Fourier Analysis Notes

The Fourier transform is a mathematical tool that decomposes a function (like a signal or image) into its constituent frequencies. Linear time-invariant (LTI) systems are a class of systems where the output is a linear combination of the input and the system's response doesn't change over time.

Here's the relationship between the two:

- **Eigenfunctions:** Complex exponentials (like sine and cosine waves) are eigenfunctions of LTI systems. This means that when an LTI system is given a complex exponential as input, the output is simply a scaled version of the same complex exponential.
- **Fourier Transform as a Decomposition:** The Fourier transform decomposes any function into a sum (or integral) of these complex exponentials.
- **LTI System Response:** Since complex exponentials are eigenfunctions, the Fourier transform of the output of an LTI system is simply the Fourier transform of the input, multiplied by a scaling factor for each frequency component. This scaling factor is called the frequency response of the system.

In essence, the Fourier transform allows us to analyze LTI systems in the frequency domain, where the system's behavior is characterized by its frequency response. This simplifies the analysis of LTI systems significantly, as convolution in the time domain becomes multiplication in the frequency domain.

Think of it like this:

Imagine an LTI system as a filter that modifies different frequencies of a signal differently. The Fourier transform breaks down the signal into its individual frequencies, and the frequency response of the system tells us how much each frequency is amplified or attenuated by the filter.

This relationship is fundamental in many areas of engineering and science, including:

- **Signal processing:** Analyzing and designing filters, understanding how signals are affected by different systems.
- **Communications:** Understanding how signals are transmitted and received over channels.
- **Control systems:** Designing systems that respond to inputs in a desired way.

In summary, the Fourier transform provides a powerful tool for analyzing LTI systems by decomposing signals into their frequency components, which are the eigenfunctions of LTI systems. This simplifies the analysis and design of such systems.

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An **eigenvector** of a matrix AA is a nonzero vector v that, when multiplied by A, results in a scalar multiple of itself:

Av=λv

where λ is the corresponding **eigenvalue**.

Example:

Consider the matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Conclusion:

- Eigenvectors: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
- Corresponding eigenvalues: 3 and 1.

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It's important to clarify the relationship between the Fourier transform and Linear Time-Invariant (LTI) systems. While the Fourier transform itself is a mathematical operation, its application is deeply intertwined with LTI systems. Here's a breakdown:

Fourier Transform and LTI Systems:

- **Fourier Transform as a Tool:**
  - The Fourier transform is a mathematical tool that decomposes a signal into its constituent frequencies. It can be applied to a wide range of signals, not just those from LTI systems.
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- **LTI Systems and Frequency Response:**
  - LTI systems are characterized by their impulse response. The Fourier transform of the impulse response is called the frequency response of the system.
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    - The frequency response tells you how the system affects different frequency components of an input signal.
- **Key Relationship:**
    - A crucial property is that when an LTI system processes an input signal, the Fourier transform of the output signal is the product of the Fourier transform of the input signal and the frequency response of the system.
    - This simplifies the analysis of LTI systems significantly. Instead of working in the time domain with convolution, you can work in the frequency domain with multiplication.
- **Therefore:**
    - The Fourier transform is exceptionally useful for analyzing LTI systems, but it is not exclusively limited to them.
    - It is very accurate to say that LTI systems are very easily analyzed by the use of the Fourier transform.
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In summary:

- The Fourier transform is a general tool for frequency analysis.
- LTI systems have a special relationship with the Fourier transform, making it a powerful tool for their analysis.

