1	0	0	0	24	min ↳ $24/6 = 4$
s_2 row → s_2	0	1	2	0	1	0	0	6	$6/1 = 6$
s_3 row → s_3	0	-1	1	0	0	1	0	1	ignore $1/-1 = -1$
s_4 row → s_4	0	0	1	0	0	0	1	2	ignore $2/0 = \infty$

step 1: find entering variable ⇒ z-row ki most -ve value i.e., x_1

step 2: find leaving variable ⇒ taking ratio of solution/x_1 and take minimum +ve value

* entering variable determines key column
* leaving variable determines key row
* their intersection determines key value

step 3: take key value as pivot element and apply Gauss-Jordan to whole table including z-row (make pivot element 1 and all elements above and below it 0)

corresponding new rows will be:

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x_1 row (old s_1 row):

$\frac{1}{6}(0 \quad 6 \quad 4 \quad 1 \quad 0 \quad 0 \quad 0 \quad 24)$

$= (0 \quad 1 \quad \frac{2}{3} \quad \frac{1}{6} \quad 0 \quad 0 \quad 0 \quad 4)$

new z-row:

$(1 \quad -5 \quad -4 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0) + 5(0 \quad 1 \quad \frac{2}{3} \quad \frac{1}{6} \quad 0 \quad 0 \quad 0 \quad 4)$

$= (1 \quad 0 \quad -\frac{2}{3} \quad \frac{5}{6} \quad 0 \quad 0 \quad 0 \quad 20)$

new S_2 row:

$(0 \quad 1 \quad 2 \quad 0 \quad 1 \quad 0 \quad 0 \quad 6) + (-1)(0 \quad 1 \quad \frac{2}{3} \quad \frac{1}{6} \quad 0 \quad 0 \quad 0 \quad 4)$

$(0 \quad 0 \quad \frac{4}{3} \quad -\frac{1}{6} \quad 1 \quad 0 \quad 0 \quad 2)$

new S_3 row:

$(0 \quad -1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1) + (1)(0 \quad 1 \quad \frac{2}{3} \quad \frac{1}{6} \quad 0 \quad 0 \quad 0 \quad 4)$

$= (0 \quad 0 \quad \frac{5}{3} \quad \frac{1}{6} \quad 0 \quad 1 \quad 0 \quad 5)$

new S_4 row:

$(0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2)$

remains as it is

- For maximization problems, we take most -ve value in z-row.
- For minimization problems, we take most +ve value in z-row.

Basic	z	x_1	↳ x_2 (enters)	s_1	s_2	s_3	s_4	Solution	Ratio or intercept
z	1	0	$-2/3$	$5/6$	0	0	0	20	
x_1	0	1	$2/3$	$1/6$	0	0	0	4	$4 \div \frac{2}{3} = 6$
S_2 (leaves)	0	0	$4/3$	$-1/6$	1	0	0	2	min ↳ $2 \div \frac{4}{3} = 1.5$
S_3	0	0	$5/3$	$1/6$	0	1	0	5	$5 \div \frac{5}{3} = 3$
S_4	0	0	1	0	0	0	1	2	$2 \div 1 = 2$

repeat the same steps 1, 2 and 3

new table formed by performing Gauss-Jordan row operations:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
S_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
S_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

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since all coefficients in z-row are +ve, optimality condition is achieved and we have our optimal values:

decision variable	optimal value
x_1	3
x_2	$\frac{3}{2}$
Z	21

* optimal values are obtained from solution column

we also get info about status of resources

resources	slack value	status
raw material M1	$S_1 = 0$	Scarce
raw material M2	$S_2 = 0$	scarce
Market limit	$S_3 = \frac{5}{2}$	abundant
Demand limit	$S_4 = \frac{1}{2}$	abundant

* scarce if = 0
* abundant if > 0

Lecture 5 online 6/2 Tuesday

- simplex method by excel solver ⇒ watch YT video
  * just for assignments, not included in exams
- graphical solution by GeoGebra Classic

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3.4.1 M-Method

If any equation does not have a slack variable we add artificial variable R. in those equations and penalise them by adding these Rs to objective function in case of minimization and subtracting in case of maximization.

Example : 3.4.1

Minimize

$$z = 4x_1 + x_2$$

subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Solution :

Min

$$z = 4x_1 + x_2$$

subj to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 - x_3 = 6 \quad \Rightarrow x_3 \text{ is surplus}$$
$$x_1 + 2x_2 + x_4 = 4 \quad \Rightarrow x_4 \text{ is slack}$$
$$x_1, x_2, x_3, x_4 \geq 0$$

since eq (i) and eq (ii) donot have slack variables, we add artifical variables R_1 and R_2 and adding them in obj function (because we are minmizing).

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Min

$$z = 4x_1 + x_2 + MR_1 + MR_2$$

subj to

$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 - x_3 + R_2 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4, R_1, R_2 \geq 0$$

we need to pick value for R which should be large but not too large. so we take $R_1 = R_2 = 100$.

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	-4	-1	0	-100	-100	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

since it's starting solution, all slacks should be zero, so to make R_1 and R_2 zero:

New z-row = old z-row + (100 × R_1 row + 100 × R_2 row)

	Basic	x_1 (enters)	x_2	x_3	R_1	R_2	x_4	Sol
	z	696	399	-100	0	0	0	900
leaves →	R_1	3	1	0	1	0	0	3
	R_2	4	3	-1	0	1	0	6
	x_4	1	2	0	0	0	1	4

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Now we need to apply the same procedure as we did in Example 3.3.1

Basic	x_1	x_2 (enters)	x_3	R_1	R_2	x_4	Sol
z	0	167	-100	-232	0	0	204
x_1	1	1/3	0	1/3	0	0	1
leaves → R_2	0	5/3	-1	-4/3	1	0	2
x_4	0	5/3	0	-1/3	0	1	3

Basic	x_1	x_2	x_3 (enters)	R_1	R_2	x_4	Sol
z	0	0	1/5	-492/5	-501/5	0	18/5
x_1	1	0	1/5	3/5	-1/3	0	3/5
x_2	0	1	-3/5	-4/5	3/5	0	6/5
leaves → x_4	0	0	1	1	0	1	1

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	0	0	0	-493/5	-501/5	0	17/5
x_1	1	0	0	2/5	-1/3	-1/5	2/5
x_2	0	1	0	-1/5	3/5	3/5	9/5
x_3	0	0	1	1	0	1	1

optimal values:

$x_1 = 2/5$, $x_2 = 9/5$, $z = 17/5$

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3.4.2 Two phase Method

Phase I: add artificial variables. Next, find basic solution s which always minimizes sum of artificial variables regardless of whether LP is minimization or maximization.

* By the end of phase I, s must be -ve or 0. if +ve, the problem doesn't have any feasible solution, no need for Phase II.

Phase II: Use the feasible solution from Phase I as a starting basic feasible solution for the original problem. Make coefficients of basic variables in z-row 0 and then continue.

Example 3.4.2 using problem in 3.4.1

Phase I:

Minimize

$$s = R_1 + R_2$$

subj to

$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 - x_3 + R_2 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4, R_1, R_2 \geq 0$$

The associated table is

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Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	-1	-1	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

As R_1 and R_2 are non zero:

New r-row = old r-row + $(1 \times R_1$ row + $1 \times R_2$ row$)$

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	7	4	-1	0	0	0	9
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

final optimum table for Phase I:

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	-1	-1	0	0
x_1	1	0	$1/5$	$3/5$	$-1/5$	0	$3/5$
x_2	0	1	$-3/5$	$-4/5$	$3/5$	0	$6/5$
x_4	0	0	1	1	-1	1	1

At this point, artificial variables have completed their mission, we can eliminate their columns from the tableu and move on to Phase II.

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Phase II:

Minimize

$$z = 4x_1 + x_2$$

subject to

$$x_1 + \tfrac{1}{5}x_3 = \tfrac{3}{5}$$
$$x_2 - \tfrac{3}{5}x_3 = \tfrac{6}{5}$$
$$x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Basic	x_1	x_2	x_3	x_4	Solution
z	-4	-1	0	0	0
x_1	1	0	$1/5$	0	$3/5$
x_2	0	1	$-3/5$	0	$6/5$
x_4	0	0	1	1	1

x_1, x_2 should be non zero:

New z-row = old z-row + $(4 \times x_1$ row $+ 1 \times x_2$ row$)$

Basic	x_1	x_2	x_3 (enters)	x_4	Solution
z	0	0	$1/5$	0	$18/5$
x_1	1	0	$1/5$	0	$3/5$
x_2	0	1	$-3/5$	0	$6/5$
leaves → x_4	0	0	1	1	1

Now just solve further to obtain optimal values.

