ROMBERG'S INTEGRATION

$$\int_0^1 \frac{dx}{1+x}$$ correct to 3 decimal places using ______.

$$\boxed{\log_e 2}$$

STEP-SIZE:-

$$h = \frac{1-0}{2} = 0.5$$

$$\frac{h}{2} = 0.25$$

$$\frac{h}{4} = 0.125$$

Using Trapezoidal:-

x y

0 1 $f(a)$

0.5	0.666
1	0.5

put x in $f(x)$ i.e., $\left(\frac{1}{1+x}\right)$

$$I(h) = \frac{0.5}{2}\left(1 + 0.5 + 2(0.666)\right)$$

$$= 0.7803$$

when $h = 0.25$

x	y
0	1
0.25	0.8
0.5	0.6666
0.75	0.5714

$I \qquad 0.5$

$$I\left(\frac{h}{2}\right) = \frac{h}{2}\left(1 + 0.5 + 2(\ldots\ldots)\right)$$

$$= \frac{0.25}{2}\left(\ldots\ldots\right)$$

$$= 0.697$$

when $h = 0.125$

$$I\left(\frac{h}{4}\right) = 0.6941$$

using Romberg's

h	$I_1 = 0.7084$		
$\frac{h}{2}$	$I_2 = 0.6970$	$I_1' = I_2 + \frac{1}{3}(I_2 - I_1) = 0.6932$	$I_1'' = I_2' + \frac{1}{3}(I_2' - I_1') = 0.6931$
$\frac{h}{4}$	$I_3 = 0.6941$	$I_2' = I_3 + \frac{1}{3}(I_3 - I_2) = 0.6931$	

$$I(h, h/2) = \frac{1}{3}\left(4I\left(\frac{h}{2}\right) - I(h)\right]$$

$$= 0.6932$$

$I(h_2, h_4)$ = Using formulae :-

0.6931

$$I(h, h/2, h/4) = \frac{1}{3}\left[4I\left(\frac{h}{2}, \frac{h}{4}\right) - I\left(h, \frac{h}{2}\right)\right]$$

$$= 0.6931$$

$$\boxed{\log_e 2 = ?}$$

$$\int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)\Big|_0^1$$

$$= \ln 2 - \ln 1$$

$$\boxed{= 0.6931}$$

$$\int_0^1 \frac{dx}{1+x^2}$$

$h = 0.5$

$\frac{h}{2} = 0.25$

$\frac{h}{4} = 0.125$

when $h = 0.5$

x	y
0	1
0.5	0.8
1	0.5

$$I(h) = \frac{h}{2}\left(1 + 0.5 + 2(0.8)\right)$$

$$= \frac{0.5}{2}\left(1.5 + 2(0.8)\right)$$

$$= 0.775$$

when $h = 0.25$

x	y

0	1
0.25	0.941
0.5	0.8
0.75	0.64
1	0.5

$$I(0.25) = \frac{0.25}{2}\left(1 + 0.5 + 2\left(0.8 + 0.941 + 0.64\right)\right)$$

$$= 1.5655$$

when $h = 0.125$

$$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left(4\, I\left(\frac{h}{2}\right) - I(h)\right)$$

$$= 1.829.$$

EXTRAPOLATION:

$$O(h^4) = \frac{1}{3}\left(4\, I_{\left(\frac{h}{2}\right)} - I_h\right)$$

$$O(h^6) = \frac{1}{15}\left(16\, I_{\left(\frac{h}{2}\right)} - I_h\right)$$

$$\vdots$$

BISECTION METHOD

Q. Find root of $x^3 - x - 1$ using Bisection.

→ $f(0) = -1$ (Put $f(x) = 0$)
$f(1) = -1$
$f(2) = 5$

Solution will lie in interval $[1, 2]$

We have to find the interval (x) where signs will be changed.