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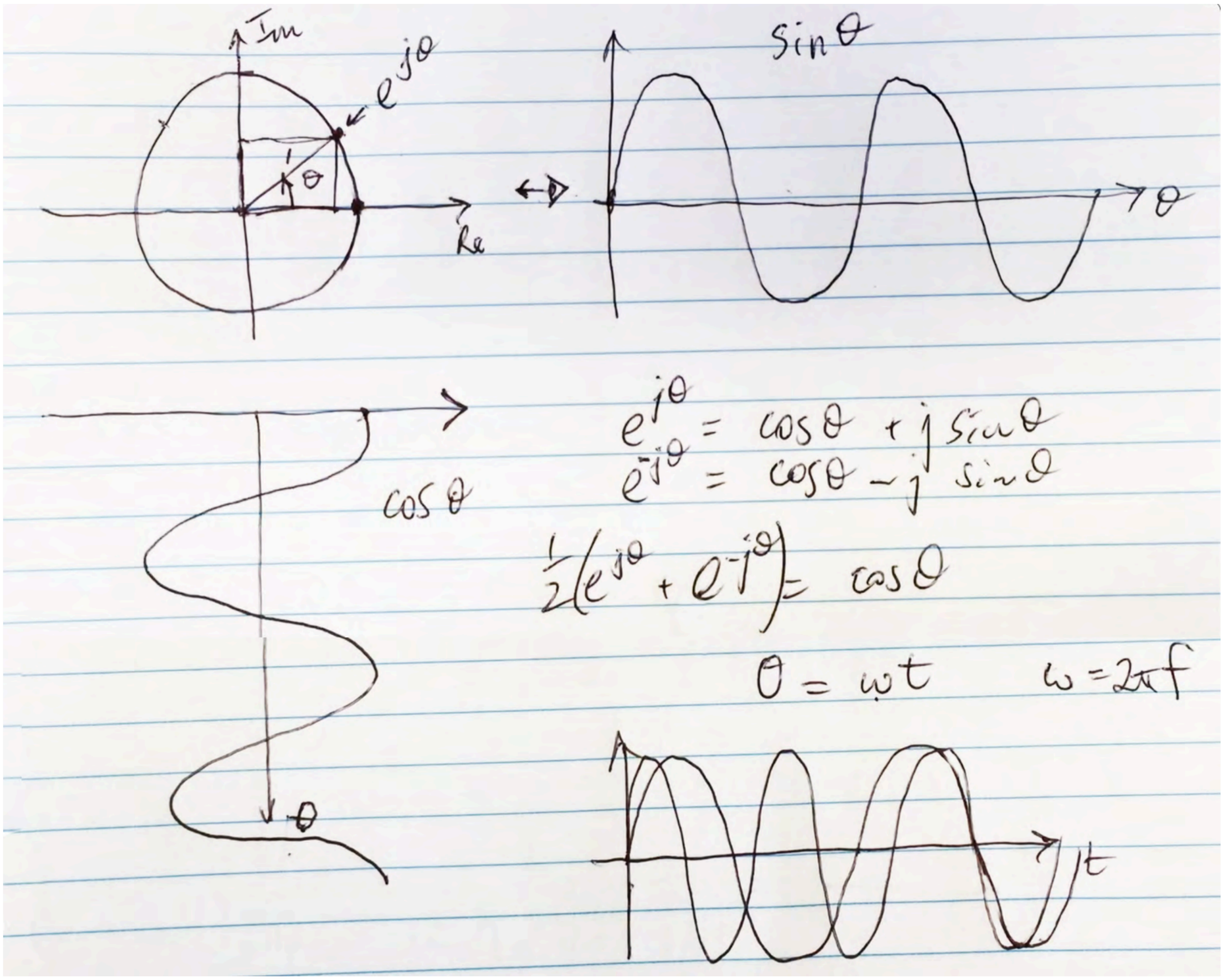
https://homepages.inf.ed.ac.uk/rbf/HIPR2/fourier.htm

https://www.southampton.ac.uk/~msn/book/new_demo/fourier/

Complex Numbers and Real Signals



The expression e^{jx} (where j is the imaginary unit, $j^2 = -1$) always lies on the **unit circle** in the complex plane because of **Euler's formula**:

$$e^{jx} = \cos x + j \sin x$$

Key Observations

1. **Magnitude Calculation**
    - The modulus (magnitude) of a complex number $z = a + jb$ is given by:

    $$|z| = \sqrt{a^2 + b^2}$$

    - Applying this to $e^{jx} = \cos x + j \sin x$:

    $$|e^{jx}| = \sqrt{\cos^2 x + \sin^2 x} = \sqrt{1} = 1$$

    - Since the magnitude is always **1**, the point e^{jx} lies on the **unit circle**.

2. **Geometric Interpretation**
    - In the complex plane, e^{jx} represents a point at angle x (in radians) from the positive real axis.
    - As x varies, e^{jx} moves along the unit circle, maintaining constant distance **1** from the origin.

Conclusion

Since e^{jx} always has a magnitude of **1**, it always lies on the unit circle, regardless of x.

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Euler's intuition behind the formula:

$$e^{jx} = \cos x + j \sin x$$

came from exploring **infinite series expansions** of exponential, sine, and cosine functions.

