Fourier Analysis

Complex Numbers

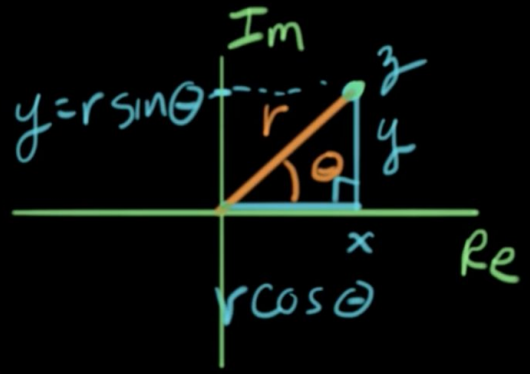
1. Relationship of complex numbers and sinusoids
2. Representation in cartesian and polar coordinates
3. Exponential representation
4. Euler's formula

Complex Numbers

imaginary $\boxed{j^2 = -1}$

$$\boxed{z = x + jy}$$

$$\boxed{z = r\angle\theta}$$



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$z = r(\cos\theta + j\sin\theta)$$

$$\boxed{z = re^{j\theta}}$$

Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

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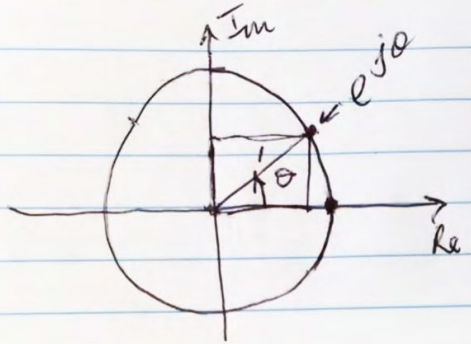
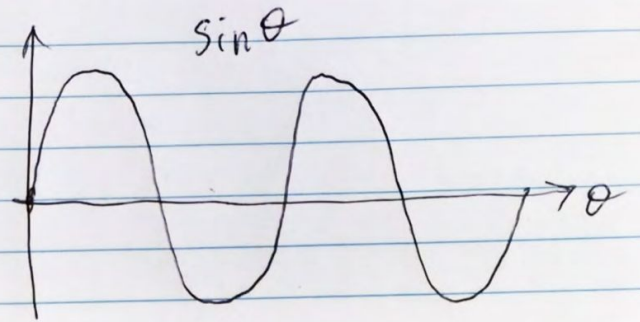
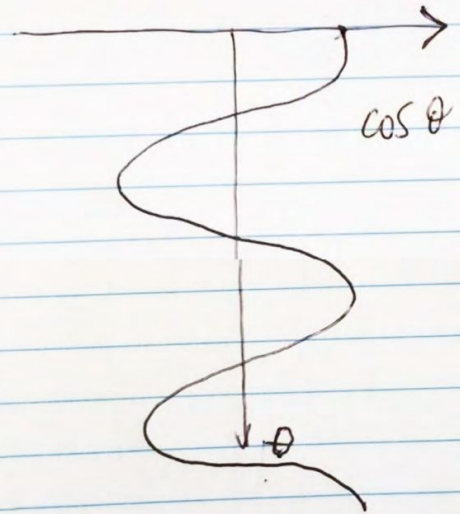


$\leftrightarrow$

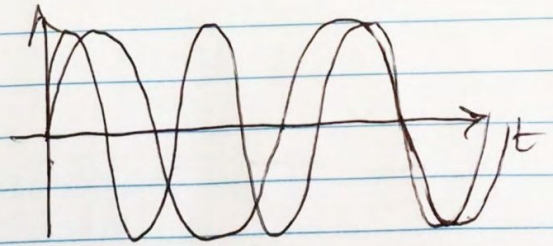




$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) = \cos\theta$$

$$\theta = \omega t \qquad \omega = 2\pi f$$



Fourier Transform

Fourier Transform is a representation of a signal in frequency domain.

1. The signal is represented as the sum (or integral) of constituent frequencies
2. By analyzing and manipulating the constituent frequencies, we can understand the signal as well as modify it in a controlled manner.
3. Fourier transform also help us understand the LTI system

Fourier Transform

Fourier Analysis allows us to

1. Understand image structure
2. Extract features
3. Perform filtering in frequency domain
4. Replace computationally expensive convolution with multiplication

Fourier Transform

1. Temporal frequency describes how quickly a signal varies over time
2. Spatial frequency describes how quickly a signal varies over space
3. A linear transform is of the form: $x = H\ell_{in}$
4. Fourier transform is a linear transformation
5. And it is reversible i.e. H is invertible $\ell_{in} = H^{-1}x$

Fourier Series

1. "French mathematician and engineer **Joseph Fourier** showed that any periodic signal could be written as an infinite sum of trigonometric functions (cosine and sine functions)"

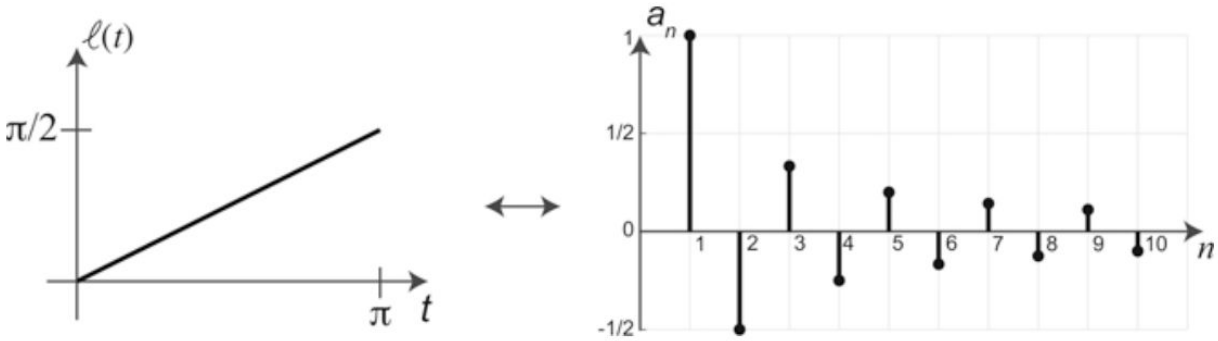
$$\ell(t) = a_1 \sin(t) + a_2 \sin(2t) + a_3 \sin(3t) + \ldots$$

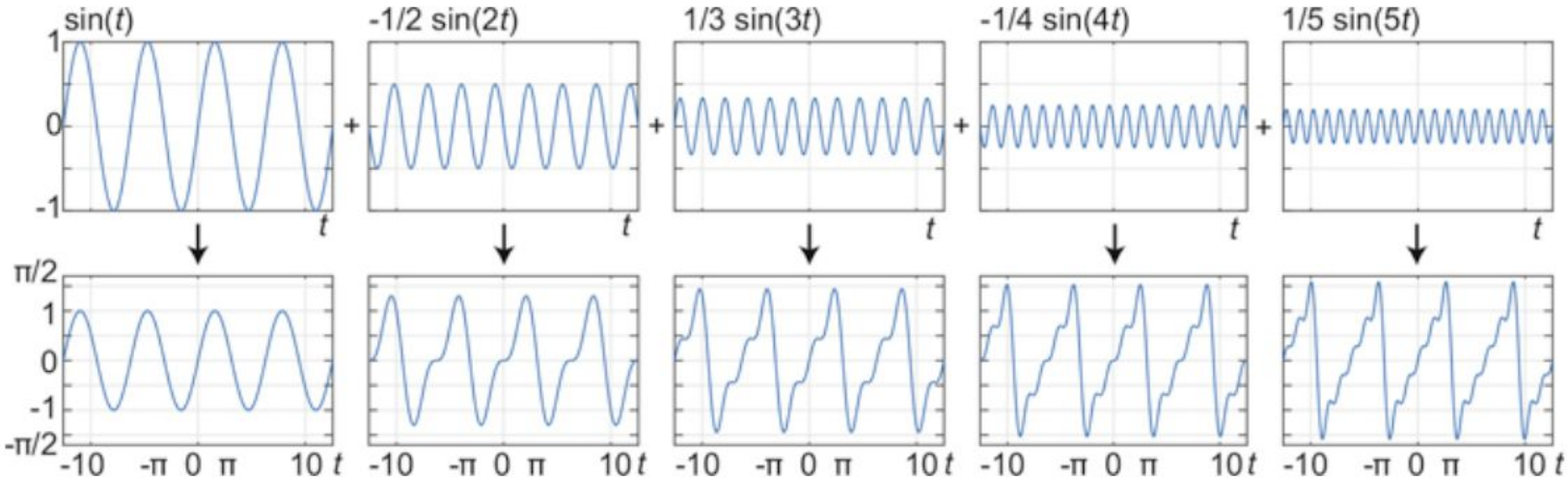
For example:

2. $\ell = t/2 =$ $$\frac{1}{2}t = \sin(t) - \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t) - \frac{1}{4}\sin(4t) + \ldots$$

Fourier Series

1.