3.5, 3.6 from book

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day / date:

4.1

Chapter # 4

Primal to dual:

1. A dual variable (y_i) is assigned to each primal (equation) constraint and a dual contraint is assigned to each primal variable i.e., there will be a separate equation for each primal variable
2. The right hand side of the primal constraints/equations provide the coefficients of the dual objective function.
3. Primal maximization will be dual minimization with all constraints equations in the form of $\geq$ and vice versa.
4. All y_i will be unrestricted.
5. The dual constraint corresponding to a primal variable is constructed by transposing the primal variable column into row. The primal objective coefficients become the dual right hand side. The remaining constraint coefficients become the dual left hand side coefficients.

Example 4.1.3

Primal:

Maximize $z = 5x_1 + 6x_2$

subject to

$x_1 + 2x_2 = 5$

$-x_1 + 5x_2 \geq 3$

$4x_1 + 7x_2 \leq 8$

x_1 unrestricted, $x_2 \geq 0$

Primal in equation form:

Substitute $x_1 = x_1^- - x_1^+$

Maximize $z = 5x_1^- - 5x_1^+ + 6x_2 + 0x_3 + 0x$

subject to

$x_1^- - x_1^+ + 2x_2 = 5$

$-x_1^- + x_1^+ + 5x_2 - x_3 = 3$

$4x_1 - 4x_1^+ + 7x_2 + x_4 = 8$

$x_1^-, x_1^+, x_2, x_3, x_4 \geq 0$

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Corresponding Dual problem:

day / date:

Minimize $z = 5y_1 + 3y_2 + 8y_3$

subject to

$$\left.\begin{array}{l} y_1 - y_2 + 4y_3 \geq 5 \\ -y_1 + y_2 - 4y_3 \geq -5 \end{array}\right\} \Rightarrow \begin{array}{l} y_1 - y_2 + 4y_3 \geq 5 \\ y_1 - y_2 + 4y_3 \leq 5 \end{array} \Rightarrow y_1 - y_2 + 4y_3 = 5$$

$$2y_1 + 5y_2 + 7y_3 \geq 6$$

$$\left.\begin{array}{l} -y_2 \geq 0 \\ y_3 \geq 0 \\ y_1, y_2, y_3 \text{ unrestricted} \end{array}\right\} \Rightarrow y_1 \text{ unrestricted},\ y_2 \leq 0,\ y_3 \geq 0$$

$\boxed{4.2}$

$V = (\ _\ _\ _\ _\ )$ → row vector

$A = \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}$

$P = \begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix}$ → column vector

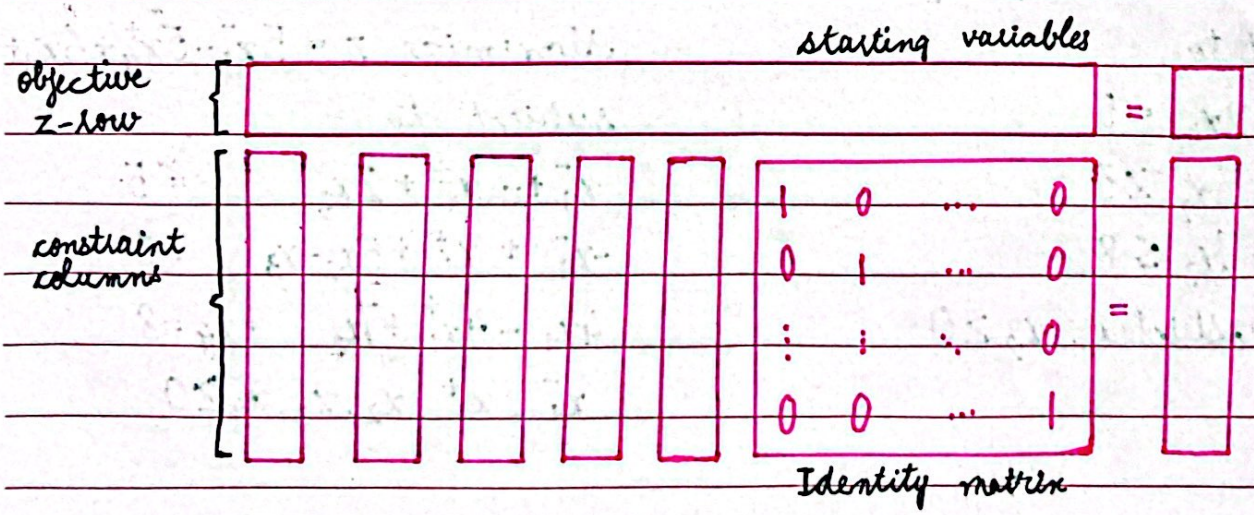
Valid operations:

VA

AP

scalar ← aA

Starting tableau:



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day / date:

after Gauss-Jordan row operations:

starting variables

objective z-row [ ] = [ ]

constraint columns [ ] = [ ]

Inverse matrix

Method 1:

$$\begin{pmatrix}\text{Optimal value of}\\ \text{dual variable } y_i\end{pmatrix} = \begin{pmatrix}\text{Optimal primal z-coefficient of starting basic variable } x_i\\ + \\ \text{Original objective coefficient of } x_i\end{pmatrix}$$

Method 2:

$$\begin{pmatrix}\text{Optimal values}\\ \text{of dual variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of}\\ \text{original objective coefficients}\\ \text{of optimal primal basic variables}\end{pmatrix} \times \begin{pmatrix}\text{Optimal}\\ \text{primal inverse}\end{pmatrix}$$

Example 4.2.1:

Primal:

Maximize $z = 5x_1 + 12x_2 + 4x_3$

Subject to

$x_1 + 2x_2 + x_3 \le 10$

$2x_1 - x_2 + 3x_3 = 8$

$x_1, x_2, x_3 \ge 0$

Primal eq. form:

original obj coefficients of optimal primal basic variable

original objective coefficient of x_i (x_4, R_1)

Maximize $z = 5x_1 + 12x_2 + 4x_3 - MR + 0$

Subject to

slack

$x_1 + 2x_2 + x_3 + x_4 = 10$

artificial

$2x_1 - x_2 + 3x_3 + R = 8$

$x_1, x_2, x_3, R \ge 0$

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day / date:

dual:

Min $w = 10y_1 + 8y_2$

subject to

x_1 $y_1 + 2y_2 \geq 5$

x_2 $2y_1 - y_2 \geq 12$

x_3 $y_1 + 3y_2 \geq 4$

x_4 $y_1 \geq 0$

R $y_2 \geq -M$ (unrestricted)

Starting table:

Basic	x_1	x_2	x_3	x_4	R	Solution
Z	-5	-12	-4	0	M	0
x_4	1	2	1	1	0	10
R_1	2	-1	3	0	1	8

Optimal Primal table:

optimal primal z-coefficient of starting basic variable x_i (x_4, R_1)

Basic	x_1	x_2	x_3	x_4	R	Solution
Z	0	0	3/5	29/5	-2/3 + M	54 4/5
x_2	0	1	-1/5	2/5	-1/5	12/5
x_1	1	0	7/5	1/5	2/5	26/5

optimal primal basic variables

optimal primal inverse

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day / date:

Method 1: * starting primal basic variables are x_4 and R and they uniquely correspond to the dual variables y_1 and y_2.

Starting primal basic variables	x_4	R
z-equation coefficients	$29/5$	$-2/5+M$
original objective coefficient	0	$-M$
dual variables	y_1	y_2
optimal dual values	$29/5+0=29/5$ (y_1)	$-2/5+M+(-M)=-2/5$ (y_2)

Method 2: * The order of the optimal primal basic variables is x_2 followed by x_1. so, same order will be followed here.