→ Find mid point $= \frac{1+2}{2} = \boxed{1.5}$

$f(1.5) = 0.875$

root will be in $[1, 1.5]$

No change in sign, find mid point

$\rightarrow$ 1.25

$f(1.25) = -0.2969$

root in $[1.25, 1.5]$

$\rightarrow$ 1.375

$f(1.375) = 0.2246$

root in $[1.25 - 1.375]$

$\rightarrow$ 1.3125

$f(1.3125) = -0.0515$

root in $[1.25 - 1.3125]$

$\rightarrow$ 1.3437

$f(1.3437) = 0.08$

$\text{root} = [1.3125 - 1.3437]$

$\rightarrow 1.3281$

$f(1.3281) = 0.014$

$\text{root} = [1.3125 - 1.3281]$

$\rightarrow 1.3203$

$f(1.3203) = -0.01$

$\text{root} = [1.3281 - 1.3203]$

$\rightarrow 1.3242$

$f(1.3242) = 0.002$

$\text{root} = [1.3203 - 1.3243]$

REGULA FALSI METHOD

1- Select interval where root lies.
2- Find root point x_2, using:

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

3- Find $f(x_2)$ and decide interval where root lie:

$$x^3 - x - 1 = 0$$

$x_0 = -1$

$x_1 = -1$

$x_2 = 5$

$[-1, 5]$ is the interval.

ITERATIVE METHOD

Q# Find real root of equation
$x^3 - x - 1 = 0$

$f(x) = x^3 - x - 1$

$f(0) = -1$
$f(1) = -1$
$f(2) = 5$

INTERVAL = (

$x = \phi(x)$

(1- $x^3 - x - 1 = 0$
$x = x^3 - 1$
$x = \phi(x)$

$\Rightarrow \phi'(x) = 3x^2$
$\phi'(1) = 3 \nless 1$
$\phi'(2) = 12 \nless 1$

(2- $x^3 - x - 1 = 0$
$x^3 - x = 1$
$x(x^2 - 1) = 1$
$x = \dfrac{1}{x^2 - 1}$

$x = \phi(x)$

$\phi(x) = (x^2 - 1)^{-1}$

③ $x^3 - x - 1 = 0$

$x^3 = 1 + x$

$x = (1+x)^{1/3}$

$x = \theta(x)$

$\theta'(x) = \frac{1}{3}(1+x)^{-2/3}$

$\theta'(1) = 0.2 < 1$

$\theta'(2) = 0.16 < 1$

$\theta'(x) = -(x^2-1)^{-2}(2x)$

$= \frac{-2x}{(x^2-1)^2}$

Put $(1, 2)$

$$\theta(x) = (1+x)^{1/3}$$

Choose random value in interval,

$$x_1 = \theta(1.5) = 1.357$$

$$x_2 = \theta(x_1) = 1.3308$$

$$x_3 = \theta(x_2) = \ldots\ldots$$

(until x repeats itself as output).

NEWTON RAPHSON METHOD

1- Select interval where root lies

2- Find $x_0 = \frac{a+b}{2}$ and then find $f(x_0)$, $f'(x_0)$

3- $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f(x)\ x^3 - x - 1$, $f'(x) = 3x^2 - 1$

SECANT METHOD

$$f(x) = x e^{x} - \cos x$$

$x_0 = 0 \quad f(0) = -1$

$x_1 = 1 \quad f(1) = 2.1779$ (Use calc in radian).

$$x_2 = x_1 - \left(\frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} \right) = 0.31467$$

Now check whether it is root or not

$f(x_2) = -0.5198$ (repeat till it is nearly equal $= 0.000\ldots$)

$$x_3 = x_2 - \left(\frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)} \right)$$

$x_3 = 0.4473$

$f(x_3) = -0.20200$

$\vdots$

Q $\Rightarrow$ $x^3 = 2x+5$ using iterative

$x - e^x = 0$ using secant

NUMERICAL

COMPUTING

GAUSS ELIMINATION METHOD

$3x+y-z=3$
$2x-8y+z=-5$
$x-2y+9z=$

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & -8 & 1 \\ 1 & -2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 8 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & +3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{array}\right]$$

PARTIAL PIVOTING:-

Look for max abs. value in 1st col.

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 0 & -26/3 & 5/3 & -7 \\ 0 & \frac{-7}{3} & \frac{28}{3} & 7 \end{array}\right] \quad \begin{array}{l} R_2 - \frac{2}{3} R_1 \\ R_3 - \frac{1}{3} R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 0 & -26/3 & 5/3 & -7 \\ 0 & 0 & \frac{231}{26} & \frac{231}{26} \end{array}\right] \quad (\text{PIVOT})$$

$$\frac{231}{26} z = \frac{231}{26}$$

$$\Rightarrow \boxed{z = 1}$$

$$\Rightarrow \frac{-26}{3} y + \frac{5}{3} z = -7$$

$$\boxed{y = \frac{-26}{3}}$$

$$\Rightarrow \boxed{x = 1}$$

Made with Goodnotes

1- Take the largest no. in first col to the top of matrix.
2- Make all the numbers below pivot zero.