Fourier Series - Demo

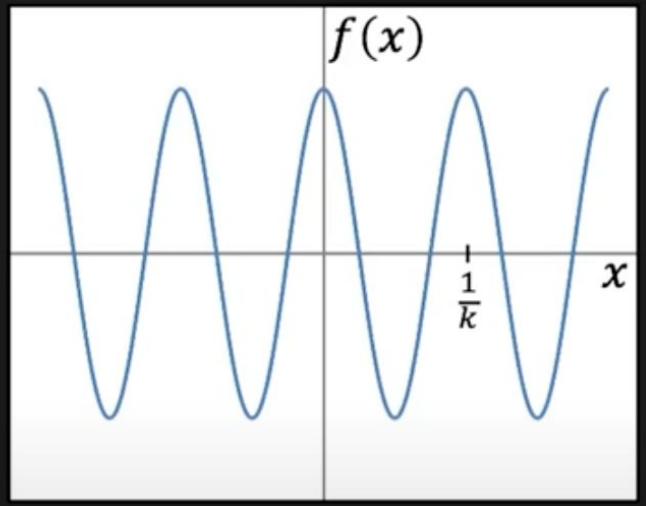
https://www.desmos.com/calculator/

https://phiresky.github.io/fourier-series-demo/

https://controlsystemsacademy.com/0018/0018.html

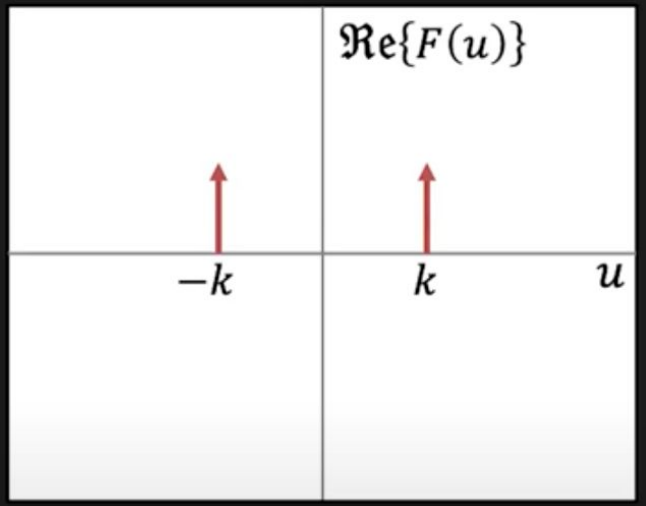
Fourier Transform Examples

Signal $f(x)$



$$f(x) = \cos 2\pi kx$$

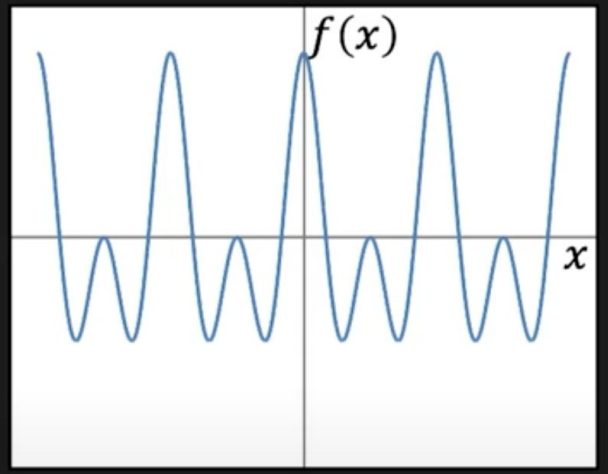
Fourier Transform $F(u)$



$$F(u) = \frac{1}{2}[\delta(u+k) + \delta(u-k)]$$

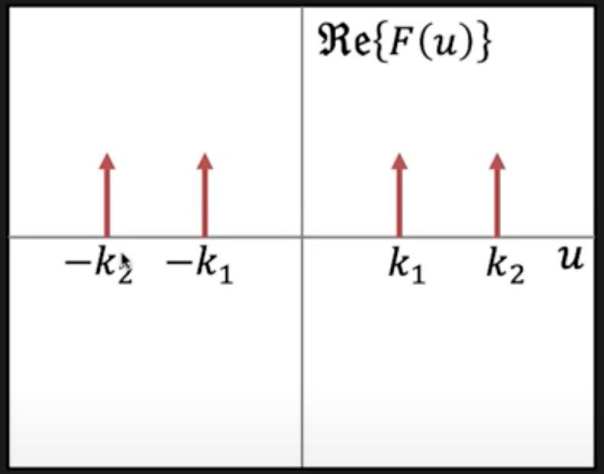
Fourier Transform Examples

Signal $f(x)$



$$f(x) = \cos 2\pi k_1 x + \cos 2\pi k_2 x$$

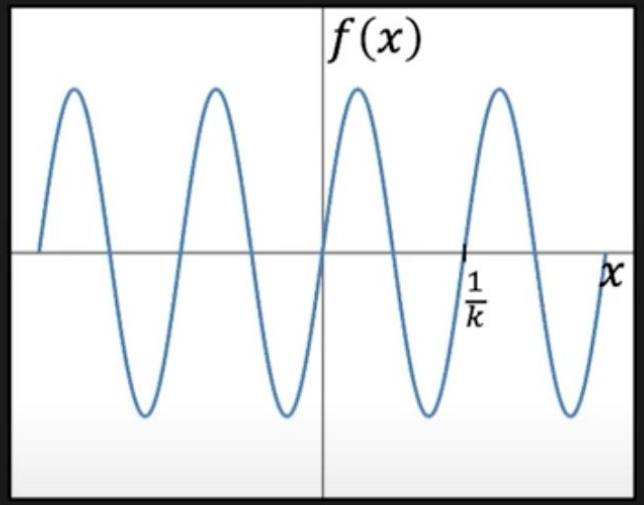
Fourier Transform $F(u)$



$$F(u) = \frac{1}{2}\left[\delta(u+k_1) + \delta(u-k_1) + \delta(u+k_2) + \delta(u-k_2)\right]$$

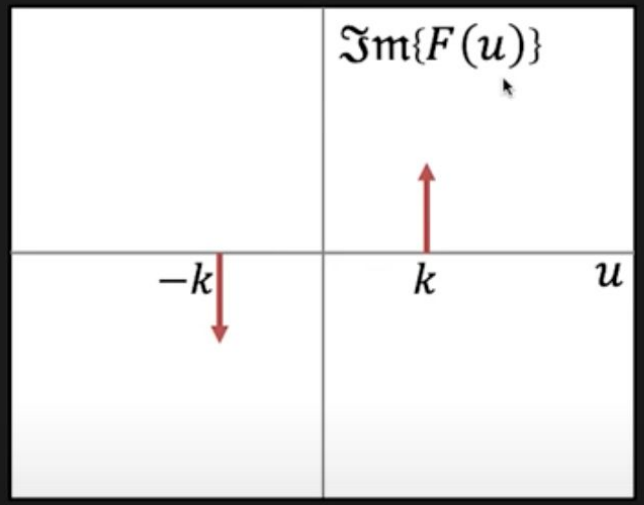
Fourier Transform Examples

Signal $f(x)$



$$f(x) = \sin 2\pi kx$$

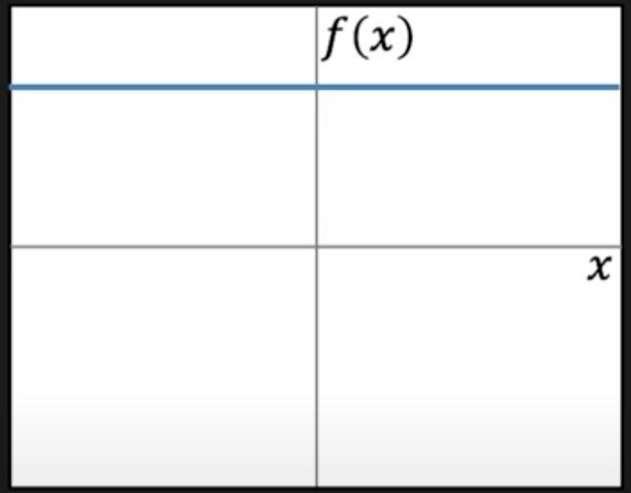
Fourier Transform $F(u)$



$$F(u) = \frac{1}{2}i[\delta(u+k) - \delta(u-k)]$$

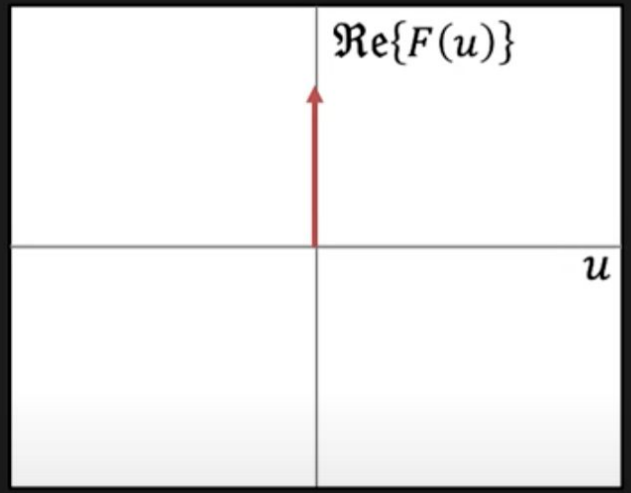
Fourier Transform Examples

Signal $f(x)$



$$f(x) = 1$$

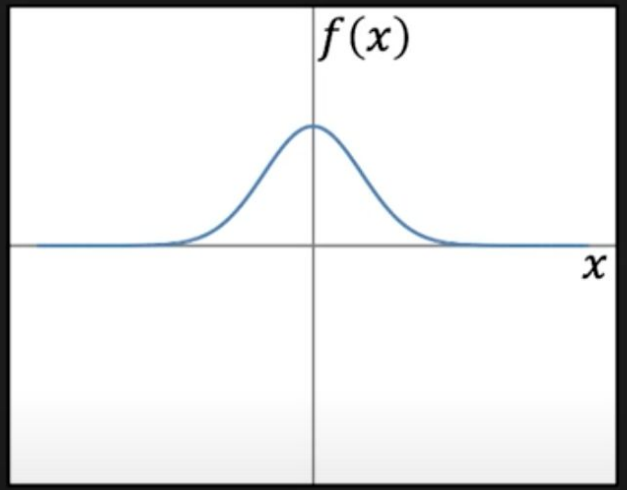
Fourier Transform $F(u)$



$$F(u) = \delta(u)$$

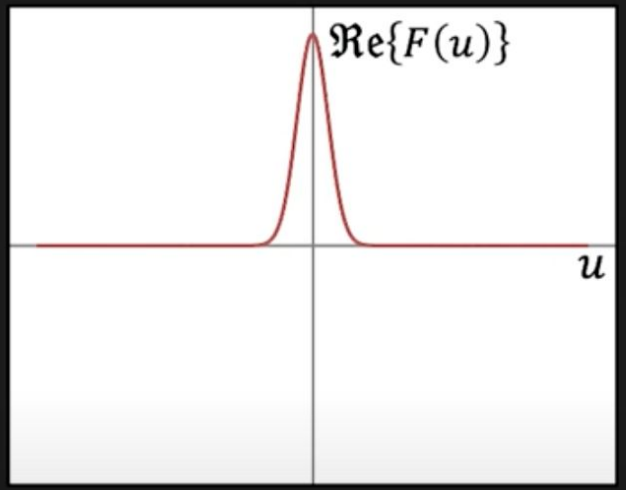
Fourier Transform Examples

Signal $f(x)$



$$f(x) = e^{-ax^2}$$

Fourier Transform $F(u)$



$$F(u) = \sqrt{\pi/a}\ e^{-\pi^2 u^2/a}$$

Fourier Transform - Continuous

1. Fourier transform and inverse Fourier transform

$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

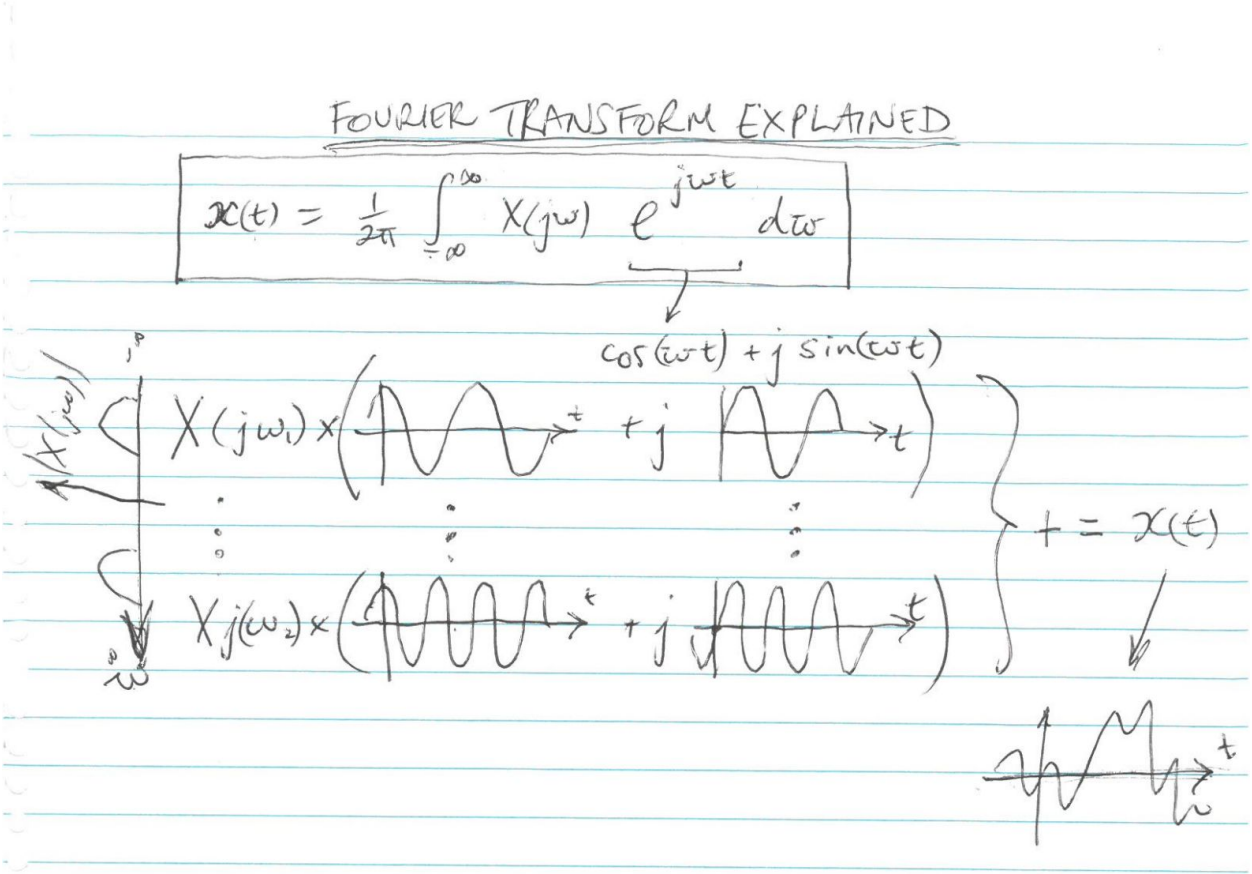
$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

Fourier Transform - Continuous

1. Fourier transform is an expansion of *f(t)* multiplied by sinusoidal terms of different frequencies
2. *t* can represent any continuous variable, and the units of the frequency variable depend on the units of *t*
3. For example, if t represents time in *seconds*, the units are *cycles/sec or Hertz (Hz)*. If t represents distance in *meters*, then the units are *cycles/meter*, and so on.

