4

Interpolation

4.1 INTERPOLATION WITH EQUAL INTERVALS

Interpolation is one of the fundamental operation in Mathematics. It has been the foundation of classical numerical analysis. Here finite differences play an important role in numerical techniques, where tabulated values of the unknown functions are available. The primary reason for this is the ease of using tables of values in hand computation. For instance, it is very convenient to look up the values of a function like sine in a table for manual computation. While using computers, it is expensive to store tabulated functions in its memory. For this, it is more convenient to use an algorithm to calculate the value of functions like sine for any arbitrary values of the argument. For instance, consider a table of values (x_i, y_i), where $i = 0, 1, \ldots, n$ of any function $y = f(x)$, the values of x being equally-spaced, i.e. $x_i = x_0 + ih$, where $i = 0, 1, \ldots, n$. The process of estimating the values of y, for any intermediate value of x is called *interpolation*. Also the method of computing the value of y, for a given value of x, lying outside the table of values of x is known as *extrapolation*. It may be noted that if the function $f(x)$ is known, the value of y corresponding to any x can be readily computed to the desired accuracy. But in practice, it may be difficult or sometimes impossible to know the function $y = f(x)$ in its exact form. Instead we can use the best known interpolation technique to get the desired values. Here various finite difference operators and their symbolic relations are very much needed to establish various interpolation formulae.

4.1.1 Finite Difference Operator

Forward Differences Operator

For a given table of values (x_k, y_k), where $k = 0, 1, 2, \ldots, n$ of a function $y = f(x)$,

with equally-spaced abscissas, we define the forward difference operator Δ as follows.

The first forward difference is usually expressed as

$$\Delta y_i = y_{i+1} - y_i,\ i = 0, 1, 2, \ldots, (n-1)$$

i.e., we have

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

where Δ is called the *forward difference operator*. Similarly, the differences of the first forward differences are called *second forward differences* and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots, \Delta^2 y_{n-1}$. Thus, in general

$$\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$$

The following Table 4.1 shows how the forward differences of all orders can be formed.

Table 4.1 Forward Difference Table

x	y	Δ	Δ^2	Δ^3
x_0	y_0			
		$\Delta y_0 = y_1 - y_0$		
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		
x_3	y_3			

Thus, the first term y_0 in the Table 4.1 is called *leading term*, while differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$, are called leading differences.

Backward Differences Operator

For a given table of values (x_k, y_k), where $k = 0, 1, 2, \ldots, n$ of a function $y = f(x)$ with equally-spaced abscissas, the first backward differences are usually expressed in terms of the backward difference operator ∇ as

$$\nabla y_i = y_i - y_{i-1},\ \text{where } i = n, (n-1), \ldots, 1$$

Thus,

$$\nabla y_1 = y_1 - y_0$$
$$\nabla y_2 = y_2 - y_1$$
$$\ldots\ldots\ldots\ldots$$
$$\nabla y_n = y_n - y_{n-1}$$

The differences of the first backward differences are called *second backward differences* and they are denoted by

$$\nabla^2 y_2, \nabla^2 y_3, \ldots, \nabla^2 y_n$$

i.e., we have

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

Thus, in general

$$\nabla^k_{y_i} = \nabla^{k-1}_{y_i} - \nabla^{k-1}_{y_{i-1}}, \text{ where } i = n, (n-1), \ldots, k$$

These backward differences are shown in Table 4.2.

Table 4.2 Backward Difference Table

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Central Differences Operator

The central difference operator δ is defined by the following relations:

$$y_1 - y_0 = \delta y_{1/2},\ y_2 - y_1 = \delta y_{3/2}, \ldots, y_n - y_{n-1} = \delta y_{n-1/2}$$

Higher-order differences are defined as follows:

$$\delta^2 y_i = \delta y_{i+(1/2)} - \delta y_{i-(1/2)}$$

$$\delta^n y_i = \delta^{n-1} y_{i+(1/2)} - \delta^{n-1} y_{i-(1/2)}$$

These central differences can be arranged as indicated in Table 4.3.

The following alternative notation may be adopted to introduce finite difference operators. Let $y = f(x)$ be a functional relation between x and y, which is also denoted by y_x. Suppose, we are given consecutive values of x differing by h, say $x, x+h, x+2h$, etc. The corresponding values of y are y_x, y_{x+h}, y_{x+2h}, etc. Thus, we can form the differences of these values, i.e.,

$$\Delta y_x = y_{x+h} - y_x = f(x+h) - f(x)$$

$$\Delta^2 y_x = \Delta y_{x+h} - \Delta y_x$$

Table 4.3 Central Difference Table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$
x_0	y_0					
		$\delta y_{1/2}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_5	y_5					

Similarly, $\nabla y_x = y_x - y_{x-h} = f(x) - f(x-h)$

and $\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = f(x+h/2) - f(x-h/2)$

Shift Operator (E)

Let $y = f(x)$ be a function of x and $x, x+h, x+2h$ etc. be the consecutive values of x.

We define an operator E such that

$$Ef(x) = f(x+h)$$

Thus the effect of E on y_x is to shift or increase the value of y_x to the next higher value y_{x+h}. If we apply the operator E twice on $f(x)$, we get

$$E^2 f(x) = E[Ef(x)] = E[f(x+h)]$$

$$E^2 f(x) = f(x+2h)$$

$$E^3 f(x) = f(x+3h)$$

In general, $E^n f(x) = f(x+nh)$

Similarly, $E^{-1} f(x) = f(x-h)$, $E^n f(x) = f(x+nh)$ for any n

Average Operator (μ)

The average operator μ is defined by

$$\mu y_x = \frac{1}{2}[y_{x+h/2} + y_{x-h/2}]$$

or

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

or

$$\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$$

Differential Operator (D)

The differential operator is defined as

$$Df(x) = \frac{d}{dx} f(x) = f'(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x) = f''(x), \text{ etc.}$$

Relation between the operators Δ, ∇, δ, E and μ:

1. We know that

$$\Delta y_x = y_{x+h} - y_x$$
$$= Ey_x - y_x$$
$$= (E-1)\, y_x$$

Hence,

$$\Delta = E - 1$$

or

$$E = 1 + \Delta$$

2. We know that

$$\Delta y_x = y_x - y_{x-h}$$
$$= y_x - E^{-1} y_x$$
$$= \left(1 - E^{-1}\right) y_x$$
$$= \left(1 - \frac{1}{E}\right) y_x$$

i.e.,

$$\nabla = 1 - \frac{1}{E}$$

i.e.,

$$E = (1 - \nabla)^{-1}$$

3. The definition of operators δ and E gives

$$\delta y_x = y_{x+h/2} - y_{x-h/2}$$
$$= E^{\frac{1}{2}} y_x - E^{-\frac{1}{2}} y_x$$
$$= (E^{1/2} - E^{-1/2}) y_x$$

Hence,

$$\delta = E^{1/2} - E^{-1/2}$$

4. The definition of operations μ and E gives

$$\mu y_x = \frac{1}{2}[y_{x+h/2} + y_{x-h/2}]$$

$$= \frac{1}{2}(E^{1/2} + E^{-1/2}) y_x$$

$$\therefore \quad \mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$$

5. We know that $Ey_x = y_{x+h} = f(x+h)$.

Using Taylor's series expansion, we have

$$Ey_x = f(x) + hf'(x) + \left(\frac{h^2}{2!}\right) f''(x) + \ldots$$
$$= f(x) + hD\, f(x) + \left(\frac{h^2}{2!}\right) D^2 f(x) + \ldots$$
$$= \left[1 + \frac{hD}{1!} + \left(\frac{h^2}{2!}\right) D^2 + \frac{h^3 D^3}{3!} + \ldots\right] f(x)$$
$$= e^{hD} y_x$$

Thus $\quad hD = \log E.$

Hence, all the operators can be expressed in terms of operator E.

Properties of operators Δ and E: The operators E and Δ satisfy the basic laws of algebra, viz., distributive law, commutative law and the law of indices.

(i) *Distributive law:*

$$\Delta[f(x) + g(x)] = [f(x+h) + g(x+h)] - [f(x) + g(x)]$$
$$= [f(x+h) - f(x)] + [g(x+h) - g(x)]$$
$$= \Delta f(x) + \Delta g(x)$$

(ii) *Commutative law:*

$$\Delta[cf(x)] = cf(x+h) - cf(x) \text{ where } c \text{ is a constant.}$$
$$= c[f(x+h) - f(x)]$$
$$= c\Delta f(x)$$

(iii) *If m and n are positive integers :*

$$\Delta^m \Delta^n f(x) = (\Delta\,\Delta \ldots m \text{ times})\, (\Delta\,\Delta \ldots n \text{ times})\, f(x)$$
$$= (\Delta\,\Delta \ldots \overline{m+n} \text{ times})\, f(x)$$
$$= \Delta^{m+n} f(x)$$

The properties (i) and (ii) are characteristics of a linear operator. Hence Δ is a linear operator.

Fundamental theorem of finite differences: If $f(x)$ is a polynomial of nth degree, then

$$\Delta^r f(x) = \begin{cases} \text{constant}, & r = n \\ 0 & r > n \end{cases}$$

In other words, nth-order difference of a polynomial of nth degree is constant and higher differences are zero.

Proof:

Let $f(x)$ be a polynomial of nth degree, given by

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \ldots + a_{n-1}x + a_n$$

where $a_0, a_1, a_2 \ldots$ are constants.

By the definition of Δ, we have

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x)\\ &= [a_0 (x+h)^n + a_1(x+h)^{n-1} + a_2(x+h)^{n-2} + a_3 (x+h)^{n-3} + \ldots\\ &\quad + a_{n-1} (x+h) + a_n] - [a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n]\\ &= [a_0 (x^n + nC_1\, x^{n-1}h + n\, C_2\, x^{n-2} h^2 + \ldots + nC_n\, x^0 h^n]\\ &\quad + a_1 [x^{n-1} + (n-1)\, C_1\, x^{n-2}\, h + \ldots + h^{n-1}]\\ &\quad + \ldots + a_n] - [a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n]\\ &= a_0nhx^{n-1} + b_2x^{n-2} + b_3\, x^{n-3} + \ldots + b_{n-1}\, x + b_n\end{aligned}$$

where $b_2, b_3 \ldots b_n$ are constants.

$\therefore\quad \Delta f(x) =$ A polynomial of degree $(n-1)$

Similarly, we can show that $\Delta^2 f(x) =$ A polynomial of degree $(n-2)$.

In general,

$$\begin{aligned}\Delta^n y_x = \Delta^n f(x) &= a_0\, n(n-1)\,(n-2) \ldots 2.1.h^n x^{n-n}\\ &= a_0\, x!\, h^n\end{aligned}$$

$\therefore$ The nth-order difference of a polynomial of nth degree is constant.

Now,
$$\begin{aligned}\Delta^{n+1} y_x &= \Delta(\Delta^n y_x)\\ &= \Delta(k)\\ &= 0\end{aligned}$$

and
$$\Delta^{n+2}\, y_x = \Delta^{n+3} y_a = \ldots = 0.$$

Note

1. The converse of the above theorem is also true.
2. If $h = 1$, then $\Delta^n(a_0x^n) = a_0(n!)$

EXAMPLE 4.1 Prove that $\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$.

Solution

$$\begin{aligned}&= \log f(x+h) - \log f(x)\\ &= \log \frac{f(x+h)}{f(x)} &\text{(i)}\end{aligned}$$

R.H.S.
$$\begin{aligned}&= \log\left[\frac{f(x) + \Delta f(x)}{f(x)}\right]\\ &= \log\left[\frac{f(x) + f(x+h) - f(x)}{f(x)}\right]\\ &= \log \frac{f(x+h)}{f(x)} &\text{(ii)}\end{aligned}$$

From equations (i) and (ii),

$$\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

EXAMPLE 4.2 Prove that $hD = \log (1+\Delta) = -\log (1-\nabla) = \sinh^{-1} (\mu\delta)$.

Solution

We know that $\Delta = E - 1$ and $hD = \log E$

From these relation, we have

$$hD = \log E = \log (1+\Delta) \qquad \text{(i)}$$

Also
$$\begin{aligned}\log E &= -\log E^{-1}\\ &= -\log (1-\nabla) &\text{(ii)}\end{aligned}$$

Also
$$\begin{aligned}\mu\delta &= \frac{1}{2}(E^{-1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})\\ &= \frac{1}{2}(E - E^{-1})\\ &= \frac{1}{2}(e^{hD} - e^{-hD}) = \sin h\,(hD)\end{aligned}$$

Therefore,
$$hD = \sinh^{-1}(\mu\delta) \qquad \text{(iii)}$$

From equation (i), (ii) and (iii), we obtain the required result.

EXAMPLE 4.3 Prove that $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$.

Solution We know that $\delta = E^{1/2} - E^{-1/2}$

$$\therefore \quad \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$$

$$= \frac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{1+\frac{(E^{1/2} - E^{-1/2})^2}{4}}$$

$$= \frac{1}{2}(E + E^{-1} - 2) + \left(\frac{E^{1/2} - E^{-1/2}}{2}\right)\sqrt{4 + E + E^{-1} - 2}$$

$$= \frac{1}{2}(E + E^{-1} - 2) + \left(\frac{E^{1/2} - E^{-1/2}}{2}\right)\sqrt{(E^{1/2} + E^{-1/2})^2}$$

$$= \frac{1}{2}(E + E^{-1} - 2 + E - E^{-1})$$

$$= \frac{1}{2}(2E - 2)$$

$$= E - 1$$

EXAMPLE 4.4 Show that $\mu\delta = \dfrac{\Delta + \nabla}{2}$.

