INTRODUCTION TO ASSIGNMENT PROBLEM

- An assignment problem is a particular case of transportation problem.
- The objective is to assign a number of resources to an equal number of activities .
- So as to minimize total cost or maximize total profit of allocation.
- The problem of assignment arises because available resources such as men, machines etc. have varying degrees of efficiency for performing different activities, therefore, cost, profit or loss of performing the different activities is different.

- Suppose that we have n jobs to be performed on m machines (one job to one machine).
- Our objective is to assign the jobs to the machines at the minimum cost (or maximum profit).
- Under the assumption that each machine can perform each job but with varying degree of efficiencies.

MATRIX FORM OF ASSIGNENMT PROBLEM

- The assignment problem can be stated that in the form of mxn matrix c_{ij} called a Cost Matrix (or) Effectiveness Matrix where c_{ij} is the cost of assigning i^{th} machine to j^{th} job.

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

MATHEMATICAL FORMULATION OF AN ASSIGNMENT PROBLEM

- Consider an assignment problem of assigning n jobs to n machines (one job to one machine). Let c_{ij} be the unit cost of assigning ith machine to the jth job and, i^{th} machine to j^{th} job.

- Let $x_{ij} = 1$, if j^{th} job is assigned to i^{th} machine.

  $x_{ij} = 0$, if j^{th} job is not assigned to i^{th} machine.

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, ..., n$$

$$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, ..., n$$

$$x_{ij} = 0 \text{ or } 1.$$

DIFFERENCE BETWEEN TRANSPORTATION PROBLEM AND ASSIGMENT PROBLEM

S.No	Transportation Problem	Assignment Problem
1	Supply at any source may be any positive quantity ai.	Supply at any source (machine)will be 1. i.e.,$a_i = 1$.
2	Demand at any destination may be any positive quantity bj.	Demand at any destination (job) will be 1. i.e.,$b_j = 1$.
3	One or more source to any number of destinations.	One source (machine) to only one destination (job).

ASSIGMENT ALGORITHM (OR) HUNGARIAN METHOD

- First check whether the number of rows is equal to number of columns, if it is so, the assignment problem is said to be balanced. Then proceed to step 1. If it is not balanced, then it should be balanced before applying the algorithm.
- **Step 1:** Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.
- **Step 2:** Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step 1 and make sure each column contains atleast one zero.

- **Step 3: (Assigning the zeros)**

  (a) Examine the rows successively until a row with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it. Cross all other zeros in the column of this encircled zero, as these will not be considered for any future assignment. Continue in this way until all the rows have been examined.

  (b) Examine the columns successively until a column with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it and cross any other zero in its row. Continue until all the columns have been examined.

- **Step 4: (Apply Optimal Test)**

  (a) If each row and each column contain exactly one encircled zero, then the current assignment is optimal.

  (b) If atleast one row or column is without an assignment (i.e., if there is atleast one row or column is without one encircled zero), then the current assignment is not optimal. Go to step 5. Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step 1 and make sure each column contains atleast one zero.

- **Step 5: Cover all the zeros by drawing a minimum number of straight lines as follows:**

  (a) Mark the rows that do not have assignment.

  (b) Mark the columns (not already marked) that have zeros in marked rows.

  (c) Mark the rows (not already marked) that have assignments in marked columns.

  (d) Repeat (b) and (c) until no more marking is required.

  (e) Draw lines through all unmarked rows and marked columns. If the number of these lines is equal to the order of the matrix then it is an optimum solution otherwise not.

- **Step 6:** Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

- **Step 7:** Repeat steps (1) to (6), until an optimum assignment is obtained.

EXAMPLE OF ASSIGMENT PROBLEMS

PROBLEM 1: Solve the following assignment problem shown in Table using Hungarian method. The matrix entries are processing time of each man in hours.

$$\begin{bmatrix} & I & II & III & IV & V \\ 1 & 20 & 15 & 18 & 20 & 25 \\ 2 & 18 & 20 & 12 & 14 & 15 \\ 3 & 21 & 23 & 25 & 27 & 25 \\ 4 & 17 & 18 & 21 & 23 & 20 \\ 5 & 18 & 18 & 16 & 19 & 20 \end{bmatrix}$$

Solution: The given problem is balanced with 5 job and 5 men.