Fourier Transform - Continuous

1.

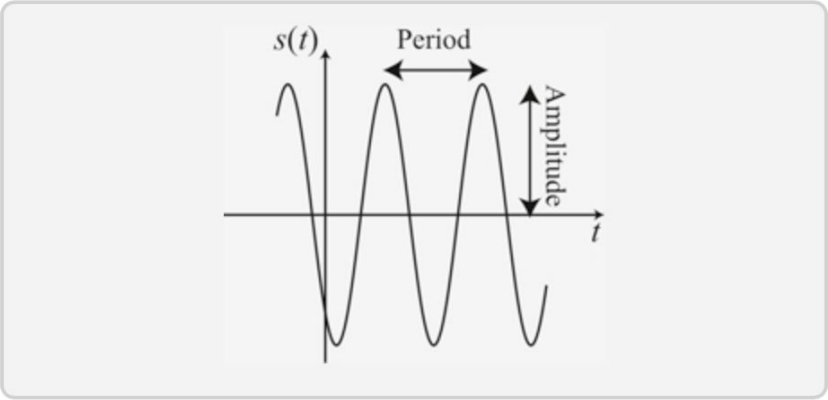


Sine Waves

Continuous time sine wave:

$$s(t) = A \sin(w\,t - \theta)$$

1. *Amplitude = A*
2. *Frequency = w = 2π/T*
3. *Phase =θ*
4. *Period T = 2π/w*



Discrete time sine wave:

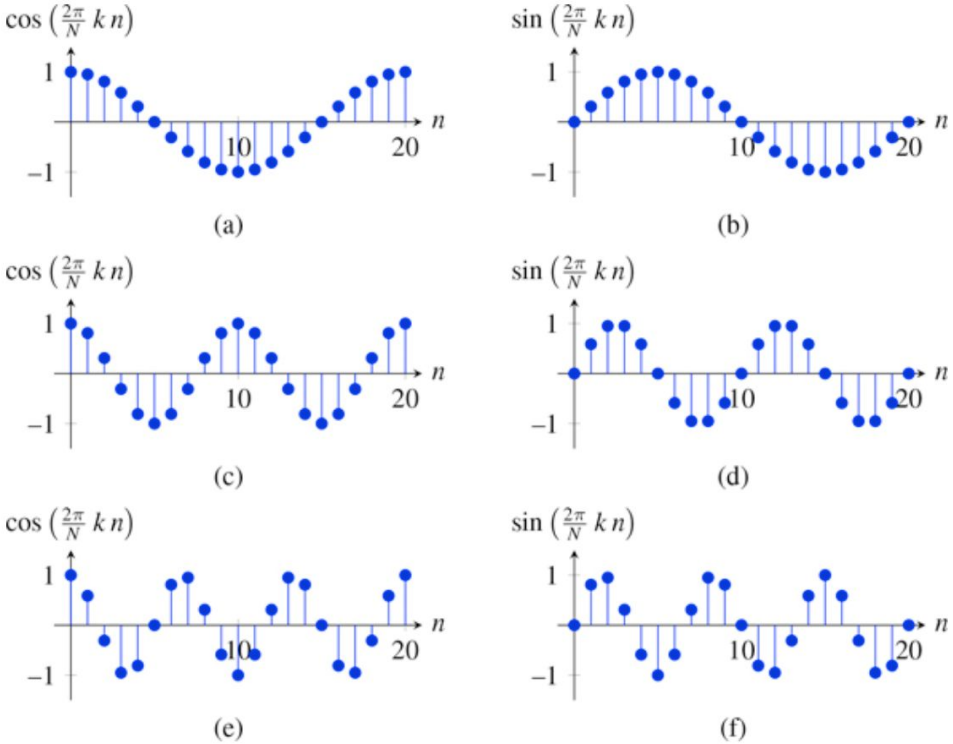
$$s[n] = A \sin(w\,n - \theta)$$

1. *w = 2π/N,* *N = 2π/w*

Sine

Following form makes the periodicity explicit: *k is frequency*

K = 1



$$s_k[n] = \sin\left(\frac{2\pi}{N} k n\right)$$

K = 2

$$c_k[n] = \cos\left(\frac{2\pi}{N} k n\right)$$

K = 3

Sine and Cosine in 2D

$$s_{u,v}[n,m] = A \sin\left(2\pi\left(\frac{un}{N} + \frac{vm}{M}\right)\right)$$

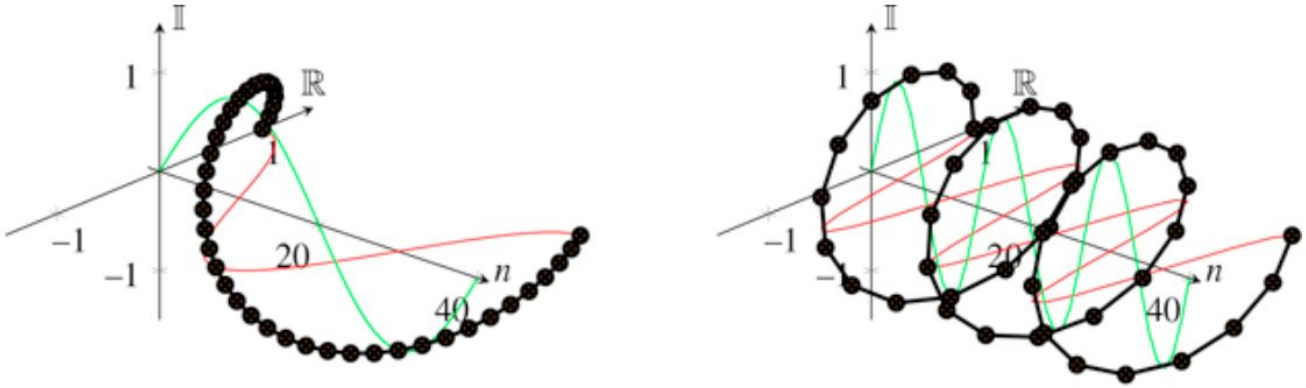
$$c_{u,v}[n,m] = A \cos\left(2\pi\left(\frac{un}{N} + \frac{vm}{M}\right)\right)$$

1. *Amplitude = A*
2. *u and v are spatial frequencies*

Complex Exponentials

Complex exponential wave (1D) : $e_u[n] = \exp\left(2\pi j \frac{u\,n}{N}\right)$

Euler's formula: $\exp(ja) = \cos(a) + j\sin(a)$



(a) (b)

Complex Exponentials

1. Complex exponential (2D) :

$$e_{u,v}[n,m] = \exp\left(2\pi j\left(\frac{un}{N} + \frac{vm}{M}\right)\right)$$

2. Complex exponentials in 2D are separable:

$$e_{u,v}[n,m] = e_u[n]\, e_v[m]$$

3. Any finite length discrete image can be decomposed as a linear combination of complex exponentials

Discrete Fourier Transform

1. Fourier Transform

$$\mathscr{L}[u,v] = \mathcal{F}\{\ell[n,m]\} = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \ell[n,m] \exp\left(-2\pi j\left(\frac{un}{N} + \frac{vm}{M}\right)\right)$$

2. Inverse Fourier Transform

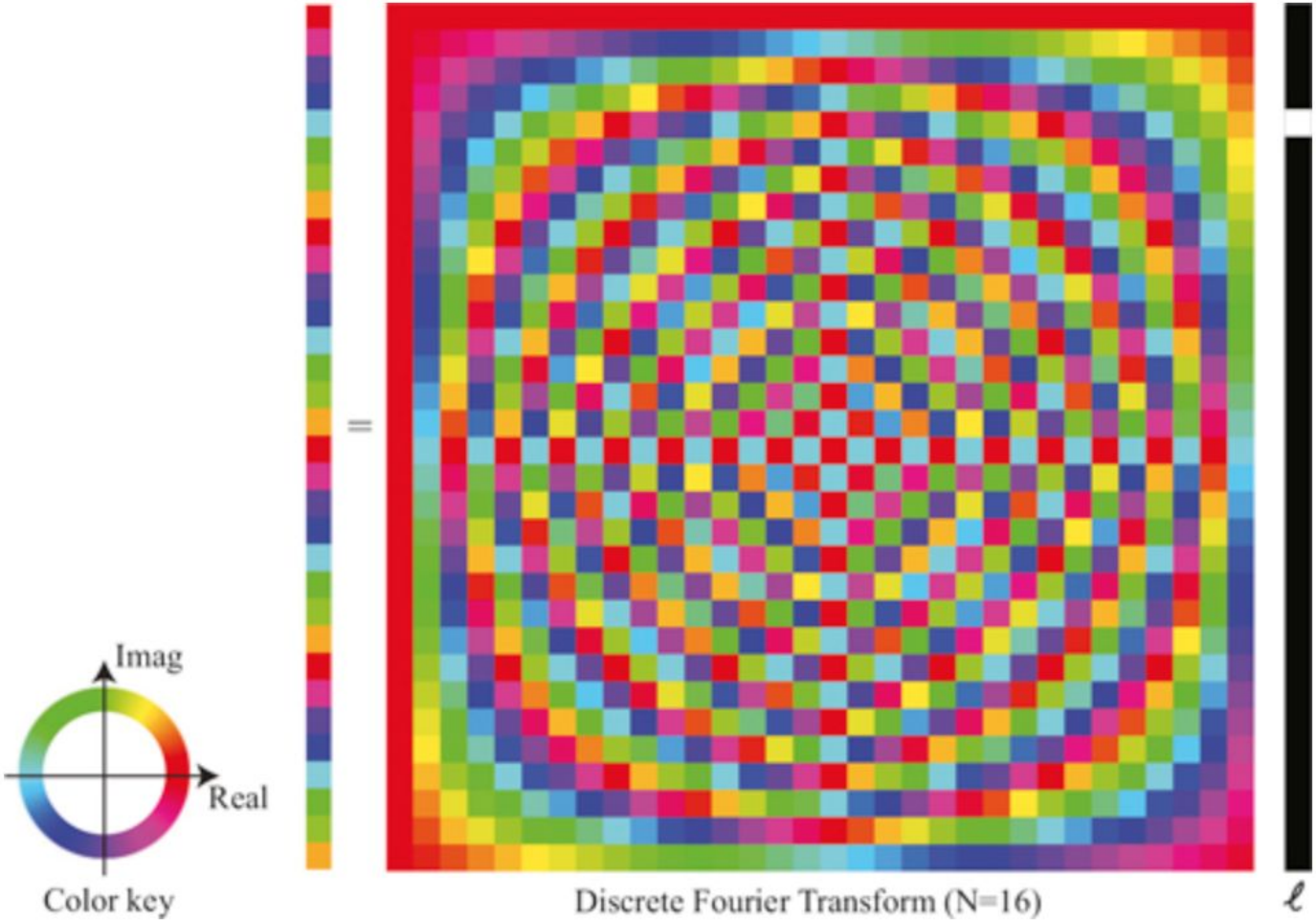
$$\ell[n,m] = \mathcal{F}^{-1}\{\mathscr{L}[u,v]\} = \frac{1}{NM}\sum_{u=0}^{N-1}\sum_{v=0}^{M-1} \mathscr{L}[u,v] \exp\left(+2\pi j\left(\frac{un}{N} + \frac{vm}{M}\right)\right)$$

