Changes in the Right-Hand Side :

Example : TOYCO assembles three types of toys **trains, trucks, and cars** using **three operations**. The daily limits on the available times for the three operations are 430,460, and 420 minutes, respectively, and the revenues per unit of toy train, truck, and car are 3$, 2$, and 5$, respectively.

The assembly times per train at the three operations are 1, 3, and 1 minutes, respectively.

The corresponding times per truck and per car are $(2, 0, 4)$ and $(1, 2, 0)$ minutes (a zero time indicates that the operation is not used).

Mathematical Modeling : Let x_1, x_2, x_3 are the daily number of units produced of trains, trucks and cars, then following is the LP model for the situation

$$\text{Maximize} \qquad z = 3x_1 + 2x_2 + 5x_3 \qquad \text{subject to}$$

$$x_1 + 2x_2 + x_3 \leq 430 \quad \text{(Operation 1)} \qquad 3x_1 + 2x_3 \leq 460 \quad \text{(Operation 2)}$$

$$x_1 + 4x_2 \geq 420 \quad \text{(Operation 3)}$$

$$x_1, x_2, x_3 \geq 0$$

The starting tableau of the problem is

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	-2	-5	0	0	0	0
x_4	1	2	1	1	0	0	430
x_5	3	0	2	0	1	0	460
x_4	1	4	0	0	0	1	420

and the optimal tableau is

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

Recommendation : The solution recommends manufacturing 100 trucks and 230 cars but no trains. The associated revenue is 1350$.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

Determination of Dual Prices : The constraints of the optimal tableau can be written as

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 430 - x_4 && \text{(Operation 1)} \\ 3x_1 + 2x_3 &= 460 - x_5 && \text{(Operation 2)} \\ x_1 + 4x_2 &= 420 - x_6 && \text{(Operation 3)} \end{aligned}$$

With this representation, the slack variables have the same units (minutes) as the operation times. Thus, we can say that a one minute decrease in the slack variable is equivalent to a one-minute increase in the operation time.
The z-equation in the optimal tableau is $z + 4x_1 + x_4 + x_5 + 0x_6 = 1350$.

The z-equation in optimal tableau

$$z = 1350 - 4x_1 + 1(-x_4) + 2(-x_5) + 0(-x_6)$$

$$\begin{aligned} z = 1350 - 4x_1 \quad & + \quad 1 \times (\text{increase in operation 1 in time}) \\ & + \quad 2 \times (\text{increase in operation 2 in time}) \\ & + \quad 0 \times (\text{increase in operation 3 in time}) \end{aligned}$$

This equation reveals that

1. a one-minute increase in operation 1 time increases z by 1$,
2. a one-minute increase in operation 2 time increases z by 2$, and
3. a one minute increase in operation 3 time does not change z.

Resource	Slack variable	z-equation Coefficient	Dual price
Operation 1	x_4	1	1$/min
Operation 2	x_5	2	2$/min
Operation 3	x_6	0	0$/min

Resource	Slack variable	z-equation Coefficient	Dual price
Operation 1	x_4	1	1\$/min
Operation 2	x_5	2	2\$/min
Operation 3	x_6	0	0\$/min

The **zero dual price** for operation 3 means that there is no economic advantage in allocating more production time to this operation. The result makes sense because the resource is already abundant, as is evident by the fact that the slack variable associated with Operation 3 is positive $(= 20)$ in the optimum solution.

As for each of Operations 1 and 2, a one minute increase will improve revenue by 1\$ and 2\$, respectively. The dual prices also indicate that, when allocating additional resources, Operation 2 may be given higher priority because its dual price is twice as much as that of Operation 1.

Determination of the feasibility ranges : The feasibility ranges in which they remain valid are to be determined.
Let D_1 , D_2, and D_3 be the changes (positive or negative) in the daily manufacturing time allocated to operations 1,2, and 3, respectively. The model can be written as follows

$$\text{Maximize} \quad z = 3x_1 + 2x_2 + 5x_3 \quad \text{subject to}$$

$$\begin{aligned}
x_1 + 2x_2 + x_3 &\leq 430 + D_1 && \text{(Operation 1)} \\
3x_1 + 2x_3 &\leq 460 + D_2 && \text{(Operation 2)} \\
x_1 + 4x_2 &\leq 420 + D_3 && \text{(Operation 3)}, \qquad x_1, x_2, x_3 \geq 0
\end{aligned}$$

The procedure is based on recomputing the optimum simplex tableau with the modified right-hand side and then deriving the conditions that will keep the solution feasible that is, the right-hand side of the optimum tableau remains nonnegative.The starting tableau is