$$A = \begin{bmatrix} 20 & 15 & 18 & 20 & 25 \\ 18 & 20 & 12 & 14 & 15 \\ 21 & 23 & 25 & 27 & 25 \\ 17 & 18 & 21 & 23 & 20 \\ 18 & 18 & 16 & 19 & 20 \end{bmatrix}$$

Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

$$A = \begin{bmatrix} 5 & 0 & 3 & 5 & 10 \\ 6 & 8 & 0 & 2 & 3 \\ 0 & 2 & 4 & 6 & 4 \\ 0 & 1 & 4 & 6 & 3 \\ 2 & 2 & 0 & 3 & 4 \end{bmatrix}$$

Subtract the smallest cost element of each Column from all the elements in the Column of the given cost matrix. See that each Column contains atleast one zero.

$$A = \begin{bmatrix} 5 & 0 & 3 & 3 & 7 \\ 6 & 8 & 0 & 0 & 0 \\ 0 & 2 & 4 & 4 & 1 \\ 0 & 1 & 4 & 4 & 0 \\ 2 & 2 & 0 & 1 & 1 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} 5 & \boxed{0} & 3 & 3 & 7 \\ 6 & 8 & \cancel{0} & \boxed{0} & \cancel{0} \\ \boxed{0} & 2 & 4 & 4 & 1 \\ \cancel{0} & 1 & 4 & 4 & \boxed{0} \\ 2 & 2 & \boxed{0} & 1 & 1 \end{bmatrix}$$

- Since each row and each column contain exactly one encircled zero, then the current assignment is optimal.
- Where the optimal assignment is as 1 to II , 2 to IV , 3 to I , 4 to V and 5 to III.
- The optimal $z = 15 + 14 + 21 + 20 + 16 = 86$ hours.

PROBLEM 2: Solve the following assignment problem shown in Table using Hungarian method. The matrix entries are processing time of each Job to each machine in hours.

$$
\begin{array}{c c}
\text{J/M} & \begin{array}{ccccc} I & II & III & IV & V \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
9 & 22 & 58 & 11 & 19 \\
43 & 78 & 72 & 50 & 63 \\
41 & 28 & 91 & 37 & 45 \\
74 & 42 & 27 & 49 & 39 \\
36 & 11 & 57 & 22 & 25
\end{bmatrix}
\end{array}
$$

Solution: The given problem is balanced with 5 job and 5 machine.

$$A = \begin{bmatrix} 9 & 22 & 58 & 11 & 19 \\ 43 & 78 & 72 & 50 & 63 \\ 41 & 28 & 91 & 37 & 45 \\ 74 & 42 & 27 & 49 & 39 \\ 36 & 11 & 57 & 22 & 25 \end{bmatrix}$$

Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

$$A = \begin{bmatrix} 0 & 13 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 46 & 11 & 14 \end{bmatrix}$$

Subtract the smallest cost element of each Column from all the elements in the Column of the given cost matrix. See that each Column contains atleast one zero.

$$A = \begin{bmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & 4 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} \cancel{0} & 13 & 49 & \boxed{0} & \cancel{0} \\ \boxed{0} & 35 & 29 & 5 & 10 \\ 13 & \boxed{0} & 63 & 7 & 7 \\ 47 & 15 & \boxed{0} & 20 & 2 \\ 25 & \cancel{0} & 46 & 9 & 4 \end{bmatrix}$$

Marking the unassigning the zeros row and crosed zero colunm of marked row,we have the following $A =$

$$\begin{array}{cc}
\begin{bmatrix}
\cancel{0} & 13 & 49 & \boxed{0} & \cancel{0} \\
\boxed{0} & 35 & 29 & 5 & 10 \\
13 & \boxed{0} & 63 & 7 & 7 \\
47 & 15 & \boxed{0} & 20 & 2 \\
25 & \cancel{0} & 46 & 9 & 4
\end{bmatrix} & \begin{array}{c} \\ \\ \checkmark \\ \\ \checkmark \end{array} \\
\begin{array}{ccccc} & \checkmark & & & \end{array} &
\end{array}$$

Crossing the marked column and unmarked row,we have the following $A =$

$$\begin{array}{cc}
\begin{bmatrix}
\cancel{0} & 13 & 49 & \boxed{0} & \cancel{0} \\
\boxed{0} & 35 & 29 & 5 & 10 \\
13 & \boxed{0} & 63 & 7 & 7 \\
47 & 15 & \boxed{0} & 20 & 2 \\
25 & \cancel{0} & 46 & 9 & 4
\end{bmatrix} & \begin{array}{c} \\ \\ \checkmark \\ \\ \checkmark \end{array} \\
\begin{array}{ccccc} & \checkmark & & & \end{array} &
\end{array}$$

Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

$$A = \begin{bmatrix} 0 & 17 & 49 & 0 & 0 \\ 0 & 39 & 29 & 5 & 10 \\ 9 & 0 & 59 & 3 & 3 \\ 47 & 19 & 0 & 20 & 2 \\ 21 & 0 & 42 & 5 & 0 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} \cancel{0} & 17 & 49 & \boxed{0} & \cancel{0} \\ \boxed{0} & 39 & 29 & 5 & 10 \\ 9 & \boxed{0} & 59 & 3 & 3 \\ 47 & 19 & \boxed{0} & 20 & 2 \\ 21 & \cancel{0} & 42 & 5 & \boxed{0} \end{bmatrix}$$

- Since each row and each column contain exactly one encircled zero, then the current assignment is optimal.
- Where the optimal assignment is as 1 to IV , 2 to I , 3 to II , 4 to III and 5 to V.
- The optimal $z = 11 + 43 + 28 + 27 + 25 = 134$ hours.

PROBLEM 3: Solve the assignment problem At the head office of a company there are five registration counters. Five persons are available for service. How should the counters be assigned to persons so as to maximize the profit?

$$\begin{bmatrix} C/P & A & B & C & D & E \\ 1 & 30 & 37 & 40 & 28 & 40 \\ 2 & 40 & 24 & 27 & 21 & 36 \\ 3 & 40 & 32 & 33 & 30 & 35 \\ 4 & 25 & 38 & 40 & 36 & 36 \\ 5 & 29 & 62 & 41 & 34 & 39 \end{bmatrix}$$

Solution: The given problem is balanced with 5 job and 5 machine.
To convert the problem as minimization we reduce the matrix by subtracting all entry by the largest value , that is 62

$$A = \begin{bmatrix} 32 & 25 & 22 & 34 & 22 \\ 22 & 38 & 35 & 41 & 26 \\ 22 & 30 & 29 & 30 & 27 \\ 37 & 24 & 22 & 26 & 26 \\ 33 & 0 & 21 & 28 & 23 \end{bmatrix}$$

Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

$$A = \begin{bmatrix} 10 & 3 & 0 & 12 & 0 \\ 0 & 16 & 13 & 19 & 4 \\ 0 & 8 & 7 & 8 & 5 \\ 15 & 2 & 0 & 4 & 4 \\ 33 & 0 & 21 & 28 & 23 \end{bmatrix}$$

Subtract the smallest cost element of each Column from all the elements in the Column of the given cost matrix. See that each Column contains atleast one zero.

$$A = \begin{bmatrix} 10 & 3 & 0 & 8 & 0 \\ 0 & 16 & 13 & 15 & 4 \\ 0 & 8 & 7 & 4 & 5 \\ 15 & 2 & 0 & 0 & 4 \\ 33 & 0 & 21 & 24 & 23 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} 10 & 3 & \cancel{0} & 8 & \boxed{0} \\ \boxed{0} & 16 & 13 & 15 & 4 \\ \cancel{0} & 8 & 7 & 4 & 5 \\ 15 & 2 & \cancel{0} & \boxed{0} & 4 \\ 33 & \boxed{0} & 21 & 24 & 23 \end{bmatrix}$$

Marking the unassigning the zeros row and crosed zero colunm of marked row,we have the following $A =$

$$\begin{array}{cc}
\begin{bmatrix}
10 & 3 & \cancel{0} & 8 & \boxed{0} \\
\boxed{0} & 16 & 13 & 15 & 4 \\
\cancel{0} & 8 & 7 & 4 & 5 \\
15 & 2 & \cancel{0} & \boxed{0} & 4 \\
33 & \boxed{0} & 21 & 24 & 23
\end{bmatrix} & \begin{matrix} \\ \checkmark \\ \checkmark \\ \\ \\ \end{matrix} \\
\begin{matrix} \checkmark & \quad & \quad & \quad & \quad \end{matrix} &
\end{array}$$