(original objective coefficients) = (Coefficient of x_2, coefficient of x_1)
$= (12, 5)$

The optimal dual values are

$$(y_1, y_2) = \begin{pmatrix}\text{Original objective} \\ \text{coefficients of } x_2, x_1\end{pmatrix} \times (\text{Optimal value})$$

$$= (12, 5)\begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix}$$

$$= \left(\frac{29}{5}, \frac{-2}{5}\right)$$

4.2.4

Constraint column construction:

$$\begin{pmatrix}\text{Constraint column} \\ \text{in iteration } i\end{pmatrix} = \begin{pmatrix}\text{Inverse in} \\ \text{iteration } i\end{pmatrix} \times \begin{pmatrix}\text{Original} \\ \text{constraint column}\end{pmatrix}$$

- x_2-column $= \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- x_1-column $= \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \times \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

* The order of basic variables will be same as in optimal table (x_2, x_1)

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day / date:

$$\begin{pmatrix}\text{Primal } z\text{-coefficient} \\ \text{of variable } x_j\end{pmatrix} = \begin{pmatrix}\text{Left hand side} \\ \text{of } j^{th} \text{ dual constraint}\end{pmatrix} - \begin{pmatrix}\text{R.H.S of } j^{th} \\ \text{dual constraint}\end{pmatrix}$$

these come from 1st dual eq

$$z\text{-coefficient of } x_1 = y_1 + 2y_2 - 5 = \frac{29}{5} + 2\left(-\frac{2}{5}\right) - 5 = 0$$

this comes from last dual eq

$$z\text{-coefficient of } R = y_2 - (-M) = -\frac{2}{5} - (-M) = -\frac{2}{5} + M$$

similarly we can compute for x_2, x_3 and x_4

$\boxed{4.4}$

- simplex → starts feasible, remain feasible until optimal
- dual simplex → starts infeasible, remain infeasible until feasible
- generalised simplex → combo of both, starts non-optimal and infeasible

Dual Simplex:

* in dual simplex, we convert all constraints into $\leq$

case 1: $x_1 + x_2 = 5$

$x_1 + x_2 \leq 5$ — (i)

$x_1 + x_2 \geq 5 \Rightarrow -x_1 - x_2 \leq -5$ — (ii)

case 2: $x_1 + x_2 \geq 6$

$-x_1 - x_2 \leq -6$ (multiply whole eq by -1)

dual feasibility condition:

leaving variable x_r ⇒ most -ve value in solution column

dual optimal condition:

for entering variable ⇒ $\min\limits_{\text{Non basic } x_j} \left\{ \left| \frac{\bar{c}_j}{\alpha_{rj}} \right., a_{rj} < 0 \right\}$

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day / date:

$c_j \rightarrow$ reduced cost of non basic variable

$\alpha_{rj} \rightarrow$ constraint coefficient of x_r-row, x_j column

Example 4.4.1:

$\text{Min } z = 3x_1 + 2x_2 + x_3$

Subj to

$3x_1 + x_2 + x_3 \geq 3$ } multiply by -1 to convert into $\leq$

$-3x_1 + 3x_2 + x_3 \geq 6$

$x_1 + x_2 + x_3 \leq 3$

$x_1, x_2, x_3 \geq 0$

Basic	x_1	x_2	x_3	x_4	x_5	x_6	solution
z	-3	-2	-1	0	0	0	0
x_4	-3	-1	-1	1	0	0	-3
x_5	3	-3	-1	0	1	0	-6 → most -ve value so x_5 is leaving
x_6	1	1	1	0	0	1	3

For entering variable:

Nonbasic variable	x_1	x_2	x_3
z-row ($\bar{c}_j$)	-3	-2	-1
x_5 row, α_{5j}	3	-3	-1
Ratio $\left\lvert \frac{\bar{c}_j}{\alpha_{5j}} \right\rvert$, $\alpha_{5j} < 0$	– ↗ ignore cos $\alpha_{5j} \not< 0$	2/3 ↑ min so x_2 is entering	1

Applying Gauss Jordan operations, we obtain the table:

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day / date:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
Z	-5	0	-1/3	0	-2/3	0	4
x_4	-4	0	-2/3	1	-1/3	0	-1 → most -ve value so x_4 is leaving
x_2	-1	1	1/3	0	-1/3	0	2
x_6	2	0	2/3	0	1/3	1	1
Ratio	5/4	-	1/2				

↑ min so x_3 is entering

After performing Gauss Jorden operations, we obtain the table:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
Z	-3	0	0	-1/2	-1/2	0	9/2
x_3	6	0	1	-3/2	1/2	0	3/2
x_2	-3	1	0	1/2	-1/2	0	3/2
x_6	-2	0	0	1	0	1	0

This table is optimal → all Z-row coefficients are -ve (since it's a minimize problem, max k case mai all coefficients should be +ve)

This table is feasible → all solution column values are +ve (should be for both min and max cases)

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Generalised Simplex:

leaving variable → solution column ki most -ve value
entering variable → pivot row (leaving variable wali row) mai basic variable ki most -ve value.

Example 4.4.2

Max $z = 2x_3$

Subj to

$-x_1 + 2x_2 - 2x_3 \geq 8$ → multiply by -1 to convert into ≤

$-x_1 + x_2 + x_3 \leq 4$

$2x_1 - x_2 + 4x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Basic	x_1	x_2 ↗ entering	x_3	x_4	x_5	x_6	Solution
z	0	0	-2	0	0	0	0
x_4	1	$\boxed{-2}$	2	1	0	0	-8 → most -ve value so x_4 is leaving
x_5	-1	1	1	0	1	0	4
x_6	2	-1	4	0	0	1	10

* since it's a max problem all z-row coefficients should've been +ve but x_3 is -2 which makes the solution non-optimal. agr sab values +ve hoteen tau dual simplex lgatay.
* infeasible cos solution column has -ve value.

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day / date:

After performing Gauss Jordan operations, we obtain the table:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
Z	0	0	-2	0	0	0	0
x_2	$-1/2$	1	-1	$-1/2$	0	0	4
x_5	$-1/2$	0	2	$1/2$	1	0	0
x_6	$3/2$	0	3	$-1/2$	0	1	14

* the problem is now feasible (no -ve value in solution column) but non-optimal (-ve value in Z-row).
* we can apply primal simplex method to determine the optimal solution but we cannot go further with generalised simplex cos to apply it, the table should be infeasible and non-optimal

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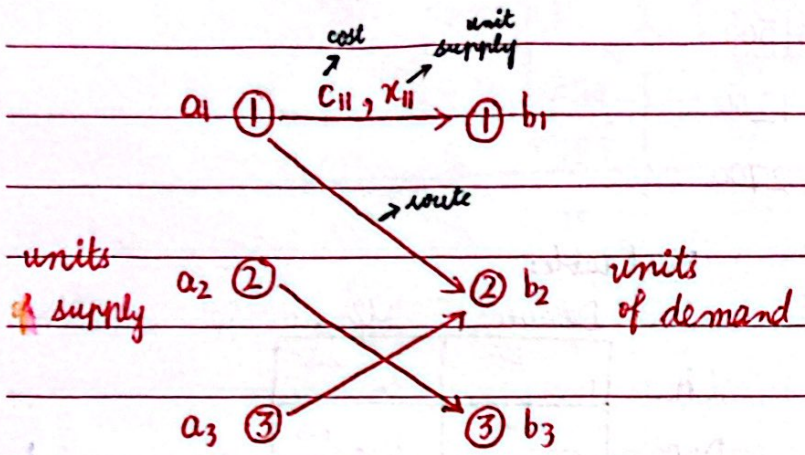
Chapter #5

day / date: 12-3-24

Transportation Model:

purpose → transportation cost ko minimum



Exp # 5.1.1

Mileage Chart

	Denver	Miami
LA	1000	2690
Detroit	1250	1350
NO	1275	850

for LA ← 8 cents per mile per ca

$\Rightarrow \frac{8}{100} \times 1000 = 0.08\$ \times 1000$

MG Transportation Model

destinations (n)

sources (m)	Denver	Miami	Supply	Dummy
LA	1000 x_{11} / 80	x_{12} / 215	~~1000~~ 0	
Detroit	x_{21} / 100	x_{22} / 108	1500	
NO	x_{31} / 102	x_{32} / 68	1200	
Dummy	0	600 / 0		
Demand	2300 / 1300	1400	3700	

600 → if there was extra 600 here

	Denver	Miami
LA	\$80	\$215
Detroit	\$100	\$108
NO	\$102	\$68

* since sum of supply and demand is equal, model is balanced, incase it is not equal, we'll add dummies.

* no of basic feasible solutions $= m + n - 1$
where $m \rightarrow$ sources , $n \rightarrow$ destinations

North west: Top left value i.e., x_{11} kay corresponding supply and demand ki min value will be replaced by x_{11}. supply wali 1000 value will be picked in this case and we'll make it 0. also, demand value mai se 1000 deduct hoga. Now that perticular row or column with 0 value in demand / supply will be removed.

