Sensitivity Analysis : In LP, the **parameters (input data)** of the model can change **within certain limits without** causing the optimum solution to change.

This is referred to as **sensitivity analysis**.

Graphical Sensitivity Analysis : Two cases will be considered :

1. Sensitivity of the optimum solution to **changes** in the **availability of the resources** (the right hand size of the constraints)

2. Sensitivity of the optimum solution to changes in **unit profit or unit cost** (coefficients of the objective function).

Example : (Changes in the right hand side of the constraints)
JOBCO produces **two products** on two machines. A unit of product 1 requires 2 hours on **machine 1** and 1 hour on **machine 2**.
For product 2, a unit requires 1 hour on **machine 1** and 3 hours on **machine 2**.
The revenues per unit of products 1 and 2 are 30$ and 20$, respectively.
The total daily processing time available for each machine is 8 hours.
Solution :
x_1 = Daily number of units of product 1
x_2 = Daily number of units of product 2.
The following is the LP model for the problem.

$$\text{Maximize} \qquad z = 30x_1 + 20x_2$$

subject to

$$2x_1 + x_2 \leq 8, \text{(machine 1)} \qquad x_1 + 3x_2 \leq 8, \text{(machine 2)} \quad x_1, x_2 \geq 0.$$

Maximize $\quad z = 30x_1 + 20x_2 \quad$ subject to

$2x_1 + x_2 \leq 8$, (machine 1) $\quad x_1 + 3x_2 \leq 8$, (machine 2) $\quad x_1, x_2 \geq 0$.

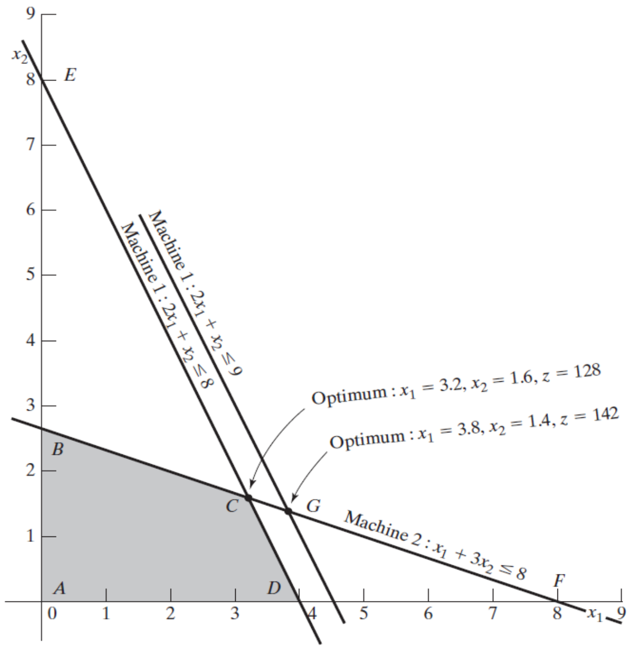

FIGURE 3.9
Graphical sensitivity of optimal solution to changes in the availability of resources (right-hand side of the constraints)

Rate of revenue change resulting from increasing machine 1 capacity by 1hour =

$$\frac{z_G - z_C}{\text{Capacity change}} = 14\$$$

Unit worth of resource : The direct link between the model resource and its output is called a unit worth of the resource. The rate of change of objective function is named as **dual** or **shadow price**.

Sensitivity Analysis

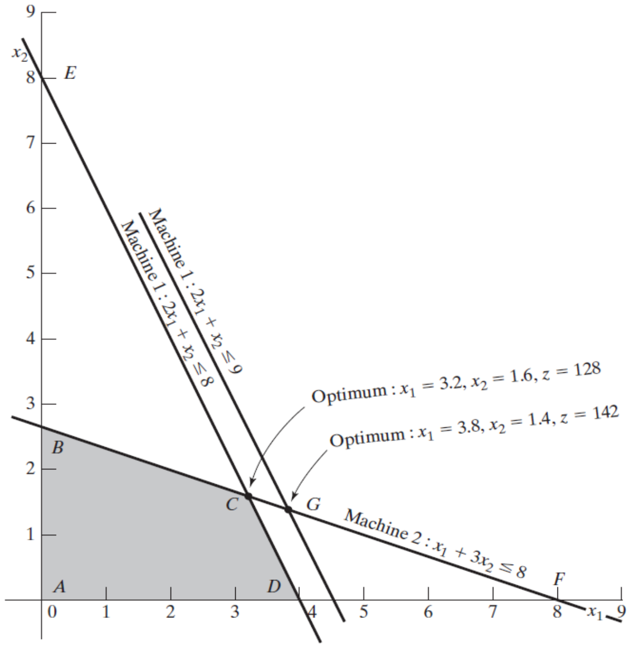

FIGURE 3.9

Graphical sensitivity of optimal solution to changes in the availability of resources (right-hand side of the constraints)