Solution We can write

$$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$$

$$= \frac{1}{2}(E - E^{-1})$$

We know that

$$\Delta = E - 1$$

$$\nabla = 1 - E^{-1}$$

$$\therefore \quad E^{-1} = 1 - \nabla$$

$$\therefore \quad \mu\delta = \frac{1}{2}(1 + \Delta - 1 + \nabla)$$

$$= \frac{1}{2}(\Delta + \nabla)$$

EXAMPLE 4.5 Evaluate $\left[\left(\dfrac{\Delta^2}{E}\right)e^x\right]\left[\dfrac{Ee^x}{\Delta^2 e^x}\right]$.

Solution

$$\left[\left(\frac{\Delta^2}{E}\right)e^x\right]\left[\frac{Ee^x}{\Delta^2 e^x}\right] = \left[(\Delta^2 E^{-1})e^x\right]\left[\frac{Ee^x}{\Delta^2 e^x}\right]$$

$$= \left[\Delta^2 e^{x-h}\right]\left[\frac{Ee^x}{\Delta^2 e^x}\right]$$

$$= \left[\Delta^2 e^x e^{-h}\right]\left[\frac{Ee^x}{\Delta^2 e^x}\right]$$

$$= e^{-h} \cdot Ee^x = e^{-h} \cdot e^{x+h}$$

$$= e^{-h} \cdot e^x \cdot e^h$$

$$= e^x$$

4.1.2 Gregory–Newton Forward Interpolation Formula

Let $y_x = f(x)$ be a function, which takes the values $y_0, y_1 \ldots y_n$, corresponding to the values $x_0, x_1, \ldots, x_n$.

Let $\quad x_i = x_0 + ih \quad (i = 0, 1, 2, \ldots, n)$

Let $y(x)$ be a polynomial of the nth degree in x and the values of x be equispace interval. Consider the polynomial of nth degree,

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \quad (4.1)$$

This polynomial contains $(n + 1)$ constants $a_0, a_1, \ldots, a_n$, which can be determined as follows:

When $x = x_0$ in (4.1)

$$y(x_0) = a_0$$

i.e.,

$$a_0 = y_0 \quad (4.2)$$

When $x = x_1$ in (4.1)

$$y(x_1) = a_0 + a_1(x_1 - x_0)$$

$$y_1 = a_0 + a_1 h \quad (4.3)$$

$$y_1 = y_0 + a_1 h \text{ (by (4.2))}$$

$$\therefore \quad y_1 - y_0 = a_1 h$$

$$\Delta y_0 = a_1 h$$

$$a_1 = \frac{\Delta y_0}{h} \quad (4.4)$$

When $x = x_2$ in (4.1)

$$y(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

i.e.,

$$y_2 = a_0 + a_1(2h) + a_2(2h^2)$$

$$y_2 = a_0 + 2ha_1 + 2h^2a_2$$

$$y_2 - y_1 = a_0 + 2ha_1 + 2h^2a_2 - a_0 - a_1h$$

$$= a_1h + 2h^2a_2$$

$$\Delta y_1 = \frac{\Delta y_0}{h} \cdot h + 2h^2 a_2$$

$$\Delta y_1 - \Delta y_0 = 2h^2a_2$$

$$\frac{\Delta^2 y_0}{2!h^2} = a_2$$

Similarly, $a_3 = \dfrac{\Delta^3 y_0}{3!h^3}$ and so on.

Substituting $a_0, a_1, a_2 \dots$ in (4.1), we get

$$y(x) = y_0 + \left(\frac{\Delta y_0}{h}\right)(x - x_0) + \left(\frac{\Delta^2 y_0}{2!h^2}\right)(x - x_0)(x - x_1)$$

$$+ \left(\frac{\Delta^3 y_0}{3!h^3}\right)(x - x_0)(x - x_1)(x - x_2) + \dots$$

$$+ \left(\frac{\Delta^n y_0}{n!h^n}\right)(x - x_0)(x - x_1)\dots(x - x_{n-1}) \qquad (4.5)$$

Let us put $x - x_0 = ph$

$$x = x_0 + ph$$

Now,

$$x - x_1 = x - x_0 + x_0 - x_1$$

$$= (x - x_0) + (x_0 - x_1)$$

$$= ph + (-h)$$

$$= (p - 1)h$$

Also

$$x - x_2 = x - x_0 + x_0 - x_2$$

$$= ph - 2h$$

$$= (p - 2)h \text{ and so on.}$$

Substituting this in (4.5), we get

$$y(x_0 + ph) = y_0 + \left(\frac{\Delta y_0}{h}\right)(ph) + \left(\frac{\Delta^2 y_0}{2!h^2}\right)[ph(p-1)h]$$

$$+ \left(\frac{\Delta^3 y_0}{3!h^3}\right)[ph\,(p-1)h\,(p-2)h] + \dots$$

$$y(x_0 + ph) = y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

$$+ \left[\frac{p(p-1)(p-2)\dots[p-(n-1)]}{n!}\right]\Delta^n y_0$$

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \left[\frac{p(p-1)(p-2)}{3!}\right]\Delta^3 y_0 + \dots$$

$$+ \left[\frac{p(p-1)\dots[p-(n-1)]}{n!}\right]\Delta^n y_0 \qquad (4.6)$$

This is known as *Gregory–Newton forward interpolation formula.*

Note

1. The first two terms of this series give the result for the ordinary linear interpolation.
2. The first three terms of this series give the result for parabolic interpolation.
3. It is used mainly for interpolating the values of y near the beginning of a set of tabular values. Equation (4.6) is also called *Newton's forward interpolation formula.*

4.1.3 Gregory–Newton Backward Interpolation Formula

Let $y_x = f(x)$ be a function, which takes the values $y_0, y_1, \dots, y_n$, corresponding to the values $x_0, x_1, \dots, x_n$. The values of x be at equal distance intervals, i.e.,

Let $x_i = x_0 + ih$, where $(i = 0, 1, 2, \dots, h)$

Let $y(x)$ be the polynomial of nth degree in x.

We assume that

$$y(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})$$
$$(x - x_{n-2}) + \dots + a_n(x - x_n)(x - x_{n-1})\dots(x - x_1) \qquad (4.7)$$

This polynomial contains $(n + 1)$ constants $a_0, a_1, \dots, a_n$ which can be determined as follows:

when $x = x_n$ in (4.7)

$$y(x_n) = a_0$$

$$y_n = a_0 \qquad (4.8$$

when $x = x_{n-1}$ in (4.7)

$$y(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n)$$

$$y_{n-1} = a_0 + a_1(x_{n-1} - x_n) \qquad (4.$$

From Eqs. (4.8) and (4.9)

$$y_n - y_{n-1} = a_0 - a_0 - a_1(x_{n-1} - x_n)$$

$$\nabla y_n = -\, a_1\,(-\,h)$$

$$\therefore \qquad a_1 = \frac{\nabla y_n}{h} \tag{4.10}$$

when $x = x_{n-2}$ in (4.7)

$$y(x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)\,(x_{n-2} - x_{n-1})$$

$$= a_0 + a_1(-\,2h) + a_2(-\,2h)\,(-\,h)$$

$$y_{n-2} = a_0 - a_1\,(2h + 2h^2 a_2) \tag{4.11}$$

From Eqs. (4.9)–(4.11)

$$y_{n-1} - y_{n-2} = a_0 + a_1(-\,h) - a_0 + a_1(2h) - 2h^2 a_2$$

$$\nabla y_{n-1} = a_1 h - 2h^2 a_2$$

$$\nabla y_{n-1} = \left(\frac{\nabla y_n}{h}\right)\cdot h - 2h^2 a_2$$

$$\nabla y_n - \nabla y_{n-1} = 2h^2 a_2$$

$$\therefore \qquad a_2 = \frac{\nabla^2 y_n}{2!\,h^2}$$

Similarly, $\quad a_3 = \dfrac{\nabla^3 y_n}{3!\,h^3}$ and so on.

Substituting $a_0, a_1, \ldots, a$ in (4.7), we get

$$y(x) = y_n + \left(\frac{\nabla y_n}{h}\right)(x - x_n) + \left(\frac{\nabla^2 y_n}{2!\,h^2}\right)(x - x_n)\,(x - x_{n-1})$$

$$+ \ldots + \left(\frac{\nabla^2 y_n}{n!\,h^n}\right)(x - x_n)\,(x - x_{n-1})\ldots(x - x_1) \tag{4.12}$$

put $\qquad x - x_n = ph$

i.e., $\qquad x = x_n + ph$

Now, $\qquad x - x_{n-1} = x - x_n + x_n - x_{n-1}$

$$= (x - x_n) + (x_n - x_{n-1})$$

$$= ph + h$$

$$= (p + 1)h$$

Also $x - x_{n-2} = (p + 2)h$ and so on.

Substituting this in (4.12), we get

$$y(x_n + ph) = y_n + \left(\frac{\nabla y_n}{h}\right)(ph) + \left(\frac{\nabla^2 y_n}{2!\,h^2}\right)(ph)\,(p+1)h$$

$$+ \left(\frac{\nabla^3 y_n}{3!\,h^3}\right)(ph)\,(p+1)\,h(p+2)h + \ldots$$

$$y_p = y_n + p\nabla y_n + \left[\frac{p(p+1)}{2!}\right](\nabla^2 y_n) + \left[\frac{p(p+1)\,(p+2)}{3!}\right](\nabla^3 y_n)$$

$$+ \left[\frac{p(p+1)\,(p+2)\ldots(p+(n-1))}{n!}\right](\nabla^n y_n) \tag{4.13}$$

This equation is known as *Gregory–Newton backward difference formula.*

Note

1. It is mainly used to interpolate the values of y near the end of a set of tabular values.
2. It is also called *Newton's backward difference formula.*

EXAMPLE 4.6 Estimate the value of sin θ at $\theta = 25°$, using the Newton–Gregory forward difference formula with the help of the following table.

θ	10	20	30	40	50
sin θ	0.1736	0.3420	0.5000	0.6428	0.7660

Solution In order to use the Newton–Gregory forward difference formula, we need the values of $\Delta^j y_0$. These coefficients can be obtained from the difference table given below.

θ	$y = \sin\theta$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	$\boxed{0.1736}$					
		$\boxed{0.1684}$				
20	0.3420		$\boxed{-\,0.0104}$			
		0.1580		$\boxed{0.0048}$		
30	0.5000		– 0.0152		$\boxed{-\,0.0004}$	–
		0.1428		0.0044		
40	0.6428		– 0.0196			
		0.1232				
50	0.7660					

Here,

$$x_0 = \theta_0 = 10$$

and

$$h = 10$$

Therefore,

$$p = \frac{x - x_0}{h} = \frac{25 - 10}{10} = 1.5$$

We have

$$y = y_0 + p(\Delta y_0) + \left[\frac{p(p-1)}{1 \cdot 2}\right](\Delta^2 y_0) + \left[\frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3}\right](\Delta^3 y_0) + \left[\frac{p(p-1)(p-2)(p-3)}{1 \cdot 2 \cdot 3 \cdot 4}\right](\Delta^4 y_0)$$

Here $y_0 = 1.1736, \quad \Delta y_0 = 0.1684, \quad \Delta^2 y_0 = -0.0104$

$$\Delta^3 y_0 = 0.0048, \quad \Delta^4 y_0 = -0.0004$$

$$\therefore \quad y_p = 1.1736 + 1.5 \times 0.1684 + \frac{(1.5)(1.5-1)}{1.2}(-0.0104) + \frac{1.5(1.5-1)(1.5-2)}{1 \cdot 2 \cdot 3}(0.0048) + (1.5)(1.5-1)\frac{(1.5-2)(1.5-3)}{1 \cdot 2 \cdot 3 \cdot 4}(-0.0004)$$

Thus, sin 25 = 0.4220, which is accurate to four decimal places.

EXAMPLE 4.7 From the following table, estimate the numbers of students who obtained marks between 40 and 45.

Marks	30–40	40–50	50–60	60–70	70–80
No. of students	31	42	51	35	31

Solution First we prepare the cumulative frequency table as follows:

Marks less than (x)	40	50	60	70	80
No. of students (y_x)	31	73	124	159	190

Now, the difference table in

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	$\boxed{31}$				
		$\boxed{42}$			
50	73		$\boxed{9}$		
		51		$\boxed{-25}$	
60	124		−16		$\boxed{37}$
		35		12	
70	159		−4		
		31			
80	190				

We shall find y_{45}.

Taking $x_0 = 40$, $x = 45$, we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5$$

Using Newton's forward interpolation formula, we get

$$y_p = 31 + (0.5 \times 42) + \left[\frac{(0.5)(0.5-1)}{2!}\right](9) + \left[\frac{(0.5)(0.5-1)(0.5-2)}{3!}\right] + \left[\frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}\right](37)$$

$$= 47.87$$

EXAMPLE 4.8 The following data gives the melting point of an alloy of lead and zinc, where y is the temperature in °C and x is the percentage of lead in the alloy.

x	40	50	60	70	80	90
y	184	204	226	250	276	304

Using Newton's interpolation formula, find the melting point of the alloy containing 84% of lead.

Solution Since the value 84 of x is near the end of the table, so we use Newton's backward formula.