$$\left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 4 & 3 & 4 & 8 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

$R_2 \longrightarrow R_2 - 4R_3$, $R_3 \rightarrow R_1 - 9R_3$

$$\left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & -5 & -20 \end{array}\right]$$

$|-1| \not> |-6|$

$$\left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 0 & -6 & -5 & -20 \\ 0 & -1 & 0 & -4 \end{array}\right]$$

SWAP

$$\left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 0 & -6 & -5 & -20 \\ 0 & -1 & 0 & -4 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 0 & -6 & -5 & -20 \\ 0 & 0 & -5 & +4 \end{array}\right] \quad R_3 \rightarrow -6R_3 + R_2$$

$$\Rightarrow \boxed{z = -\frac{4}{5}}$$

$$\Rightarrow -6y = -20 + 5z = -20 + 5\left(\frac{-4}{5}\right)$$

$$\boxed{= \frac{-24}{-6} = +4}$$

$$\Rightarrow 9x = 7 - 4z - 3y = 7 - 4\left(-\frac{4}{5}\right) - 3(4) \Big/ 9$$

$$\boxed{= -0.2}$$

COMPLETE PIVOTING

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 9 & 8 \\ 2 & -8 & 1 & -5 \\ 3 & 1 & -1 & 3 \end{array}\right] \qquad R_3 \Leftrightarrow R_1$$

$$= \left[\begin{array}{ccc|c} 9 & -2 & 1 & 8 \\ 1 & -8 & 2 & -5 \\ -1 & 1 & 3 & 3 \end{array}\right] \qquad C_1 \Leftrightarrow C_3$$

$$= \left[\begin{array}{ccc|c} 9 & -2 & 1 & 8 \\ 0 & -7 & 5 & -2 \\ 0 & \frac{7}{7} & \frac{28}{7} & \frac{35}{7} \end{array}\right] \qquad \begin{array}{l} R_2 \rightarrow R_2 + R_3 \\ R_3 \rightarrow 9R_3 + R_1 \\ R_3 \rightarrow \frac{R_3}{7} \end{array}$$

$$= \left[\begin{array}{ccc|c} 9 & -2 & 1 & 8 \\ 0 & -7 & 5 & -2 \\ 0 & 1 & 4 & 5 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 9 & -2 & 1 & 8 \\ 0 & -7 & 5 & -2 \\ 0 & 0 & 33 & 33 \end{array}\right] \quad R_3 \to +7R_3 + R_2$$

$\Rightarrow \boxed{x = 1}$ $\qquad$ (C_1 was swapped with C_3)

$$\Rightarrow \quad y = \frac{-2-5}{(-7)} = \frac{-7}{7} \boxed{= +1}$$

$$\Rightarrow \quad z = \left(8 - 1 + 2\,(+1)\right) / 9$$

$$\boxed{= 1}$$

1- Take the biggest number from the whole matrix and mark it as pivot.

2- Make all entries below pivot 0.

$$\left[\begin{array}{ccc|c} 28 & 4 & -1 & 32 \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 28 & 4 & -1 & 32 \\ 0 & 80 & 281 & 640 \\ 0 & 11 & -16 & -13 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_2 \to 28R_2 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 28 & -1 & 4 & 32 \\ 0 & 281 & 80 & 640 \\ 0 & -16 & 11 & -13 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 28 & -1 & 4 & 32 \\ 0 & 281 & 80 & 640 \\ 0 & 0 & 273.18 & 411.6875 \end{array}\right] \quad R_3 \to \frac{R_3 \times 281 + R_2}{16}$$

$$= \left[\begin{array}{ccc|c} 28 & -1 & 4 & 32 \\ 0 & 281 & 80 & 640 \\ 0 & 0 & 273.18 & 411.6875 \end{array}\right]$$

$$\Rightarrow \quad z = \frac{411.6875}{273.18} \boxed{= 1.507019}$$

$$\Rightarrow \quad 281y + 80z = 640$$

$$y = \frac{640 - 80z}{281} = \boxed{1.8485}$$

$$\Rightarrow \quad 28x - 1y + 4z = 32$$

$$x = \frac{32 + y - 4z}{28} = \boxed{0.9935}$$

DO-LITTLE:

$$\begin{bmatrix} 2 & 5 & -3 \\ 4 & 7 & 1 \\ 3 & -5 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\Rightarrow u_{11} = 2$ $\Rightarrow u_{12} = 5$ $\Rightarrow u_{13} = -3$