Matrix Formulation of 1D DFT

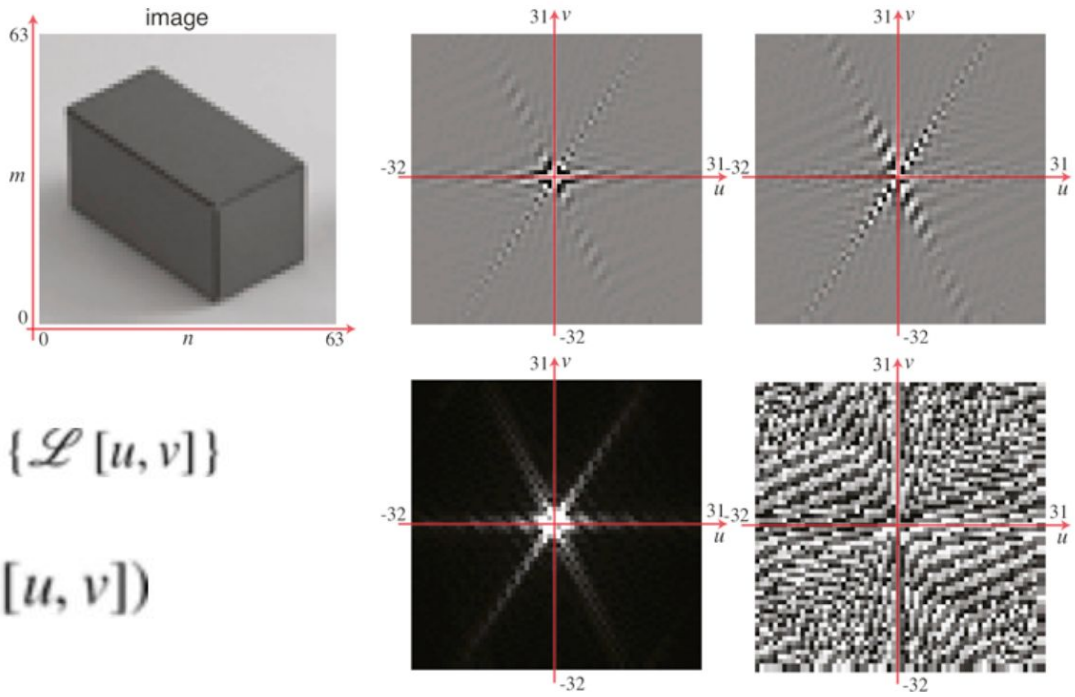
$$\mathbf{F} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \exp\left(-2\pi j \frac{1}{N}\right) & \exp\left(-2\pi j \frac{2}{N}\right) & \dots & \exp\left(-2\pi j \frac{N-1}{N}\right) \\ 1 & \exp\left(-2\pi j \frac{2}{N}\right) & \exp\left(-2\pi j \frac{4}{N}\right) & \dots & \exp\left(-2\pi j \frac{2(N-1)}{N}\right) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \exp\left(-2\pi j \frac{(N-1)}{N}\right) & \exp\left(-2\pi j \frac{(N-1)2}{N}\right) & \dots & \exp\left(-2\pi j \frac{(N-1)(N-1)}{N}\right) \end{bmatrix} \quad (16.20)$$

Discrete Fourier Transform

1. The DFT's fundamental purpose is to decompose a signal into its constituent frequency components.
2. Each row of the DFT matrix acts as a "filter" that isolates a specific frequency within the input signal.
3. In essence, you're "correlating" the input signal with a complex sinusoid of a particular frequency.
4. The multiplication of the input signal with a row of the DFT matrix can be thought of as a correlation.
5. If the input signal contains a strong component at the frequency represented by a particular row, the result of the multiplication will have a large magnitude.
6. This indicates that there's a strong "match" between the input signal and that specific frequency.


Imag
Real
Color key
=
Discrete Fourier Transform (N=16)
ℓ

Visualizing the Fourier Transformation



$$\mathscr{L}[u,v] = Re\{\mathscr{L}[u,v]\} + j\,Imag\{\mathscr{L}[u,v]\}$$

$$\mathscr{L}[u,v] = A[u,v]\ \exp(j\,\theta[u,v])$$

Figure 16.7: DFT of an image and visualization of (top) the real and imaginary components, and (bottom) the amplitude and phase of the Fourier transform.

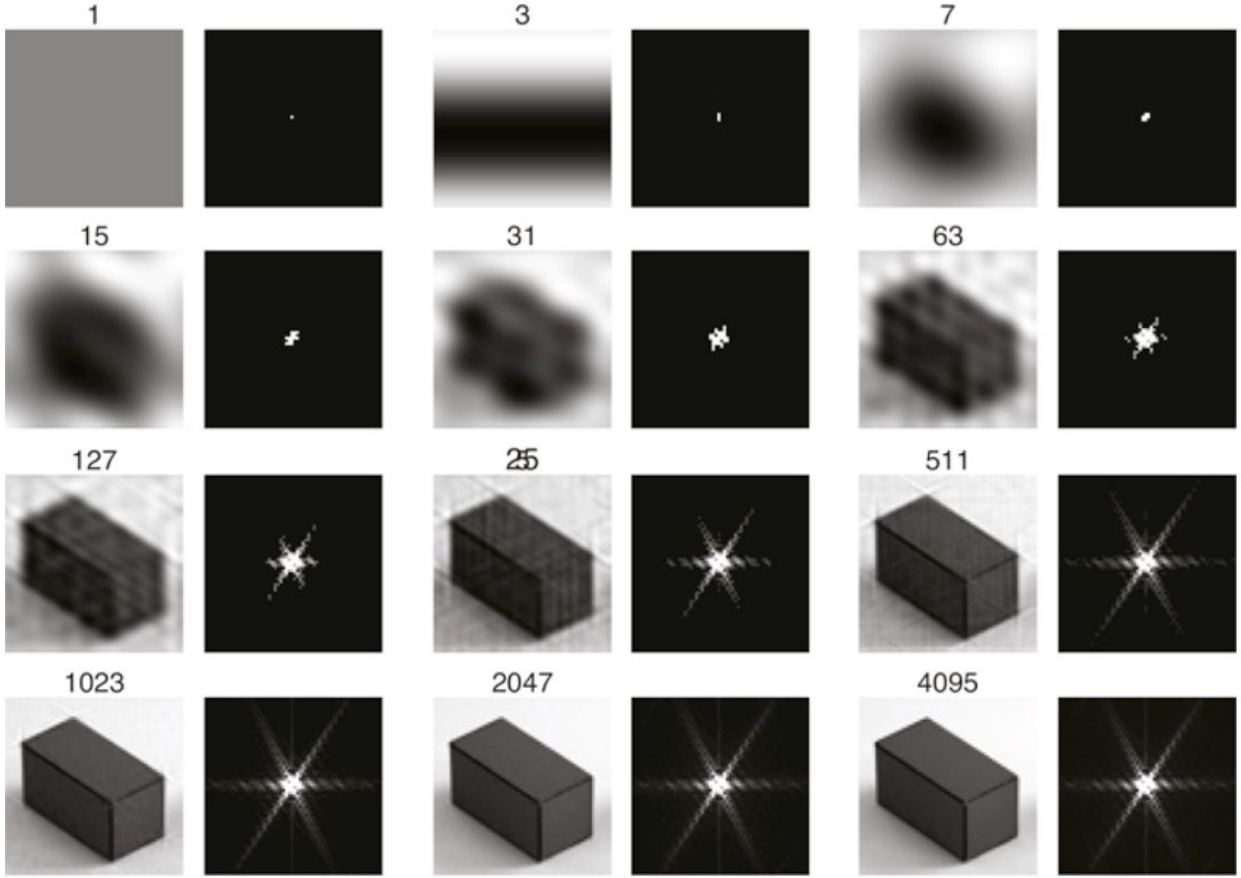


Figure 16.8: Reconstructing an image from the N Fourier coefficients of the largest amplitude. The right frame shows the location, in the Fourier domain, of the N Fourier coefficients, which when inverted, give the image at the left.

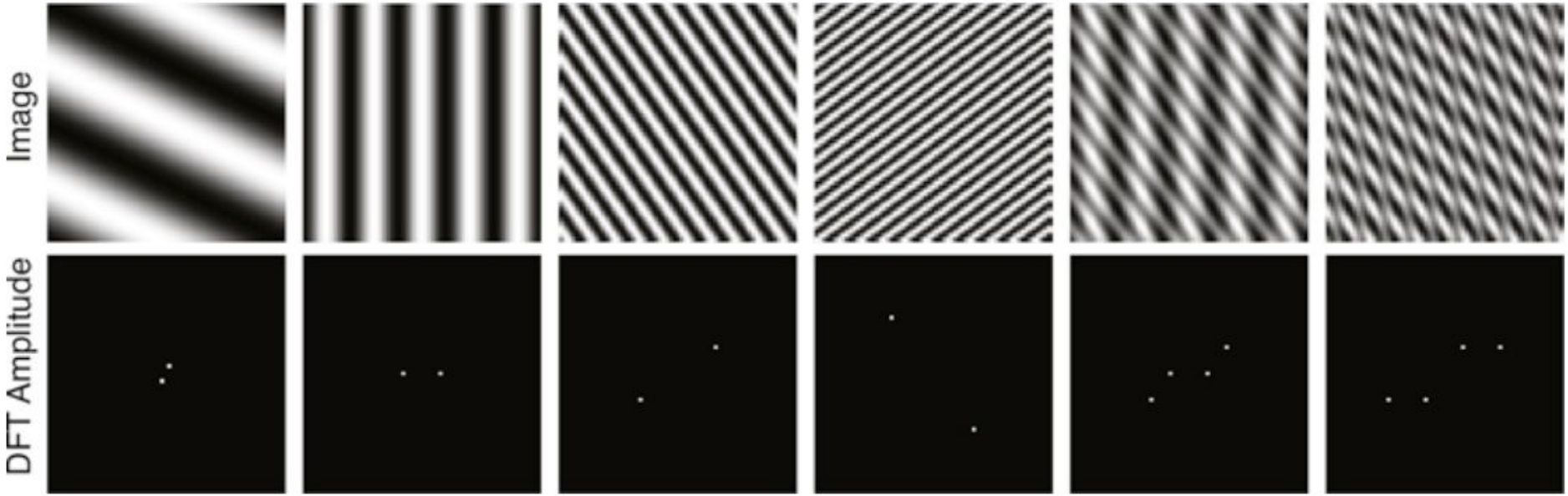


Figure 16.9: Some 2D Fourier transform pairs. Images are 64 × 64 pixels. The waves are cosine with frequencies (1, 2), (5, 0), (10, 7), (11, –15). The last two examples show the sum of two waves and the product.