The backward differences are calculated and tabulated below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
40	184					
		20				
50	204		2			
		22		0		
60	206		2		0	
		24		0		$\boxed{0}$
70	250		2		$\boxed{0}$	
		26		$\boxed{0}$		
80	276		$\boxed{2}$			
		$\boxed{28}$				
90	$\boxed{304}$					

Newton's backward formula gives

$$y_p = y_n + p(\nabla y_n) + \left[\frac{p(p+1)}{2!}\right](\nabla^2 y_n) + \left[\frac{p(p+1)(p+2)}{3!}\right](\nabla^3 y_n)$$

$$+ \left[\frac{p(p+1)(p+2)(p+3)}{4!}\right](\nabla^4 y_n) + \ldots$$

Here, $x_n = 90$, $y_n = 304$, $\nabla y_n = 28$, $\nabla^2 y_n = 2$, $h = 10$

$$\therefore \quad p = \frac{x - x_n}{h}$$

$$= \frac{84 - 90}{10}$$

$$= -0.6$$

$$\therefore \quad y_p = 304 + (-0.6 \times 28) + \left[\frac{(-0.6)(-0.6+1)}{2!}\right](2) + 0$$

$$= 304 - 16.8 - 0.24$$

$$= 287 \text{ nearly}$$

EXAMPLE 4.9 The following are data from the steam table.

Temperature (°C)	140	150	160	170	180
Pressure (kgf km²)	3.685	4.854	6.302	8.076	10.225

Using Newton's formula, find the pressure of the steam for a temperature of 142°C.

Solution Since 142 is near the beginning of the table. Hence, we use Newton's forward formula.

The forward differences are calculated and tabulated below:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	$\boxed{3.685}$				
		$\boxed{1.169}$			
150	4.854		$\boxed{0.279}$		
		1.448		$\boxed{0.047}$	
160	6.302		0.326		$\boxed{0.002}$
		1.1774		0.049	
170	8.076		0.395		
		2.149			
180	10.225				

We have

$$y = y_0 + p(\Delta y_0) + \left[\frac{p(p-1)}{1 \cdot 2}\right](\Delta^2 y_0) + \left[\frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3}\right](\Delta^3 y_0)$$

$$+ \left[\frac{p(p-1)(p-2)(p-3)}{1 \cdot 2 \cdot 3 \cdot 4}\right](\Delta^4 y_0)$$

Here, $x_0 = 140$, $y_0 = 3.685$, $\Delta y_0 = 1.169$, $\Delta^2 y_0 = 0.279$,
$\Delta^3 y_0 = 0.047$, $\Delta^4 y_0 = 0.002$

Now,

$$p = \frac{x - x_0}{h}$$

$$= \frac{142 - 140}{10}$$

$$= 0.2$$

$$\therefore \quad y_p = 3.685 + (0.2)(1.169) + \left[\frac{(0.2)(0.2-1)}{2!}\right](0.279)$$

$$+ \left[\frac{(0.2)(0.2-1)(0.2-2)}{1 \cdot 2 \cdot 3}\right](0.047)$$

$$+ \left[\frac{(0.2)(0.2-1)(0.2-2)(0.2-3)}{1 \cdot 2 \cdot 3 \cdot 4}\right](0.002)$$

$$= 3.685 + 0.2338 - 0.02232 + 0.002256 - 0.0000672$$

$$= 3.899$$

EXERCISES

4.1 Using Newton's forward formula, find the value of $f(1.6)$, if

x	1	1.4	1.8	2.2
$f(x)$	3.49	4.82	5.96	6.5

[*Ans.* 5.54]

4.2 Using Newton's backward formula, find the value of $e^{-1.9}$, from the following table of values of e^{-x}.

x	1.00	1.25	1.50	1.75	2.00
e^{-x}	0.3679	0.2865	0.2231	0.1738	0.1353

[*Ans.* 0.1496]

4.3 Estimate the values of $f(22)$ and $f(42)$, from the following available data.

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

[*Ans.* 352, 219]

4.4 The following table gives the valus of sin θ.

θ	30°	31°	32°	33°	34°
sin θ	0.5000	0.5150	0.5299	0.5446	0.5592

Find the value of sin 32° 15′.

[*Ans.* 0.5336]

4.5 Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4.

4.6 The following table gives the values of tan x, for $0.10 \le x \le 0.30$.

x	0.10	0.15	0.20	0.25	0.30
$y = \tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Find (i) tan 0.12, (ii) tan 0.26.

[*Ans.* 0.12057, 0.26602]

4.7 Using Newton's formula, find y, when $x = 27$, from the following data.

x	10	15	20	25	30
y	35.4	32.2	29.1	26.0	23.1

[*Ans.* 24.8]

4.8 Estimate exp (1.85), from the following table.

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
exp(x)	5.474	6.050	6.689	7.389	8.166	9.025	9.974

[*Ans.* 6.359819]

4.9 The amount A of a substance remaining in a reacting system after an interval of time t in a certain chemical experiment is given below.

t (min)	2	5	8	11
A (gm)	94.8	87.9	81.3	75.1

Obtain the value of A where $t = 9$ min, using Newton's backward difference interpolation formula.

[*Ans.* 74.925]

4.10 If $u_{-1} = 10$, $u_1 = 8$, $u_2 = 10$, $u_4 = 50$, find u_0 and u_3.

[*Ans.* 10, 22]

4.11 Given $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$ and $y_4 = 100$, $y_5 = 8$ without forming the difference table, find $\Delta^5 y_0$.

[*Ans.* 755]

4.12 The area A of a circle of diameter d is given for the following values.

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

Calculate the area of a circle of a diameter 105.

[*Ans.* 0.8666]

4.2 CENTRAL DIFFERENCE INTERPOLATION FORMULA

In this section, we shall discuss central difference formulae, which are applicable for interpolation near the middle of the table, instead of beginning and end of tabulated values.

Suppose that $y = f(x)$ be the functional relation between x and y. If x takes the values $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$ and the corresponding values of y are $y_{-2}, y_{-1}, y_0, y_1, y_2$. By using the relation $\delta = \Delta E^{-1/2}$, we can form the difference table in the two notations as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2} (= \delta y_{-3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} (= \delta^2 y_{-1})$		
x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
		$\Delta y_{-1} (= \delta y_{-1/2})$		$\Delta^3 y_{-2} (= \delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1} (= \delta^2 y_0)$		$\Delta^4 y_{-2} (= \delta^4 y_0)$
		$\Delta y_0 (= \delta y_{1/2})$		$\Delta^3 y_{-1} (= \delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0 (= \delta^2 y_1)$		
		$\Delta y_1 (= \delta y_{3/2})$			
$x_0 + 2h$	y_2				

4.2.1 Gauss's Forward Interpolation Formula

The Newton–Gregory formula for forward interpolation, with origin at $x = x_0$, is

$$y_p = y_0 + p \cdot \Delta y_0 + \left[\frac{p(p-1)}{1 \cdot 2}\right](\Delta^3 y_0) + \left[\frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3}\right](\Delta^3 y_0) + \ldots$$

where

$$p = \frac{x - x_0}{h} \tag{4.14}$$

From the forward difference table, we have

$$\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$$

$$\therefore \quad \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 - \Delta^3 y_{-1} = \Delta^4 y_{-1}$$

$$\therefore \quad \Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$$

Also,

$$\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$$

$$\therefore \quad \Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

Similarly,

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{ etc.}$$

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \ldots$, from the above results, we get

$$y_p = y_0 + p\,\Delta y_0 + \left[\frac{p(p-1)}{1 \cdot 2}\right](\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \left[\frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3}\right]$$

$$(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \left[\frac{p(p-1)(p-2)(p-3)}{1 \cdot 2 \cdot 3 \cdot 4}\right](\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$

$$y_p = y_0 + p\Delta y_0 + \left[\frac{p(p-1)}{1 \cdot 2}\right](\Delta^2 y_{-1}) + \left[\frac{p(p-1)}{1 \cdot 2} + \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3}\right](\Delta^3 y_{-1})$$

$$+ \left[\frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} + \frac{p(p-1)(p-2)(p-3)}{1 \cdot 2 \cdot 3 \cdot 4}\right](\Delta^4 y_{-1}) + \ldots$$

$$+ (\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$

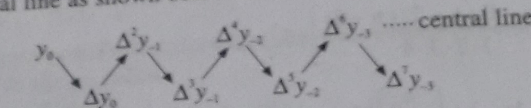
$$\therefore \quad y_p = y_0 + p\,\Delta y_0 + \left[\frac{p(p-1)}{1 \cdot 2}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{1 \cdot 2 \cdot 3}\right](\Delta^3 y_{-1})$$

$$+ \left[\frac{(p+1)p\,(p-1)(p-2)}{1 \cdot 2 \cdot 3 \cdot 4}\right](\Delta^4 y_{-2}) + \ldots$$

This formula is called *Gauss's forward formula of interpolation* as it is used to interpolate the values of the function for the value $p (0 < p < 1)$ measured forwardly from the origin.

Note

This formula contains odd differences below the central line and even differences on the central line as shown below.



4.2.2 Gauss's Backward Interpolation Formula

The Newton's Gregory formula for forward interpolation, with origin at $x = x_0$, is

$$y_p = y_0 + p \cdot \Delta y_0 + \left[\frac{p(p-1)}{1 \cdot 2}\right](\Delta^2 y_0) + \left[\frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3}\right](\Delta^3 y_0) + \ldots \tag{4.15}$$

where

$$p = \frac{x - x_0}{h}$$

From the backward difference table, we have

$$\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$$

$$\therefore \quad \Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$$

$$\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$$

$$\therefore \quad \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

Also,

$$\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{ etc.}$$

Substituting for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \ldots$ from the above results, we get

$$y_p = y_0 + p\,(\Delta y_{-1} + \Delta^2 y_{-1}) + \left[\frac{p(p-1)}{1 \cdot 2}\right](\Delta^2 y_{-1} + \Delta^3 y_{-1})$$

$$+ \left[\frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3}\right](\Delta^3 y_{-1} + \Delta^4 y_{-1})$$

$$+ \left[\frac{p\,(p-1)(p-2)(p-3)}{1 \cdot 2 \cdot 3 \cdot 4}\right](\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$

$$= y_0 + p(\Delta y_{-1}) + \left[p + \frac{p(p-1)}{1 \cdot 2}\right](\Delta^2 y_{-1}) + \left[\frac{p(p-1)}{1 \cdot 2} + \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3}\right]$$

$$(\Delta^3 y_{-1}) + \left[\frac{p(p-1)(p-2)}{1\cdot2\cdot3} + \frac{p(p-1)(p-2)(p-3)}{1\cdot2\cdot3\cdot4}\right](\Delta^4 y_{-1})$$

$$+\left[\frac{p(p-1)(p-2)(p-3)}{1\cdot2\cdot3\cdot4}\right](\Delta^5 y_{-1}) + \ldots$$

$$= y_0 + p(\Delta y_{-1}) + \left(\frac{p}{1\cdot2}\right)(2+p-1)(\Delta^2 y_{-2}) + \left[\frac{p(p-1)}{1\cdot2\cdot3}\right](3+p-2)$$

$$(\Delta^3 y_{-1}) + \left[\frac{p(p-1)(p-2)}{1\cdot2\cdot3\cdot4}\right](4+p-3)(\Delta^4 y_{-1})$$

$$+\left[\frac{p(p-1)(p-2)(p-3)}{1\cdot2\cdot3\cdot4}\right](\Delta^5 y_{-1}) + \ldots$$

i.e., $$y_p = y_0 + p(\Delta y_{-1}) + \left[\frac{(p+1)p}{1\cdot2}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)p(p-1)}{1\cdot2\cdot3}\right](\Delta^3 y_{-1})$$

$$+\left[\frac{(p+1)\,p(p-1)(p-2)}{1\cdot2\cdot3\cdot4}\right](\Delta^4 y_{-1})$$

$$+\left[\frac{p(p-1)(p-2)(p-3)}{1\cdot2\cdot3\cdot4}\right](\Delta^5 y_{-1}) + \ldots$$

i.e., $$y_p = y_0 + \left(\frac{p}{1}\right)(\Delta y_{-1}) + \left[\frac{(p+1)p}{1\cdot2}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{1\cdot2\cdot3}\right]$$

$$(\Delta^3 y_{-2} + \Delta^4 y_{-2}) + \left[\frac{(p+1)\,p(p-1)(p-2)}{1\cdot2\cdot3\cdot4}\right](\Delta^4 y_{-2} + \Delta^5 y_{-2})$$

$$+\left[\frac{p(p-1)(p-2)(p-3)}{1\cdot2\cdot3\cdot4}\right](\Delta^5 y_{-2} + \Delta^6 y_{-2}) + \ldots$$

$$y_p = y_0 + \left(\frac{p}{1}\right)(\Delta y_{-1}) + \left[\frac{(p+1)p}{1\cdot2}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{1\cdot2\cdot3}\right](\Delta^3 y_{-2})$$

$$+\left[\frac{p(p+1)(p-1)}{1\cdot2\cdot3\cdot4}\right](4+p-2)(\Delta^4 y_{-2}) + \ldots$$

$$\therefore\quad y_p = y_0 + (p)\Delta y_{-1} + \left[\frac{(p+1)p}{1\cdot2}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{1\cdot2\cdot3}\right](\Delta^3 y_{-2})$$

$$+\left[\frac{(p+2)(p+1)\,p(p-1)}{1\cdot2\cdot3\cdot4}\right](\Delta^4 y_{-2}) + \ldots$$

This formula is called *Gauss's backward formula of interpolation*, as it is used to interpolate the value of the function for a negative value of $p(-1 < p < 0)$.