$\Rightarrow l_{21} u_{11} = 4$

$l_{21} = 2$

$\Rightarrow l_{21} u_{12} + u_{22} = 7$

$u_{22} = -3$

$\Rightarrow l_{21} u_{13} + u_{23} = 1$

$u_{23} = 7$

$\Rightarrow l_{31} u_{11} = 3$

$l_{31} = 1.5$

$\Rightarrow l_{31} u_{12} + l_{32} u_{22} = -5$

$l_{32} = 4.1667$

$\Rightarrow l_{31} u_{13} + l_{32} u_{23} + u_{33} = -2$

$u_{33} = -26.6667$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3/2 & 25/6 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$$

$\Rightarrow y_1 = 7$ $\qquad \Rightarrow y_2 = -17$

$\Rightarrow y_3 = \dfrac{187}{3}$

$UX = Y$

$$\begin{bmatrix} 2 & 5 & -3 \\ 0 & -3 & 7 \\ 0 & 0 & -80/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -17 \\ 187/3 \end{bmatrix}$$

$\Rightarrow \boxed{z = -187/80}$ $\qquad \Rightarrow \boxed{y = 17/80}$

$\Rightarrow \boxed{x = -43/80}$

CROUT'S:

$$\begin{bmatrix} 2 & 5 & -3 \\ 4 & 7 & 1 \\ 3 & -5 & -2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow l_{11} = 2$

$\Rightarrow l_{11}u_{12} = 5$

$u_{12} = 5/2$

$\Rightarrow l_{11}u_{13} = -3$

$u_{13} = -3/2$

$\Rightarrow l_{21} = 4$

$\Rightarrow l_{21}u_{12} + l_{22} = 7$

$l_{22} = -3$

$\Rightarrow l_{21}u_{13} + l_{22}u_{23} = 1$

$u_{23} = -7/3$

$\Rightarrow l_{31} = 3$

$\Rightarrow l_{31}u_{12} + l_{32} = -5$

$l_{32} = -25/2$

$\Rightarrow l_{31}u_{13} + l_{32}u_{23} +$

$l_{33} = -2$

$l_{33} = -80/3$

$LY = B$

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & -3 & 0 \\ 3 & -25/2 & -80/3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$$

$\Rightarrow y_1 = 7/2$

$\Rightarrow y_2 = 17/3$

$\Rightarrow y_3 = \dfrac{-187}{80}$

$UX = Y$

$$\begin{bmatrix} 1 & 5/2 & -3/2 \\ 0 & 1 & -7/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7/2 \\ 17/3 \\ -187/80 \end{bmatrix}$$

$\Rightarrow \boxed{z = -187/80}$

$\Rightarrow \boxed{y = 17/80}$

$\Rightarrow \boxed{x = -43/80}$

CHOLESKY:

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

Applicable only iff a matrix is symmetric and positive definite.

POSITIVE DEFINITE: if det of any 1x1, 2x2 or 3x3 highlighted matrix >0 its positive definite

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$$

$\det|4| > 0$

$$\det\begin{vmatrix} 4 & 2 \\ 2 & 17 \end{vmatrix} > 0$$

$$\det\begin{vmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{vmatrix} > 0$$

Now we can use Cholesky's method.

SOLUTION:

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & u_{12} & u_{13} \\ 0 & l_{22} & u_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$\Rightarrow l_{11} = 2$

$\Rightarrow l_{11}u_{12} = 2$

$u_{12} = 1$

$\Rightarrow l_{11}u_{13} = 14$

$u_{13} = 7$

$\Rightarrow l_{21} l_{11} = 2$

$l_{21} = 1$

$\Rightarrow l_{21}u_{12} + l_{22}^2 = 17$

$l_{22} = 4$

$\Rightarrow l_{21}u_{13} + l_{22}u_{23} = -5$

$u_{23} = -5$

$\Rightarrow l_{11} l_{31} = 14$

$l_{31} = 7$

$\Rightarrow l_{31}u_{12} + l_{32} l_{22} = -5$

$l_{32} = -3$

$\Rightarrow l_{31}u_{13} + l_{32}u_{23} + l_{33}^2 = 83$

$l_{33} = 5$

$LY = B$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

$\Rightarrow y_1 = 7$

$\Rightarrow y_2 = -27$

$\Rightarrow y_3 = 5$

$UX = Y$

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}$$

$\boxed{\Rightarrow \quad x = 3}$ $\boxed{\Rightarrow \quad y = -6}$ $\boxed{\Rightarrow z = 1}$