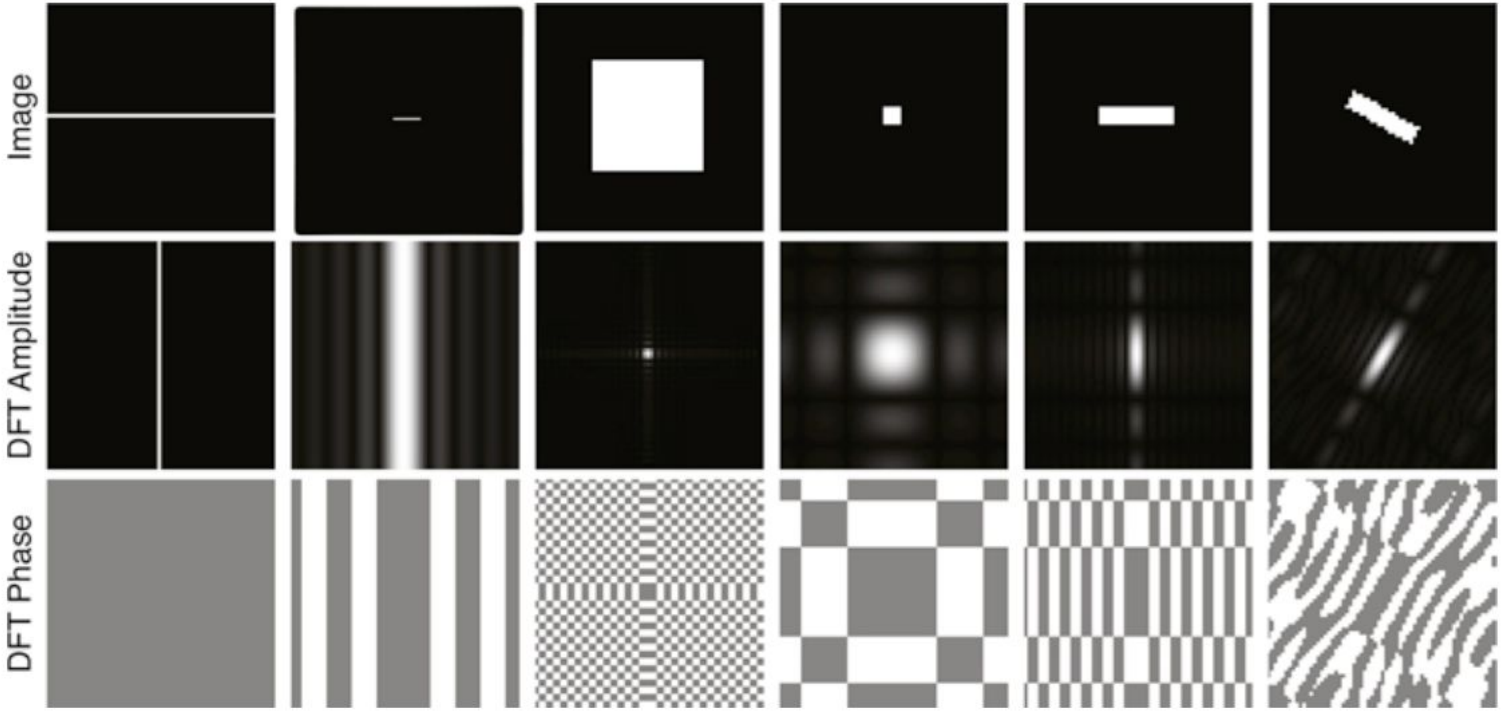


Figure 16.10: Some two-dimensional Fourier transform pairs. Note the trends visible in the collection of transform pairs: As the support of the image in one domain gets larger, the magnitude in the other domain becomes more localized. A line transforms to a line oriented perpendicularly to the first. Images are 64×64 pixels, origin is in the center.

“If we multiply an image with a complex exponential, its Fourier transform is translated”

“Multiplying a signal by a wave is called **signal modulation** and it is one of the basic operations in communications.”

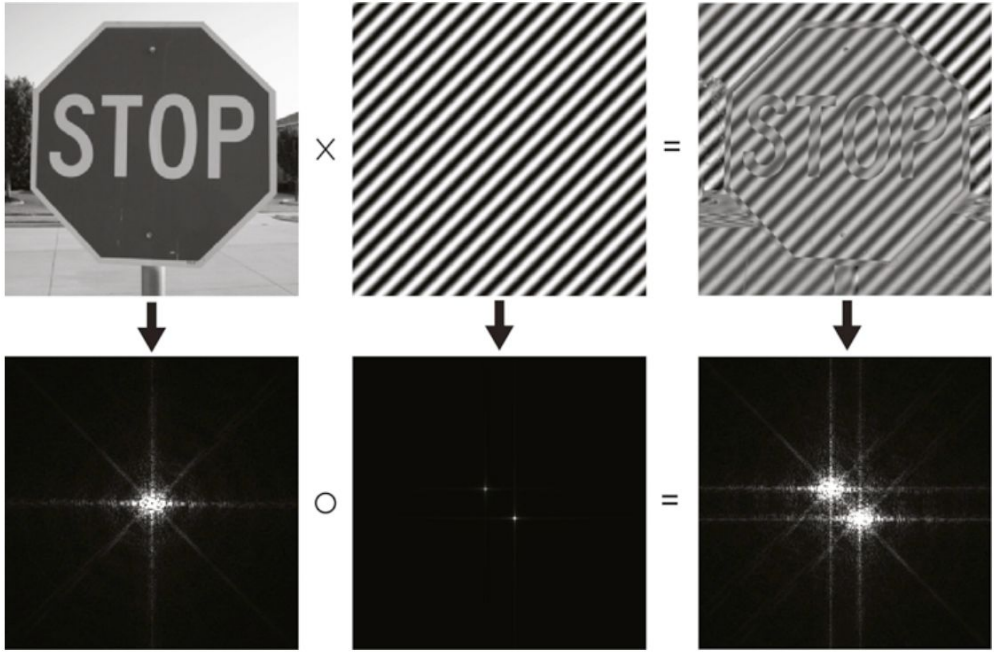


Figure 16.14: Modulation in space. Multiplying an image by a cosine wave results in a new image with a Fourier transform with two copies of the Fourier transform of the original image. Only the magnitude of the Fourier transforms are shown.

Modulation - Signals

1. Signal modulation is the process of modifying a carrier signal (typically a high-frequency wave) to encode information from a message signal.
2. Efficient Transmission: Low-frequency signals (like audio) cannot travel long distances efficiently. Modulation shifts them to a higher frequency suitable for transmission.
3. Multiplexing: Multiple signals can be transmitted simultaneously by using different carrier frequencies (e.g., FM radio stations).
4. Noise Resistance: Higher frequency signals are less affected by environmental noise and interference.
5. Antenna Size Reduction: Higher frequencies allow for practical antenna sizes (lower frequencies require very large antennas).

Modulation - Computer Vision

Structured Light Systems (Depth Sensing)

How it works:

1. A projector emits a modulated light pattern onto a scene.
2. A camera captures the deformed pattern, and depth information is extracted by analyzing how the light is distorted.

Example:

1. Microsoft Kinect and iPhone Face ID use infrared light modulation for depth sensing.
2. Time-of-flight (ToF) cameras use amplitude modulation to measure depth.

Modulation - Computer Vision

Optical Flow and Motion Estimation

1. Phase-based methods in optical flow estimate motion by analyzing changes in phase-modulated signals.
2. Modulation in phase correlation helps track object movement in a video.
3. Used in autonomous vehicles (for detecting motion) and video stabilization.

Image compression (JPEG uses Discrete Cosine Transform modulation)

Convolution Theorem

1.

Convolution and Fourier Transform		
Spatial Domain		Frequency Domain
$g(x) = f(x) * h(x)$	$\longleftrightarrow$	$G(u) = F(u)\,H(u)$
Convolution		Multiplication
$g(x) = f(x)\,h(x)$	$\longleftrightarrow$	$G(u) = F(u) * H(u)$
Multiplication		Convolution

1.

Convolution and Fourier Transform

Convolution: $g(x) = f(x) * h(x) = \int_{-\infty}^{\infty} f(\tau)h(x-\tau)\,d\tau$

Fourier Transform of $g(x)$:

$$G(u) = \int_{-\infty}^{\infty} g(x)e^{-i2\pi ux}dx$$

$$G(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)h(x-\tau)e^{-i2\pi ux}d\tau dx$$

$$G(u) = \underbrace{\int_{-\infty}^{\infty} f(\tau)e^{-i2\pi u\tau}d\tau}_{F(u)} \; \underbrace{\int_{-\infty}^{\infty} h(x-\tau)e^{-i2\pi u(x-\tau)}\,dx}_{H(u)}$$