Note

This formula contains odd differences above the central line and even differences on the line as shown below:

$$\begin{array}{ccccccccccc} & & \nabla y_{-1} & & & \nabla^3 y_{-2} & & & \nabla^5 y_{-3} & & \\ & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & \searrow & \\ y_0 & \ldots & & \nabla^2 y_{-1} & \ldots & & \nabla^4 y_{-2} & \ldots & & \nabla^6 y_{-3} & \end{array}$$

EXAMPLE 4.10 Apply Gauss's forward formula to obtain $f(x) = x \sin x + 2$, at $x = 3.5$, from the following table.

x	2	3	4	5
$f(x)$	3.818	2.423	−1.027	−2.794

Solution

Here $h = 1$

Taking $x_0 = 3$

$$\therefore\quad p = \frac{x - x_0}{h} = \frac{3.5 - 3}{1} = 0.5$$

Therefore, the central difference table is

p	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
−1	2	3.818			
			−1.395		
0	3	$\boxed{2.423}$		$\boxed{-2.055}$	
			$\boxed{-3.45}$		$\boxed{3.738}$
1	4	−1.027		1.683	
			−1.767		
2	5	−2.794			

Gauss's forward formula is

$$y_p = y_0 + p(\Delta y_0) + \left[\frac{p(p-1)}{2!}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{3!}\right](\Delta^3 y_{-1})$$

$$+\left[\frac{(p+1)\,p(p-1)(p-2)}{4!}\right](\Delta^4 y_{-2}) + \ldots$$

$$= 2.423 + 0.5(-3.45) + \left[\frac{(0.5)(0.5-1)}{2!}\right](-2.055)$$

$$+\left[\frac{(0.5+1)(0.5)(0.5-1)}{3!}\right](3.738)$$

$$= 0.254$$

EXAMPLE 4.11 Given the table:

x	1.5	2.5	3.5	4.5
$y = xe^x$	8.963	24.364	66.230	180.034

Find the value of y, at $x = 2.25$, by Gauss's backward difference formula.

Solution

Here $h = 0.5$

Taking $x_0 = 2.5$

$$\therefore \qquad p = \frac{x - x_0}{h}$$

$$= -0.5$$

Let us form the central difference table

p	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
−1	1.5	8.963			
			$\boxed{15.401}$		
0	2.5	$\boxed{24.364}$		$\boxed{26.465}$	
			41.866		45.473
1	3.5	66.230		71.938	
			113.804		
2	4.5	180.034			

Gauss's backward formula is

$$y_p = y_0 + p(\Delta y_{-1}) + \left[\frac{p(p+1)}{2!}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{3!}\right](\Delta^3 y_{-2}) + \ldots \quad \text{(i)}$$

$$= 24.364 + (-0.5)(15.401) + \left[\frac{(-0.5)(-0.5+1)}{2!}\right](26.465)$$

$$= 13.355$$

EXAMPLE 4.12 Using Gauss's forward formula, evaluate y_{30}, given that $y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070$, $y_{33} = 16.3432$ and $y_{37} = 15.5154$.

Solution

Taking $x_0 = 29$, $h = 4$, we have to find the value of y for $x = 30$.

$$\therefore \qquad p = \frac{x - x_0}{h}$$

$$= \frac{30 - 29}{4}$$

$$= 0.25$$

Therefore, the central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$
21	−2	18.4708				
			−0.6564			
25	−1	17.8144		−0.0510		
			−0.7074		−0.0074	
29	0	$\boxed{17.1070}$		$\boxed{-0.0564}$		$\boxed{-0.0022}$
			$\boxed{-0.7638}$		$\boxed{-0.0076}$	
33	1	16.3432		−0.0640		
			−0.8278			
37	2	15.5154				

Gauss's forward formula is

$$y_p = y_0 + p(\Delta y_0) + \left[\frac{p(p-1)}{1\cdot 2}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{1\cdot 2\cdot 3}\right](\Delta^3 y_{-1})$$

$$+\left[\frac{(p+1)\,p(p-1)(p-2)}{1\cdot 2\cdot 3\cdot 4}\right](\Delta^4 y_{-2}) + \ldots$$

$$= 17.1070 + (0.25)(-0.7638) + \left[\frac{(0.25)(-0.75)}{2}\right](0.0564)$$

$$+\left[\frac{(1.25)(0.25)(-0.75)}{6}\right](-0.0076) +$$

$$\left[\frac{(1.25)(0.25)(-0.75)(-1.75)}{24}\right]\times(-0.0022)$$

$$= 17.1070 - 0.19095 + 0.00529 + 0.0003 - 0.00004$$

$$= 16.9216 \text{ approximately}$$

EXAMPLE 4.13 Interpolate by means of Gauss's backward formula the sales of concern for the year 1966, given that:

Year	1931	1941	1951	1961	1971	1981
Sales (in lakhs of rupees)	12	15	20	27	39	52

Solution Since we have to use Gauss's backward formula, we choose $x_0 = 1971$.

Here $h = 10$

$$\therefore \qquad p = \frac{x - x_0}{h}$$

At year 1966,

$$p = \frac{1966 - 1971}{10}$$

$$= \frac{-5}{10}$$

$$= -0.5$$

We construct the following central difference table:

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
−4	12					
		3				
−3	15		2			
		5		0		
−2	20		2		3	
		7		3		−10
−1	27		5		−7	
		$\boxed{12}$		$\boxed{-4}$		
0 →	$\boxed{39}$		$\boxed{1}$			
		13				
1	52					

Here $y_0 = 39, \quad \Delta y_{-1} = 12, \quad \Delta^2 y_{-1} = 1, \quad \Delta^3 y_{-2} = -4$

Gauss's backward formula is

$$y_p = y_0 + p\,\Delta y_{-1} + \left[\frac{(p+1)p}{1\cdot 2}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{1\cdot 2\cdot 3}\right](\Delta^3 y_{-2}) + \ldots$$

$$= 39 + (-0.5)\,(12) + \left[\frac{(-0.5+1)\,(-0.5)}{1\cdot 2}\right](1)$$

$$+ \left[\frac{(-0.5+1)\,(-0.5)\,(-0.5-1)}{1\cdot 2\cdot 3}\right](-4) + \ldots$$

$$= 39 - 6 - 0.125 - 0.2$$

$$= 32.625$$

Therefore, sales in the year 1966 is of Rs. 32.625 lakh.

EXAMPLE 4.14 Apply Gauss's-forward formula to find the value of $f(x)$, at $x = 3.75$, from the following table:

x	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	24.145	22.043	20.225	18.644	17.262	16.047

Solution

Taking $x_0 = 3.5$ and $h = 0.5$

$$\therefore \qquad p = \frac{x - 3.5}{0.5}$$

Therefore, the central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
−2	24.145					
		−2.102				
−1	22.043		0.284			
		−1.818		−0.047		
0	20.225		0.237		0.009	
		−1.581		−0.038		−0.003
1	18.644		0.199		0.006	
		−1.382		−0.032		
2	17.262		0.167			
		−1.215				
3	16.047					

Gauss's forward formula is

$$y_p = y_0 + p(\Delta y_0) + \left[\frac{p(p-1)}{1\cdot 2}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{1\cdot 2\cdot 3}\right](\Delta^3 y_{-1})$$

$$+ \left[\frac{(p+1)\,p(p-1)\,(p-2)}{1\cdot 2\cdot 3\cdot 4}\right](\Delta^4 y_{-2})$$

$$+ \left[\frac{(p+2)\,(p+1)\,p(p-1)\,(p-2)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\right](\Delta^5 y_{-2}) + \ldots$$

when

$$x = 3.75$$

$$p = \frac{3.75 - 3.5}{0.5}$$

$$= 0.5$$

Hence,

$$y_0 = 20.225,\quad \Delta^2 y_{-1} = 0.237,\quad \Delta^4 y_{-2} = 0.009,$$
$$\Delta y_0 = -1.581,\quad \Delta^3 y_{-1} = -0.038,\quad \Delta^5 y_{-2} = -0.003$$

$$\therefore \quad y_p = 20.225 + (0.5)(-1.581) + \left[\frac{(0.5)(0.5-1)}{1\cdot 2}\right](0.237)$$
$$+ \left[\frac{(0.5+1)(0.5)(0.5-1)}{1\cdot 2\cdot 3}\right](-0.038)$$
$$+ \left[\frac{(0.5+1)(0.5)(0.5-1)(0.5-2)}{1\cdot 2\cdot 3\cdot 4}\right](0.009)$$
$$+ \left[\frac{(0.5+2)(0.5+1)(0.5)(0.5-1)(0.5-2)}{1\cdot 2\cdot 3\cdot 4}\right](-0.003)$$
$$= 20.225 - 0.7905 - 0.02925 + 0.002375 + 0.0002109375 - 0.00003515625$$

$= 19.407$, correct to three decimal places.

EXAMPLE 4.15 Apply the best Gauss interpolation formula to compute the population of a town, for the year 1974, given that:

Year (x)	1939	1949	1959	1969	1979	1989
Population in thousand(y)	12	15	20	27	39	52

Solution Since the year 1974 of x is middle of the year 1969 to 1979. So we use any one of the Gauss interpolation formula:

Taking

$$x_n = 1969 \text{ and } h = 10$$

$$\therefore \quad p = \frac{1974 - 1969}{10} = 0.5$$

Therefore, the central difference table is:

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1939	– 3	12					
			3				
1949	– 2	15		2			
			5		0		
1959	– 1	20		2		3	
			7		3		– 10
1969	0	27		5		– 7	
			12		– 4		
1979	1	39		1			
			13				
1989	2	52					

Here

$$y_0 = 27,\ \Delta y_{-1} = 7,\quad \Delta^2 y_{-1} = 5,$$
$$\Delta^3 y_{-1} = 3,\quad \Delta^4 y_{-1} = -7,\quad \Delta^5 y_{-1} = -10$$

Gauss's backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \left[\frac{(p+1)p}{2!}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{3!}\right](\Delta^3 y_{-2})$$
$$+ \left[\frac{(p+2)(p+1)\,p(p-1)}{4!}\right](\Delta^4 y_{-1})$$
$$+ \left[\frac{(p+2)(p+1)\,p(p-1)(p-2)}{5!}\right](\Delta^5 y_{-1}) + \ldots$$
$$= 27 + (0.5)(7) + \left[\frac{(1.5)(0.5)}{2}\right](5) + \left[\frac{(1.5)(0.5)(-0.5)}{6}\right]$$
$$+ \left[\frac{(2.5)(1.5)(-0.5)}{24}\right](-7)$$
$$+ \left[\frac{(2.5)(1.5)(0.5)(-0.5)(-1.5)}{120}\right](-10) + \ldots$$
$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172$$
$$= 32.345$$

EXERCISES

4.13 Apply a central difference formula to obtain $f(32)$, given that $f(25) = 0.2707, f(30) = 0.3027, f(35) = 0.3386, f(40) = 0.3794$.

[*Ans.* 0.3165]

4.14 The values of e^{-x} at $x = 1.72$ to 1.76 are given in the following table:

x	1.72	1.73	1.74	1.75	1.76
e^{-x}	0.17907	0.17728	0.17552	0.17377	0.17204

Find the value of $e^{-1.7425}$.

[*Ans.* 0.17508]

4.15 Using Gauss's interpolation formula, find (a) sin 61° 24′, (b) sin 63° 48′, from the following table:

x	60°	61°	62°	63°	64°	65°
sin$x$	0.86603	0.87462	0.88295	0.89101	0.89879	0.90631

[*Ans.* (a) 0.87798, (b) 0.89259

4.16 Given that $\sqrt{12500} = 111.803399$, $\sqrt{12510} = 111.848111$, $\sqrt{12520} = 111.892806$, $\sqrt{12530} = 111.937483$, show by Gauss's backward formula that $\sqrt{12516} = 111.874930$.

4.17 Use Gauss's forward formula to get y_{30}, given that, $y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070$, $y_{33} = 16.3432$, $y_{37} = 15.5154$.

[Ans. 16.9216]

4.18 Given $f(2) = 10$, $f(1) = 8$, $f(0) = 5$, $f(1) = 10$, find $f(1/2)$, by Gauss's forward formula.

[Ans. 0.06]

4.19 The population of a town is given below. Apply Gauss's backward formula to get the population in 1926.

Year (x)	1911	1921	1931	1941	1951
Population in thousands(y)	15	20	27	39	52

[Ans. 22.898]

4.20 Use Gauss's interpolation formula to get y_{16}, given

x	5	10	15	20	25
e^{-x}	26.782	19.951	14.001	8.762	4.163

[Ans. 12.901]

4.2.3 Stirling's Formula

The average of the two Gauss formulae i.e., forward formula and backward formula, gives the Stirling's formula.