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day / date:

New table:

	Denver	Miami	Supply
Det	1300 (100)	x_{22} (108)	1500
NO	x_{31} (102)	x_{32} (68)	1200
Demand	1300 (0) min	1400	2700

Next table:

	Mia	Supply
Det	200 (108)	200 (0) min
NO	1200 (68) x_{32}	1200
Demand	1400	1400

Final table:

	Denver	Miami
LA	1000 (80)	x_{12} (215)
Det	1300 (100)	200 (108)
NO	x_{31} (102)	1200 (68)

Transportation cost (Minimum cost):

$(1000 \times 80) + (1300 \times 100) + (200 \times 108) + (1200 \times 68)$

$= 80{,}000 + 130{,}000 + 21{,}600 + 81{,}600$

$= 313{,}200$

Question: 1-13 , 22-29

Question #6 : demands ghalat likhi hain ⇒ 36, 42, 30 are correct

	C1	C2	C3	supply
P1	600	700	400	25
P2	320	300	350	40
P3	500	480	450	30
dummy	1000	1000	—	13
demand	36	42	30	108

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Least cost method: $m+n-1 = 6$ solutions

day / date:

	D 1	2	3	4	supply
F 1	10	2 (15)	20	11 (0)	~~15~~ 0
2	12	7	9 (15)	20 (10)	25
3	4 (5)	14	16	18 (5)	10
Demand	5	~~15~~ 0	15	15	50

* pooray table mai se least dekhein gy baki process is same.

since both values are same, we can eliminate either row or column.

↓

	1	3	4	supply
1	10	20	11	0
2	12	9	20	25
3	4 (5)	16	18	~~10~~ 5
Demand	~~5~~ 0	15	15	30

↓

	3	4	supply
1	20	11	0
2	9 (15)	20	~~25~~ 10
3	16	18	5
Demand	~~15~~ 0	15	15

↓

	4	supply
1	11 (0)	0
2	20	10
3	18	5
demand	15	15

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day / date:

	4	supply
2	20	10
3	5 (18)	~~8~~ 9
D	~~15~~ 10	10

↓

	4	supply
2	10 (20)	10
	10	

Basic feasible solution;

$x_{12} = 15$, $x_{31} = 5$, $x_{23} = 15$, $x_{14} = 0$, $x_{34} = 5$, $x_{24} = 10$

Transportation cost :

$2 \times 15 + 4 \times 5 + 9 \times 15 + 0 \times 11 + 5 \times 18 + 10 \times 20 = \$475$

By northwest method: \$ 520

Question # 5.8

	1	2	3	supply
R1	\$18	18	-	6
R2	30	80	90	5
R3	22	20	12	6
Dummy				2
Demand	4	8	7	19

10/cents / 1000 gallons

$\frac{10¢}{1000} \times \frac{1}{100} \times 10^6 \times 1000 \times$ mileage

↓ to convert 10 cents into dollars

↓ for dollars

↓ cos all costs are in million

↓ cos all costs are in 1000 dollars

$= \frac{mil}{10} = 0.1$ mileage

$= 100 \times$ miles

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H.W Q22-29

day / date:

5.9 :

	1	2	3	supply
R1	\$18	18	-	6
R2	30	80	90	5
R3	22	20	12	6
Dummy	-	50	50	2
Demand	4	8	7	19

$\frac{5¢}{100} \times 10^6 \times \frac{1}{10^3}$

Now apply both northwest and least cost methods.

for

Chapter #5

Youtube Channel : ETIS

video lectures : 12 onwards

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day / date: 28 / 3

Chapter #6

Network Model:

1. natural gas pipeline network
2. shortest route b/w cities
3. time schedule for construction projects
4. minimum cost flow schedule for refineries

Network definition:

nodes : network consists of set of nodes and linked by arcs

directed in oriented arc: which has some specific direction

path: set of arcs joining two distinct nodes, passing through other nodes

cycle or loop: a path that connect a node back

connected network: If every two distinct nodes are linked

tree: a cycle free connected network

spanning tree: links all nodes of network

6.2 Minimal Spanning tree Algo:

step 0 : set $C_0 = \{\phi\}$

$\overline{C}_0 = \{N\}$

step 1 : choose any node i from $\overline{C}_0$ and set

$C_i = \{i\}$, $\overline{C}_i = N - \{i\}$

step 2 : select the next node from $\overline{C}_i$ with the shortest arc and repeat the process until $\overline{C}_k = \{\phi\}$

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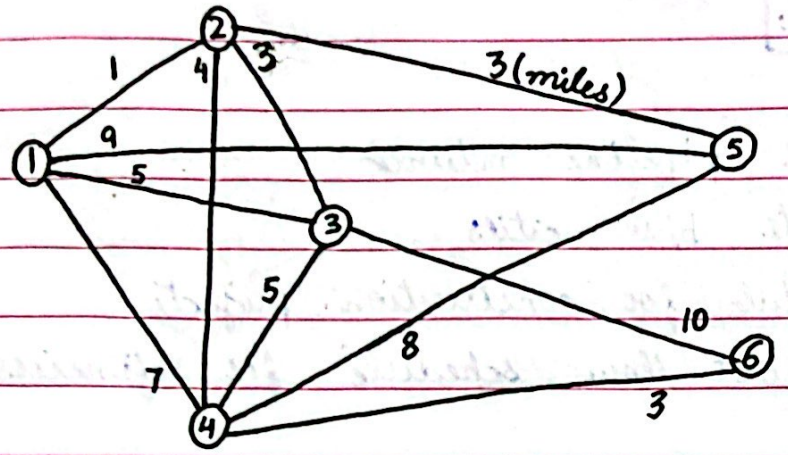
day / date:

Example 6.2.1



1→2 , 1 min ①	2→3 , 6	5→4 , 8	4→3 , 5 min ⑤
1→5 , 9	2→4 , 4 min ③	5→1 , 9	4→6 , 3 min ④
1→3 , 5	2→5 , 3 min ②		
1→4 , 7			3→6 , 10

$N = \{1, 2, 3, 4, 5, 6\}$

step 0 : $C_0 = \{\phi\}$

$\overline{C_0} = \{1, 2, 3, 4, 5, 6\}$

step 1 : $C_1 = \{1\}$, $\overline{C_1} = \{2, 3, 4, 5, 6\}$

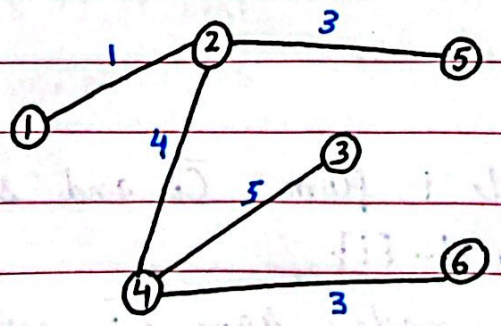
$C_2 = \{1, 2\}$, $\overline{C_2} = \{3, 4, 5, 6\}$

$C_3 = \{1, 2, 5\}$, $\overline{C_3} = \{3, 4, 6\}$

$C_4 = \{1, 2, 5, 4\}$, $\overline{C_4} = \{3, 6\}$

$C_5 = \{1, 2, 5, 4, 6\}$, $\overline{C_5} = \{3\}$

$C_6 = \{1, 2, 5, 4, 6, 3\}$, $\overline{C_6} = \{\phi\}$



minimum distance of spanning tree : $1+3+4+3+5 = 16$

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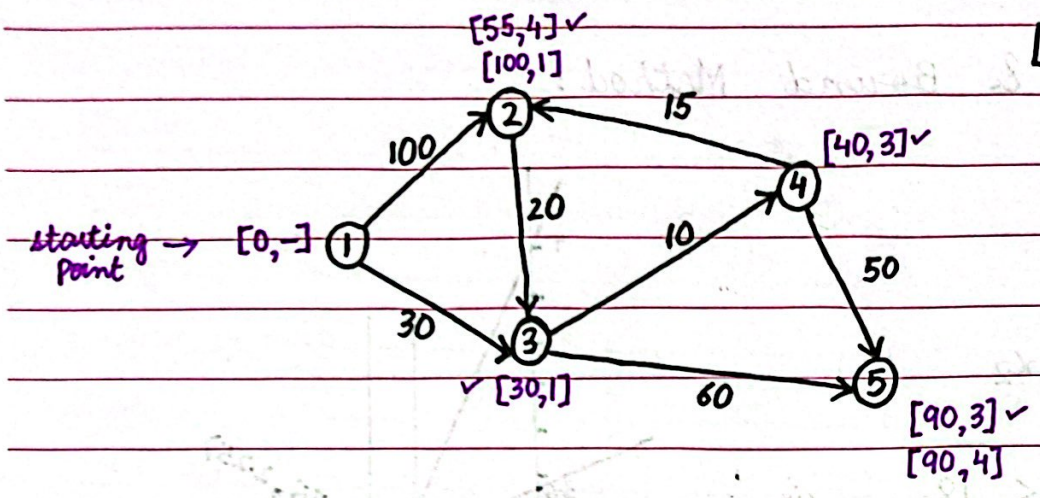
6.3.2