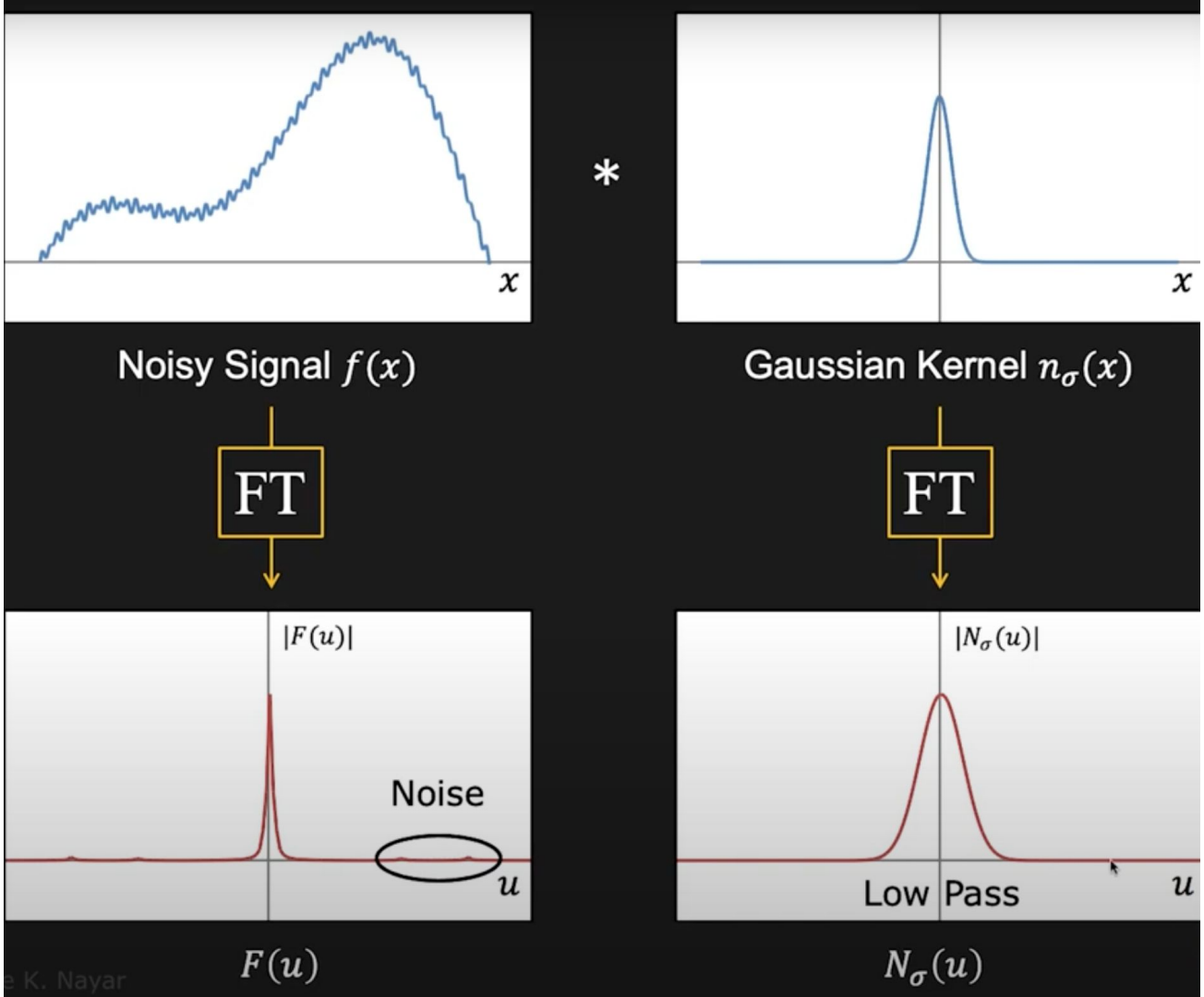
ee K. Nayar

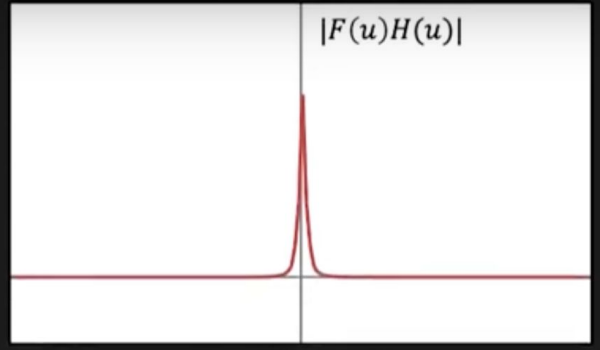
Convolution Using Fourier Transform

$$g(x) = f(x) * h(x)$$

$$\uparrow \text{IFT} \qquad \downarrow \text{FT} \qquad \downarrow \text{FT}$$

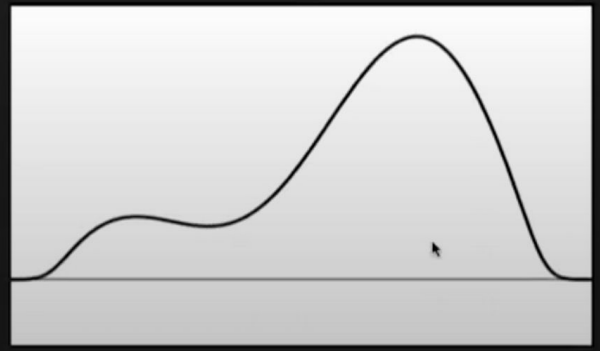
$$G(u) = F(u) \times H(u)$$


x
*
x
Noisy Signal $f(x)$
Gaussian Kernel $n_\sigma(x)$
FT
FT
$|F(u)|$
Noise
u
$|N_\sigma(u)|$
Low Pass
u
$F(u)$
$N_\sigma(u)$



$F(u)H(u)$

IFT



Gaussian Blurred Signal $g(x)$

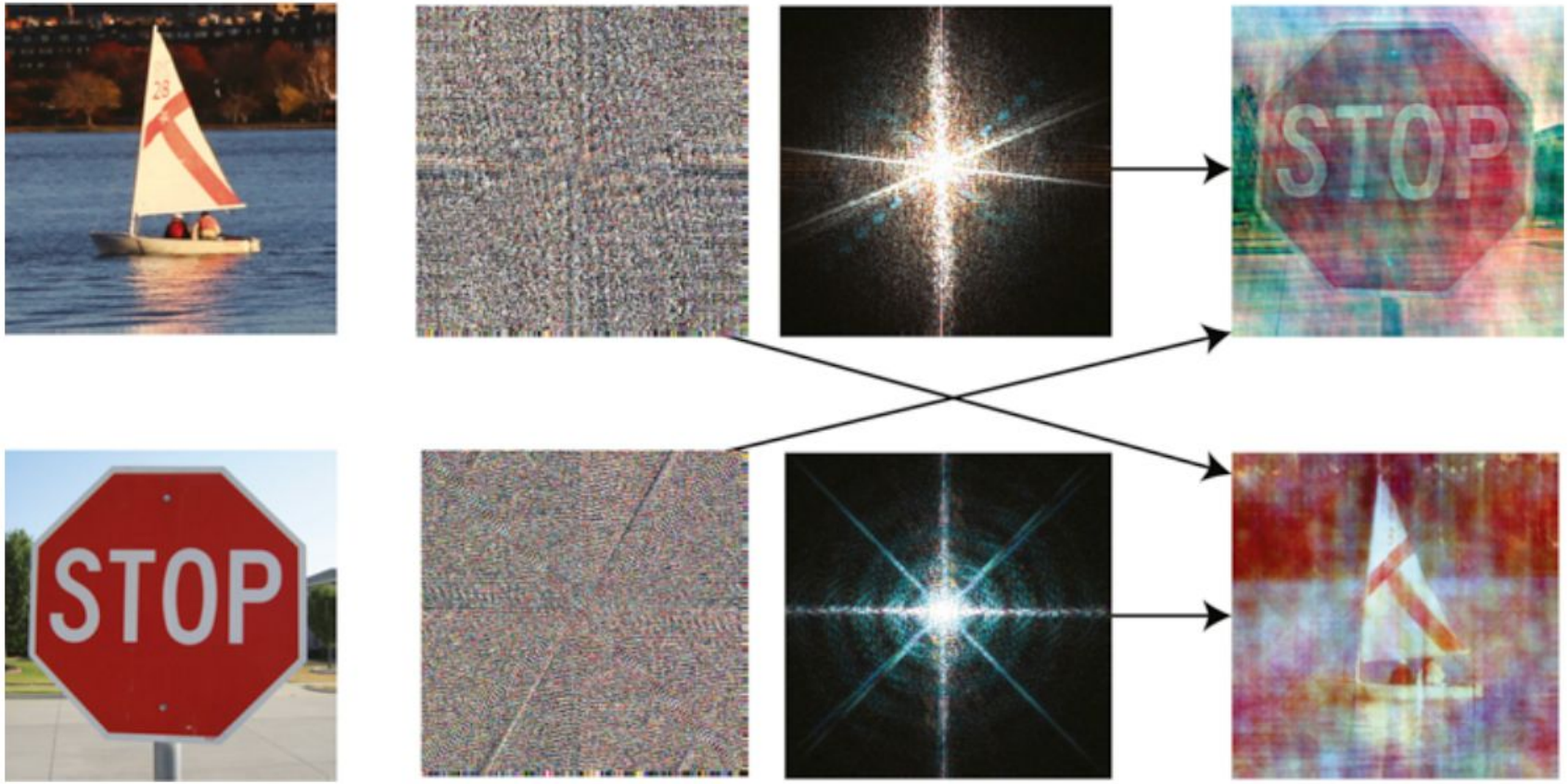


Figure 16.15: Swapping the amplitude and the phase of the Fourier Transform of two images. Each color channel is processed in the same way.

“phase seems to be carrying most of the image information”

“The phase carries the information of where the image contours are by specifying how the phases of the sinusoids must line up in order to create the observed contours and edges”

“The amplitude is great for capturing images that contain strong periodic patterns.”

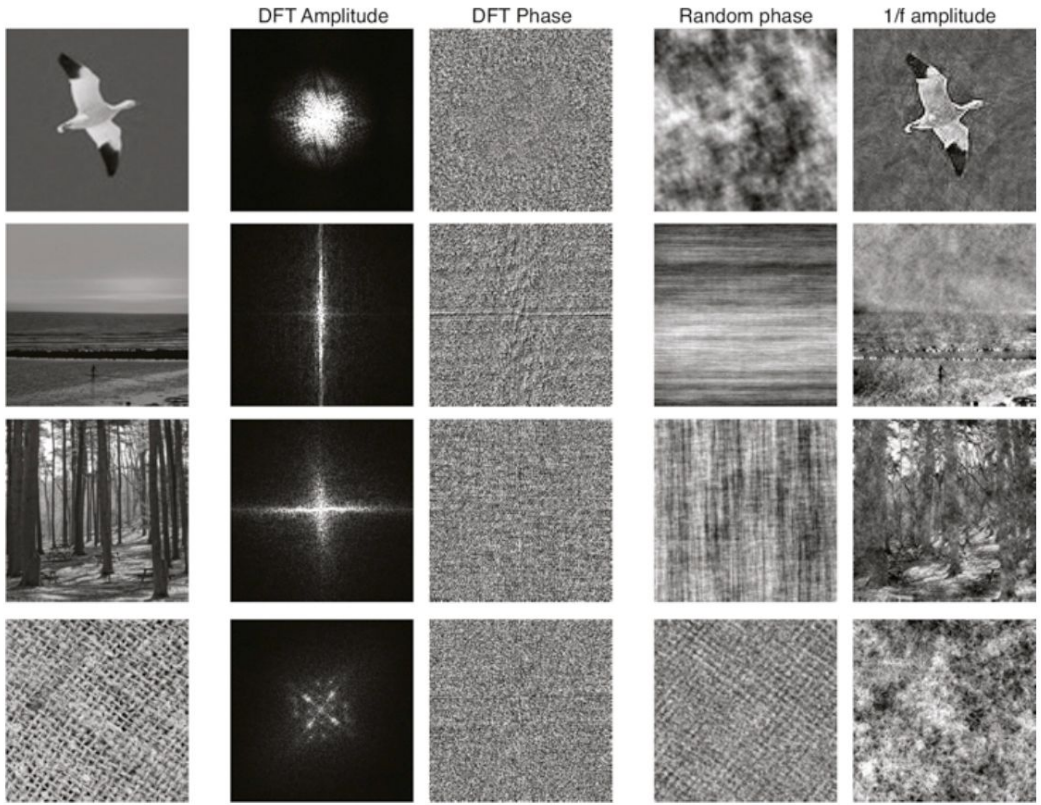


Figure 16.16: The relative importance of phase and amplitude depends on the image. Each row shows one image, its Fourier transform (amplitude and phase), and the resulting images obtained by applying the inverse Fourier transform to a signal with the original amplitude and randomized phase, and a signal with the original phase and a generic fixed $1/f$ amplitude. Note that for the first image, the phase seems to be the most important component.

