The Two-Phase Method

- The method solves the LP in two phases:
- Phase I attempts to find a starting basic feasible solution, and, if one is found,
- Phase II is invoked to solve the original problem.
- **Phase I:**
- Put the problem in equation form and add the necessary artificial variables to the constraints (exactly as in the M-method) to secure a starting basic solution.
- **Form a new objective function that minimizes the sum off all artificial variables, regardless of whether the LP is maximization or minimization.** If the minimum value of the sum is positive, the LP problem has no feasible solution. Otherwise, proceed to Phase II.
- **Phase II:**
- Use the feasible solution from Phase I as a starting basic feasible solution for the original problem.

Minimize $z = 4x_1 + x_2$

$$
\begin{aligned}
3x_1 + \ \ x_2 &= 3 \\
4x_1 + 3x_2 &\geq 6 \\
x_1 + 2x_2 &\leq 4 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Minimize $z = 4x_1 + x_2$

$$
\begin{aligned}
3x_1 + \ \ x_2 \qquad\qquad &= 3 \\
4x_1 + 3x_2 - x_3 \qquad &= 6 \\
x_1 + 2x_2 \qquad + x_4 &= 4 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
3x_1 + \ \ x_2 \qquad + R_1 \qquad\qquad &= 3 \\
4x_1 + 3x_2 - x_3 \qquad + R_2 \qquad &= 6 \\
x_1 + 2x_2 \qquad\qquad + x_4 &= 4 \\
x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0
\end{aligned}
$$

Phase I

$$\text{Minimize } r = R_1 + R_2$$

$$\begin{aligned} 3x_1 + x_2 + R_1 &= 3 \\ 4x_1 + 3x_2 - x_3 + R_2 &= 6 \\ x_1 + 2x_2 + x_4 &= 4 \end{aligned}$$

$$x_1, x_2, x_3, x_4, R_1, R_2 \geq 0$$

Basic	x1	x2	x3	R1	R2	x4	Solution
r	0	0	0	-1	-1	0	0
R1	3	1	0	1	0	0	3
R2	4	3	-1	0	1	0	6
x4	1	2	0	0	0	1	4

Basic	x1	x2	x3	R1	R2	x4	Solution
r	0	0	0	-1	-1	0	0
R1	3	1	0	1	0	0	3
R2	4	3	-1	0	1	0	6
x4	1	2	0	0	0	1	4
r	7	4	-1	0	0	0	9
R1	3	1	0	1	0	0	3
R2	4	3	-1	0	1	0	6
x4	1	2	0	0	0	1	4

Basic	x1	x2	x3	R1	R2	x4	Solution
r	0	0	0	-1	-1	0	0
R1	3	1	0	1	0	0	3
R2	4	3	-1	0	1	0	6
x4	1	2	0	0	0	1	4
r	7	4	-1	0	0	0	9
R1	3	1	0	1	0	0	3
R2	4	3	-1	0	1	0	6
x4	1	2	0	0	0	1	4
r	0	$\frac{5}{3}$	-1	$-\frac{7}{3}$	0	0	2
x1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
R2	0	$\frac{5}{3}$	-1	$-\frac{4}{3}$	1	0	2
x4	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3
r	0	0	0	-1	-1	0	0
x1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
x4	0	0	1	1	-1	1	1

basic	x1	x2	x3	R1	R2	x4	Solution
r	0	0	0	-1	-1	0	0
x1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
x4	0	0	1	1	-1	1	1

Phase II

$$\text{Minimize } z = 4x_1 + x_2$$

$$x_1 + \tfrac{1}{5}x_3 = \tfrac{3}{5}$$

$$x_2 - \tfrac{3}{5}x_3 = \tfrac{6}{5}$$

$$x_3 + x_4 = 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

basic	x1	x2	x3	x4	Solution
r	0	0	0	0	0
x1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x4	0	0	1	1	1

Phase II

$$\text{Minimize } z = 4x_1 + x_2$$

$$\begin{aligned} x_1 + \tfrac{1}{5}x_3 &= \tfrac{3}{5} \\ x_2 - \tfrac{3}{5}x_3 &= \tfrac{6}{5} \\ x_3 + x_4 &= 1 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Basic	x1	x2	x3	x4	Solution
z	-4	-1	0	0	0
x1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x4	0	0	1	1	1

Minimize $z = 4x_1 + x_2$

$$x_1 + \tfrac{1}{5}x_3 = \tfrac{3}{5}$$
$$x_2 - \tfrac{3}{5}x_3 = \tfrac{6}{5}$$
$$x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Basic	x1	x2	x3	x4	Solution
z	-4	-1	0	0	0
x1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x4	0	0	1	1	1
z	0	0	$\frac{1}{5}$	0	$\frac{18}{5}$
x1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x4	0	0	1	1	1
z	**0**	**0**	**0**	$\mathbf{-\frac{1}{5}}$	$\mathbf{\frac{17}{5}}$
x1	1	0	0	$-\frac{1}{5}$	$\mathbf{\frac{2}{5}}$
x2	0	1	0	$\frac{3}{5}$	$\mathbf{\frac{9}{5}}$
x3	0	0	1	1	**1**