DIJKSTRA'S ALGORITHM

day / date:



$[U_j, i] = [U_i + d_{ij}, i]$

U_i : shortest distance from node 1 to i

d_{ij} : distance of arc (i,j)

i : parent node

Node	Label	Permanent Distance
1	[0,-]	[0,-]
2	[100,1], [55,4] ✓	[55,4]
3	[30,1]	[30,1]
4	[40,3]	[40,3]
5	[90,3] ✓, [90,4]	[90,3] or [90,4]

shortest distance:

1 → 2 : 1 → 3 → 4 → 2

1 → 3 : 1 → 3

1 → 4 : 1 → 3 → 4

1 → 5 : 1 → 3 → 5 or 1 → 3 → 4 → 5

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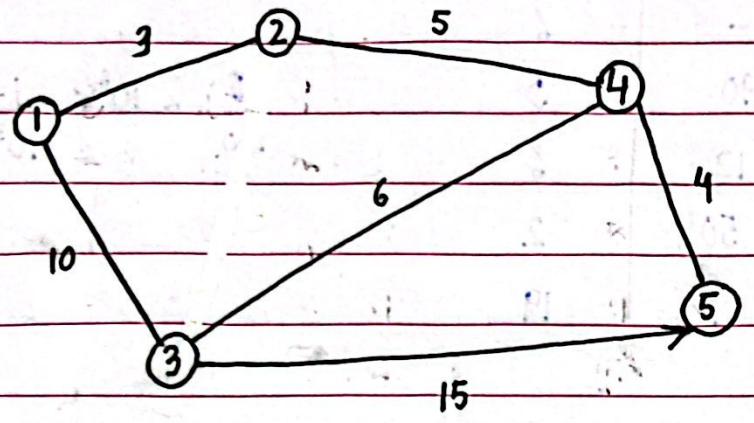
Floyd' algorithm

day / date: 16-4 Tuesday



Iteration 0 :

D_0

	1	2	3	4	5
1	-	3	10	∞	∞
2	3	-	∞	5	∞
3	10	∞	-	6	15
4	∞	5	6	-	4
5	∞	∞	-	4	-

S_0

	1	2	3	4	5
1	-	2	3	4	5
2	1	-	3	4	5
3	1	2	-	4	5
4	1	2	3	-	5
5	1	2	3	4	-

Iteration 1 : $K = 1$

$d_{23} = d_{32} = d_{12} + d_{13} = 10 + 3 = 13$

D_1

	1	2	3	4	5
1	-	3	10	∞	∞
2	3	-	13	5	∞
3	10	13	-	6	15
4	∞	5	6	-	4
5	∞	∞	-	4	-

S_1

	1	2	3	4	5
1	-	2	3	4	5
2	1	-	1	4	5
3	1	1	-	4	5
4	1	2	3	-	5
5	1	2	3	4	-

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Iteration 2 : $K=2$

$d_{14} = d_{41} = d_{12} + d_{24} = 3+5 = 8$

	1	2	3	4	5
1	-	3	10	8	∞
2	3	-	13	5	∞
3	10	13	-	6	15
4	8	5	6	-	4
5	∞	∞	-	4	-

	1	2	3	4	5
1	-	2	3	2	5
2	1	-	1	4	5
3	1	1	-	4	5
4	2	2	3	-	5
5	1	2	3	4	-

Iteration 3 : $K=3$

$d_{15} = d_{13} + d_{35} = 25$

$d_{25} = d_{23} + d_{35} = 28$

	1	2	3	4	5
1	-	3	10	8	25
2	3	-	13	5	28
3	10	13	-	6	15
4	8	5	6	-	4
5	∞	∞	-	4	-

,

	1	2	3	4	5
1	-	2	3	2	3
2	1	-	1	4	3
3	1	1	-	4	5
4	2	2	3	-	5
5	1	2	3	4	-

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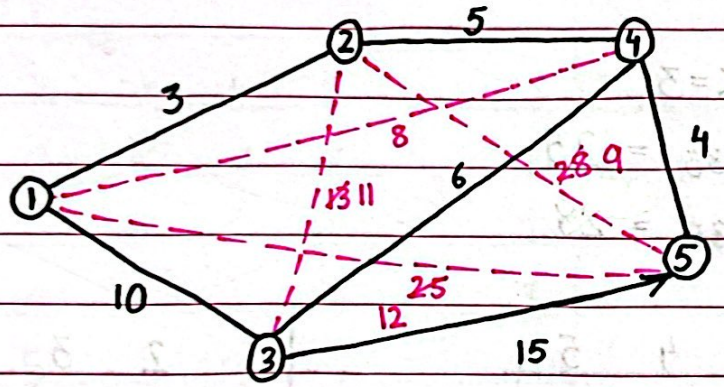
Iteration 4 : $k=4$

	1	2	3	4	5
1	-	3	10	8	12
2	3	-	11	5	9
3	10	11	-	6	10
4	8	5	6	-	4
5	12	9	10	4	-

	1	2	3	4	5
1	-	2	3	2	4
2	1	-	4	4	4
3	1	4	-	4	4
4	2	2	3	-	5
5	4	4	4	4	-



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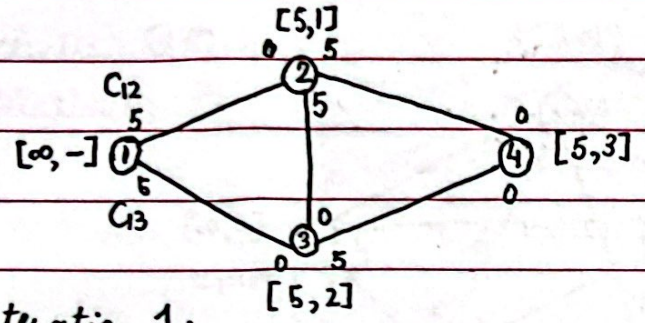
6.4 day / date:

Maximum flow Algo:

find breakthrough path with maximum flow b/w source node and sink node.

Arc flow C_{ij} from $i \to j$
Arc flow C_{ji} from $j \to i$



* jitne outgoing paths honge, utni iterations hongi

Iteration 1:

$i = 1$

step 1: $a_1 = \infty$, label node 1 $[\infty, -]$

step 2: $S_1 - \{2,3\} \neq \emptyset$

step 3: $K=2$, because $C_{ij} = \max\{C_{12}, C_{13}\}$

$= \max\{5,5\} = 5$

$\boxed{\text{set } a_2 = 5}$

label node 2 $[5,1]$

step 2: $i = 2$

$S_2 = \{3,4\} \neq \emptyset$

step 3:

$k = 3$ $\quad C_{ij} = \max\{C_{23}, C_{24}\}$

$\max\{5,5\} = 5$

$a_3 = 5$

label node 3 $[5,2]$

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day / date:

step 2: $i=3$

step 3: $S_3 = \{4\} \neq \emptyset$

$k=4$, $\boxed{a_4 = 5}$

label node 4 $[5,3]$

step 5: $N = \{1,2,3,4\}$

$f_1 = \min\{a_1, a_2, a_3, a_4\}$

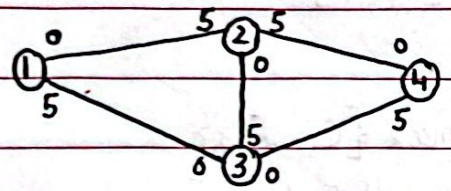
$= \min\{\infty, 5, 5, 5\}$

$\boxed{f_1 = 5}$

path 1-2-3-4

ⓐ $\{C_{ij} - f_1,\ C_{ji} + f_1\}$ $i \to j$

$\{C_{ij} + f_1,\ C_{ij} - f_1\}$ $j \to i$



* the path you followed, only those residues will be updated

Situation 2:

$i=1$

step 1: $a_1 = \infty$, labll node 1 $[\infty, -]$

step 2: $S_1 = \{3\} \neq \emptyset$

step 3: $k=3$, $a_3 = 5$

label 3 $[5,1]$

step 2: $i=3$

$S_3 = \{2\} \neq \emptyset$ $k=2$, $a_2 = 5$

$i=2$

$S_2 = \{4\} \neq \emptyset$

$K=4$, $a_4 = 5$

$f = \min\{\infty, 5, 5, 5\} = 5$

$f_2 = 5$

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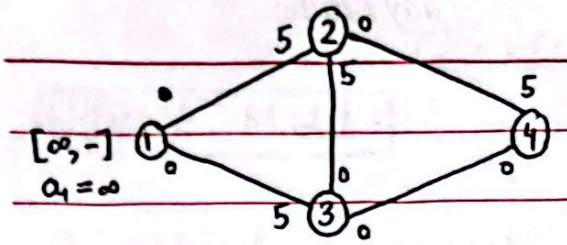
day / date:

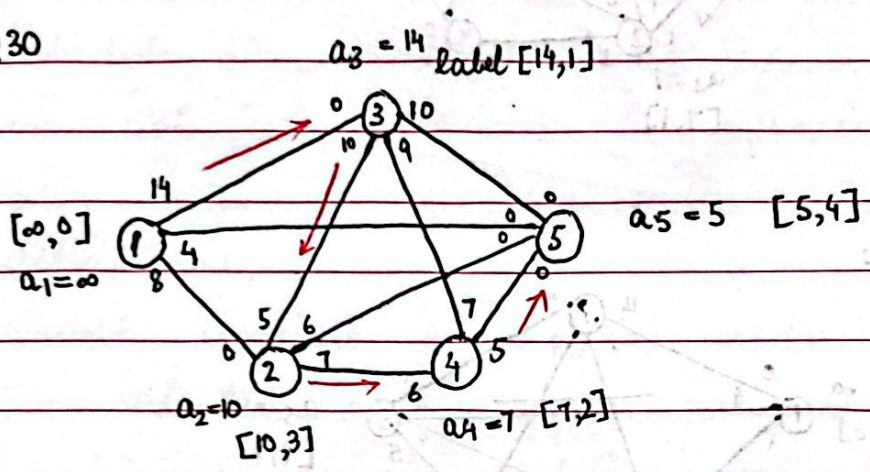


max flow $= f_1 + f_2$

$= 5 + 5 = 10$

Exercise Q30

Iteration 1:



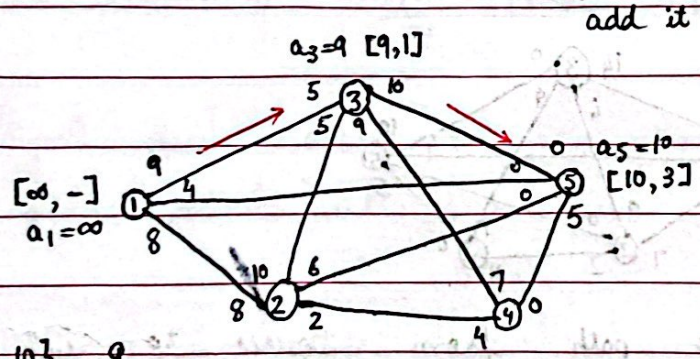
$f_1 = \min\{a_1, a_2, a_3, a_4, a_5\}$

$f_1 = \min\{\infty, 14, 10, 7, 5\}$

$f_1 = 5$

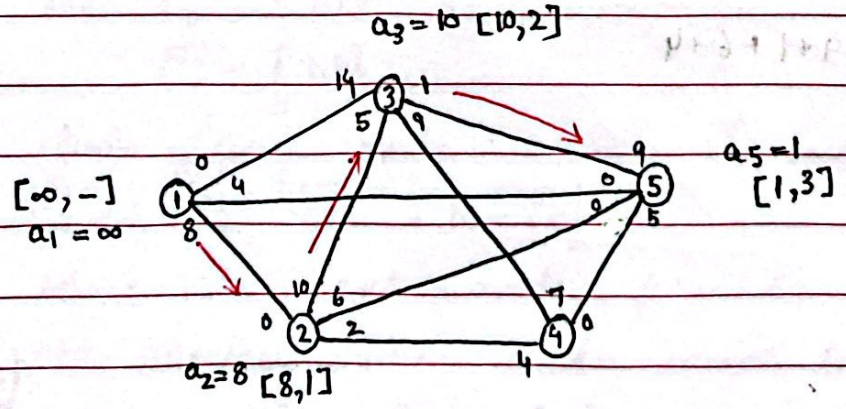
(now we subtract 5 from one side and add it to the other side of our path)

Iteration 2:



$f_2 = \min[9, 10] = 9$

Iteration 3:



$f_3 = 1$

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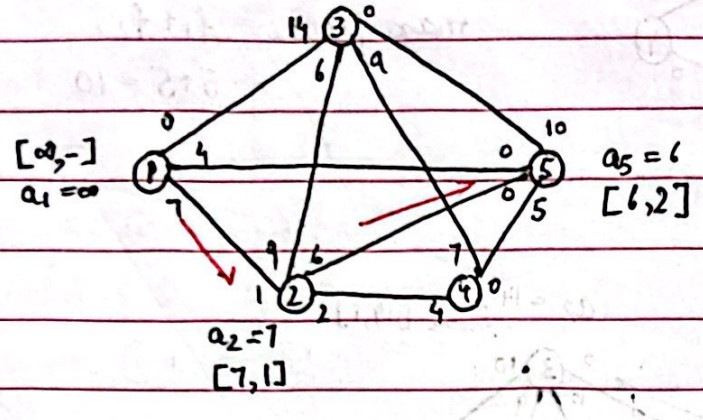
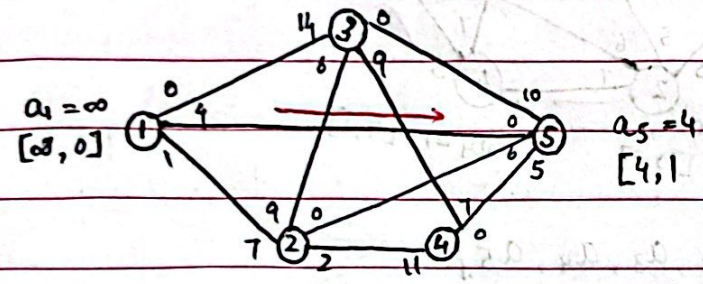
Iteration 4:

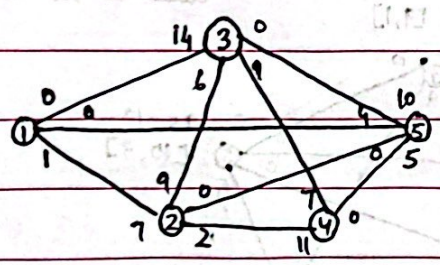


$f_4 = 6$

Iteration 5:



Iteration 6:



now there's no path from source to sink (1 to 5) so stop

flow = f_1, f_2, f_3, f_4, f_5

$= 5 + 9 + 1 + 6 + 4$

$= 25$

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Chapter # 9

day / date: 25 April Thursday

9.2.1 Branch & Bound Method:

Q9.56 a)

Max

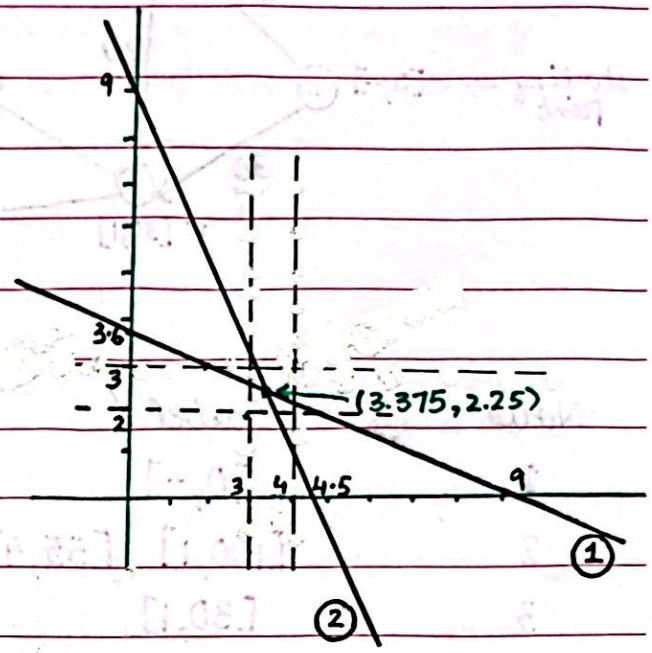
$$z = 3x_1 + 2x_2$$

s.t

$$2x_1 + 5x_2 \le 18 \quad \text{—(1)}$$

$$4x_1 + 2x_2 \le 18 \quad \text{—(2)}$$

$$x_1, x_2 \ge 0 \text{ and are integers}$$



solving algebraically

eq(1)