The dual price of 14.00\$/hr remains valid for changes (increases or decreases) in machine 1 capacity that move its constraint parallel to itself to any point on the line segment BF. Because $\frac{z_F - z_B}{16 - 2.67} = 13.99 \approx 14\$$

This means that the range of applicability of the given **dual price** can be computed as follows :

Minimum machine 1 capacity (at $B = (0, 2.67)$)
$= (2)(0) + (1)(2.67) = 2.67\text{hr}$

Maximum machine 1 capacity (at $B = (8, 0)$) $= 2 \times 8 + 1 \times 0 = 16\text{hr}$

We can thus conclude that the dual price of 14.00\$/hr will remain valid for the range

$$2.67hr \leq \text{Machine 1 capacity} \leq 16hrs$$

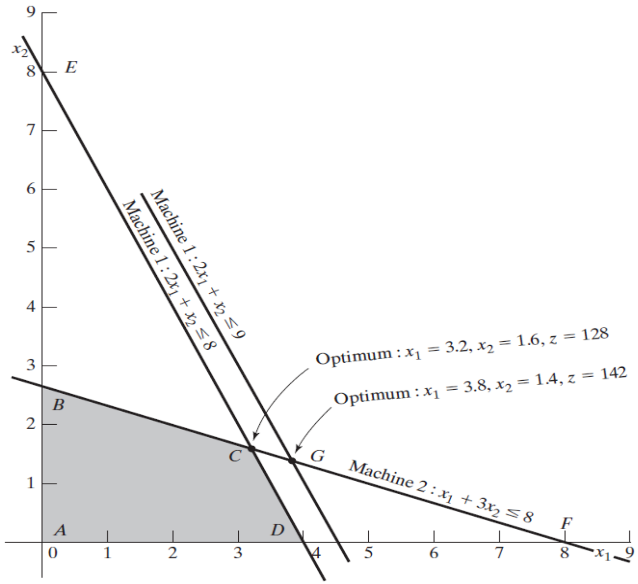

FIGURE 3.9
Graphical sensitivity of optimal solution to changes in the availability of resources (right-hand side of the constraints)

Changes outside this range will produce a different dual price (worth per unit).

Using similar computations, you can verify that the dual price for machine 2 capacity is 2.00\$/hr and it remains valid for changes (increases or decreases) that move its constraint parallel to itself to any point on the line segment DE, which yields the following limits :
Minimum machine 2 capacity (at $D = (4, 0)) = 1 \times 4 + 3 \times 0 = 4$hr
Maximum machine 2 capacity (at $E = (8, 0)) = 1 \times 0 + 3 \times 8 = 24$hr
We can thus conclude that the dual price of 2.00\$/hr will remain valid for the range

$$4hr \le \text{Machine 2 capacity} \le 24hrs$$

The computed limits for machine 1 and 2 are referred to as the **feasibility ranges**.

Question 1 : If JOBCO can increase the capacity of both machines, which machine should receive higher priority ?

Answer : The dual prices for machines 1 and 2 are 14.00\$/hr and 2.00\$/hr. This means that each additional hour of machine 1 will increase revenue by 14.00\$, as opposed to only 2.00\$ for machine 2. Thus, priority should be given to machine 1.

Question 2 : A suggestion is made to increase the capacities of machines 1 and 2 at the additional cost of 10\$/hr. Is this advisable ?

Answer : For machine 1, the additional net revenue per hour is $14.00 - 10.00 = 4.00\$$ and for machine 2, the net is $2.00\$ - 10.00\$ = -8.00\$$.
Hence, only the capacity of machine 1 should be increased.

Question 3 : If the capacity of machine 1 is increased from the present 8 hours to 13 hours, how will this increase impact the optimum revenue?

Answer : The dual price for machine 1 is 14.00\$ and is applicable in the range $(2.67, 16)$ hr. The proposed increase to 13 hours falls within the feasibility range. Hence, the increase in revenue is $14.00\$(13-8) = 70.00\$$, which means that the total revenue will be increased to (current revenue + change in revenue) $= 128 + 70 = 198\$$.

Question 4 : Suppose that the capacity of machine 1 is increased to 20 hours, how will this increase impact the optimum revenue?