1. The Power Series Approach

Euler knew that the function e^x could be expanded as a **Maclaurin series**:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Similarly, sine and cosine functions also had known Maclaurin series expansions:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

2. Substituting $x \to jx$ in e^x

Since Euler was investigating **complex exponentials**, he replaced x with jx in the power series for e^x
:

$$e^{jx} = 1 + jx + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \frac{(jx)^4}{4!} + \dots$$

Expanding each term:

$$e^{jx} = 1 + jx + \frac{j^2x^2}{2!} + \frac{j^3x^3}{3!} + \frac{j^4x^4}{4!} + \dots$$

Using $j^2 = -1$, $j^3 = -j$, and $j^4 = 1$, we rearrange terms:

$$e^{jx} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + j\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

From the known series expansions of cosine and sine:

$$e^{jx} = \cos x + j \sin x$$

3. Geometric Intuition

Euler also realized that in the complex plane, multiplication by e^{jx} corresponds to a **rotation by angle** x. This aligns perfectly with how cosine and sine describe circular motion, reinforcing that:

- $\cos x$ is the **real** part (horizontal movement).
- $\sin x$ is the **imaginary** part (vertical movement).

Conclusion

Euler's intuition came from recognizing patterns in **power series expansions** and verifying that complex exponentials naturally describe **rotations** in the complex plane.

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Multiplying by e^{jx} corresponds to a **rotation by angle x in the complex plane** because of **Euler's formula**:

$$e^{jx} = \cos x + j\sin x$$

1. Geometric Interpretation

Consider a complex number $z = re^{j\theta}$, which is in **polar form**, where:

- r is the magnitude (distance from origin).
- θ is the angle from the positive real axis.

Now, if we **multiply** z by e^{jx}:

$$z' = z \cdot e^{jx} = re^{j\theta} \cdot e^{jx}$$

Using exponent properties:

$$z' = re^{j(\theta + x)}$$

This means the new complex number **maintains the same magnitude** r but its angle increases by x, which is a **rotation by x radians**.

2. Example with a Simple Complex Number

Let's take $z = 1$ (which is at $1 + j0$, lying on the real axis). Multiplying by e^{jx}:

$$1 \cdot e^{jx} = \cos x + j\sin x$$

This result shows that 1 is rotated by x radians, moving it counterclockwise along the unit circle.

3. Why This Works

Since multiplication in polar form just adds angles, multiplying by e^{jx} shifts any complex number by x radians **without changing its magnitude**.