Gauss's forward formula is

$$y_p = y_0 + p(\Delta y_0) + \left[\frac{p(p-1)}{2!}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{3!}\right](\Delta^3 y_{-1})$$
$$+\left[\frac{(p+1)\,p(p-1)\,(p-2)}{4!}\right](\Delta^4 y_{-2})$$
$$+\left[\frac{(p+2)\,(p+1)\,p(p-1)\,(p-2)}{5!}\right](\Delta^5 y_{-2}) + \ldots \tag{4.16}$$

Gauss's backward formula is

$$y_p = y_0 + p(\Delta y_{-1}) + \left[\frac{(p+1)p}{2!}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{3!}\right](\Delta^3 y_{-2})$$
$$+\left[\frac{(p+2)\,(p+1)\,p(p-1)}{4!}\right](\Delta^4 y_{-2}) + \ldots \tag{4.17}$$

Adding Eqs. (4.16) and (4.17), we obtain

$$2y_p = 2y_0 + p(\Delta y_0 + \Delta y_{-1}) + \left[\frac{p(p-1)}{2!} + \frac{(p+1)^p}{2!}\right](\Delta^2 y_{-1})$$
$$+\left[\frac{(p+1)\,p(p-1)}{3!}\right](\Delta^3 y_{-1} + \Delta^3 y_{-2})$$
$$+\left[\frac{(p+2)\,(p+1)\,p(p-1)}{4!}\right](\Delta^4 y_{-2}) + \ldots$$

i.e.,
$$2y_p = 2y_0 + p(\Delta y_0 + \Delta y_{-1}) + \left[\frac{2p^2}{2!}\right](\Delta^2 y_{-1}) + \left[\frac{p(p^2-1^2)}{3!}\right]$$
$$(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \left[\frac{2p^2(p^2-1^2)}{4!}\right](\Delta^4 y_{-2}) + \ldots$$

$\therefore$
$$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \left[\frac{p^2}{2!}\right](\Delta^2 y_{-1}) + \left[\frac{p(p^2-1^2)}{3!}\right]$$
$$\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \left[\frac{p^2(p^2-1^2)}{4!}\right](\Delta^4 y_{-2}) + \ldots$$

This formula is known as Stirling's formula of interpolation, as it is used to interpolate the values of the function for the value $p\left(-\frac{1}{2} \le p \le \frac{1}{2}\right)$. Moreover, good estimates will be got if $-\frac{1}{4} \le p \le \frac{1}{4}$.

Note

1. This formula involves means of the odd differences just above and below the central line and even differences on this line as shown below:

$$y_0 \begin{pmatrix} \Delta y_{-1} \\ \cdots\cdots \\ \Delta y_0 \end{pmatrix} \ldots \Delta^2 y_{-1} \ldots \begin{pmatrix} \Delta^3 y_{-2} \\ \cdots\cdots \\ \Delta^3 y_{-1} \end{pmatrix} \ldots \Delta^4 y_{-2} \ldots \begin{pmatrix} \Delta^5 y_{-3} \\ \cdots\cdots \\ \Delta^5 y_{-2} \end{pmatrix} \Delta^6 y_{-3} \ldots\ldots \text{ central line}$$

$\uparrow$ Mean $\quad\quad \uparrow$ Mean $\quad\quad \uparrow$ Mean

2. p must lie between $-\frac{1}{2}$ to $\frac{1}{2}$ while using this formula.

4.2.4 Bessel's Formula

Gauss's forward formula is

$$y_p = y_0 + p(\Delta y_0) + \left[\frac{p(p-1)}{1\cdot 2}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{3!}\right](\Delta^3 y_{-1})$$
$$+\left[\frac{(p+1)\,p(p-1)\,(p-2)}{4!}\right](\Delta^4 y_{-2}) + \ldots \tag{4.18}$$

From the forward difference table

$$y_1 - y_0 = \Delta y_0$$
$$\therefore \quad y_0 = y_1 - \Delta y_0$$
$$\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$$
$$\therefore \quad \Delta^2 y\text{-}1 = \Delta^2 y_0 - \Delta^3 y_{-1}$$

Similarly, $\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^3 y_{-2}$ etc.

Equation (4.18) can be written as

$$y_p = \left(\frac{y_0}{2} + \frac{y_0}{2}\right) + p(\Delta y_0) + \frac{1}{2}\left[\frac{p(p-1)}{1\cdot 2}\right](\Delta^2 y_{-1}) + \frac{1}{2}\left[\frac{p(p-1)}{1\cdot 2}\right](\Delta^2 y_{-1})$$
$$+\left[\frac{(p+1)\,p(p-1)}{1\cdot 2\cdot 3}\right](\Delta^3 y_{-1}) + \ldots$$

Using the above results, we obtain

$$y_p = \frac{y_0}{2} + \frac{1}{2}(y_1 - \Delta y_0) + p\,(\Delta y_0) + \frac{1}{2}\left[\frac{p(p-1)}{2!}\right](\Delta^2 y_{-1})$$
$$+\frac{1}{2}\left[\frac{p(p-1)}{2!}\right](\Delta^2 y_0 - \Delta^3 y_{-1})$$
$$+\left[\frac{(p+1)\,p(p-1)}{3!}\right](\Delta^3 y_{-1}) + \ldots$$
$$= \frac{y_0 + y_1}{2} + \left(p - \frac{1}{2}\right)(\Delta y_0) + \frac{1}{2}\left[\frac{p(p-1)}{2!}\right](\Delta^2 y_{-1} + \Delta^2 y_0)$$
$$+\left[\frac{p(p-1)}{2!}\right]\left(-\frac{1}{2} + \frac{p+1}{3}\right)(\Delta^3 y_{-1}) + \ldots$$
$$= \frac{y_0 + y_1}{2} + \left(p - \frac{1}{2}\right)(\Delta y_0) + \left[\frac{p(p-1)}{2!}\right]\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+\left[\frac{\left(p - \frac{1}{2}\right)p(p-1)}{3!}\right](\Delta^3 y_{-1})$$
$$+\left[\frac{(p+1)\,p(p-1)\,(p-2)}{4!}\right]\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \ldots$$

This formula is known as *Bessel's formula*. While using this formula p must be in the range $\frac{1}{4} \le p \le \frac{3}{4}$.

Note

1. This formula involves odd differences below the central line and means of the even differences on and below the line as shown below:

$$\begin{array}{ccccc} \left.\begin{matrix} y_0 \\ y_1 \end{matrix}\right\} & \Delta y_0 & \left.\begin{matrix} \Delta^2 y_{-1} \\ \Delta^2 y_0 \end{matrix}\right\} & \Delta^3 y_{-1} & \left.\begin{matrix} \Delta^4 y_{-2} \\ \Delta^4 y_{-1} \end{matrix}\right\} \\ \uparrow & & \uparrow & & \uparrow \\ \text{Mean} & & \text{Mean} & & \text{Mean} \end{array}$$

2. In this formula, the coefficients of all the odd order differences are zero, when $p = 1/2$. This special case of Bessel's formula is known as the *formula for interpolating to halves*.

4.2.5 Laplace–Everett's Formula

Gauss's forward formula is

$$y_p = y_0 + \frac{p}{1}(\Delta y_0) + \left[\frac{p(p-1)}{1\cdot 2}\right](\Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)}{1\cdot 2\cdot 3}\right](\Delta^3 y_{-1})$$
$$+\left[\frac{(p+1)\,p(p-1)\,(p-2)}{1\cdot 2\cdot 3\cdot 4}\right](\Delta^4 y_{-2})$$
$$+\left[\frac{(p+2)\,(p+1)\,p(p-1)\,(p-2)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\right](\Delta^5 y_{-2}) + \ldots \tag{4.19}$$

Let us eliminate the odd differences in this formula, by using the relations.

$$\Delta y_0 = y_1 - y_0$$
$$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$$
$$\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \text{ etc.}$$

Then (4.19), becomes

$$y_p = y_0 + p(y_1 - y_0) + \left[\frac{p(p-1)}{1\cdot 2}\right](\Delta^2 y_{-1})$$
$$+\left[\frac{(p+1)(p-1)}{1\cdot 2\cdot 3}\right](\Delta^2 y_0 - \Delta^2 y_{-1})$$
$$+\left[\frac{(p+1)\,p(p-1)(p-2)}{1\cdot 2\cdot 3\cdot 4}\right](\Delta^4 y_{-2})$$
$$+\left[\frac{(p+2)(p+1)\,p(p-1)(p-2)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\right](\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \ldots$$
$$= (1-p)\,y_0 + py_1 + p(p-1)\left[\frac{1}{1\cdot 2} - \frac{p+1}{1\cdot 2\cdot 3}\right](\Delta^2 y_{-1})$$
$$+\left[\frac{(p+1)\,p(p-1)}{1\cdot 2\cdot 3}\right](\Delta^2 y_0)$$
$$+(p+1)\,p(p-1)(p-2)\left[\frac{1}{1\cdot 2\cdot 3\cdot 4} - \frac{(p+2)}{5}\right](\Delta^4 y_{-2})$$
$$+\left[\frac{(p+2)(p+1)\,p(p-1)(p-2)}{5!}\right](\Delta^4 y_{-1}) + \ldots$$
$$= (1-p)\,y_0 + py_1 - \left[\frac{p(p-1)(p-2)}{3!}\right](\Delta^2 y_{-1})$$
$$+\left[\frac{(p+1)\,p(p-1)}{3!}\right](\Delta^2 y_0)$$
$$-\left[\frac{(p+1)\,p(p-1)(p-2)(p-3)}{5!}\right](\Delta^4 y_{-2})$$
$$+\left[\frac{(p+2)(p+1)\,p(p-1)(p-2)}{5!}\right](\Delta^4 y_{-1}) + \ldots \tag{4.20}$$

In order to get a convenient form of the above formula, we put $1 - p = q$ i.e., $p = 1 - q$, and change the terms in (4.20) with a negative sign.
Hence, (4.20) becomes

$$y_p = qy_0 + \left[\frac{q(q^2-1^2)}{3!}\right](\Delta^2 y_{-1}) + \left[\frac{q(q^2-1^2)(q^2-2^2)}{5!}\right](\Delta^4 y_{-2}) + \ldots + py_1$$
$$+\left[\frac{p(p^2-1^2)}{3!}\right](\Delta^2 y_0) + \left[\frac{p(p^2-1^2)(p^2-2^2)}{5!}\right](\Delta^4 y_{-1}) + \ldots$$

This formula is known as *Laplace–Everett formula.*

Note

1. This formula is an extensively used interpolation formula and uses only even order differences as shown below:

$$\begin{array}{ccccccc} y_0 & \rightarrow & \Delta^2 y_{-1} & \rightarrow & \Delta^4 y_{-2} & \rightarrow & \Delta^6 y_{-3} \\ & — & & — & & — & \\ y_1 & & \Delta^2 y_0 & & \Delta^4 y_{-1} & & \Delta^6 y_{-2} \end{array}$$

2. This formula is to be used when p lies between 0 and 1; accurate results are got when $p \le \frac{3}{4}$.

Relation between Bessel's and Everett's Formula

There is a close relation between Bessel's formula and Everett's formula. It is possible to deduce one from the other by suitable rearrangement.

Let us start with Bessel's formula

$$y_p = y_0 + p(\Delta y_0) + \left[\frac{p(p-1)}{2!}\right]\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] + \left[\frac{p(p-1)(p-1/2)}{3!}\right]$$
$$(\Delta^3 y_{-1}) + \left[\frac{(p+1)\,p(p-1)(p-2)}{4!}\right]\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \ldots \tag{4.21}$$

In (4.21), let us express the odd order differences in terms of the corresponding lower even order differences

$$y_1 - y_0 = \Delta y_0$$
$$\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$$

We have

$$y_p = y_0 + p(y_1 - y_0) + \left[\frac{p(p-1)}{2!}\right]\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] + \left[\frac{p(p-1)(p-1/2)}{3!}\right]$$
$$(\Delta^2 y_0 - \Delta^2 y_{-1}) + \left[\frac{(p+1)\,p(p-1)(p-2)}{4!}\right]\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \ldots$$

On simplification, we have

$$y_p = (1-p)y_0 + \left[\frac{p(p-1)}{4} - \frac{(p-1)\,p(p-1/2)}{6}\right](\Delta^2 y_{-1}) + \ldots$$
$$+ py_1 + \left[\frac{p(p-1)}{4} + \frac{p(p-1)(p-1/2)}{6}\right](\Delta^2 y_0) + \ldots$$

i.e., $y_p = qy_0 + \left[\frac{q(q^2-1^2)}{3!}\right](\Delta^2 y_{-1}) + \ldots + py_1 + \left[\frac{p(p^2-1^2)}{3!}\right](\Delta^2 y_0) + \ldots$

which is *Everett's formula* truncated after second differences

Thus, we find that Bessel's formula truncated after third differences is equivalent to Everett's formula truncated after second differences.