Answer : The proposed change is outside the range $(2.67, 16)$ hr for which the dual price of 14.00\$ remains applicable. Thus, we can only make an immediate conclusion regarding an increase up to 16 hours. Beyond that, further calculations are needed to find the answer. Remember that falling outside the feasibility range does not mean that the problem has no solution. It only means that we do not have sufficient information to make an immediate decision.

Question 5 : We know that the change in the optimum objective value equals (dual price $\times$ change in resource) so long as the change in the resource is within the feasibility range. What about the associated optimum values of the variables?

Answer : The optimum values of the variables will definitely change. However, the level of information we have from the graphical solution is not sufficient to determine the new values. We will learn while discussing, the sensitivity problem algebraically, provides this detail.

Changes in the Objective Coefficients : Changes in revenue units (i.e., objective-function coefficients) will change the slope of the line z.

However, as can be seen from the figure, the optimum solution will remain at point C so long as the objective function lies between lines BF and DE, the two constraints that define the optimum point. This means that there is a range for the coefficients of the objective function that will keep the optimum solution unchanged at C.

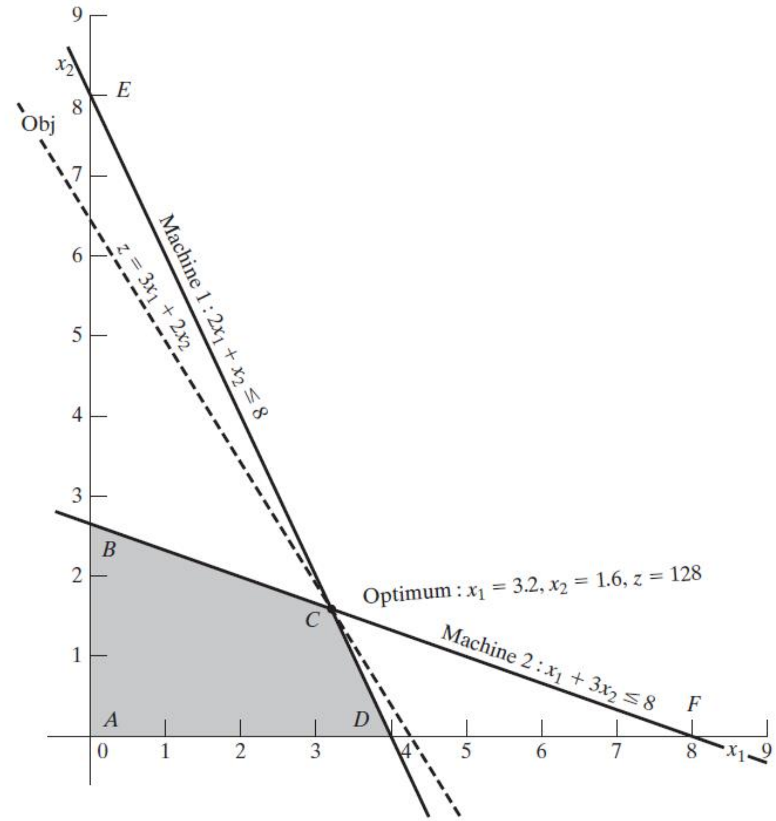

FIGURE 3.10

Graphical sensitivity of optimal solution to changes in the revenue units (coefficients of the objective function)

Changes in the Objective Coefficients : Write objective function $z = c_1x_1 + c_2x_2$

The optimum solution will remain at the point as long as the objective function $z = c_1x_1 + c_2x_2$ remains between two lines $x_1 + 3x_2 = 8$ and $2x_1 + x_2 = 8$, which means