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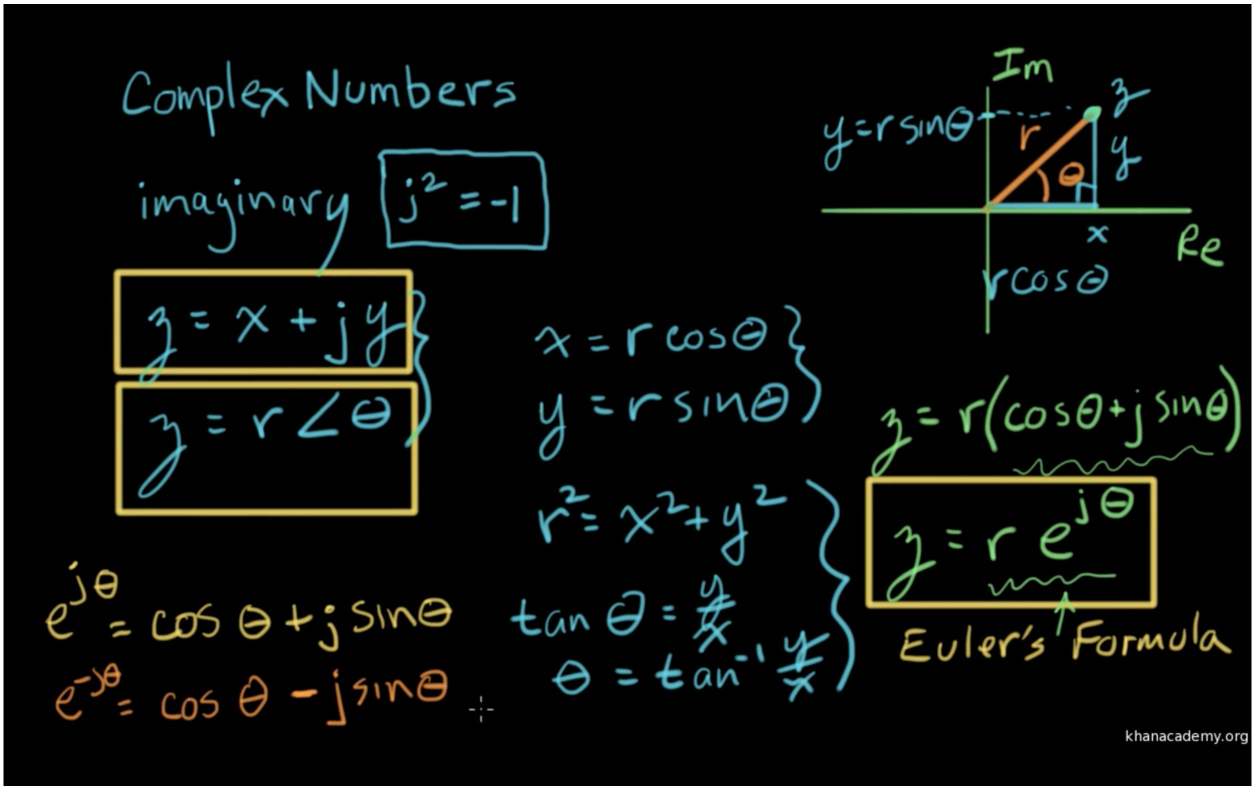
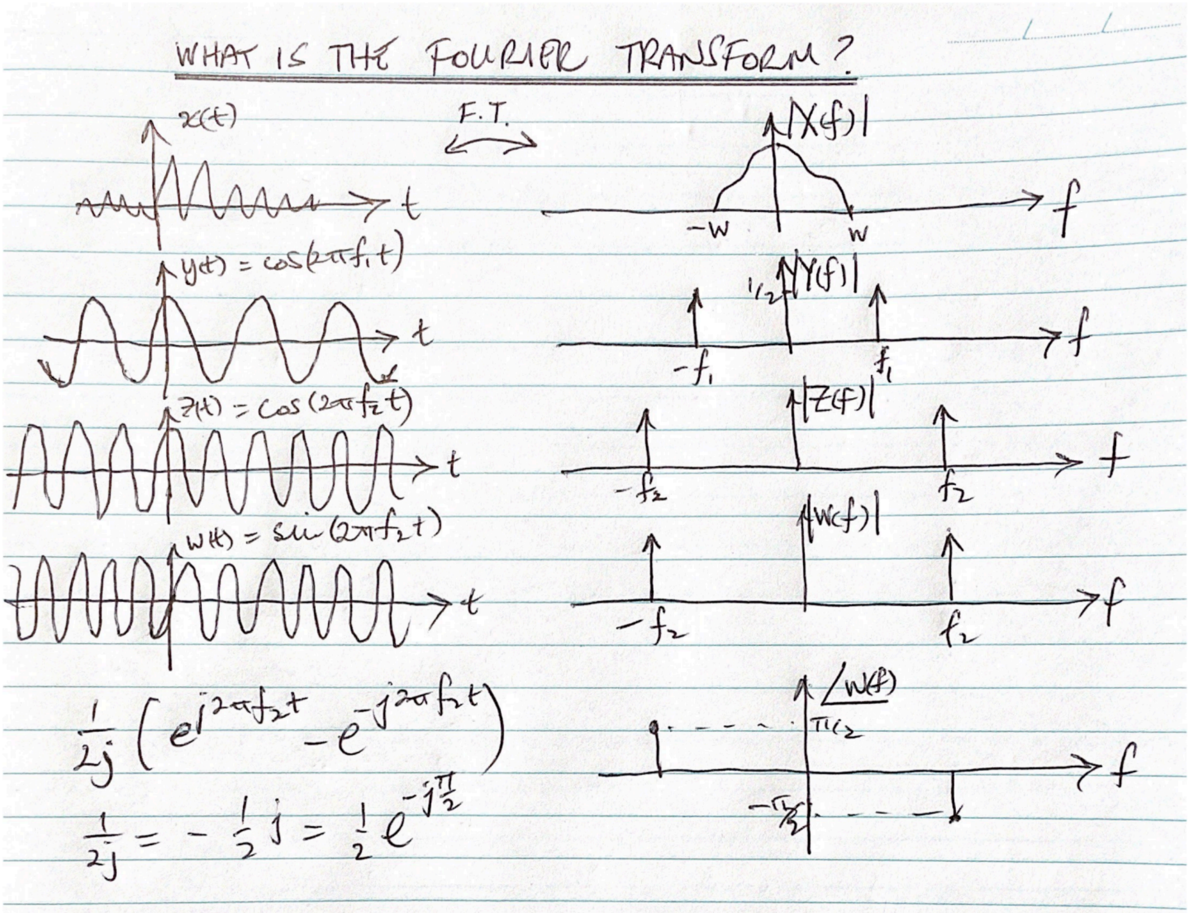