$x_1 = 0 \Rightarrow x_2 = 3.6 \quad (0, 3.6)$

$x_2 = 0 \Rightarrow x_1 = 9 \quad (9, 0)$

eq(2)

$x_1 = 0 \Rightarrow x_2 = 9 \quad (0, 9)$

$x_2 = 0 \Rightarrow x_1 = 4.5 \quad (4.5, 0)$

For LP1:

$4x_1 + 2x_2 = 18$ —(2)

$-4x_1 - 10x_2 = -36$ —(1) x-2

$\boxed{x_2 = 2.25}$

$\boxed{x_1 = 3.375}$

For LP2:

$\boxed{x_1 = 3}$

put in eq(1)

bcz it's intersecting line (1) in graph

$\boxed{x_2 = 2.4}$

For LP4:

$\boxed{x_2 = 2}$

$\boxed{x_1 = 3}$

For LP3:

$\boxed{x_1 = 4}$

put in eq(2)

$\boxed{x_2 = 1}$

For LP5:

$\boxed{x_2 = 3}$

put in eq(1)

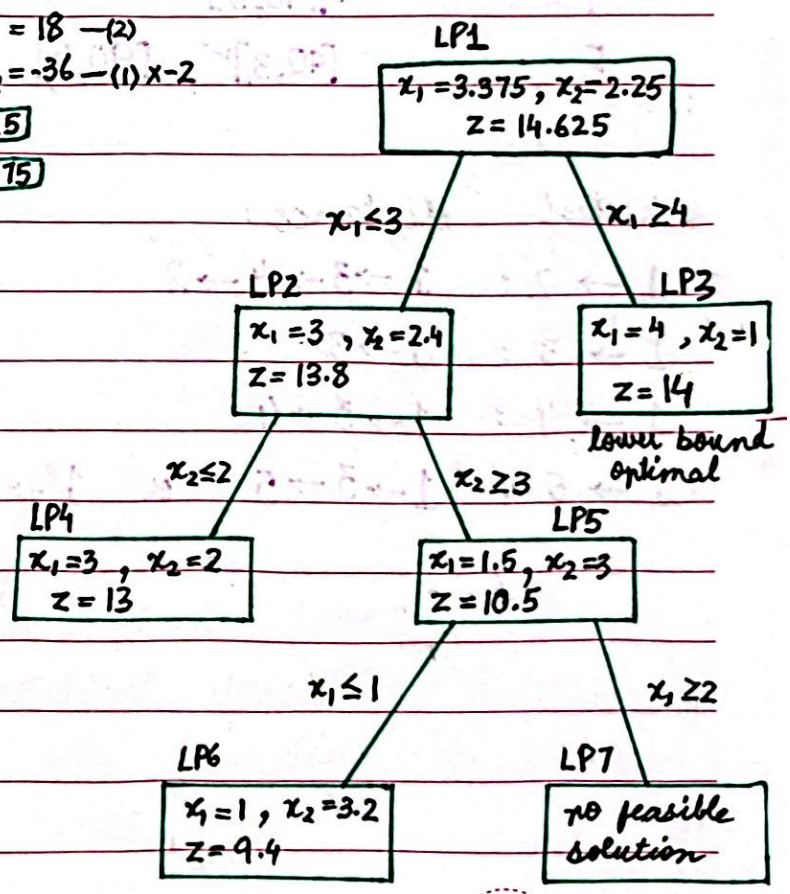
$\boxed{x_1 = 1.5}$

For LP6:

$\boxed{x_1 = 1}$

put in eq(1)

$\boxed{x_2 = 3.2}$



* as we go down to the LP level our solution region also gets shorter.

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day / date: 30/4 Tuesday

9.2.2 Cutting plane Algorithm:

Example 9.2.2

$$\text{Max } z = 7x_1 + 10x_2 \quad \Rightarrow \quad z - 7x_1 \underbrace{- 10x_2}_{} = 0$$

most -ve value iss liye table mai pahlay x_2 ayega

subject to

$$-x_1 + 3x_2 \le 6$$
$$7x_1 + x_2 \le 35$$
$$x_1, x_2 \ge 0 \text{ and integers}$$

Basic	x_1	x_2	x_3	x_4	solution
z	0	0	$63/22$	$31/22$	$133/2 = 66 + \frac{1}{2}$
x_2	0	1	$7/22$	$1/22$	$7/2 = 3 + 1/2$
x_1	1	0	$-1/22$	$3/22$	$9/2 = 4 + 1/2$

we choose variable with greatest fractional value, since both are $\frac{1}{2}$, we pick any

picking x_2:

$$x_2 + 7/22\, x_3 + 1/22\, x_4 = 3 + 1/2$$
$$x_2 - 3 = -7/22\, x_3 - 1/22\, x_4 + 1/2 \le 0$$
$$-7/22\, x_3 - 1/22\, x_4 + 1/2 \le 0$$
$$-7/22\, x_3 - 1/22\, x_4 \le -1/2$$
$$-7/22\, x_3 - 1/22\, x_4 + s_1 = -1/2 \qquad \text{(Cut I)}$$

Basic	x_1	x_2	x_3 ↓ entering	x_4	s_1	solution
z	0	0	$63/22$	$31/22$	0	$66\ 1/2$
x_1	0	1	$7/22$	$1/22$	0	$3\ 1/2$
x_2	1	0	$-1/22$	$3/22$	0	$4\ 1/2$
s_1	0	0	$-7/22$	$-1/22$	1	$-1/2$ ← leaving

optimal but infeasible so we apply dual simplex

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Basic	x_1	x_2	x_3	x_4	s_1	Solution
Z	0	0	0	1	9	62
x_2	0	1	0	0	1	3
x_1	1	0	0	$1/7$	$-1/7$	$4\ 4/7$
x_3	0	0	1	$1/7$	$-22/7$	$1\ 4/7$

since we have fractional values, we repeat process until we get integer values

x_1 source row:

$$x_1 + 1/7\, x_4 - 1/7\, s_1 = 4 + 4/7$$

$$x_1 + (0 + 1/7)\, x_4 + (-1 + 6/7)\, s_1 = 4 + 4/7$$

$$x_1 - 4 = -1/7\, x_4 - 6/7\, s_1 + 4/7 \leq 0$$

$$-1/7\, x_4 - 6/7\, s_1 \leq -4/7$$

$$-1/7\, x_4 - 6/7\, s_1 + s_2 = -4/7 \quad \text{(Cut II)}$$

Basic	x_1	x_2	x_3	x_4	S_1	S_2	Solution
Z	0	0	0	1	9	0	62
x_2	0	1	0	0	1	0	3
x_1	1	0	0	$1/7$	$-1/7$	0	$4\ 4/7$
x_3	0	0	1	$1/7$	$-22/7$	0	$1\ 4/7$
S_2	0	0	0	$-1/7$	$-6/7$	1	$-4/7$

optimal but infeasible so we apply dual simplex

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day / date:

Basic	x_1	x_2	x_3	x_4	S_1	S_2	Solution
z	0	0	0	0	3	7	58
x_2	0	1	0	0	1	0	3
x_1	1	0	0	0	-1	1	4
x_3	0	0	1	0	-4	1	1
x_4	0	0	0	1	6	-7	4

The optimum solution contains all integers so we stop.