Made with Goodnotes

GAUSS JACOBIE METHOD

$a_1x + b_1y + c_1z = d_1 \Rightarrow x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$

$a_2x + b_2y + c_2z = d_2 \Rightarrow y = \frac{1}{b_2}(d_2 - a_2x - c_2z)$

$a_3x + b_3y + c_3z = d_3 \Rightarrow z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$

CONDITION:

$|a_1| \geq |b_1| + |c_1|$

$|b_2| \geq |a_2| + |c_2|$

$|c_3| \geq |a_3| + |b_3|$

1^{st} iteration:-

(put all other variables = 0)

$$x_1' = \frac{d_1}{a_1}$$

$$y_1' = \frac{d_2}{b_2}$$

$$z_1' = \frac{d_3}{c_3}$$

2^{nd} iteration:

(put x_1', y_1', z_1' in equations).

$$8x + 2y - 3z = 5$$
$$x + 5y + 2z = 3$$
$$2x + 4y + 7z = 4$$

$$x = \frac{1}{8}(5 - 2y + 3z)$$

$$y = \frac{+1}{5}(3 - x - 2z)$$

$$z = \frac{1}{7}(4 - 2x - 4y)$$

$$x' = \frac{5}{8} \qquad y' = \frac{3}{5} \qquad z' = \frac{4}{7}$$

2nd iteration:

$$x = \frac{1}{8}\left(5 - 2\left(\frac{3}{5}\right) + 3\left(\frac{4}{7}\right)\right) = 0.6892$$

$y = 0.2464$ $z = 0.05$

Repeat till x, y, z repeat their values.

GAUSS SEIDAL METHOD

Same as Jacobie.

In first iteration put only $y, z = 0$ to find x.

For y, put the found value of x and $z = 0$.

For z, put x, y

Now repeat and update x, y, z.

Order of Convergence.

BISECTION METHOD:

In general,

$$x_{n+1} = \frac{x_{n-1} + x_n}{2} \quad \text{——— } (A)$$

Let $f(x)=0$ be an equation and α be the exact root of $f(x)$

$\Rightarrow f(\alpha)=0$

x_n may differ from α by error e_n

$$x_n = e_n + \alpha$$

$$x_{n-1} = e_{n-1} + \alpha$$

$$x_{n+1} = e_{n+1} + \alpha$$

Substituting in (A)

$$\alpha + e_{n+1} = \frac{\alpha + e_n + \alpha + e_{n-1}}{2}$$

$$e_{n+1} = 1 + \frac{e_{n-1} + e_n}{2}$$

$$e_{n+1} = \frac{e_n}{2}\left[1 + \frac{e_{n-1}}{e_n}\right]$$

$$e_{n+1} \approx \frac{e_n}{2} \qquad \left(\text{neglect } 1+\frac{e_{n-1}}{e_n}\right)$$

$$e_{n+1} \approx A e_n \qquad (A = 1/2)$$

$$e_{n+1} \approx A e_n^k \qquad (\text{where } k = 1)$$

Rate of convergence is linear.

REGULA-FALSI :

In general,

$$x_{n+1} = \frac{x_{n-1} f(x_n) - f(x_{n-1}) x_n}{f(x_n) - f(x_{n-1})}$$

$$x_{n+2} = \frac{x_n f(x_{n+1}) - f(x_n) x_{n+1}}{f(x_{n+1}) - f(x_n)} \quad \text{— } A$$