Complex Numbers
imaginary
$\boxed{j^2 = -1}$
$z = x + jy$
$z = r\angle\theta$
$x = r\cos\theta$
$y = r\sin\theta$
$r^2 = x^2 + y^2$
$\tan\theta = \frac{y}{x}$
$\theta = \tan^{-1}\frac{y}{x}$
$e^{j\theta} = \cos\theta + j\sin\theta$
$e^{-j\theta} = \cos\theta - j\sin\theta$
Im
Re
$y = r\sin\theta$
$r\cos\theta$
$z = r(\cos\theta + j\sin\theta)$
$z = re^{j\theta}$
Euler's Formula
khanacademy.org



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To understand -(1/2)j in polar coordinates, let's break it down:

Understanding the Complex Number

- "j" represents the imaginary unit, which is √(-1).
- Therefore, -(1/2)j is a purely imaginary number.
- In the complex plane, this number lies on the negative imaginary axis.

Converting to Polar Coordinates

Polar coordinates represent a point with a radius (r) and an angle (θ).

- **Radius (r):**
  - The radius is the distance from the origin to the point.
  - In this case, the distance is 1/2.
  - Therefore r = 0.5
- **Angle (θ):**

- The angle is measured counterclockwise from the positive real axis.
- Since -(1/2)j lies on the negative imaginary axis, the angle is 270 degrees, or 3π/2 radians.
- However, it is also common to express this angle as -π/2 radians.

Therefore:

- -(1/2)j in polar coordinates is (1/2, 3π/2) or (1/2, -π/2).

In summary:

- The magnitude is 0.5
- The angle is -π/2 radians or 3π/2 radians.

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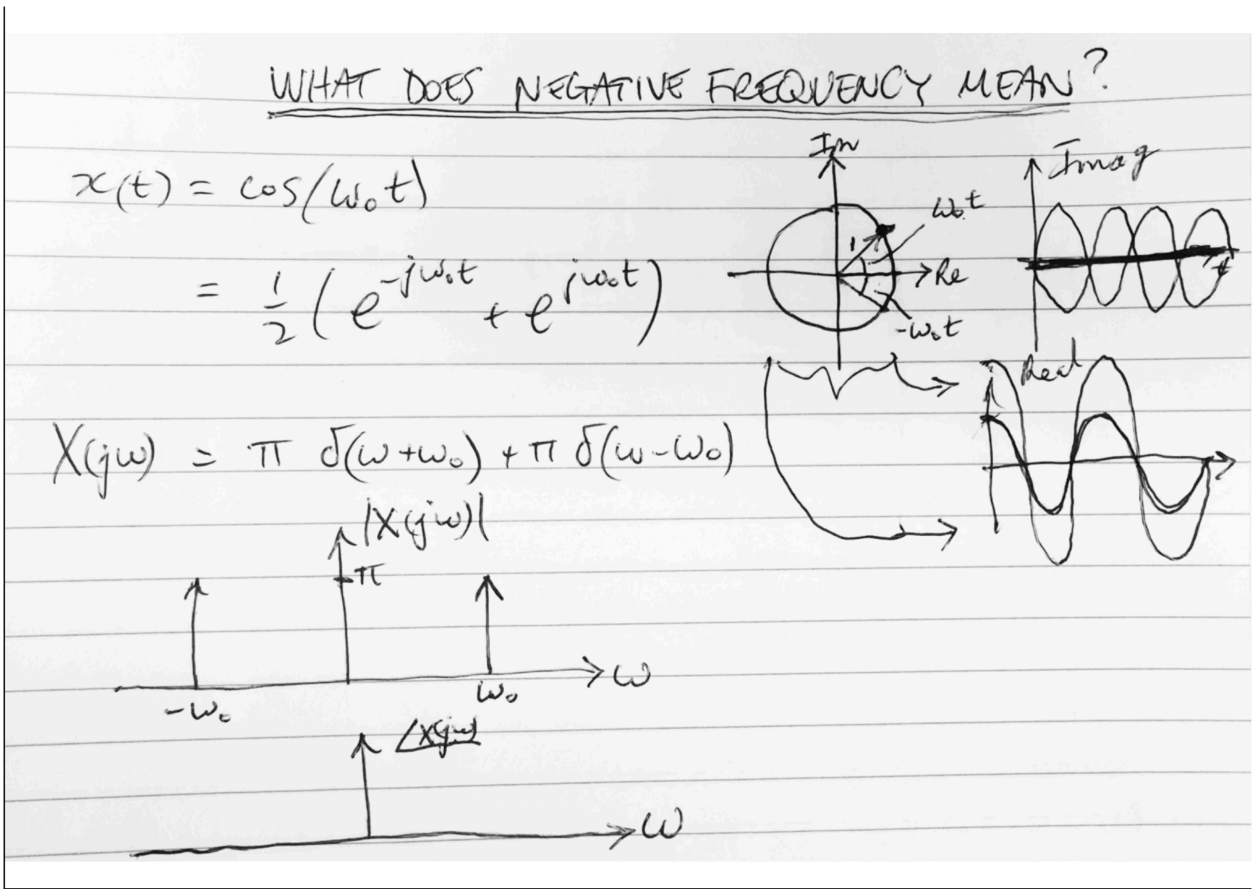