							Solution			
Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS	D_1	D_2	D_3
z	-3	-2	-5	0	0	0	0	0	0	0
x_4	1	2	1	1	0	0	430	1	0	0
x_5	3	0	2	0	1	0	460	0	1	0
x_4	1	4	0	0	0	1	420	0	0	1

								Solution		
Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS	D_1	D_2	D_3
z	-3	-2	-5	0	0	0	0	0	0	0
x_4	1	2	1	1	0	0	430	1	0	0
x_5	3	0	2	0	1	0	460	0	1	0
x_4	1	4	0	0	0	1	420	0	0	1

We will obtained the same tableau but it is augmented and the columns of slack variables and D_1, D_2 and D_3 are identical to their corresponding slacks. The optimum tableau is

								Solution		
Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS	D_1	D_2	D_3
z	4	0	0	1	2	0	1350	1	2	0
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	$\frac{1}{2}$	$-\frac{1}{4}$	0
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	0	$\frac{1}{2}$	0
x_6	2	0	0	-2	1	1	20	-2	1	1

								Solution		
Basic	x_1	x_2	x_3	x_4	x_5	x_6	RHS	D_1	D_2	D_3
z	4	0	0	1	2	0	1350	1	2	0
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	-	$\frac{1}{4}$0	100	$\frac{1}{2}$	$-\frac{1}{2}$	0
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	0	$\frac{1}{2}$	0
x_6	2	0	0	-2	1	1	20	-2	1	1

The new optimal tableau provides the following optimal solution

$$\begin{aligned} z &= 1350 + D_1 + 2D_2 \\ x_2 &= 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2 \\ x_3 &= 230 + \frac{1}{2}D_2, \qquad x_6 = 20 - 2D_1 + D_2 + D_3 \end{aligned}$$

As shown earlier, the new z-value confirms that the dual prices for operations 1,2, and 3 are 1,2, and 0, respectively. Apply the feasibility condition we obtain the following

$$\begin{aligned} x_2 &= 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2 \geq 0, \qquad x_3 = 230 + \frac{1}{2}D_2 \geq 0 \\ x_6 &= 20 - 2D_1 + D_2 + D_3 \geq 0 \end{aligned}$$

Algebraic Sensitivity Analysis

Any simultaneous changes $D_1 D_2$, and D_3 that satisfy these inequalities will keep the solution feasible.

$$
\begin{aligned}
x_2 &= 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2 \geq 0 \\
x_3 &= 230 + \frac{1}{2}D_2 \geq 0 \qquad x_6 = 20 - 2D_1 + D_2 + D_3 \geq 0
\end{aligned}
$$

Suppose that the manufacturing time available for operations 1,2, and 3 are 480,440, and 410 minutes respectively. Then, $D_1 = 480 - 430 = 50$, $D_2 = 440 - 460 = 50$, and $D_3 = 410 - 420 = -10$. Substituting in the feasibility conditions, we get

$$
\begin{aligned}
x_2 &= 100 + \frac{1}{2}(50) - \frac{1}{4}(-20) = 130 > 0 \qquad \text{(feasible)} \\
x_3 &= 230 + \frac{1}{2}(-20) > 0 \qquad \text{(feasible)} \\
x_6 &= 20 - 2(50) + (-20) + (-10) < -110 < 0 \qquad \text{(infeasible)}
\end{aligned}
$$

Hence the solution is infeasible. Alternatively, if the changes in the resources are such that $D_1 = -30$, $D_2 = -12$, and $D_3 = 10$, then

$$
\begin{aligned}
x_2 &= 100 + \frac{1}{2}(-30) - \frac{1}{4}(-12) = 88 > 0 \qquad \text{(feasible)} \\
x_3 &= 230 + \frac{1}{2}(-12) = 224 > 0 \qquad \text{(feasible)} \\
x_6 &= 20 - 2(-30) + (-12) + (10) < 78 > 0 \qquad \text{(feasible)}
\end{aligned}
$$

Case I : Change in operation 1 time from 460 to $460 + D_1$ minutes. This change is equivalent to setting $D_2 = D_3 = 0$ in the simultaneous conditions, which yields

$$\begin{aligned} x_2 &= 100 + \frac{1}{2}D_1 \geq 0 \quad \Rightarrow \quad D_1 \geq -200 \\ x_3 &= 230 > 0 \\ x_6 &= 20 - 2D_1 \geq 0 \quad \Rightarrow \quad D_1 \leq 10 \end{aligned}$$

We have $-200 \leq D_1 \leq 10$.