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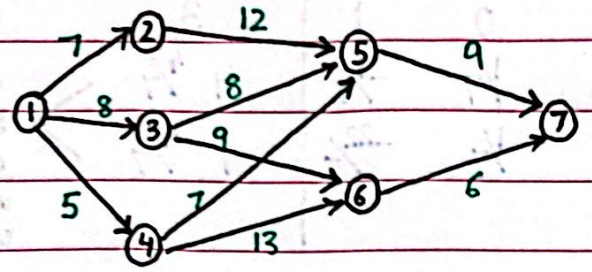
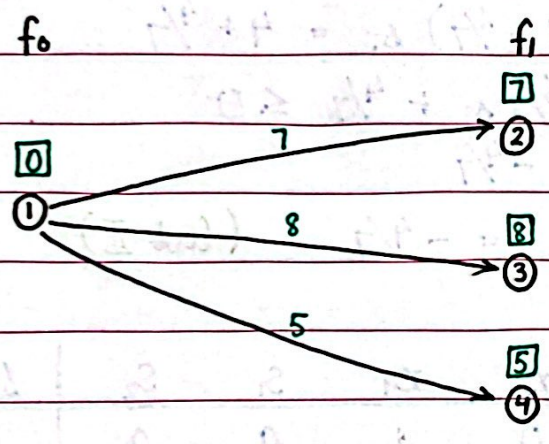
Chapter # 12
DYNAMIC PROGRAMMING

12.1.1

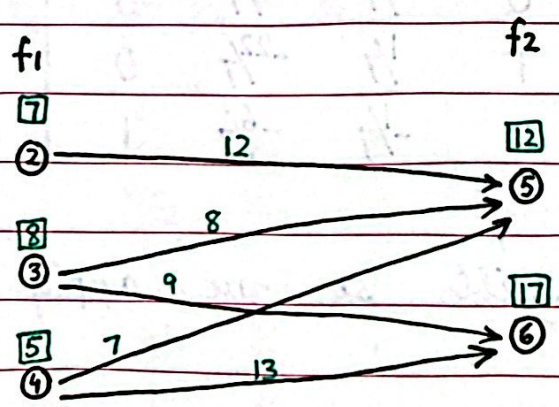


stage 1 :



2,3,4 states

stage 2 :



states 5,6

2→5 19
3→5 16
4→5 12 min

2→6 –
3→6 17 min
4→6 18

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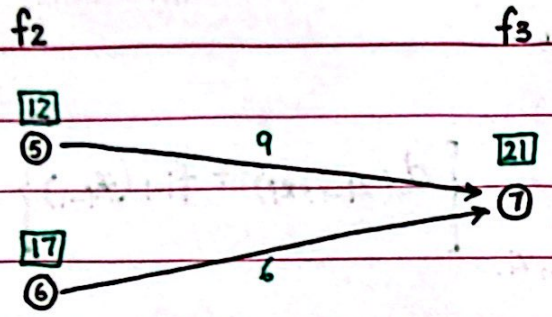
stage 3:

day / date:



state 7

5 → 7 (21) min

6 → 7 (23)

path : 1-4-5-7

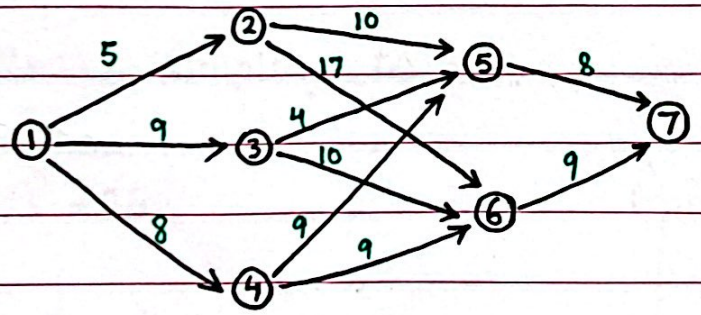
recursive eq :

$$f_0(x_0 = 1) = 0$$

$$f_i(x_i) = \min_{\substack{\text{all feasible} \\ (x_{i-1}, x_i)}} \{ d(x_{i-1}, x_i) + f_{i-1}(x_{i-1}) \} \quad i = 1, 2, 3, \ldots$$

exercise

Question # 1



To city	shortest distance	from	
2	5	1	
3	9	1	stage 1
4	8	1	
5	$\min\{\overset{2}{5+10}, \overset{3}{9+4}, \overset{4}{8+9}\} = 13$	3	
6	$\min\{\overset{2}{5+17}, \overset{3}{9+10}, \overset{4}{8+9}\} = 17$	4	stage 2
7	$\min\{\overset{5}{13+8}, \overset{6}{17+9}\} = 21$	5	stage 3

path : 1-3-5-7

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12.2 Backward Recursion:

$$f_4(x_4 = 7) = 0$$

$$f_i(x_i) = \min_{\substack{\text{all feasible} \\ \text{routes } (x_{i-1}, x_i)}} \left\{ d(x_{i-1}, x_i) + f_{i-1}(x_{i-1}) \right\}, \quad i = 1, 2, 3$$

$f_3 \rightarrow f_2 \rightarrow f_1$

To city	shortest distance	from	
5	8	7	stage 1
6	9	7	
2	$\min\{\overset{5}{8+10}, \overset{6}{9+17}\} = 18$	5	stage 2
3	$\min\{\overset{5}{8+4}, \overset{6}{9+10}\} = 12$	5	
4	$\min\{\overset{5}{8+9}, \overset{6}{9+9}\} = 17$	5	
1	$\min\{\overset{2}{18+5}, \overset{3}{12+9}, \overset{4}{17+8}\} = 21$	3	stage 3

path : 1-3-5-7

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Chapter # 19

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Monte Carlo Simulation

Steps:

step 1: clearly define the problem.

a. Identify the objective of the problem.

b. Identify the main factor which have the greatest effect on the objective function.

step 2:

a. specify the variables and parameters of the model.

b. formulate the appropriate decision rules.

c. Identify the type of distribution.

d. specify the manner in which time will change.

Example 19.1.1

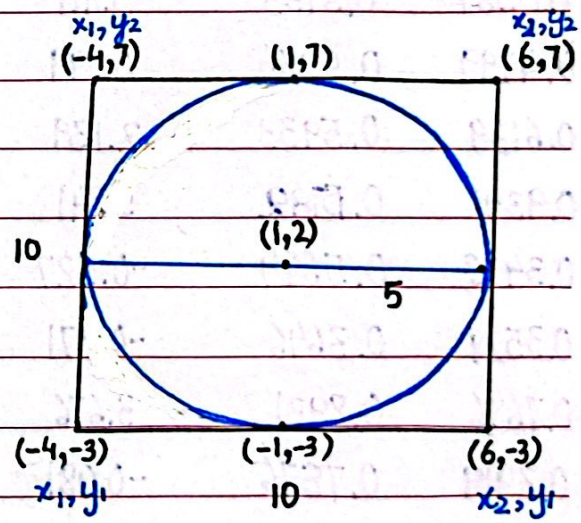
use Monte-Carlo sampling to estimate area of the circle having equation:

$(x-1)^2 + (y-2)^2 = 25$

solution:

$(x-h)^2 + (y-k)^2 = r^2$

centre $= (h, k) = (1,2)$



if m out of n sampled points fall within the circle then

$$\text{Approximate area of the circle} = \frac{m}{n}\left(\text{Area of the square}\right) = \frac{m}{n}(10 \times 10) = 100\frac{m}{n}$$

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To ensure all points are equally divided in the square, we take :

$$f_1(x) = \frac{1}{10} \quad , \quad \overset{x_1}{-4} \le x \le \overset{x_2}{6}$$

$$f_2(y) = \frac{1}{10} \quad , \quad \overset{y_1}{-3} \le y \le \overset{y_2}{7}$$

A pair of 0-1 random numbers R_1 and R_2 can be used to generate random points (x, y) in the square as :

$$x = \overset{x_1}{-4} + \overset{x_2 - x_1}{[6-(-4)]} R_1$$

$$y = \overset{y_1}{-3} + \overset{y_2 - y_1}{[7-(-3)]} R_2$$

	R_1	R_2	x	y	$(x-1)^2+(y-2)^2$			Point lies in circle
1.	0.0589	0.6733	-3.411	3.733	22.46	<	25	✓
2.	0.4799	0.9486	0.799	6.486	22.16	<	25	✓
3.	0.6139	0.5933	2.139	2.933	2.16	<	25	✓
4.	0.9341	0.1782	5.341	-1.218	29.19	≮	25	✗
5.	0.3473	0.5644	-0.527	2.644	2.74	<	25	✓
6.	0.3529	0.3646	-0.471	0.646	3.99	<	25	✓
7.	0.7676	0.8931	3.676	5.931	22.61	<	25	✓
8.	0.3919	0.7876	-0.081	4.876	9.43	<	25	✓
9.	0.5199	0.6358	1.199	3.358	1.88	<	25	✓
10.	0.7472	0.8954	3.472	5.954	21.74	<	25	✓

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Point fall inside the circle if

$(x-1)^2+(y-2)^2 < 25$

as 9 out of 10 points lie inside the circle

Area of circle $= \frac{9}{10}(10 \times 10) = 90$

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