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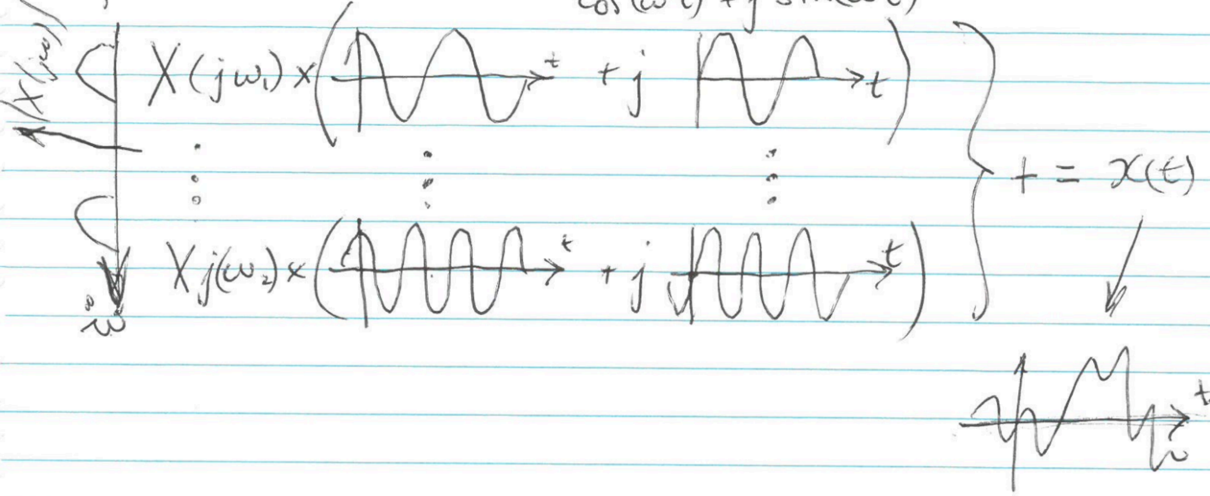
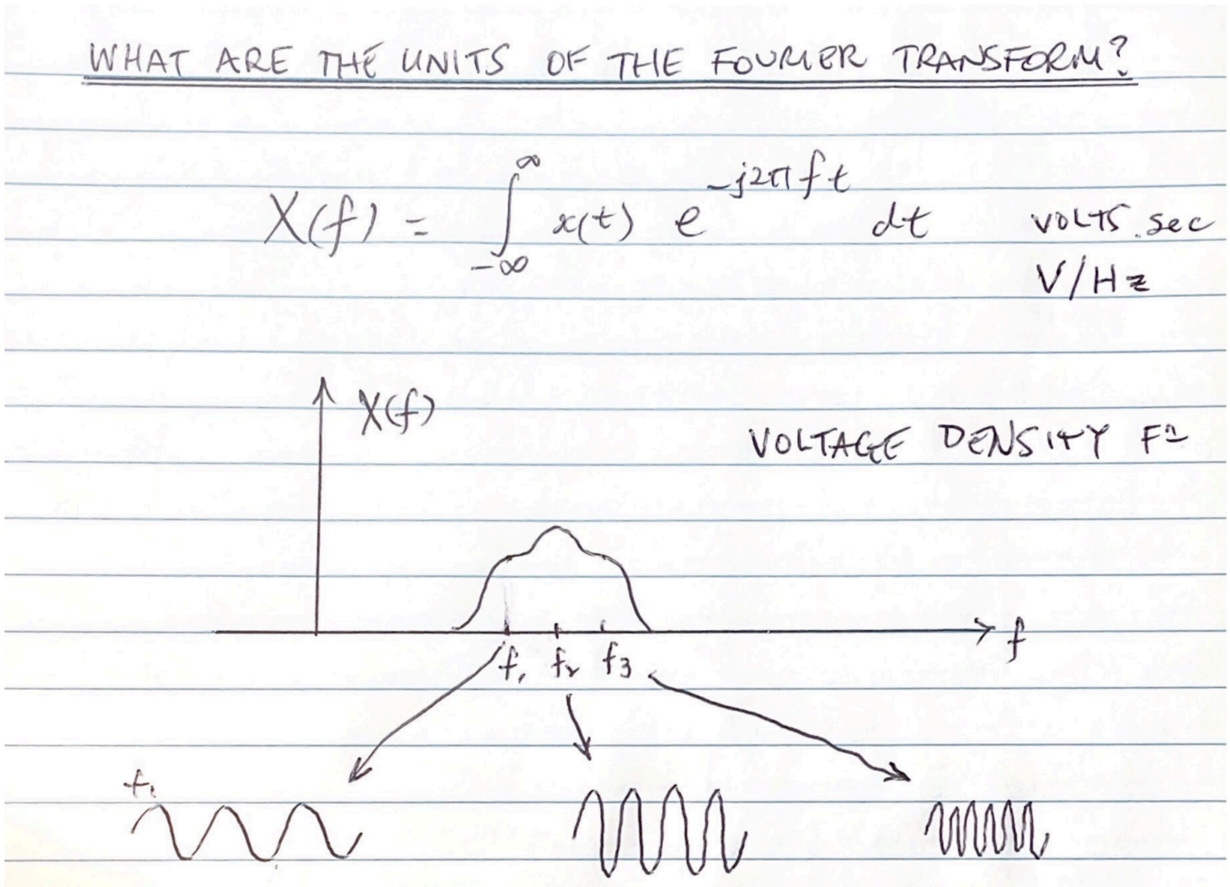
FOURIER TRANSFORM EXPLAINED