EXAMPLE 4.16 Find the value of cos 51° 42′, by using Stirling's formula, from the table given.

x	50°	51°	52°	53°	54°
$y = \cos x$	0.6428	0.6293	0.6157	0.6018	0.5878

Solution

Here, $h = 1°, x_0 = 52°$

$$\therefore \quad p = \frac{51°\,42' - 52°}{1°}$$

$$= -\frac{18'}{60'}$$

$$= -0.3$$

The central difference table will be:

p	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
– 2	50°	0.6428				
			– 0.0135			
– 1	51°	0.6293		– 0.0001		
			– 0.0136		– 0.0002	
0	52°	0.6157		– 0.0003		0.0004
			– 0.0139		0.0002	
1	53°	0.6018		– 0.0001		
			– 0.0140			
2	54°	0.5878				

By Stirling's formula, we have

$$y_p = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \left[\frac{p^2}{2}\Delta^2 y_{-1}\right] + \left[\frac{p(p^2-1^2)}{3!}\right]$$

$$\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \left[\frac{p^2(p^2-1^2)}{4!}\right](\Delta^4 y_{-2}) + \ldots$$

$$= 0.6157 + (-0.3)\left[\frac{(-0.0136) + (-0.0139)}{2}\right] + \left[\frac{(-0.3)^2}{2}\right](-0.0003)$$

$$+ \left[\frac{(-0.3)(-0.3^2 - 1^2)}{3!}\right]\left[\frac{(-0.0002)(-0.0002)}{2}\right]$$

$$+ \left[\frac{(-0.3)^2(-0.3^2 - 1^2)}{4!}\right](0.0004)$$

$$= 0.619825$$

EXAMPLE 4.17 The following table gives the values of the probability integral $f(x) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-x^2}\,dx$ for certain equidistant values of x. Find the value of this integral, when $x = 0.5437$.

x	$f(x)$
0.51	0.5292437
0.52	0.5378987
0.53	0.5464641
0.54	0.5549392
0.55	0.5633233
0.56	0.5716157
0.57	0.5798158

Solution The central difference table is:

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0.51	0.5292437				
		0.0086550			
0.52	0.5378987		– 0.0000896		
		0.0085654		-7×10^{-7}	
0.53	0.5464641		– 0.0000903		0
		0.0084751		-7×10^{-7}	
0.54	0.5549392		– 0.0000910		0
		0.0087841		-7×10^{-7}	
0.55	0.5633233		– 0.0000917		-8.306×10^{-4}
		0.0082841		-8.313×10^{-4}	
0.56	0.5716157		– 0.000923		
		0.0082001			
0.57	0.5798158				

Here, $x_0 = 0.54$

Take $x = 0.5437$

Given $h = 0.01$

Stirling's formula is

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \left[\frac{p^2}{2!}\right](\Delta^2 y_{-1}) + \left[\frac{p(p^2-1)}{3!}\right]\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \left[\frac{p^2(p^2-1^2)}{4!}\right](\Delta^4 y_{-2}) + \ldots$$

i.e., $p = 0.37$

$$y_p = (0.5549392 + 0.37)\left(\frac{0.0084751 + 0.0087841}{2}\right) + \left(\frac{0.37^2}{2}\right)(-0.0000910) + \left[\frac{0.37\,(0.37^2-1)}{3!}\right]\left[\frac{(-7\times10^{-7}) + (-7\times10^{-7})}{2}\right] + \text{negligible terms}$$

$$= 0.5549392 + (3.192952 \times 10^{-0.3}) - (9.1 \times 10^{-0.5})$$

$$= 0.5549392 + 0.003192952 - 0.00000622895 + 0.000000037$$

$$y(0.5437) = 0.55812596$$

EXAMPLE 4.18 Employ Stirling's formula to compute $y_{12.2}$, from the following table ($y_x = 1 + \log_{10} \sin x$).

$x°$	10	11	12	13	14
$10^5 u_x$	23967	28060	31788	35209	38368

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	0.23967				
		0.04093			
11	0.28060		– 0.00365		
		0.03728		0.00058	
12	0.31788		– 0.00307		– 0.00013
		0.03421		– 0.00045	
13	0.35209		– 0.00062		
		0.03159			
14	0.38368				

Hence, $x = 12.2$

Take $x = 12$

$$\therefore \quad p = \frac{12.2 - 12}{1} = 0.2$$

Stirling's formula is

$$y_p = y_0 + \frac{p}{1}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \left(\frac{p^2}{2!}\right)(\Delta^2 y_{-1})\left[\frac{p(p^2-1)}{3!}\right]\left[\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2}\right] + \left[\frac{p^2(p^2-1^2)}{4!}\right](\Delta^4 y_{-2}) + \ldots$$

$$\therefore \quad y_{0.2} = 0.31788 + 0.2\left(\frac{0.03728 + 0.03421}{2}\right) + \left[\frac{(0.2)^2}{2}\right](-0.00307) + \left[\frac{(0.2)\,(0.2^2-1)}{6}\right]\left(\frac{0.00058 - 0.00045}{2}\right) + \left[\frac{(0.2)^2\,(0.2^2-1)}{24}\right](-0.00013)$$

$$= 0.31788 + 0.00715 - 0.00006 - 0.000002 + 0.0000002$$

$$= 0.32497$$

EXAMPLE 4.19 Apply Bessel's formula to compute $y_{12.3}$ from the following table, when $y_x = 2x + \cos x$

x	10	11	12	13	14
y_x	20.9848	22.9816	24.9781	26.9743	28.9702

Solution

Take $x_0 = 12$

Here, $h = 1$

$$\therefore \quad p = \frac{12.3 - 12}{1} = 0.3$$

Let us form the central difference table

p	x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
– 2	10	20.9848				
			1.9968			
– 1	11	22.9816		– 0.0003		
			1.9965		0	
0	12	24.9781		– 0.0003		0
			1.9962		0	
1	13	26.9743		– 0.0003		
			1.9959			
2	14	28.9702				

Bessel's formula is

$$y_p = \left[\frac{y_0+y_1}{2}\right] + (p-1/2)(\Delta y_0) + \left[\frac{p(p-1)}{2!}\right]\left(\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}\right)$$
$$+\left[\frac{(p-1/2)\,p(p-1)}{3!}\right](\Delta^3 y_{-1}) + \ldots$$
$$= \left(\frac{24.9781+26.9743}{2}\right) + (0.3-1/2)(1.9962) + \left[\frac{(0.3)(0.3-1)}{2!}\right]$$
$$\left(\frac{-0.0003-0.0003}{2}\right)+0$$
$$= 25.5769$$

EXAMPLE 4.20 Probability distribution function values of a normal distribution are given below:

x	0.2	0.6	1.0	1.4	1.8
y_x	0.39104	0.33322	0.24197	0.14973	0.07895

Using a suitable interpolation formula, find the value of $p(1.2)$.

Solution

Here, $h = 0.4$

Take $x_0 = 1$

$$\therefore \quad p = \frac{1.2-1}{0.4} = 0.5$$

Let us form the central difference table.

p	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
−2	0.2	0.39104				
			−0.05782			
−1	0.6	0.33322		−0.03343		
			−0.09125		0.03244	
0	1.0	$\boxed{0.24197}$		$\boxed{-0.00099}$		$\boxed{-0.00999}$
			$\boxed{-0.09224}$		$\boxed{0.02245}$	
1	1.4	$\boxed{0.14973}$		$\boxed{-0.02146}$		
			−0.07078			
2	1.8	0.07895				

At $x = 1.2$, we get $p = 0.5$.

This shows that Bessel's formula is the most suitable one. Bessel's formula is

$$y_p = \left[\frac{y_0+y_1}{2} + (p-1/2)\right](\Delta y_0) + \left[\frac{p(p-1)}{2!}\right]\left[\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}\right]$$
$$+\left[\frac{(p-1/2)\,p(p-1)}{3!}\right](\Delta^3 y_{-1}) + \left[\frac{(p+1)\,p(p-1)(p-2)}{4!}\right]$$
$$\left(\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}\right)+\ldots$$
$$= \left[\frac{0.24197+0.14973}{2}\right] + (0.5-1/2)(-0.09224) + \left[\frac{(0.5)(0.5-1)}{2!}\right]$$
$$\left(-\frac{0.00099+0.02146}{2}\right) + \left[\frac{(0.5-1/2)(0.5)(0.5-1)}{3!}\right](0.02245)$$
$$+\left[\frac{(0.5+1)(0.5)(0.5-1)(0.5-2)}{4!}\right]\left(\frac{-0.00999+0}{2}\right)$$
$$= 0.19445 \text{ approximately}$$

EXAMPLE 4.21 Given the table:

x	$\log x$
310	2.4913617
320	2.5051500
330	2.5185139
340	2.5314789
350	2.5440680
360	2.5563025

Find log 337.5, by Everett formula.

Solution

Here, $x = 337.5$ and $h = 10$

Take $x_0 = 330$

$$p = \frac{337.5-330}{10} = \frac{7.5}{10} = 0.75$$

As $p > \frac{1}{2}$ and $p = 0.75$, $q = 0.25$, we can use Laplace–Everett formula.

p	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
−2	310	2.4913617					
			0.0137883				
−1	320	2.5051500		−4244			
			0.0133639		225		
0	330	$\boxed{2.5185139}$		$\boxed{-3989}$		$\boxed{-25}$	
			0.0129650		230		8
1	340	$\boxed{2.5314789}$		$\boxed{-3759}$		$\boxed{-17}$	
			0.0125891		213		
2	350	2.5440680		−3546			
			0.0122345				
3	360	2.5563025					

Laplace–Everett's formula is

$$y_p = qy_0 + \left[\frac{q(q^2-1^2)}{3!}\right](\Delta^2 y_{-1}) + \left[\frac{q(q^2-1^2)(q^2-2^2)}{5!}\right](\Delta^4 y_{-2}) + \ldots + py_1$$

$$+ \left[\frac{p(p^2-1)}{3!}\right](\Delta^2 y_0) + \left[\frac{p(p^2-1)(p^2-2^2)}{5!}\right](\Delta^4 y_{-1}) + \ldots$$

$$\therefore \quad y_{(337.5)} = (0.25 \times 2.5185139) + \left[\frac{0.25\,(0.0625-1)}{6}\right](-0.0003989)$$

$$+ \left[\frac{0.25\,(0.0625-1)\,(0.0625-4)}{120}\right](-0.0000025)$$

$$+ (0.75 \times 2.5314789) + \left[\frac{0.75\,(0.5625-1)}{6}\right](-0.0003759)$$

$$+ \left[\frac{0.75\,(0.5625-1)\,(0.5625-4)}{120}\right](-0.0000017)$$

$$= 0.629628475 + 0.000015582 - 0.0000001923 + 1.898609175$$

$$+ 0.00002055 - 0.00000001598$$

$$= 2.5282736 \text{ nearly}$$

EXAMPLE 4.22 Everett's formula, find $f(1.15)$, given $f(1) = 1.000$, $f(1.10) = 1.049$, $f(1.20) = 1.096$, $f(1.30) = 1.140$.

Solution The difference table is

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
1.0	1.000			
		0.049		
1.1	$\boxed{1.049}$		$\boxed{-0.002}$	
		0.047		−0.001
1.2	$\boxed{1.096}$		$\boxed{-0.003}$	
		0.044		
1.3	1.14			

Everett's formula is

$$y_p = qy_0 + \left[\frac{q(q^2-1^2)}{3!}\right](\Delta^2 y_{-1}) + \left[\frac{q(q^2-1^2)(q^2-2^2)}{5!}\right](\Delta^4 y_{-2}) + py_1$$

$$+ \left[\frac{p(p^2-1^2)}{3!}\right](\Delta^2 y_0) + \left[\frac{p(p^2-1^2)(p^2-2^2)}{5!}\right](\Delta^4 y_{-1}) + \ldots$$

Here, $x_0 = 1.1$

Take $x = 1.15$

$$\therefore \quad p = \frac{x - x_0}{h}$$

$$= \frac{1.15 - 1.1}{0.1}$$

$$= 0.5$$

$$q = 0.5$$

$$\therefore \quad y(1.15) = (0.5)(1.049) + \left[\frac{(0.5)((0.5)^2-1)}{3!}\right](-0.002) + 0.5\,(1.096)$$

$$+ \left[\frac{0.5\,(0.5^2-1)}{3!}\right](-0.003)$$

$$= 0.548 + 0.0002 + 0.5245 + 0.0001$$

$$= 1.0728$$

EXAMPLE 4.23 Apply Everett's formula to obtain u_{25}, given $u_{20} = 2854$, $u_{24} = 3162$, $u_{28} = 3544$ and $u_{32} = 3992$.

Solution The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	2854			
		308		
24	$\boxed{3162}$		$\boxed{74}$	
		382		− 12
28	$\boxed{3544}$		$\boxed{62}$	
		448		
32	3992			

Laplace–Everett's formula is

$$y_p = qy_0 + \left[\frac{q(q^2-1^2)}{3!}\right](\Delta^2 y_{-1}) + \left[\frac{q(q^2-1^2)(q^2-2^2)}{5!}\right](\Delta^4 y_{-2}) + \ldots$$

$$+ py_1 + \left[\frac{p(p^2-1^2)}{3!}\right](\Delta^2 y_0) + \left[\frac{p(p^2-1^2)(p^2-2^2)}{5!}\right](\Delta^4 y_{-1}) + \ldots$$

Here, $x = 25$

Take $x_0 = 24$

$$\therefore \quad p = \frac{25-24}{4}$$

$$= 0.25$$

$$q = 0.75$$

$$\therefore \quad y(25) = 0.75\,(3162) + \left[\frac{0.7\,(0.7^2-1^2)}{3!}\right](74) + (0.25)\,(3544)$$

$$+ \left[\frac{(0.2)\,(0.5^2-1^2)}{3!}\right](62)$$

$$= 2367.097 + 884.45$$

$$= 3251.547$$

EXERCISES

4.21 Find cos 0.806595, by Stirling's formula, given

cos 0.8050 = 0.6931
cos 0.8065 = 0.6920
cos 0.8080 = 0.6909
cos 0.8055 = 0.6928

cos 0.8070 = 0.6917
cos 0.8060 = 0.6924
cos 0.8075 = 0.6913

[*Ans.* 0.6919]

4.22 Using Stirling's formula, find $f(1.63)$.

Given:

x	1.50	1.60	1.70	1.80	1.90
$f(x)$	17.609	20.412	23.045	25.527	27.875

[*Ans.* 21.21933]

4.23. Using Bessel's formula, Find $f(5)$.

Given:

x	0	4	8	12
$f(x)$	143	158	177	199

[*Ans.* 162.4]

4.24 Using Bessel's formula, estimate $3\sqrt{46.24}$, given the following table of $y = 3\sqrt{x}$.

x	41	45	49	53
y	3.4482	3.5569	3.6593	3.7563

[*Ans.* 3.58931]

4.25 Find y, when $x = 0.543$, using Stirling's formula.

Given:

x	0.51	0.52	0.53	0.54	0.55	0.56	0.57
y	0.529	0.538	0.546	0.555	0.563	0.572	0.580

[*Ans.* 0.58045]

4.26 Using Laplace–Everett's formula, find log 337.5, given:

x	310	320	330	340	350	360
log x	2.4913	2.5051	2.5185	2.5315	2.5441	2.5563

[*Ans.* 2.5283]

4.27 Using Everett's formula, find $y(34)$, given $y(20) = 11.4699$, $y(25) = 12.7834$, $y(30) = 13.7648$, $y(35) = 14.4982$, $y(40) = 15.0463$.