Case II : Change in operation 2 time from 430 to $430 + D_2$ minutes. This change is equivalent to setting $D_1 = D_3 = 0$ in the simultaneous conditions, which yields

$$\begin{aligned} x_2 &= 100 - \frac{1}{4}D_2 \geq 0 \quad \Rightarrow \quad D_2 \leq 400 \\ x_3 &= 230 + \frac{1}{2}D_2 \geq 0 \quad \Rightarrow \quad D_2 \geq -460 \\ x_6 &= 20 + D_2 \geq 0 \quad \Rightarrow \quad D_2 \leq -20 \end{aligned}$$

We have $-20 \leq D_2 \leq 400$.

Case III : Change in operation 3 time from 420 to $420 + D_3$ minutes. This change is equivalent to setting $D_1 = D_2 = 0$ in the simultaneous conditions, which yields

$$\begin{aligned} x_2 &= 100 \geq 0 \\ x_3 &= 230 \\ x_6 &= 20 + D_3 \geq 0 \quad \Rightarrow \quad D_3 \geq -20 \end{aligned}$$

Resource	Dual price	Feasibility range	Minimum	Current	Maximum
Operation 1	1	$-200 \leq D_1 \leq 10$	230	430	440
Operation 2	2	$-20 \leq D_2 \leq 400$	440	460	860
Operation 3	0	$-20 \leq D_3 < \infty$	400	420	∞

Remark : It is important to notice that the dual prices will remain applicable for any simultaneous changes that keep the solution feasible, even if the changes violate the individual ranges.
For example, the changes $D_1 = 30$, $D_2 = -12$, and $D_3 = 100$, will keep the solution feasible even though $D_1 = 30$ violates the feasibility range $-200 \leq D_I \leq 10$, as the following computations show :

$$\begin{aligned}
x_2 &= 100 + \frac{1}{2}(30) - \frac{1}{4}(-12) = 118 > 0 \qquad \text{(feasible)} \\
x_3 &= 230 + \frac{1}{2}(-12) = 224 > 0 \qquad \text{(feasible)} \\
x_6 &= 20 - 2(30) + (-12) + (100) = 48 > 0 \qquad \text{(feasible)}
\end{aligned}$$

Summary :

1. The dual prices remain valid so long as the changes $D_i, i = 1, 2, ..., m$, in the right-hand sides of the constraints satisfy all the feasibility conditions when the changes are simultaneous or fall within the feasibility ranges when the changes are made individually.

2. For other situations where the dual prices are not valid because the simultaneous feasibility conditions are not satisfied or because the individual feasibility ranges are violated, the recourse is to either re-solve the problem with the new values of D_i or apply the post-optimal analysis.

Question : In the TOYCO model, suppose that the changes D_1, D_2, and D_3 are made simultaneously in the three operations.

1. If the availabilities of operations 1,2, and 3 are changed to 438,500, and 410 minutes, respectively, use the simultaneous conditions to show that the current basic solution remains feasible, and determine the change in the optimal revenue by using the optimal dual prices.

2. If the availabilities of the three operations are changed to 460,440, and 380 minutes, respectively, use the simultaneous conditions to show that the current basic solution becomes infeasible.

3.6.3 Algebraic Sensitivity Analysis—Objective Function

In Section 3.6.1, we used graphical sensitivity analysis to determine the conditions that will maintain the optimality of the solution of a two-variable LP. In this section, we extend these ideas to the general LP problem.

Definition of reduced cost. To facilitate the explanation of the objective function sensitivity analysis, first we need to define *reduced costs*. In the TOYCO model (Example 3.6-2), the objective z-equation in the optimal tableau can be written as

$$z = 1350 - 4x_1 - x_4 - 2x_5$$

The optimal solution does not produce toy trains ($x_1 = 0$). The reason can be seen from the z-equation, where a unit increase in x_1 (above its current zero value) decreases z by \$4—namely, $z = 1350 - 4 \times (1) - 1 \times (0) - 2 \times (0) = \$1346$.

production of toy train incurs cost because it consumes resources (operations time). Thus, from the standpoint of optimization, the "attractiveness" of x_1 depends on the cost of consumed resources relative to revenue. This relationship defines the so-called **reduced cost** and is formalized in the LP literature as

$$\begin{pmatrix}\text{Reduced cost}\\ \text{per unit}\end{pmatrix} = \begin{pmatrix}\text{Cost of consumed}\\ \text{resources per unit}\end{pmatrix} - (\text{Revenue per unit})$$

To appreciate the significance of this definition, in the original TOYCO model the revenue per unit for toy trucks (= \$2) is less than that for toy trains (= \$3). Yet the optimal solution recommends producing toy trucks ($x_2 = 100$ units) and no toy trains ($x_1 = 0$). The reason is that the cost of the resources used by one toy truck (i.e., operations time) is smaller than its unit price. The opposite applies in the case of toy trains.