$$\frac{1}{3} \le \frac{c_1}{c_2} \le \frac{2}{1}$$

or

$$0.333 \le \frac{c_1}{c_2} \le 2$$

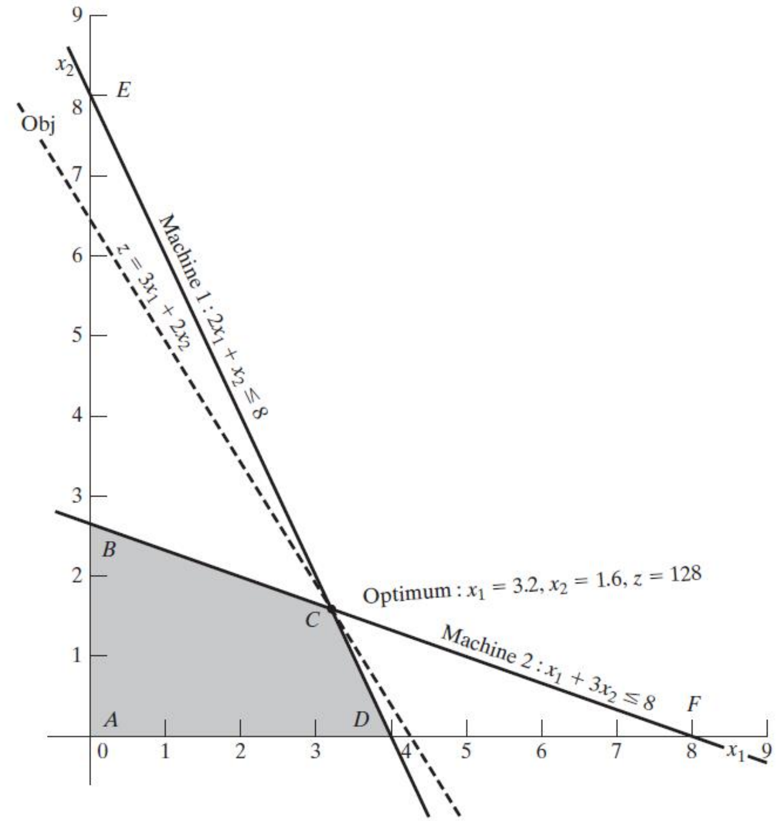

FIGURE 3.10

Graphical sensitivity of optimal solution to changes in the revenue units (coefficients of the objective function)

This information can give answers of questions regarding optimum solution.

Changes in the Objective Coefficients :

$$0.333 \leq \frac{c_1}{c_2} \leq 2$$

Question 1 : Suppose that the unit revenues for products 1 and 2 are changed to 35$ and 25$, respectively. Will the current **optimum remain** the same ?
Answer : The objective function will change to $z = 35x_1 + 25x_2$ and the slope of the line will be $\frac{c_1}{c_2} = \frac{35}{25} = 1.4$ which is in the optimality range.

Remark : When the ratio falls outside this range, additional calculations are needed to find the new optimum.

Remark : Notice that although the values of the variables at the optimum point C remain unchanged, the optimum value of z changes to 152.00$.

Changes in the Objective Coefficients :

$$0.333 \leq \frac{c_1}{c_2} \leq 2$$

Question 2 : Suppose that the unit revenue of product 2 is fixed at its current value of $c_2 = 20.00\$$. What is the associated range for c_1, the unit revenue for product 1 that will keep the optimum unchanged ?
Answer : Put $c_2 = 20$ in the inequality

$$0.333 \times 20 \leq c_1 \leq 2 \times 20 \qquad 6.67 \leq c_1 \leq 40$$

This range is referred to as the **optimality range** for c_1 and it implicitly assumes that c_2 is fixed at 20.00\$.
Optimality range for c_2 can be obtained by fixing $c_1 = 30\$$

$$2 \leq \frac{c_2}{c_1} \leq 0.333, \qquad 15 \leq c_2 \leq 90.$$

The results lay the foundation for the development of sensitivity analysis for the general LP problem.

Question : A company produces two products, A and B. The unit revenues are 2$ and 3$, respectively.
Two raw materials, $M1$ and $M2$, used in the manufacture of the two products have respective daily availabilities of 8 and 18 units.
One unit of A uses 2 units of $M1$ and 2 units of $M2$, and 1 unit of B uses 3 units of MI and 6 units of $M2$.

1. Determine the dual prices of MI and $M2$ and their **feasibility ranges**.

2. Suppose that 4 additional units of $M1$ can be acquired at the cost of 30 cents per unit. Would you recommend the additional purchase ?

3. What is the most the company should pay per unit of $M2$?

4. If $M2$ availability is increased by 5 units, determine the associated optimum revenue.

Question : A company produces two products, A and B. The unit revenues are 2$ and 3$, respectively.
Two raw materials, $M1$ and $M2$, used in the manufacture of the two products have respective daily availabilities of 8 and 18 units.
One unit of A uses 2 units of $M1$ and 2 units of $M2$, and 1 unit of B uses 3 units of MI and 6 units of $M2$.

1. Determine the optimality condition for $\frac{c_A}{c_B}$ that will keep the optimum unchanged.

2. Determine the optimality ranges for c_A and c_B, assuming that the other coefficient is kept constant at its present value.

3. What is the most the company should pay per unit of $M2$?

4. If the unit revenues c_A and c_B are changed simultaneously to 5$ and 4$, respectively, determine the new optimum solution.