[*Ans.* 14.3684]

4.28 Apply Bessel's formula to get the value of $y(45)$, given:

x	40	44	48	52
y	51.08	63.24	70.88	79.84

[*Ans.* 65.0175]

4.29 Using Bessel's formula, estimate $(46.24)^{1/3}$, given:

x	60	61	62	63	64	65
$y(x)$	7782	7853	7924	7993	8062	8129

[*Ans.* 3.5893]

4.30 Apply central difference formula to find $f(12)$, given:

x	5	10	15	20
$f(x)$	54.14	60.54	67.72	75.88

[*Ans.* 63.30]

4.31 Using a suitable interpolation formula, obtain the value of y, when $x = 15$, given:

x	10	12	14	16	18	20
$y(x)$	51.21	60.24	75.32	96.02	119.78	151.45

[*Ans.* 85.189]

4.32 The values of an elliptic integral $F(\phi)$ for certain values of the amplitude ϕ are given in the table below. Find the value of the integral, when $\phi = 23.5°$, by using Bessel's formula.

ϕ	$F(\phi)$
21°	0.370634373
22°	0.388705151
23°	0.406834931
24°	0.425026420
25°	0.443282329
26°	0.461605362

[*Ans.* 0.415922792]

4.33 Using Stirling's formula, find y_{35}, given $y_{20} = 512$, $y_{30} = 439$, $y_{35} = 395$, $y_{40} = 346$, $y_{50} = 243$.

[*Ans.* $y_{35} = 395$]

4.34 Employ Laplace–Everett's formula to evaluate $y(26)$ and $y(27)$, given:

x	15	20	25	30	35	40
$y(x)$	12.849	16.351	19.524	22.396	24.999	27.356

[*Ans.* 20.121, 20.707]

4.35 Using Bessel's formula, obtain y_{25}; given $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$.

[*Ans.* 32.945]

4.3 INTERPOLATION WITH UNEQUAL INTERVALS

So far we have discussed several interpolation formulae which are being applicable only for equally-spaced values of the argument. If the values of x are given at unequal intervals, it is convenient to introduce the idea of divided differences. Before deriving this formula, we shall first define these differences.

Let the function $y = f(x)$, take the values $f(x_0)$, $f(x_1)$, $f(x_2)$ ... $f(x_n)$, corresponding to the values $x_0, x_1, x_2 \ldots x_n$ of the argument x, where $x_1 - x_0$, $x_2 - x_1 \ldots x_n - x_{n-1}$ need not be necessarily equal.

The first divided differences of $f(x)$ for the arguments x_0 is defined as

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and is denoted by

$$f(x_0, x_1) \text{ or by } \Delta f(x_0) \text{ or by } (x_0, x_1)$$

$$\therefore \qquad (x_0, x_1) = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly,

$$(x_1, x_2) = \frac{y_2 - y_1}{x_2 - x_1}$$

and

$$(x_2, x_3) = \frac{y_3 - y_2}{x_3 - x_2} \text{ etc.}$$

The second difference for x_0, x_1, x_2 is defined as

$$(x_0, x_1, x_2) = \frac{(x_1, x_2) - (x_0, x_1)}{x_2 - x_0}$$

In general,

$$(x_0, x_1 \ldots x_n) = \left[\frac{1}{n!\,h^n}\right](\Delta^n y_0)$$

for equally-spaced values of x.

Note

1. The divided differences are symmetrical in their arguments, i.e., $(x_0, x_1) = (x_1, x_0)$.
2. The nth divided differences of a polynomial of the nth degree are constant, i.e., $\Delta^n y_0$ will be constant.

4.3.1 Newton's Divided Difference Formula

Let $y_0, y_1 \ldots y_n$ be the values of $y = f(x)$, corresponding to the arguments $x_0, x_1 \ldots x_n$.

Then, from the definition of divided differences, we have

$$(x, x_0) = \frac{y - y_0}{x - x_0}$$

So that

$$y = y_0 + (x - x_0)\,(x, x_0)$$

Also

$$(x, x_0, x_1) = \frac{(x, x_0) - (x_0, x_1)}{x - x_1} \qquad (4.22)$$

i.e.,

$$(x, x_0) = (x_0, x_1) + (x - x_1)\,(x, x_0, x_1)$$

Substitute (x, x_0) in (4.22), we obtain

$$y = y_0 + (x - x_0)\,(x_0, x_1) + (x - x_0)\,(x - x_1)\,(x, x_0, x_1) \qquad (4.23)$$

Proceeding in this manner, we get

$$y = f(x) = y_0 + (x - x_0)\,(x_0, x_1) + (x - x_0)\,(x - x_1)\,(x_0, x_1, x_2) + (x - x_0)\,(x - x_1)$$
$$(x - x_2)\,(x_0, x_1, x_2) + \ldots + (x - x_0)\,(x - x_1) \ldots (x - x_n)\,(x, x_0, x_1 \ldots x_n) \qquad (4.24)$$

which is called *Newton's general interpolation formula with divided differences.*

EXAMPLE 4.24 Given the values:

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Evaluate $f(9)$, by using Newton's divided difference formula.

Solution The divided difference table is:

x	y	Δ	$\Delta^2 y$	$\Delta^3 y$
5	$\boxed{150}$			
		121		
7	392		24	
		265		1
11	1452		32	
		457		1
13	2366		42	
		709		
17	5202			

Newton's divided difference interpolation formula is

$$y = f(x) = y_0 + (x - x_0)\,(x_0, x_1) + (x - x_0)\,(x - x_1)\,(x_0, x_1, x_2) + (x - x_0)\,(x - x_1)$$
$$(x - x_2)\,(x_0, x_1, x_2, x_3) + \ldots + (x - x_0)\,(x - x_1) \ldots (x - x_n)\,(x, x_0, x_1 \ldots x_n)$$

Taking $x = 9$, we have

$$f(9) = 150 + (9 - 5) \times 121 + (9 - 5)\,(9 - 7) \times 24$$
$$+ (9 - 5)\,(9 - 7)\,(9 - 11) \times 1$$
$$= 150 + 484 + 192 - 16$$
$$= 810$$

EXAMPLE 4.25 Certain corresponding values of x and $\log_{10} x$ are given below.

x	300	304	305	307
$\log_{10} x$	2.4771	2.4829	2.4843	2.4871

Find $\log_{10} 310$, by Newton's divided difference formula.

Solution The divided difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
300	2.4771			
		$\frac{2.4829 - 2.4771}{304 - 300} = 0.00145$		
304	2.4829		0.00001	
		$\frac{2.4843 - 2.4829}{305 - 304} = 0.0014$		0.00000142
305	2.4843		0	
		$\frac{2.4871 - 2.4843}{307 - 305} = 0.0014$		
307	2.4871			

Newton's divided difference interpolation formula is

$$y = f(x) = y_0 + (x - x_0)\,(x_0, x_1) + (x - x_0)\,(x - x_1)\,(x_0, x_1, x_2) + (x - x_0)$$
$$(x - x_1)\,(x - x_2)\,(x_0, x_1, x_2, x_3) + \ldots + (x - x_0)$$
$$(x - x_1) \ldots (x - x_n)\,(x, x_0, x_1, x_2 \ldots x_n)$$

$$f(310) = 2.4771 + (310 - 300)\,(0.00145) + (310 - 300)\,(310 - 304)$$
$$(0.00001) + (310 - 300)\,(310 - 304)\,(310 - 305)\,(0.00000142)$$
$$= 2.49228$$

EXAMPLE 4.26 Find the third difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$.

Solution Given $y = f(x) = x^3 - 2x$

when,

$$x = 2, f(2) = 4$$

$$x = 4, f(4) = 56$$

$$x = 9, f(9) = 711$$

$$x = 10, f(10) = 980$$

The divided difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	4			
		26		
4	56		15	
		131		1
9	711		23	
		269		
10	980			

$\therefore$

$$\Delta^3 y = 1$$

EXAMPLE 4.27 Using the following table, find $y'(4)$ and the maximum value of y.

x	0	1	2	5
y	2	3	12	147

Solution Since x values are not equally-spaced, so we apply Newton's divided difference formula.

The divided difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	2			
		1		
1	3		4	
		9		1
2	12		9	
		45		
5	147			

Newton's divided difference interpolation formula is

$$f(x) = y_0 + (x - x_0)(x_0, x_1) + (x - x_0)(x - x_1)(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)(x_0, x_1, x_2, x_3) + \ldots + (x - x_0)(x - x_1) \ldots (x - x_n)(x, x_0, x_1, \ldots x_n) \quad \text{(i)}$$

Here,

$$(x_0, x_1) = 1$$

$$(x_0, x_1, x_2) = 4$$

$$(x_0, x_1, x_2, x_3) = 1$$

and

$$y_0 = 2$$

Also

$$x_0 = 0$$

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 5$$

Substituting these values in (i), we get

$$y = f(x) = 2 + (x - 0)1 + (x - 0)(x - 1)4 + (x - 0)(x - 1)(x - 2) \cdot 1$$

$$y(x) = x^3 + x^2 - x + 2$$

$$y'(x) = 3x^2 + 2x - 1$$

$$y'(4) = 55$$

We know that $y(x)$ is maximum if $y'(x) = 0$ and $y''(x) < 0$.

$\therefore$ $y'(x) = 3x^2 + 2x - 1 = 0 \Rightarrow (-1, 1/3)$

Also $y''(x) = 6x + 2 \Rightarrow y'(-1) < 0.$

$\therefore$ The maximum value of y occur at -1.

Maximum value of y is

$$y(-1) = (-1)^3 + (-1)^2 - (1) + 2 = 3$$

EXAMPLE 4.28 Find the interpolating polynomial, for the following data:

x	0	1	2	4
y	1	1	2	5

Solution The divided difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		0		
1	1		$\frac{1}{2}$	
		1		$-\frac{1}{12}$
2	2		$\frac{1}{6}$	
		$\frac{3}{2}$		
4	5			

Newton's divided difference formula gives

$$y = f(x) = 1 + (x-0)(0) + (x-0)(x-1)(1/2) + (x-0)(x-1)(x-2)(-1/12)$$

$$y(x) = -\frac{x^3}{12} + \frac{3x^2}{4} - \frac{2}{3}x + 1$$

EXERCISES

4.36 Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$ and $\log_{10} 661 = 2.8202$, find by divided difference formula the value of $\log_{10} 656$.

[*Ans.* 2.8168]

4.37 By means of Newton's divided difference formula, find the value of $f(8)$, given:

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

[*Ans.* 448]

4.38 If $f(x) = \frac{1}{x^2}$, find the divided differences $f(a, b)$, $f(a, b, c)$ and $f(a, b, c, d)$.

$$\left[\text{Ans. } -\frac{(a+b)}{a^2b^2}, \frac{ab+bc+ca}{a^2b^2c^2}, -\frac{(abc+bcd+acd+abd)}{a^2\,b^2\,c^2d^2}\right]$$

4.39 Determine $f(x)$ as a polynomial in x for the following data.

x	– 4	– 1	0	2	5
$f(x)$	1245	33	5	9	1335

[*Ans.* $3x^4 - 5x^3 + 6x^2 - 14x + 5$]

4.40 Find the function $f(x)$ from the following table.

x	0	1	2	4	5	7
$f(x)$	0	0	– 12	0	600	7308

[*Ans.* $x - (x-1)(x-4)(x^2 + 2x - 5)$]

4.41 Construct a table of divided difference for the following data.

x	0	2	3	5	6
y	1	19	55	241	415

[*Ans.* 75]

4.42 Find $y(x = 20)$, by using Newton's divided differences, given:

x	12	18	22	24	32
$y(x)$	146	836	19481	2796	9236

[*Ans.* 1305.36]

4.43 Find the polynomial equation of degree four passing through the points (8, 1515), (7, 778), (5, 138), (4, 43) and (2, 3).

[*Ans.* $y = x^4 - 10x^3 + 36x^2 - 36x - 5$]

4.44 Using divided difference table, find $f(x)$, which takes the values 1, 4, 40, 85 as $x = 0, 1, 3, 4$.

[*Ans.* $x^3 + x^2 + x + 1$]

4.45 Find $y(x = 5.60275)$ from the following table.

x	5.600	5.602	5.605	5.607	5.608
y	0.77556	0.776826	0.778712	0.779965	0.780591

[*Ans.* 0.777298926]

4.3.2 Lagrange's Interpolation Formula

Lagrange's interpolation formula can be used only when the values of the independent variable x are unequally-spaced, also the difference of y are not small. Now we shall derive such a formula as folllows.