Let $f(x)=0$ be an equation and α be the exact root of $f(x)$

$\Rightarrow f(\alpha)=0$

x_n may differ from α by error e_n

$$x_n = e_n + \alpha$$

$$x_{n-1} = e_{n-1} + \alpha$$

$$x_{n+1} = e_{n+1} + \alpha$$

Substituting in A

$$e_{n+2} + \alpha = \frac{(\alpha+e_n)f(\alpha+e_{n+1}) - (\alpha+e_{n+1})f(\alpha+e_n)}{f(\alpha+e_{n+1}) - f(\alpha+e_n)}$$

$$e_{n+2} + \alpha = \frac{f(\alpha+e_{n+1})\alpha + f(\alpha+e_{n+1})e_n - f(\alpha+e_n)(\alpha) - f(\alpha+e_n)(e_{n+1})}{f(\alpha+e_{n+1}) - f(\alpha+e_n)}$$

$$e_{n+2} + \alpha = \frac{\alpha\left(f(\alpha+e_{n+1}) - f(\alpha+e_n)\right) + f(\alpha+e_{n+1})e_n - f(\alpha+e_n)(e_{n+1})}{f(\alpha+e_{n+1}) - f(\alpha+e_n)}$$

$$e_{n+2}+\cancel{\alpha} = \cancel{\alpha} + \frac{f(\alpha+e_{n+1})e_n - f(\alpha+e_n)(e_{n+1})}{f(\alpha+e_{n+1}) - f(\alpha+e_n)}$$

(TAYLOR SERIES)

$$e_{n+2} = \left(e_n\left[f(\alpha)+(e_{n+1})f'(\alpha)+\frac{(e_{n+1})^2}{2!}f''(\alpha)\ldots\ldots\right] - e_{n+1}\left[f(\alpha)+e_n f'(\alpha)+\frac{e_n^2}{2!}f''(\alpha)\ldots\ldots\right] \right) \div$$

$$\left(\left[f(\alpha)+(e_{n+1})f'(\alpha)+\frac{(e_{n+1})^2}{2!}f''(\alpha)\ldots\ldots\right] - \left[f(\alpha)+e_n f'(\alpha)+\frac{e_n^2}{2!}f''(\alpha)\ldots\ldots\right] \right)$$

$f(\alpha=0)$

$$e_{n+2} = \left(\cancel{(e_n)(e_{n+1})f'(\alpha)} + \frac{e_n(e_{n+1})^2}{2!}f''(\alpha)\ldots\ldots - \cancel{(e_{n+1})(e_n)f'(\alpha)} + (e_{n+1})\frac{(e_n^2)}{2!}f''(\alpha)\ldots\ldots \right) \div$$

$$\left(e_{n+1}f'(\alpha) - e_n f'(\alpha) + \frac{(e_{n+1})^2}{2!}f''(\alpha) - \frac{(e_n)^2}{2!}f''(\alpha) \right)$$

Neglecting higher powers,

$$e_{n+2} = \frac{\frac{e_n e_{n+1}}{2} f''(\alpha)(e_{n+1} - e_n)}{f'(\alpha)(e_{n+1} - e_n) + f''(\alpha)(e_{n+1}^2 - e_n^2)}$$

$$e_{n+2} = \frac{\frac{(e_n e_{n+1})}{2} f''(\alpha)\cancel{(e_{n+1} - e_n)}}{[f'(\alpha) + f''(\alpha)(e_{n+1} + e_n)]\cancel{(e_{n+1} - e_n)}}$$

$$e_{n+2} = \frac{e_n e_{n+1}}{2} \frac{f''(\alpha)}{f'(\alpha)} \left[1 + \frac{f''(\alpha)(e_{n+1} + e_n)}{2f'(\alpha)}\right]^{-1}$$