Quiz

Match the Fourier transform magnitudes with the corresponding

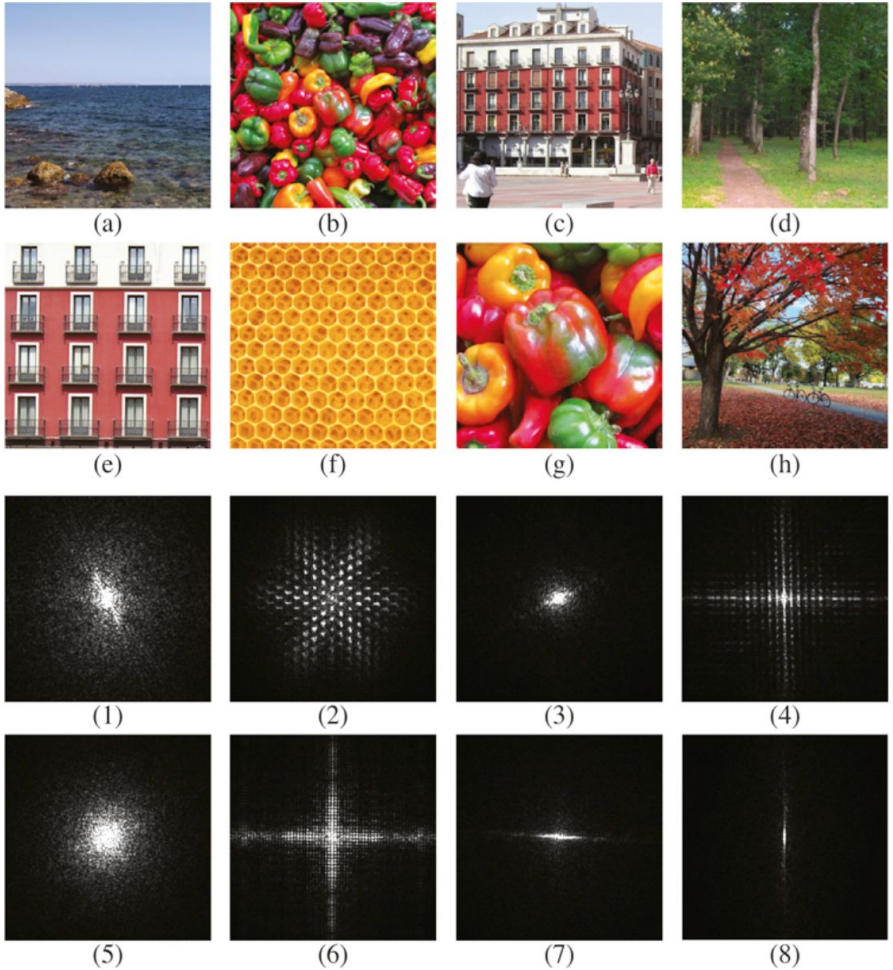
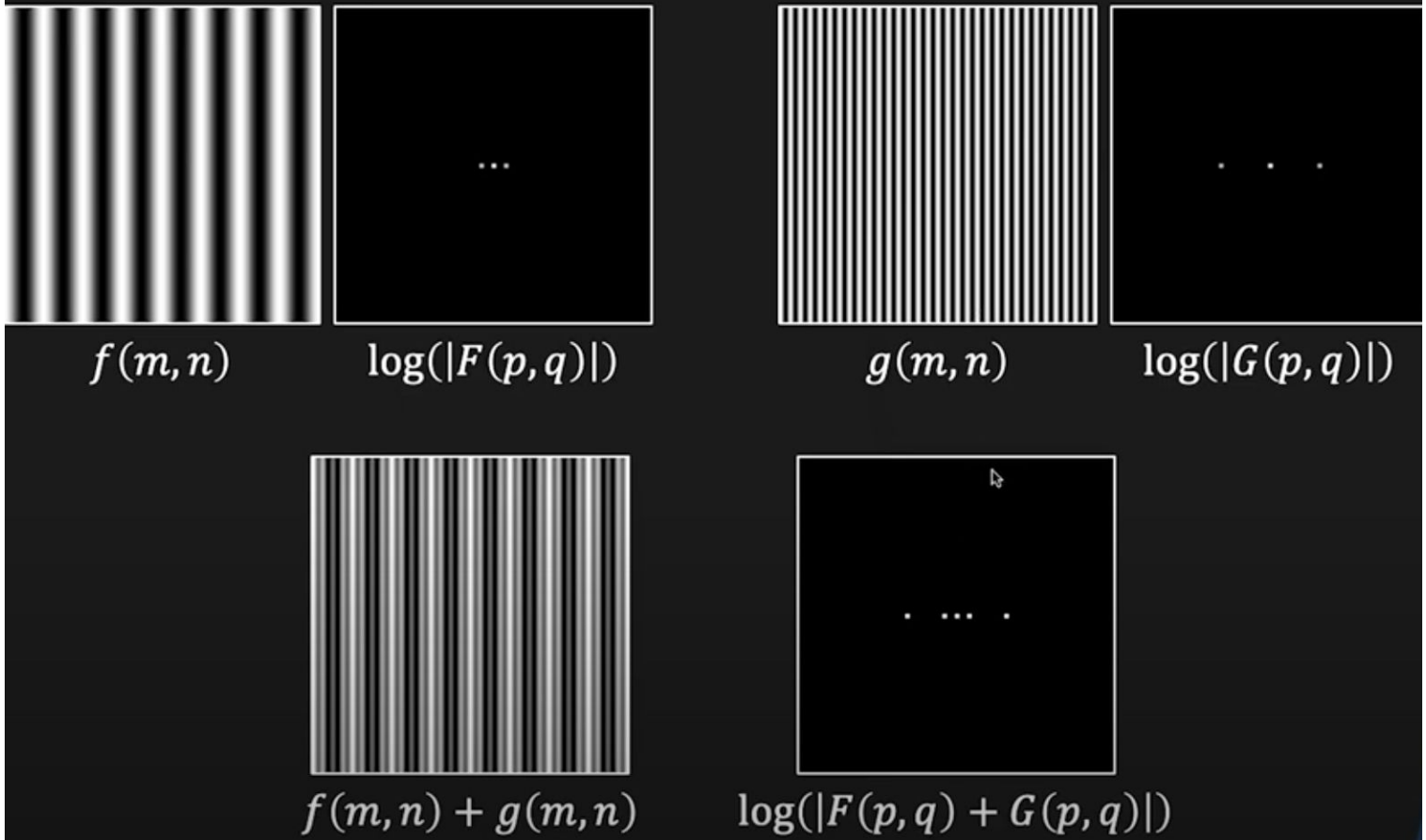


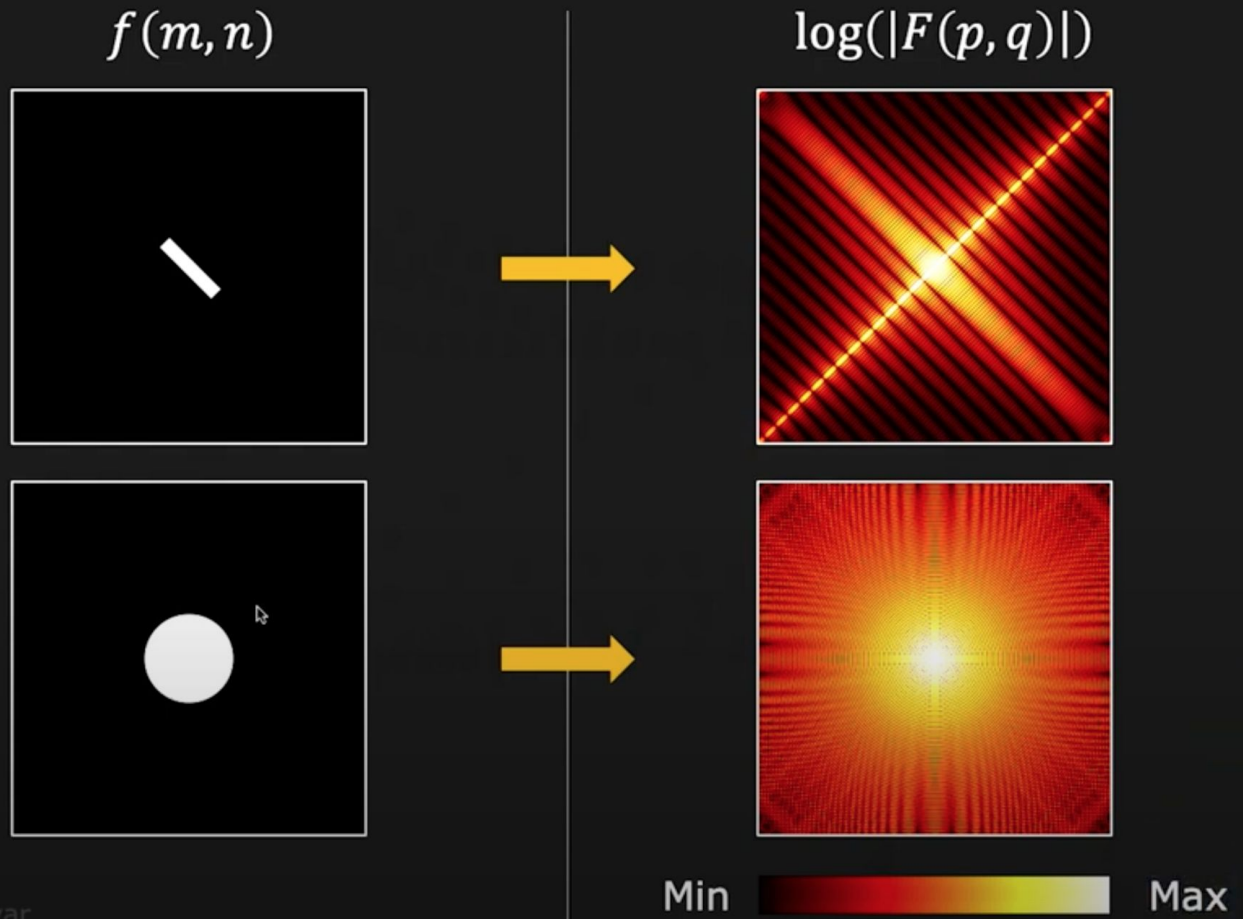
Figure 16.17: The Fourier transform matching game: Match each image (a-h) with its corresponding Fourier transform magnitude (1-8).

1.

