FINALS

NUMERICAL SOLUTION OF ODEs

PICARD'S METHOD:

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

Q. Solve $\frac{dy}{dx} = 1 + xy$ with $x_0 = 0, y_0 = 0$ upto 3^{rd} approximation.

1^{st}:

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$y_1 = 0 + \int_0^x (1 + x(0))\,dx$$

$$y_1 = 0 + |x|_0^x$$

$$y_1 = 0 + x$$

$$y_1 = x$$

2nd:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$= 0 + \int_{0}^{x} (1 + x^2)\, dx$$

$$= 0 + \left| x + \frac{x^3}{3} \right|_0^x$$

$$= x + \frac{x^3}{3}$$

3rd:

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

$$y_3 = 0 + \int_{0}^{x} \left(1 + x\left(x + \frac{x^3}{3}\right)\right) dx$$

$$y_3 = \int_{0}^{x} 1 + x^2 + \frac{x^4}{3} \quad dx$$

$$y_3 = x^2 + \frac{x^3}{3} + \frac{x^5}{15}$$

Q. $y=1$ when $x=0$

$y=?$ when $x=0.2$

$$\frac{dy}{dx} = x-y$$

1st: $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$

$$y_1 = 1 + \int_0^x (x-1)\,dx$$

$$y_1 = 1 + \left| \left(\frac{x^2}{2} - x \right) \right|_0^x$$

$$y_1 = \frac{x^2}{2} - x + 1$$

$$\Rightarrow y_1(0.2) = \frac{0.2^2}{2} - (0.2) + 1$$

$$= 0.82$$

2^{nd}:

$$y_2 = y_0 + \int_{x}^{x_0} f(x, y_1)\, dx$$

$$y_2 = 1 + \int_0^x \left(x - \left(\frac{x^2}{2} - x + 1\right)\right) dx$$

$$y_2 = 1 + \int_0^x \left(-\frac{x^2}{2} + 2x - 1\right) dx$$

$$y_2 = \left(-\frac{x^3}{6} - x + \frac{2x^2}{2}\right) + 1$$

$$y_2 = x^2 - \frac{x^3}{6} - x + 1$$

$$y_2(0.2) = 1 - 0.2 + (0.2)^2 - \frac{(0.2)^3}{6}$$

$$= 0.8387$$

3^{rd}:

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

$$y_3 = 1 + \int_0^x x - \left(x^2 - \frac{x^3}{6} - x + 1\right) dx$$

Made with Goodnotes

$$y_3 = 1 + \int_0^x \left(-x^2 + \frac{x^3}{6} + 2x - 1\right) dx$$

$$y_3 = 1 + \left(-\frac{x^3}{3}\right) + \frac{x^4}{24} + x^2 - x$$

$$y_3 = \frac{x^4}{24} - \frac{x^3}{3} + x^2 - x + 1$$

$$y_3(0.2) = 0.8374$$

TAYLOR SERIES:

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' \ldots\ldots$$

$\because x_n = x_{n-1} + h$

Q. $\frac{dy}{dx} = x + y$

$x_0 = 1$

$y_0 = 0$

$h = 0.1$

$y = ?$ when $x = 1.2$

$$\boxed{x_1 = x_0 + h = 1 + 0.1 = 1.1}$$

$y' = x + y \qquad y'_0 = 1 + 0 = 1$

$y'' = 1 + y' \qquad y''_0 = 1 + 1 = 2$

$y''' = y'' \qquad y'''_0 = 2$

$y'''' = y''' \qquad y''''_0 = 2$

By Taylor Series,

$$y_1 = y(x_1) = y_0 + hy'_0 + \frac{h^2}{2!} y'' + \frac{h^3}{3!} y'''_0 \ldots\ldots$$