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, \underbrace{e^{j\omega t}}_{\cos(\omega t) + j\sin(\omega t)}\, d\omega}$$



https://www.youtube.com/watch?v=7Tk6BAJ3mm8



In probability theory, a density function, more formally known as a probability density function (PDF), is a crucial tool for describing the likelihood of a continuous random variable taking on a given value.[1] Here's a breakdown:

Key Concepts:

- **Continuous Random Variable:**
    - This is a variable that can take on any value within a given range (e.g., height, temperature, time).
- **Probability:**
    - In simple terms, it's the chance of something happening.
- **Density:**
    - In this context, it refers to the "concentration" of probability at a particular value.

What a PDF Does:

- A PDF, denoted as f(x), provides a way to describe the relative likelihood that a continuous random variable X will take on a specific value x.

- It's important to understand that f(x) itself is not the probability of X = x. For a continuous variable, the probability of it taking on any single, exact value is actually zero.
- Instead, the PDF is used to find the probability that X falls within a certain range of values. This is done by calculating the area under the PDF curve over that range.
- In essence, the higher the PDF value at a particular point, the more likely it is that the random variable will take on a value close to that point.

Key Properties of a PDF:

- **Non-negativity:**
    - $f(x) \geq 0$ for all x. This means the PDF can never be negative.
- **Total area under the curve equals 1:**
    - The integral of f(x) over the entire range of possible values for X must equal 1. This reflects the fact that the total probability of all possible outcomes must be 1.

In simpler terms:

Imagine you're measuring the rainfall in a city. A PDF could describe the likelihood of getting different amounts of rainfall. Areas where the PDF is high indicate amounts of rainfall that are more common, while areas where it's low indicate less common amounts.

I hope that helps.

