SECTION 3.4
ARTIFICIAL STARTING SOLUTION

- M TECHNIQUE
- TWO PHASE METHOD

ARTIFICIAL STARTING SOLUTION

- The simplex algorithm requires an initial basic feasible solution (IBFS). Such a solution is found by using the slack variables as our basic variables.
- If an LP has any ≥ or equality constraints, an IBFS may not be readily apparent.
- In this case artificial variables are used that play the role of slacks at the first iteration.
- Two closely related methods, the M-method and the two-phase method are used to solve the LPs that include artificial variables.

$$\text{Minimize } z = 4x_1 + x_2$$

$$
\begin{aligned}
3x_1 + x_2 &= 3 \\
4x_1 + 3x_2 &\geq 6 \\
x_1 + 2x_2 &\leq 4 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

$$\text{Minimize } z = 4x_1 + x_2$$

$$
\begin{aligned}
3x_1 + x_2 \phantom{{}- x_3 + x_4} &= 3 \\
4x_1 + 3x_2 - x_3 \phantom{{}+ x_4} &= 6 \\
x_1 + 2x_2 \phantom{{}- x_3} + x_4 &= 4 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

M Method

Minimize $z = 4x_1 + x_2$

$$
\begin{aligned}
3x_1 + x_2 &= 3 \\
4x_1 + 3x_2 &\geq 6 \\
x_1 + 2x_2 &\leq 4 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Minimize $z = 4x_1 + x_2$

$$
\begin{aligned}
3x_1 + x_2 \phantom{{}- x_3 + x_4} &= 3 \\
4x_1 + 3x_2 - x_3 \phantom{{}+ x_4} &= 6 \\
x_1 + 2x_2 \phantom{{}- x_3} + x_4 &= 4 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

Basic	x1	x2	x3	R1	R2	x4	Solution
z	-4	-1	0	-100	-100	0	0
R1	3	1	0	1	0	0	3
R2	4	3	-1	0	1	0	6
x4	1	2	0	0	0	1	4

$$\text{Artificial variable objective function coefficient} = \begin{cases} -M, \text{ in maximization problems} \\ M, \text{ in minimization problems} \end{cases}$$

$$\text{Minimize } z = 4x_1 + x_2 + MR_1 + MR_2$$

$$\begin{aligned} 3x_1 + x_2 \quad + R_1 \quad &= 3 \\ 4x_1 + 3x_2 - x_3 \quad + R_2 \quad &= 6 \\ x_1 + 2x_2 \quad + x_4 &= 4 \\ x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0 \end{aligned}$$

Basic	x1	x2	x3	R1	R2	x4	Solution
z	-4	-1	0	-100	-100	0	0
R1	3	1	0	1	0	0	3
R2	4	3	-1	0	1	0	6
x4	1	2	0	0	0	1	4

$$\text{Artificial variable objective function coefficient} = \begin{cases} -M, \text{ in maximization problems} \\ M, \text{ in minimization problems} \end{cases}$$

$$\text{Minimize } z = 4x_1 + x_2 + MR_1 + MR_2$$

$$\begin{aligned} 3x_1 + x_2 + R_1 &= 3 \\ 4x_1 + 3x_2 - x_3 + R_2 &= 6 \\ x_1 + 2x_2 + x_4 &= 4 \\ x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0 \end{aligned}$$

Check consistency

Basic	x1	x2	x3	R1	R2	x4	Solution
z	696	399	-100	0	0	0	900
R1	3	1	0	1	0	0	3
R2	4	3	-1	0	1	0	6
x4	1	2	0	0	0	1	4

Basic	x1	x2	x3	R1	R2	x4	Solution
z	696	399	-100	0	0	0	900
R1	3	1	0	1	0	0	3
R2	4	3	-1	0	1	0	6
x4	1	2	0	0	0	1	4
z	0	167	-100	-232	0	0	204
x1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
R2	0	$\frac{5}{3}$	-1	$-\frac{4}{3}$	1	0	2
x4	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3
z	0	0	$\frac{1}{5}$	$-\frac{492}{5}$	$-\frac{501}{5}$	0	$\frac{18}{5}$
x1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
x4	0	0	1	1	-1	1	1
z	**0**	**0**	**0**	$-\frac{493}{5}$	**-100**	$-\frac{1}{5}$	$\frac{17}{5}$
x1	1	0	0	$\frac{2}{5}$	0	$-\frac{1}{5}$	$\frac{2}{5}$
x2	0	1	0	$\frac{-1}{5}$	0	$\frac{3}{5}$	$\frac{9}{5}$
x3	0	0	1	1	-1	1	**1**

Remarks. If the final simplex iteration includes at least one artificial variable with a positive value, this will indicate that the LP does not have a feasible solution (i.e., the constraints cannot be satisfied simultaneously).