$$= 0 + (0.1)(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2)$$

$$= 0.11034$$

$$\boxed{\simeq 0.1103}$$

$x_1 = 1.1 \quad , y_1 = 0.1103 \quad , h = 0.1$

$$\boxed{x_2 = x_1 + h = 1.2}$$

$y' = x+y$ $\qquad$ $y_1' = 1.1 + 0.1103 = 1.2$

$y'' = 1+y'$ $\qquad$ $y_1'' = 1+1.2 = 2.2$

$y''' = y''$ $\qquad$ $y_1''' = 2.2103$

By Taylor Series:-

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' \ldots\ldots$$

$$= 0.1103 + (0.1)(1.2103) + \frac{(0.1)^2}{2}(2.2103)$$

$$+ \frac{(0.1103)^3}{3!}(2.2103)$$

$\boxed{y_2 = 0.2428}$ at $\boxed{x = 1.2}$

Q. $y' = 2y + 3e^x$ $\qquad$ $x_0 = 0$, $y_0 = 1$

$y = ?$ when $x = 0.1$ and $x = 0.2$

Made with Goodnotes

$h = x_1 - x_0 = 0.1 - 0.0$

$= 0.1$

$y' = 2y + 3e^x$ $\qquad$ $y_0' = 2(1) + 3e^0 = 5$

$y'' = 2y' + 3e^x$ $\qquad$ $y_0'' = 2(5) + 3e^0 = 13$

$y''' = 2y'' + 3e^x$ $\qquad$ $y_0''' = 2(13) + 3 = 29$

Using Taylor Series,

$$y_1 = y(0.1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0'''$$

$$= 1 + (0.1)(5) + \frac{(0.1)^2}{2!}(13) + \frac{(0.1)^3}{3!}(29)$$

$$= 1.5701$$

$x_2 = 0.2$

$y' = 2y + 3e^x$ $\qquad$ $y_1' = 2(1.57) + 3e^{(0.2)}$

$y_1' = 6.4555$

$y'' = 2y' + 3e^{x}$

$y_1'' = 2(6.45) + 3e^{0.1}$
$= 16.2265$

$y''' = 2y'' + 3e^{x}$

$y_1''' = 2(16.2265) + 3e^{0.1}$
$= 35.7685$

$h = x_2 - x_1$
$\boxed{= 0.1}$

Using Taylor series,

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' \ldots\ldots$$

$$= 1.5701 + (0.1)(6.4555) + \frac{(0.1)^2}{2!}(16.2265)$$

$$+ \frac{(0.1)^3}{3!}(35.7685)$$

$$\boxed{= 2.3029}$$

Made with Goodnotes

Q. $y' = 1 + xz$ $\qquad\qquad z' = -xy$

$y_0 = 0, \ x_0 = 0$ $\qquad\qquad z_0 = 1, \ x_0 = 0$

$y = ?$ when $x = 0.3$

$x_1 = x_0 + h$

$\boxed{h = 0.3}$

$y' = 1 + xz$	$y'_0 = 1$	$z' = -xy$	$z'_0 = 0$
$y'' = z + xz'$	$y''_0 = 1$	$z'' = -(y + xy')$	$z''_0 = 0$
$y''' = z' + z' + xz''$ $= 2z'$	$y'''_0 = 0$	$z''' = -y' - y' - xy''$	$z'''_0 = -2$

$$y_1 = y_0 + hy'_0 + \frac{h^2 y''_0}{2!} + \frac{h^3 y'''_0}{3!} \cdots$$

$$y_1 = 0 + (0.3) + \frac{(0.3)^2 (1)}{2!} + \frac{(0.3)^3 \, 0}{3!}$$

$\boxed{y_1 = 0.345}$

$$z_1 = z_0 + h z_0' + \frac{h^2 z_0''}{2!} \ldots \ldots$$

$$z_1 = 1 + 0 + 0 \ldots$$

$$\boxed{z_1 = 1}$$

EULER'S METHOD:

$$y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1})$$

$$h = \frac{x_n - x_0}{n}$$

Q. $\frac{dy}{dx} = \frac{y-x}{y+x}$ $\quad y(0) = 1$

$\Rightarrow x_0 = 0$

$y_0 = 1$

$y = ?$ when $x_n = 0{\cdot}1$

Let, $n = 5$

Then, $h = \frac{x_n - x_0}{n} = \frac{0{\cdot}1 - 0}{5} = 0{\cdot}02$

Using Euler's method,

$$y_1 = y_0 + h\,f(x_0, y_0) = 1 + (0{\cdot}02)\left(\frac{1-0}{1+0}\right)$$

$$\boxed{= 1{\cdot}02}$$

$$y_2 = y_1 + hf(x_1, y_1) = 1.02 + (0.02)\left(\frac{1.02 - .02}{1.02 + .02}\right)$$

$$\boxed{= 1.0392}$$

$$y_3 = 1.0392 + (0.02)\frac{(1.0392 - 0.04)}{(1.0392 + 0.04)}$$

$$\boxed{= 1.0577}$$

$$\boxed{y_4 = 1.0756}$$

$$y(0.1) = 1.0928$$

$$\boxed{y_5 = 1.0928}$$

Q. $\frac{dy}{dx} = x + y$

$y(0) = 1$ $\Rightarrow x_0 = 0$, $y_0 = 1$

$x_n = 1$

Let $n = 10$,

Then $h = \frac{x_n - x_0}{n} = 0.1$

x	y	$y' = x+y$	$y_{n+1} + (0.1)\, y' = y_n$
0.0	1.00	1.00	$1.00 + (0.1)(1.00) = 1.10$
0.1	1.1	1.20	$1.10 + (0.1)(1.20) = 1.22$
0.2	1.22	1.42	$1.22 + (0.1)(1.42) = 1.36$
0.3	1.36	1.66	$1.36 + (0.1)(1.66) = 1.53$
0.4	1.53	1.93	$1.53 + (0.1)(1.93) = 1.72$
0.5	1.72	2.22	$1.72 + (0.1)(2.22) = 1.94$
0.6	1.94	2.54	$1.94 + (0.1)(2.54) = 2.19$
0.7	2.19	2.89	$2.19 + (0.1)(2.89) = 2.48$
0.8	2.48	3.28	$2.48 + (0.1)(3.28) = 2.81$
0.9	2.81	3.71	$2.81 + (0.1)(3.71) = 3.18$
1.0	3.18		

$$\boxed{y(1.0) = 3.18}$$

MODIFIED-EULER'S METHOD:

First find,

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

Then,

$$y_1^{n+1} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)}) \right]$$

Repeat till values are same.

Q. $\frac{dy}{dx} = \log(x+y)$ $\quad h = 0.2$

$y_0 = 2 \quad , \quad x_0 = 1$

$y = ?$ when $x = 1.2 \, , \, x = 1.4$

INITIAL-APPROXIMATION:

$$y_1^0 = y_0 + hf(x_0, y_0) = 2 + (0.2)(1.0986)$$

$$\boxed{= 2.2197}$$

1^{st} Iteration:

$$y_1^1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^0) \right]$$

$$= 2 + \frac{0.2}{2}\left[1.0986 + f(1.2, 2.2197) \right]$$

$$\boxed{= 2.2328}$$

2^{nd} Iteration:-

$$y_1^2 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^1) \right]$$

$$= 2 + \frac{0.2}{2}\left[1.0986 + f(1.2, 2.2328)\right]$$

$$\boxed{= 2.2332}$$

3rd Iteration:

$$y_1^3 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^2)\right]$$

$$= 2 + \frac{0.2}{2}\left[1.0988 + f(1.2, 2.2332)\right]$$

$$\boxed{= 2.2332}$$

$$\therefore\ y_1^2 = y_1^3 = 2.2332$$

$$\boxed{y_1 = y(1.2) = 2.2332}$$

$y = ?$ when $x = 1.4$

INITIAL-APPROXIMATION:-

$$y_2^0 = y_1 + h f(x_1, y_1)$$

$$= 2.2332 + 0.2(1.2335) = \boxed{2.4779}$$

1^{st} ITERATION:

$$y_2^1 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^0)\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + f(1.4, 2.4779)\right]$$

$$\boxed{= 2.4921}$$

2^{nd} ITERATION:

$$y_2^2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^1)\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + f(1.4, 2.4921)\right]$$

$$\boxed{= 2.4924}$$

3^{rd} ITERATION:

$$y_2^3 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^2)\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + f(1.4, 2.4924)\right]$$

$$\boxed{= 2.4924}$$

$$\therefore \quad y_2^{2} = y_2^{3}$$

$$\boxed{y_2 = y(1.4) = 2.4924}$$

IMPROVED-EULER'S METHOD:

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f\left[x_n + h, y_n + hf(x_n, y_n)\right]\right)$$

Q. $y' = x+y \qquad x_0 = 0, \quad y_0 = 1$

$y = ?$ when $x = 0.2, \; x = 0.4$

$$y_1 = y_0 + \frac{h}{2}\left(f(0,1) + f\left[0+h, 1+h(f(0,1))\right]\right)$$

$$\boxed{= 1.24}$$

$$y_2 = y_1 + \frac{0.2}{2}\left(f(0.2, 1.24) + f(0.2+0.2, 1+h(f(0.2, 1.24)))\right)$$

$$\boxed{y_2 = 1.5528}$$

RUNGE-KUTTA METHOD:

1st order: Same as Euler method.

2nd order:

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = hf(x_n, y_n) \quad , \quad k_2 = hf(x_n + h, y_n + k_1)$$

3rd order:

$$y_n = y_{n-1} + K$$

$$k_1 = hf(x_0, y_0) \quad , \quad k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1) \quad , \quad K = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

4th order:

$$y_n = y_{n-1} + \frac{k}{6}$$

$$k = k_1 + 2k_2 + 2k_3 + k_4$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

2^{nd} order:

Q. $\dfrac{dy}{dx} = x^2+y$ $\qquad x_0=0, y_0=1$

$y=?$ when $x=0.02$

$h=0.01$

Using R.K 2^{nd} order:

$K_1 = hf(x_0,y_0) = 0.01$
$K_2 = hf(x_0+h, y_n+k_1) = 0.010$
$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$

$= 1+\frac{1}{2}(0.01+0.01)$

$\boxed{= 1.010}$

Again,
$K_1 = hf(x_1,y_1) = 0.01$
$K_2 = hf(x_1+h, y_1+k_1) = 0.01$

$$y_2 = y_1 + \frac{1}{2}(k_1+k_2)$$

$$\boxed{= 1.020}$$

3^{rd} order:

Q. $y' = 3x+y^2$ $\qquad x_0=0 \;,\; y_0 = 1$

$y = ?$ when $x = 0.1$

$$k_1 = hf(x_0,y_0) = 0.1 f(0,1)$$
$$= 0.1$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = 0.125$$

$$k_3 = hf(x_0+h, y_0+2k_2-k_1) = 0.1263$$

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$= \frac{1}{6}(0.1 + 4\times 0.125 + 0.1623)$$

$$= 0.127$$

$$y_1 = y_0 + K$$
$$= 1 + 0.127$$
$$\boxed{= 1.127}$$

4^{th} order:

Q. $y' = x + y$ $x_0 = 0$ $y_0 = 1$

$h = 0.1$

find y when $x = 0.2$

$$k_1 = hf(x_0, y_0) = 0.1$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1105$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.12105$$

$$y_1 = y_0 + \frac{k}{6} = y_0 + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$\boxed{= 1.1103}$$

$y(0.2) = y_2 = ?$

Find k_1, k_2, k_3, k_4

$$y_2 = y_1 + \frac{k}{6} = y_1 + \left(\frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\right)$$

$$\boxed{= 1.2423}$$

PREDICTOR-CORRECTOR

MILNE'S METHOD:

If $y(x_1), y(x_2)$ and $y(x_3)$ are not given, find them using any of the previous methods.