2D Fourier Transform: Example 1



2D Fourier Transform: Example 2

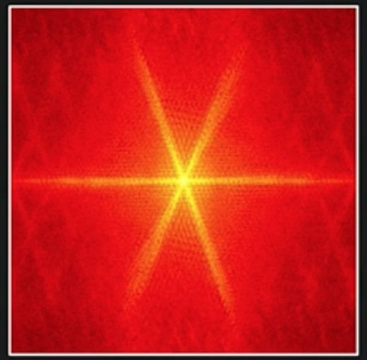


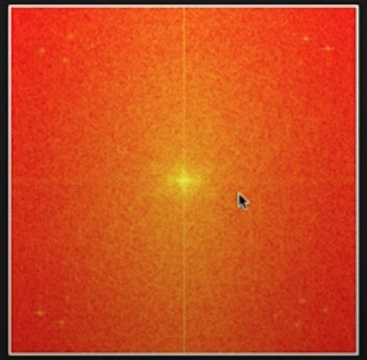
K. Navar

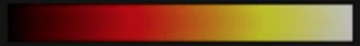
2D Fourier Transform: Example 3

$f(m,n)$

$\log(|F(p,q)|)$





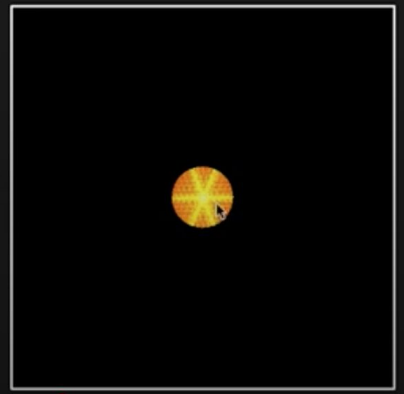
Min  Max

K. Nayar

Low Pass Filtering

$f(m,n)$

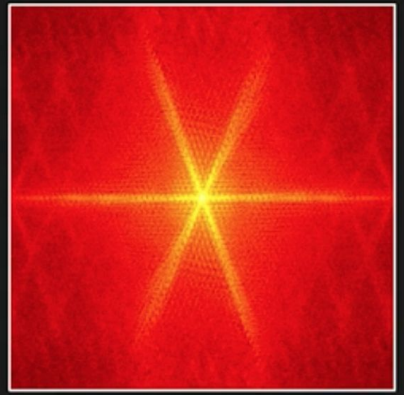
$\log(|F(p,q)|)$

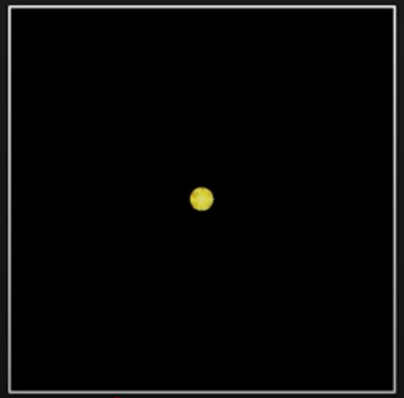


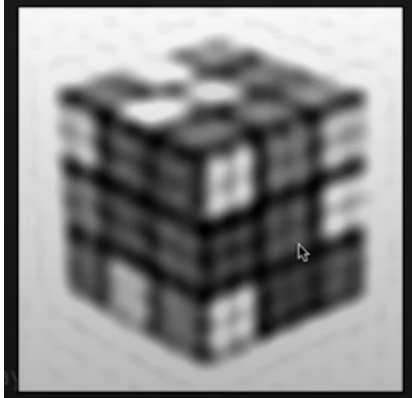
Low Pass Filtering

$f(m,n)$

$\log(|F(p,q)|)$



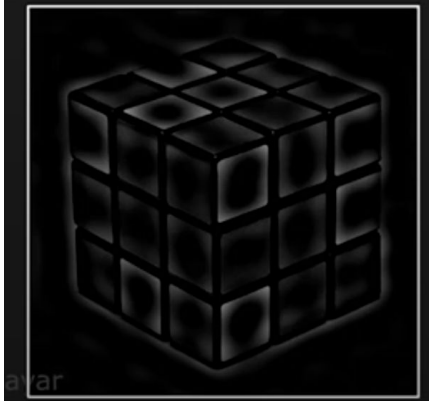


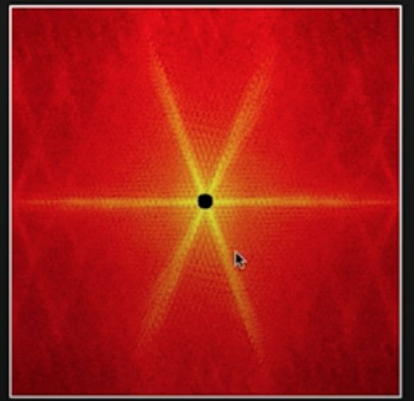


High Pass Filtering

$f(m,n)$

$\log(|F(p,q)|)$

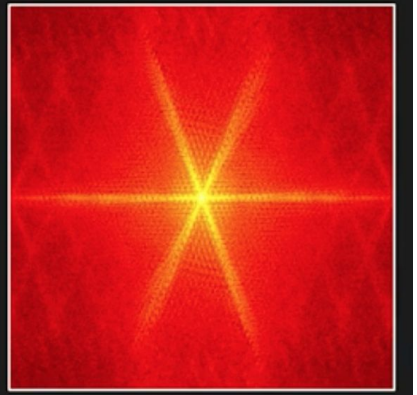


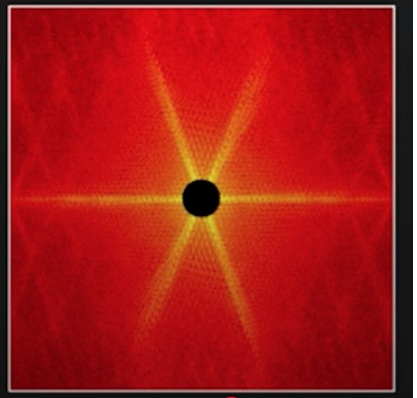


High Pass Filtering

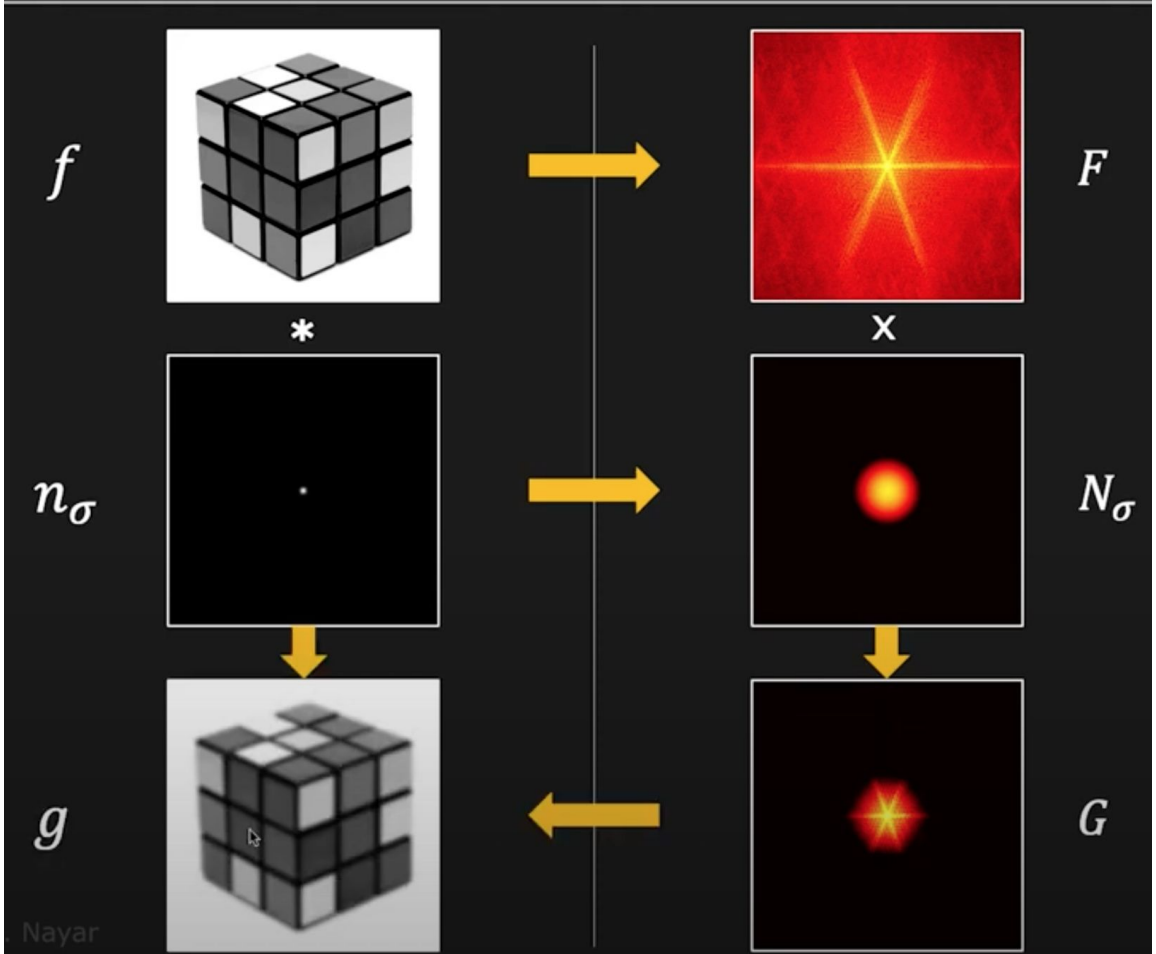
$f(m,n)$

$\log(|F(p,q)|)$



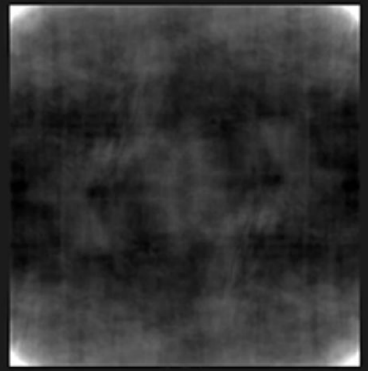


Gaussian Smoothing



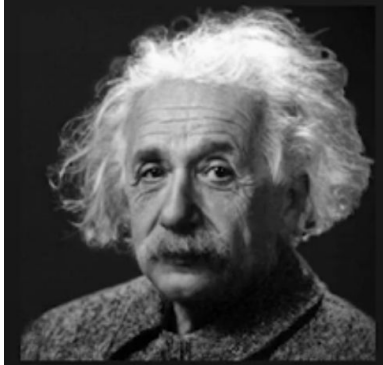
Importance of Phase

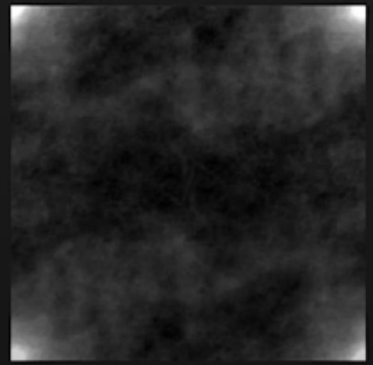
Original Image

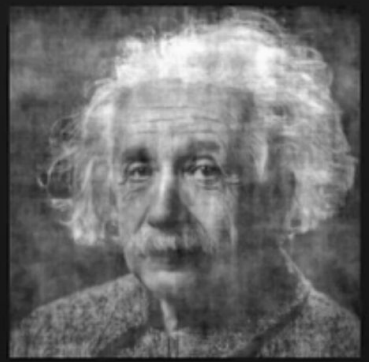


Magnitude Preserved,
Phase Set to Zero

Phase Preserved,
Magnitude Set to Averag
of Natural Images

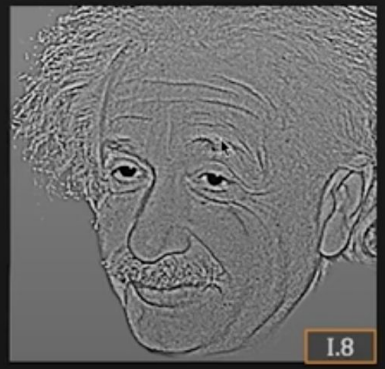






Hybrid Images

Low Freq Only



High Freq Only

Hybrid (Sum) Image

Conclusion

1. The Fourier transform is an indispensable tool for
   a. linear systems analysis,
   b. image analysis,
   c. for efficient filter output computation.
2. “ it's easy to analyze images according to spatial frequency, and this represents some progress in interpreting the image over merely a pixel representation”
3. “But the Fourier transform has a major drawback as an image representation: it's too global! Every sinusoidal component covers the entire image.”

Conclusion

1. “The Fourier transform **tells us a little about what is happening in the image** (based on the spatial frequency content), but it tells us nothing about **where it is happening**.”
2. “Conversely, a pixel representation gives great spatial localization, but a pixel value by itself doesn't help us learn much about what's going on in the image.
3. We seek a representation that's somewhere in between those two extremes.”

References

1. Foundations of Computer Vision - Chapter 16
1. https://www.iaincollings.com/signals-and-systems
2. Columbia University https://fpcv.cs.columbia.edu