Crossing the marked column and unmarked row,we have the following $A =$

$$\begin{bmatrix} 10 & 3 & \cancel{0} & 8 & \boxed{0} \\ \boxed{0} & 16 & 13 & 15 & 4 \\ \cancel{0} & 8 & 7 & 4 & 5 \\ 15 & 2 & \cancel{0} & \boxed{0} & 4 \\ 33 & \boxed{0} & 21 & 24 & 23 \end{bmatrix}$$

Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

$$A = \begin{bmatrix} 14 & 3 & 0 & 8 & 0 \\ 0 & 12 & 9 & 11 & 0 \\ 0 & 4 & 3 & 0 & 1 \\ 19 & 2 & 0 & 0 & 4 \\ 37 & 0 & 21 & 24 & 23 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} 14 & 3 & \boxed{0} & 8 & \cancel{0} \\ \cancel{0} & 12 & 9 & 11 & \boxed{0} \\ \boxed{0} & 4 & 3 & \cancel{0} & 1 \\ 19 & 2 & \cancel{0} & \boxed{0} & 4 \\ 37 & \boxed{0} & 21 & 24 & 23 \end{bmatrix}$$

- Since each row and each column contain exactly one encircled zero, then the current assignment is optimal.
- Where the optimal assignment is as 1 to C , 2 to E , 3 to A , 4 to D and 5 to B.
- The Maximize profit is $z = 40 + 36 + 40 + 36 + 62 = 214$.

PROBLEM 4: Solve the assignment problem

$$\begin{array}{cc} \text{M/J} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 7 & 5 & 8 & 4 \\ 5 & 6 & 7 & 4 \\ 8 & 7 & 9 & 8 \end{bmatrix} \end{array}$$

Solution: The given problem is not balanced with 3 men and 4 job.
To convert the problem as balanced by adding a dummy row with 0 cost we have.

$$A = \begin{bmatrix} 7 & 5 & 8 & 4 \\ 5 & 6 & 7 & 4 \\ 8 & 7 & 9 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

$$A = \begin{bmatrix} 3 & 1 & 4 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} 3 & 1 & 4 & \boxed{0} \\ 1 & 2 & 3 & \cancel{0} \\ 1 & \boxed{0} & 2 & 1 \\ \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

Marking the unassigning the zeros row and crosed zero colunm of marked row,we have the following $A =$

$$\begin{array}{c c}
\begin{bmatrix} 3 & 1 & 4 & \boxed{0} \\ 1 & 2 & 3 & \cancel{0} \\ 1 & \boxed{0} & 2 & 1 \\ \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix} & \begin{array}{c} \checkmark \\ \checkmark \\ \\ \\ \end{array} \\
\begin{array}{cccc} & & & \checkmark \end{array} &
\end{array}$$

Crossing the marked column and unmarked row,we have the following $A =$

$$\begin{bmatrix} 3 & 1 & 4 & \boxed{0} \\ 1 & 2 & 3 & \cancel{0} \\ 1 & \boxed{0} & 2 & 1 \\ \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

$$A = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} 2 & \cancel{0} & 2 & \boxed{0} \\ \boxed{0} & 1 & 2 & \cancel{0} \\ 1 & \boxed{0} & 2 & 2 \\ \cancel{0} & \cancel{0} & \boxed{0} & 1 \end{bmatrix}$$

- Since each row and each column contain exactly one encircled zero, then the current assignment is optimal.
- Where the optimal assignment is as 1 to D , 2 to A , 3 to B , d to C .
- The optimal solution is $z = 4 + 5 + 7 + 0 = 16$.

QUESTION TO ANSWER

PROBLEM 1: Solve the following assignment problem .

$$\begin{bmatrix} \text{J/M} & A & B & C & D \\ 1 & 1 & 4 & 6 & 3 \\ 2 & 9 & 7 & 10 & 9 \\ 3 & 4 & 5 & 11 & 7 \\ 4 & 8 & 7 & 8 & 5 \end{bmatrix}$$

Answer:1 to A, 2 to C, 3 to B, 4 to D . Min cost =21

PROBLEM 2: Solve the assignment problem

$$\begin{array}{c|c} \text{M/J} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \end{bmatrix} \end{array}$$

Answer:1 to A, 2 to B, 3 to C, 4 to D . Min cost =50

PROBLEM 3: Solve the following assignment problem to get maximum profit .

J/M	A	B	C	D
1	35	27	28	37
2	28	34	29	40
3	35	24	32	28
4	24	32	25	28

Answer:1 to A, 2 to D, 3 to C, 4 to B . Max profit $=139$