With the given definition of *reduced cost*, we can see that an unprofitable variable (such as x_1) can be made profitable in two ways:

1. By increasing the unit revenue.
2. By decreasing the unit cost of consumed resources.

In most situations, the price per unit is dictated by market conditions and may be difficult to increase at will. On the other hand, reducing the consumption of resources is a more viable option because the manufacturer may be able to reduce cost by making the production process more efficient.

Determination of the optimality ranges. We now turn our attention to determining the conditions that will keep a solution optimal. The development is based on the definition of *reduced cost*.

In the TOYCO model, let d_1, d_2, and d_3 represent the change in unit revenues for toy trucks, trains, and cars, respectively. The objective function then becomes

$$\text{Maximize } z = (3 + d_1)x_1 + (2 + d_2)x_2 + (5 + d_3)x_3$$

We first consider the general situation in which all the objective coefficients are changed *simultaneously*.

With the simultaneous changes, the z-row in the starting tableau appears as:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$-3 - d_1$	$-2 - d_2$	$-5 - d_3$	0	0	0	0

When we generate the simplex tableaus with the same sequence of entering and leaving variables used in the original model (before the changes d_j are made), the optimal iteration will appear as follows (convince yourself that this is indeed the case by carrying out the simplex row operations):

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	$4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1$	0	0	$1 + \frac{1}{2}d_2$	$2 - \frac{1}{4}d_2 + \frac{1}{2}d_3$	0	$1350 + 100d_2 + 23d_3$
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	$-\frac{1}{4}$	0	0	-2	1	1	20

The new optimal tableau is the same as in the *original* optimal tableau, except for the *reduced costs* (z-equation coefficients). This means that *changes in the objective-function coefficients can affect the optimality of the problem only*. (Comonly.)

You really do not need to carry out the simplex row operation to compute the new reduced costs. An examination of the new z-row shows that the coefficients of d_j are taken directly from the constraint coefficients of the optimum tableau. A convenient way for computing the new reduced cost is to add a new top row and a new leftmost column to the optimum tableau, as shown by the shaded areas in the following illustration.

		d_1	d_2	d_3	0	0	0	
	Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
1	z	4	0	0	1	2	0	1350
d_2	x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
d_3	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
0	x_6	2	0	0	-2	1	1	20

The entries in the top row are the change d_j associated with variable x_j. For the leftmost column, the top element is 1 in the z-row followed by d_i basic variable x_i. Keep in mind that $d_i = 0$ for slack variable x_i.

To compute the new reduced cost for any variable (or the value of z), multiply the elements of its column by the corresponding elements in the leftmost column, add them up, and subtract the top-row element from the sum. For example, for x_1, we have

$$\text{Reduced cost for } x_1 = [4 \times 1 + (-\tfrac{1}{4}) \times d_2 + \tfrac{3}{2} \times d_3 + 2 \times 0] - d_1$$
$$= 4 - \tfrac{1}{4}d_2 + \tfrac{3}{2}d_3 - d_1$$

The current solution remains optimal so long as the new reduced costs (z-equation coefficients) remain nonnegative (maximization case). We thus have the following *simultaneous optimality conditions* corresponding to nonbasic x_1, x_4, and x_5:

$$4 - \tfrac{1}{4}d_2 + \tfrac{3}{2}d_3 - d_1 \geq 0$$
$$1 + \tfrac{1}{2}d_2 \geq 0$$
$$2 - \tfrac{1}{4}d_2 + \tfrac{1}{2}d_3 \geq 0$$

Remember that the reduced cost for a basic variable is always zero, as the modified optimal tableau shows.