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1 D DFT Matrix Formulation

DFT Definition

For a discrete signal $x[n]$ of length N, the DFT is given by:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1$$

Matrix Form

This can be rewritten in matrix notation as:

$$X = W_N x$$

where:

- X is the **DFT vector** $[X[0], X[1], \ldots, X[N-1]]^T$
- x is the **signal vector** $[x[0], x[1], \ldots, x[N-1]]^T$
- W_N is the **DFT matrix** of size $N \times N$, whose elements are defined as:

$$W_N[k, n] = e^{-j2\pi kn/N}, \quad 0 \leq k, n < N$$

Explicit DFT Matrix

For $N = 4$, the DFT matrix W_4 is:

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j2\pi/4} & e^{-j4\pi/4} & e^{-j6\pi/4} \\ 1 & e^{-j4\pi/4} & e^{-j8\pi/4} & e^{-j12\pi/4} \\ 1 & e^{-j6\pi/4} & e^{-j12\pi/4} & e^{-j18\pi/4} \end{bmatrix}$$

which simplifies to:

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Thus, computing the DFT of x reduces to a simple **matrix-vector multiplication**:

$$X = W_N x$$

Similarly, the **Inverse DFT (IDFT)** can be expressed as:

$$x = W_N^{-1} X = \frac{1}{N} W_N^H X$$

where W_N^H is the **conjugate transpose** (Hermitian) of W_N.

This matrix formulation is useful for understanding the **Fast Fourier Transform (FFT)** and for implementing the DFT efficiently in numerical computations.

Great question! The idea that frequencies beyond \(N/2 \) correspond to **negative frequencies** comes from the **periodicity of the discrete Fourier transform (DFT)** and how sinusoidal signals behave in the frequency domain. Let's break this down step by step.

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The **DFT frequencies increase up to k=N/2 and then start decreasing** due to the symmetry of the Discrete Fourier Transform (DFT). This is because the DFT inherently represents **both positive and negative frequencies**, and those frequencies "wrap around" at k=N/2.

The DFT inherently assumes periodicity, and the **negative frequencies wrap around** due to aliasing.

The DFT works with **complex exponentials** $e^{-j2\pi kn/N}$**.**

These exponentials have **both positive and negative frequency components**.

The indices beyond N/2 behave **like negative frequencies** because of the periodicity of the complex exponential:

$$e^{-j2\pi(N-k)n/N} = e^{j2\pi kn/N}$$

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To understand what each row of a DFT matrix represents, it's essential to connect it to the concept of frequency analysis. Here's a detailed breakdown:

Core Concept: Frequency Decomposition

- The DFT's fundamental purpose is to decompose a signal into its constituent frequency components.
- Each row of the DFT matrix acts as a "filter" that isolates a specific frequency within the input signal.

Detailed Explanation:

1. **Basis Functions:**
    - Each row of the DFT matrix corresponds to a specific "basis function." These basis functions are complex sinusoids (combinations of sine and cosine waves).
    - The frequency of these sinusoids increases as you move down the rows of the matrix.
2. **Frequency Bins:**
    - Each row effectively represents a "frequency bin." When you multiply the input signal vector by a row of the DFT matrix, you're calculating how much of that specific frequency is present in the signal.
    - In essence, you're "correlating" the input signal with a complex sinusoid of a particular frequency.
3. **Row-by-Row Breakdown:**
    - **Row 0 (DC Component):**
        - The first row of the DFT matrix consists of all 1s.
        - This row calculates the "DC component" of the signal, which is the average value of the signal. It tells you the signal's overall average level.
        - Thus this row represents the 0 frequency bin.
    - **Subsequent Rows (Frequency Components):**
        - The subsequent rows contain complex exponentials that oscillate at increasing frequencies.
        - Each row corresponds to a specific frequency. The higher the row number, the higher the frequency.
        - When you perform the matrix multiplication, each row determines the amplitude and phase of that particular frequency component in the input signal.

            - The value of the complex numbers in each row, change in a sinusoidal fashion, and that rate of change defines the frequency that the row represents.
4. **Correlation:**
    - The multiplication of the input signal with a row of the DFT matrix can be thought of as a correlation.
    - If the input signal contains a strong component at the frequency represented by a particular row, the result of the multiplication will have a large magnitude.
    - This indicates that there's a strong "match" between the input signal and that specific frequency.

In simpler terms:

- Imagine you have a musical chord. The DFT matrix is like a set of "listening devices," each tuned to a different musical note.
- Each row of the matrix is one of those "listening devices."
- When you apply the DFT, each "listening device" tells you how loud its specific note is in the chord.

Therefore, each row of the DFT matrix is a tool to measure the strength of a specific frequency component within a signal.

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