Ignore $\left[1 + \frac{f''(\alpha)(e_{n+1} + e_n)}{2f'(\alpha)}\right]^{-1}$

$$e_{n+2} \simeq \frac{e_n e_{n+1}}{2} \frac{f''(\alpha)}{f'(\alpha)}$$

$$e_{n+2} \simeq M e_n e_{n+1} \quad \text{———} \quad B$$

We know,

$e_{n+1} = A e_n^k$ $\qquad$ $e_{n+2} = A e_{n+1}^k$

$$e_n = \left(\frac{e_{n+1}}{A}\right)^{1/k}$$

From B

$$A e_{n+1}^{k} = M e_n e_{n+1}$$

$$A e_{n+1}^{k} = M\left(\frac{e_{n+1}}{A}\right)^{1/k} \cdot e_{n+1}$$

$$A e_{n+1}^{k} = M\left(e_{n+1}\right)^{1/k} \cdot e_{n+1} \cdot A^{-1/k}$$

$$A e_{n+1}^{\boxed{k}} = M \cdot A^{-1/k} \cdot e_{n+1}^{\boxed{1/k+1}}$$

$$\frac{1}{k} + 1 = k$$

$$k^2 = k + 1$$

$$k^2 - k - 1 = 0$$

$$k = \frac{1 \pm \sqrt{5}}{2}$$

$$k = \frac{1 + \sqrt{5}}{2} \qquad \text{(Taking +)}$$

$$\boxed{k = 1.618}$$

SECANT:-

In general,

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{— } A$$

Let $f(x)=0$ be an equation and α be the exact root of $f(x)$

$\Rightarrow f(\alpha)=0$

x_n may differ from α by error e_n

$x_n = e_n + \alpha$

$x_{n-1} = e_{n-1} + \alpha$

$x_{n+1} = e_{n+1} + \alpha$

Substituting in A

$$\alpha + e_{n+1} = \alpha + e_n - \frac{f(\alpha+e_n)(\cancel{\alpha}+e_n - \cancel{\alpha} + e_{n-1})}{f(\alpha+e_n) - f(\alpha+e_{n-1})}$$

$$\alpha + e_{n+1} = \alpha + e_n - \frac{f(\alpha+e_n)(e_n - e_{n-1})}{f(\alpha+e_n) - f(\alpha+e_{n-1})} \quad \text{— } B$$

Made with Goodnotes

TAYLOR-SERIES EXPANSION

$$f(\alpha+e_n)=f(\alpha)+e_n f'(\alpha)+\frac{e_n^2}{2!}f''(\alpha)\ldots\ldots$$

$$f(\alpha+e_{n-1})=f(\alpha)+e_{n-1} f'(\alpha)+\frac{e_{n-1}^2}{2!}f''(\alpha)\ldots\ldots$$

$$f(\alpha+e_n)-f(\alpha+e_{n-1})$$

$$=(e_n-e_{n-1})f'(\alpha)+\left(\frac{e_n^2-e_{n-1}^2}{2!}\right)f''(\alpha)$$

$$=(e_n-e_{n-1})\left[f'(\alpha)+\frac{f''(\alpha)}{2}(e_n+e_{n-1})\right]$$

Substituting in B

$$\cancel{\alpha}+e_{n+1}=\cancel{\alpha}+e_n-\frac{\cancel{(e_n-e_{n-1})}\left[f(\alpha)+e_nf'(\alpha)+e_n^2/2\,f''(\alpha)\ldots\right]}{\cancel{(e_n-e_{n-1})}\left[f'(\alpha)+\frac{f''(\alpha)}{2}(e_n+e_{n-1})\right]}$$

$$f(\alpha)=0$$

$$e_{n+1}=e_n-\frac{\left[e_nf'(\alpha)+e_n^2/2\,f''(\alpha)\ldots\ldots\right]}{f'(\alpha)\left[1+\left(\frac{e_n+e_{n-1}}{2}\right)\frac{f''(\alpha)}{f'(\alpha)}\right]}$$

Made with Goodnotes

$$e_{n+1} = e_n - \frac{\cancel{f'(\alpha)}\left[e_n + e_n^2/2!\,\frac{f''(\alpha)}{f'(\alpha)}\right]}{\cancel{f'(\alpha)}\left[1 + \left(\frac{e_n + e_{n-1}}{2}\right)\frac{f''(\alpha)}{f'(\alpha)}\right]}$$

$$= e_n - \left[e_n + \frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}\right]\left[1 + \frac{(e_n + e_{n-1})f''(\alpha)}{2f'(\alpha)}\right]^{-1}$$

Binomial theorem full entry martay hoye

$$= e_n - \left[e_n + \frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}\right]\left[1 - \frac{(e_n + e_{n-1})f''(\alpha)}{2f'(\alpha)}\right]$$

$$= e_n - \left[e_n - \frac{e_n(e_n + e_{n-1})}{2}\frac{f''(\alpha)}{f'(\alpha)} + \frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} \ldots\ldots\right]$$