To illustrate the use of these conditions, suppose that the objective function of TOYCO is changed from $z = 3x_1 + 2x_2 + 5x_3$ to $z = 2x_1 + x_2 + 6x_3$. Then, $d_1 = 2 - 3 = -\$1$, $d_2 = 1 - 2 = -\$1$, and $d_3 = 6 - 5 = \$1$. Substitution in the given conditions yields

$$4 - \tfrac{1}{4}d_2 + \tfrac{3}{2}d_3 - d_1 = 4 - \tfrac{1}{4}(-1) + \tfrac{3}{2}(1) - (-1) = 6.75 > 0 \text{ (satisfied)}$$

$$1 + \tfrac{1}{2}d_2 = 1 + \tfrac{1}{2}(-1) = .5 > 0 \qquad \text{(satisfied)}$$

$$2 - \tfrac{1}{4}d_2 + \tfrac{1}{2}d_3 = 2 - \tfrac{1}{4}(-1) + \tfrac{1}{2}(1) = 2.75 > 0 \qquad \text{(satisfied)}$$

The results show that the proposed changes will keep the current solution ($x_1 = 0$, $x_2 = 100, x_3 = 230$) optimal (with a new value of $z = 1350 + 100d_2 + 230d_3 = 1350 + 100 \times -1 + 230 \times 1 = \$1480$). If any condition is not satisfied, a new solution must be determined (see Chapter 4).

The preceding discussion has dealt with the maximization case. The only difference in the minimization case is that the reduced costs (z-equations coefficients) must be ≤ 0 to maintain optimality.

The *optimality ranges* dealing with changing d_j *one at a time* can be developed from the simultaneous optimality conditions.[12] For example, suppose that the objective coefficient of x_2 only is changed to $2 + d_2$—meaning that $d_1 = d_3 = 0$. The simultaneous optimality conditions thus reduce to

$$\left.\begin{array}{l} 4 - \tfrac{1}{4}d_2 \geq 0 \Rightarrow d_2 \leq 16 \\ 1 + \tfrac{1}{2}d_2 \geq 0 \Rightarrow d_2 \geq -2 \\ 2 - \tfrac{1}{4}d_2 \geq 0 \Rightarrow d_2 \leq 8 \end{array}\right\} \Rightarrow -2 \leq d_2 \leq 8$$

In a similar manner, you can verify that the individual changes $(3 + d_1)$ and $(5 + d_3)$ for x_1 and x_3 yield the optimality ranges $d_1 \le 4$ and $d_3 \ge -\frac{8}{3}$, respectively.

The given individual conditions can be translated to total unit revenue ranges. For example, for toy trucks (variable x_2), the total unit revenue is $2 + d_2$, and its optimality range $-2 \le d_2 \le 8$ translates to

$$\$0 \le (\text{Unit revenue of toy truck}) \le \$10$$

It assumes that the unit revenues for toy trains and toy cars remain fixed at \$3 and \$5, respectively.

It is important to notice that the changes d_1, d_2, and d_3 may be within their allowable individual ranges without satisfying the simultaneous conditions and vice versa. For example, consider $z = 6x_1 + 8x_2 + 3x_3$. Here $d_1 = 6 - 3 = \$3$, $d_2 = 8 - 2 = \$6$, and $d_3 = 3 - 5 = -\$2$, which are all within the permissible individual ranges $(-\infty < d_1 \le 4, -2 \le d_2 \le 8$, and $-\frac{8}{3} \le d_3 < \infty)$. However, the corresponding simultaneous conditions yield

$$4 - \tfrac{1}{4}d_2 + \tfrac{3}{2}d_3 - d_1 = 4 - \tfrac{1}{4}(6) + \tfrac{3}{2}(-2) - 3 = -3.5 < 0 \quad \text{(not satisfied)}$$

$$1 + \tfrac{1}{2}d_2 = 1 + \tfrac{1}{2}(6) = 4 > 0 \quad \text{(satisfied)}$$

$$2 - \tfrac{1}{4}d_2 + \tfrac{1}{2}d_3 = 2 - \tfrac{1}{4}(6) + \tfrac{1}{2}(-2) = -.5 < 0 \quad \text{(not satisfied)}$$

Remarks. The *feasibility ranges* presented in Section 3.6.2 and the *optimality ranges* developed in Section 3.6.3 work fine so long as the sensitivity analysis situation calls for changing the parameters of the problem *one at a time*, a rare occurrence in practice. The fact of the matter is that this limited usefulness is dictated by how far mathematics allows us to go before the results become too unwieldy. *So, what should one do in practice to carry out meaningful sensitivity analyses that entail making simultaneous changes anywhere in the model?* The good news is that advances in computing and in mathematical programming languages (e.g., AMPL) now make it possible to solve huge LPs rather quickly. Thus, a viable option is to solve complete LP scenarios *completely*, and then compare the answers. Of course, a great deal of thought must be given to constructing viable scenarios that will allow testing model changes in a systematic and logical manner.