Let $y = f(x)$ be a function, which takes the values $y_0, y_1, y_2 \ldots y_n$ corresponding to $x_0, x_1, \ldots x_n$. Since then are $n + 1$ values of y corresponding to $n + 1$ values of x, we can represent the function $f(x)$ by a polynomial in x of degree n.

Let this polynomial be of the form

$$y = f(x) = a_0(x - x_1)(x - x_2) \ldots (x - x_n) + a_1(x - x_0)(x - x_2)(x - x_3) \ldots (x - x_n) + a_2(x - x_0)(x - x_1)(x - x_3) \ldots (x - x_n) + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}) \quad (4.25)$$

Taking $x = x_0$ and $y = y_0$ in Eq. (4.25), we obtain

$$y_0 = f(x_0) = a_0(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)$$

i.e.,
$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)}$$

Similarly, taking $x = x_1$ and $y = y_1$ in Eq. (4.25), we obtain

$$y_1 = f(x_1) = a_0(0) + a_1(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)$$

i.e.,
$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_3)}$$

In like manner, we obtain that

$$a_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_4)\dots(x_3 - x_n)}$$

$$a_3 = \frac{y_3}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)\dots(x_3 - x_n)}$$

$$\vdots \qquad \vdots$$

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1)(x_n - x_2)\dots(x_n - x_{n-1})}$$

Substituting the values of $a_0, a_1, a_2, \dots a_n$ in Eq. (4.25), we get

$$y = f(x) = \left[\frac{(x - x_1)(x - x_2)\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)}\right](y_0)$$
$$+ \left[\frac{(x - x_0)(x - x_2)\dots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)}\right](y_1)$$
$$\vdots$$
$$+ \left[\frac{(x - x_0)(x - x_1)\dots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\dots(x_n - x_{n-1})}\right](y_n)$$

This is known as *Lagrange's interpolation formula for unequal intervals.*

Note

Lagrange's formula can also be applied for equal intervals. But compared to other equal interval interpolation formula, it is difficult to apply.

EXAMPLE 4.29 Find the value of $f(x) = x \log x$ at $x = 5$ from the following data, by using Lagrane'gs formula.

x	3	7	9	12
y	1.4313	5.9156	8.5881	12.9501

Solution Since the independent variable x are at unequal interval, by Lagrange's interpolation formulae, we have

$$y = f(x) = \left[\frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}\right](y_0)$$
$$+ \left[\frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}\right](y_1)$$
$$+ \left[\frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}\right](y_2)$$
$$+ \left[\frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}\right](y_3)$$

Here,
$$x_0 = 3, x_1 = 7, x_2 = 9, x_3 = 12$$
$$y_0 = 1.4313, \quad y_1 = 5.9156, \quad y_2 = 8.5881, y_3 = 12.9501$$

When $x = 5$, we obtain

$$y = \left[\frac{(5-7)(5-9)(5-12)}{(3-7)(3-9)(3-12)}\right](1.4313)$$
$$+ \left[\frac{(5-3)(5-9)(5-12)}{(7-3)(7-9)(7-12)}\right](5.9156)$$
$$+ \left[\frac{(5-3)(5-7)(5-12)}{(9-3)(9-7)(9-12)}\right](8.5881)$$
$$+ \left[\frac{(5-3)(5-7)(5-9)}{(12-3)(12-7)(12-9)}\right](12.9501)$$

$$y(5) = 0.371077 + 8.28184 + 6.67963 + 1.534826$$
$$= 3.508113$$

EXAMPLE 4.30 Given $y(15) = 150$, $y(7) = 392$, $y(11) = 1452$, $y(13) = 2366$ $y(17) = 5202$, find $y(9)$ by using Lagrange's formula.

Solution

Here $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17, y_0 = 150, y_1 = 392, y_2 = 1452,$ $y_3 = 2366, y_4 = 5202$

Lagrange's formula is

$$y = f(x) = \left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right](y_0)$$
$$+\left[\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right](y_1)$$
$$+\left[\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right](y_2)$$
$$+\left[\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right](y_3)$$

Taking $x = 9$, we have

$$y = f(a) = \left[\frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}\right](150)$$
$$+\left[\frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}\right](392)$$
$$+\left[\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}\right](1452)$$
$$+\left[\frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}\right](2366)$$
$$+\left[\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}\right](5202)$$
$$= -16.66 + 209.066 + 1290.666 - 788.666 + 115.6$$
$$= 810.006$$

EXAMPLE 4.31 Find the parabola passing through the points (0, 1), (1, 3) and (3, 55), using Lagrange's interpolation formula.

Solution

Here, $x_0 = 0, x_1 = 1, x_2 = 3, y_0 = 1, y_1 = 3, y_2 = 55$

By Lagrange's formula

$$y = f(x) = \left[\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right](y_0)$$
$$+\left[\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right](y_1)$$

$$+\left[\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right](y_2)$$
$$+\left[\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right](y_3)$$

Substituting the above value

$$y = f(x) = \left[\frac{(x-1)(x-3)}{(0-1)(0-3)}\right](1) + \left[\frac{(x-0)(x-3)}{(1-0)(1-3)}\right](3) + \left[\frac{(x-0)(x-1)}{(3-0)(3-1)}\right](55)$$
$$= \left[\frac{(x-1)(x-3)}{3}\right] + \left[\frac{x(x-3)}{(-2)}\right](3) + \left[\frac{x(x-1)}{6}\right](55)$$
$$= \left[\frac{x^2-3x-x+3}{3}\right] - \left[\frac{3x^2-9x}{2}\right] + \left[\frac{55x^2-55x}{6}\right]$$
$$= \frac{2x^2-6x-2x+6-9x^2+27x+55x^2-55x}{6}$$
$$= \frac{48x^2-36x+6}{6}$$
$$y = 8x^2 - 6x + 1$$

EXERCISES

4.46 Using Lagrange's interpolation formula, find $y(10)$, from the following table.

x	5	6	9	11
y	12	13	14	16

[*Ans.* 14.66666]

4.47 Given $u_1 = 22, u_2 = 30, u_4 = 82, u_7 = 106, u_8 = 206$, find u_6.

[*Ans.* 83.515]

4.48 Find $f(0)$, given

x	−1	−2	2	4
$f(x)$	−1	−9	11	69

[*Ans.* 1]

4.49 Interpolate y at $x = 5$, given

x	1	2	3	4	7
y	2	4	8	16	128

[*Ans.* 32.9]

4.50 Use Lagrange's formula to find the value of y at $x = 6$ from the following data.

$$x = 3, 7, 9, 10$$

and the corresponding value of

$$y = 168, 120, 72, 63$$

[*Ans.* 147]

4.51 The following data are taken from the steam table.

Temperature °C	100	150	180	200
Density (kg/m^3)	958	917	887	865

Use Lagrange's formula to find the density of saturated water when the temperature of the saturated steam is 110°C.

[*Ans.* 951]

4.52 The observed values of a function $f(x)$ for different values of x are given in the following table,

x	14	17	31	35
$f(x)$	68.7	64.0	44.0	39.1

Find the value of $f(x)$ corresponding to $x = 27$, by applying Lagrange's formula.

[*Ans.* 49.3]

4.3.3 Inverse Interpolation

So far we have studied, given a set of values of x and y, we required to find the value of y corresponding to the value of x not present in the table. Sometimes we need to find the value of x corresponding to the value of y which is not in the table.

For this purpose, Lagrange's interpolation will be very useful. We interchange x and y in the Lagrange's interpolation formula, i.e., take y as independent variable and x as dependent variable. Hence, we obtain.

$$x = \left[\frac{(y-y_1)(y-y_2)\ldots\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots\ldots(y_0-y_n)}\right](x_0)$$

$$+\left[\frac{(y-y_0)(y-y_2)\ldots\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots\ldots(y_1-y_n)}\right]$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\left[\frac{(y-y_0)(y-y_1)\ldots\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1')\ldots\ldots(y_n-y_{n-1})}\right](x_n)$$

EXAMPLE 4.32 Use Lagrange's inverse interpolation formula to obtain the value of t, when $A = 85$ from the following table.

t	2	5	8	14
A	94.8	87.9	81.3	68.7

Solution
Here,

$$x_0 = 2 \quad x_1 = 5 \quad x_2 = 8 \quad x_3 = 14$$

$$y_0 = 94.8 \quad y_1 = 87.9 \quad y_2 = 81.3 \quad y_3 = 68.7$$

The inverse interpolation formula is

$$x(y) = \left[\frac{(y-y_1)(y-y_2)\ldots\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots\ldots(y_0-y_n)}\right](x_0)$$

$$+\left[\frac{(y-y_0)(y-y_2)\ldots\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots\ldots(y_1-y_n)}\right](x_1)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\left[\frac{(y-y_0)(y-y_1)\ldots\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots\ldots(y_n-y_{n-1})}\right](x_n)$$

Taking $y = 85$, we have

$$x(85) = \left[\frac{(85-87.9)(85-81.3)(85-68.7)}{(94.8-87.9)(94.8-81.3)(94.8-68.7)}\right](2)$$

$$+\left[\frac{(85-94.8)(85-81.3)(85-68.7)}{(87.9-94.8)(87.9-81.5)(87.9-68.7)}\right](5)$$

$$+\left[\frac{(85-94.8)(85-87.9)(85-68.7)}{(81.3-94.8)(81.3-87.9)(81.3-68.7)}\right](8)$$

$$+\left[\frac{(85-94.8)(85-87.9)(85-81.3)}{(68.7-94.8)(68.7-87.9)(68.7-81.3)}\right](14)$$

$$= 6.5928$$

EXAMPLE 4.33 Lagrange's formula inversely to obtain the value of ϕ, when $F(\phi) = 0.3887$, from the following table.

ϕ	21°	23°	25°
$F(\phi)$	0.3706	0.4068	0.4433

Solution

Here, $\phi_0 = 21°$ $\phi_1 = 23°$ $\phi_2 = 25°$

$F_0(\phi) = 0.3706$ $F_1(\phi) = 0.4068$ $F_2(\phi) = 0.4433$

The inverse interpolation formula is

$$x = \left[\frac{(y-y_1)(y-y_2).....(y-y_n)}{(y_0-y_1)(y_0-y_2).....(y_0-y_n)}\right](x_0)$$

$$+\left[\frac{(y-y_0)(y-y_2).....(y-y_n)}{(y_1-y_0)(y_1-y_2).....(y_1-y_n)}\right](x_1)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\left[\frac{(y-y_0)(y-y_1)(y-y_2).....(y-y_{n-1})}{(y_n-y_0)(y_n-y_1).....(y_n-y_{n-1})}\right](x_n)$$

Taking $y = 0.3887$, we obtain

$$x(0.3887) = \phi = \left[\frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)}\right](21)$$

$$+\left[\frac{(0.3887-0.3706)(0.3887-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)}\right](23)$$

$$+\left[\frac{(0.3887-0.3706)(0.3887-0.4068)}{(0.4433-0.3706)(0.4433-0.4068)}\right](25)$$

$$= 7.885832 + 17.202739 - 3.086525$$

$$= 22.0020°$$

$$= 22°$$

EXAMPLE 4.34 Find the value of x corresponding to $y = 1000$, by using inverse interpolation, from the given data.

x	3	5	7	9
y	6	24	58	108

Solution

Here, $x_0 = 3,\ x_1 = 5,\ x_2 = 7,\ x_3 = 9$

$y_0 = 6,\ y_1 = 24,\ y_2 = 58,\ y_3 = 108$

When $y = 100$, substituting above values in inverse interpolation formula, we get

$$x = \left[\frac{(100-24)(100-58)(100-108)}{(6-24)(6-58)(6-108)}\right](3)$$

$$+\left[\frac{(100-6)(100-58)(100-108)}{(24-6)(24-58)(24-108)}\right](5)$$

$$+\left[\frac{(100-6)(100-24)(100-108)}{(58-6)(58-24)(25-108)}\right](7)$$

$$+\left[\frac{(100-6)(100-24)(100-58)}{(108-6)(108-24)(108-58)}\right](9)$$

$$= 0.802413 + (-3.071895) + 4.52561 + 6.30352$$

$$= 8.559657$$

EXERCISES

4.53 Apply Lagrange's formula inversely to obtain the root of the equation $f(x) = 0$, given $f(30) = -30, f(34) = -13, f(38) = 3, f(42) = 18$. [*Ans.* 37.23]

4.54 Find the age-corresponding to the annuity value 13.6, given

Age (x)	30	35	40	45	50
Annuity value (y)	15.9	14.9	14.1	13.3	12.5

[*Ans.* 43]

4.55 The following table gives the values of the probability integral

$$f(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\,dx$$

corresponding to certain values of x. For what value of x is this integral equal to 0.5.

x	0.46	0.47	0.48	0.49
$f(x)$	0.4846555	0.4937452	0.5027498	0.5116683

[*Ans.* 3.898668]

4.56 Apply inverse interpolation, from the data given below, find the value of x, when $y = 13.5$.

x	93.0	96.2	100.0	104.2	108.7
y	11.38	12.80	14.70	17.07	19.91

[*Ans.* 97.6557503]