Ignored higher powers above ^

$$= e_n - \left[e_n - \cancel{\frac{f''(\alpha)}{f'(\alpha)}\left(\frac{e_n^2}{2}\right)} - \frac{f''(\alpha)}{f'(\alpha)}\frac{e_n e_{n-1}}{2} + \cancel{\frac{e_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}}\right]$$

$$= \cancel{e_n} - \cancel{e_n} + \frac{f''(\alpha)}{f'(\alpha)}\frac{(e_n e_{n-1})}{2}$$

$$e_{n+1} = A(e_n e_{n-1}) \qquad A = \frac{f''(\alpha)}{2f'(\alpha)}$$

Generally,

$$e_{n+1} = A e_n^k$$

$$e_n = A e_{n-1}^k$$

$$\Rightarrow e_{n-1} = e_n^{1/k} A^{-1/k}$$

$$A e_n^k = A e_n e_{n-1}$$

$$A e_n^k = A e_n e_n^{1/k} A^{-1/k}$$

$$A e_n^{\boxed{k}} = A e_n^{\boxed{1/k+1}} A^{-1/k}$$

$$\Rightarrow k = \frac{1}{k} + 1$$

$$k^2 = 1 + k$$

$$k^2 - k - 1 = 0$$

$$k = \frac{1+\sqrt{5}}{2}$$

$$\boxed{k = 1.618}$$

NEWTON-RAPHSON'S:

Let $f(x)=0$ be an equation and α be the exact root of $f(x)$

$\Rightarrow f(\alpha)=0$

x_n may differ from α by error e_n

$$x_n = e_n + \alpha$$

$$x_{n-1} = e_{n-1} + \alpha$$

$$x_{n+1} = e_{n+1} + \alpha$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$e_{n+1} + \cancel{\alpha} = e_n + \cancel{\alpha} - \frac{\left[f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) \right]}{\left[f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha) \right]}$$

$$e_{n+1} = e_n - \frac{\left[f(\alpha) + e_n f'(\alpha) + e_n^2/2\, f''(\alpha) \right]}{\left[f'(\alpha) + e_n f''(\alpha) + e_n^2/2\, f'''(\alpha) \right]}$$

Since e_n is small, e_n^2, e_n^3 will be very small, neglect them.

$$e_{n+1} = e_n - \frac{e_n f'(\alpha)}{f'(\alpha) + e_n f''(\alpha)}$$

$$e_{n+1} = \frac{\cancel{e_n f'(\alpha)} + e_n^2 f''(\alpha) - \cancel{e_n f'(\alpha)}}{f'(\alpha) + e_n f''(\alpha)}$$

$$e_{n+1} = \frac{e_n^2 f''(\alpha)}{f'(\alpha)\left[1 + \frac{e_n f''(\alpha)}{f'(\alpha)}\right]}$$

$$\frac{e_{n+1}}{e_n^2} = \frac{f''(\alpha)}{f'(\alpha)}\left[1 + \frac{e_n f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

BINOMIAL:-

$$\frac{e_{n+1}}{e_n^2} = \frac{f''(\alpha)}{f'(\alpha)}\left(1 - \frac{e_n f''(\alpha)}{f'\alpha} + \dots\dots\right)$$

$$\frac{e_{n+1}}{e_n^2} = \frac{f''(\alpha)}{f'(\alpha)} - e_n\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 + \dots\dots$$

$$e_{n+1} \approx \frac{f''(\alpha)}{f'(\alpha)} e_n^2$$

$e_{n+1} \simeq A e_n^2$

$A \boxed{k}$ $e_{n+1} = A e_n^{\boxed{2}}$

$\boxed{k = 2}$

Rate of convergence is quadratic

ITERATIVE :-

In general,

$x_{n+1} = f(x_n)$ —— (A)

Let $f(x) = 0$ be an equation and α be the exact root of $f(x)$

$\Rightarrow f(\alpha) = 0$

x_n may differ from α by error e_n

$x_n = e_n + \alpha$ $\qquad \alpha = f(\alpha)$

$x_{n-1} = e_{n-1} + \alpha$

$x_{n+1} = e_{n+1} + \alpha$

Substituting in A

$$\alpha + e_{n+1} = f(\alpha + e_n)$$

$$\cancel{\alpha} + e_{n+1} = \cancel{f(\alpha)} + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) \ldots\ldots$$

$$e_{n+1} = e_n f'(\alpha)$$ (Ignore higher powers)

$$e_{n+1} \simeq A e_n \qquad A = f'(\alpha)$$

$$A e_{n+1}^{\boxed{k}} = A e_n^{\boxed{1}}$$

$$\boxed{\Rightarrow k = 1}$$

Rate of convergence of iterative method is linear.