Find,

$f_0 = f(x_0, y_0)$ $\quad$ $f_1 = f(x_1, y_1)$

$f_2 = f(x_2, y_2)$ $\quad$ $f_3 = f(x_3, y_3)$

By Milne's predictor method:

$$y_4 = y_0 + \frac{4}{3}h(2f_1 - f_2 + 2f_3)$$

Then find,

$f_4 = f(x_4, y_4)$

By milne's corrector method:

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

Q. Use R.K method of 4^{th} order, $y = ?$ when $x = 0.1, 0.2, 0.3$

$y' = xy + y^2$, $x_0 = 0$, $y_0 = 1$

Continue the solution at $x = 0.4$ using Milne's method.

$y(0.1)$:

$K_1 = hf(x_0, y_0) = (0.1)(f(0,1)) = 0.1$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1155$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1172$

$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1360$

$K = (K_1 + 2K_2 + 2K_3 + K_4) = 0.7014$

$\therefore y_1 = y(0.1) = y_0 + \frac{K}{6} = 1.1169$

Made with Goodnotes

$y(0.2) = 1.2774$ (By R.K methods)
$y(0.3) = 1.5041$

Applying Milne's method,

$x_0 = 0.0$	$y_0 = 1$	$f_0 = 1$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2774$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5041$	$f_3 = 2.7132$

By Predictor method,

$$y_4 = y_0 + \frac{4}{3}h(2f_1 - f_2 + 2f_3) = 1.8334$$

$\Rightarrow$

$x_4 = 0.4$, $y_4 = 1.8334$, $f_4 = 4.0988$

By Corrector method,

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) = \boxed{1.8387}$$

ADAM'S BASHFORTH:

Same as milne's method,

Predictor formula:

$$y_4 = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0)$$

Corrector formula:

$$y_4 = y_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1)$$

Q. $y' = x - y^2$ $\qquad y = ?$ at $x = 0.8$

$x_0 = 0.0$	$y_0 = 0$	$f_0 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$f_1 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$f_2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$f_3 = 0.5690$

By predictor formula,

$$y_4 = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0)$$

$$y_4 = 0.3050$$

$x_4 = 0.8$ $y_4 = 0.3050$, $f_4 = 0.7070$

By corrector formula,

$$y_4 = y_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1)$$

$$= 0.3046$$

Repeat corrector till y_4 has same values.

BVPs

Finite Difference:

$$h = \frac{x_n - x_0}{n}$$

$$y_i' = \frac{1}{2h}\left[y_{i+1} - y_{i-1} \right]$$

$$y_i'' = \frac{1}{h^2}\left[y_{i+1} - 2y_i + y_{i-1} \right]$$

Q. $y'' = x + y$

$x_0 = 0 \qquad y_0 = 0$

$x_n = 1 \qquad y_n = 0$

$n = 4$

$$h = \frac{x_n - x_0}{4} = \frac{1}{4}$$

$x_0 = 0 \quad , \quad x_1 = \frac{1}{4} \quad , \quad x_2 = \frac{2}{4} \quad , \quad x_3 = \frac{3}{4}$

$x_4 = 1$

$$y''_i = \frac{1}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right] = x_i + y_i$$

$$= 16\left[y_{i+1} - 2y_i + y_{i-1}\right] = x_i + y_i$$

$i = 1$

$$16(y_2 - 2y_1 + y_0) = x_1 + y_1$$

$$\Rightarrow 16y_2 - 33y_1 = \frac{1}{4}$$

$i = 2$

$$\Rightarrow 16y_3 - 33y_2 + 16y_1 = \frac{1}{2}$$

$i = 3$

⋮

Solve these equ. to find values of y_1, y_2, y_3.