Topic: Solve a problem using GeoGebra.

Example 3.2-1

Consider the following LP with two variables:

$$\text{Maximize } z = 2x_1 + 3x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\le 4 \\ x_1 + 2x_2 &\le 5 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Figure 3.2 provides the graphical solution space for the problem.

3.4 ARTIFICIAL STARTING SOLUTION

As demonstrated in Example 3.3-1, LPs in which all the constraints are $(\leq)$ with nonnegative right-hand sides offer a convenient all-slack starting basic feasible solution. Models involving $(=)$ and/or $(\geq)$ constraints do not.

The procedure for starting "ill-behaved" LPs with $(=)$ and $(\geq)$ constraints is to use **artificial variables** that play the role of slacks at the first iteration. The artificial variables are then disposed of at a later iteration. Two closely related methods are introduced here: the *M*-method and the two-phase method.

3.4.1 *M*-Method[6]

The *M*-method starts with the LP in equation form (Section 3.1). If equation i does not have a slack (or a variable that can play the role of a slack), an artificial variable, R_i, is added to form a starting solution similar to the all-slack basic solution. However, because the artificial variables are not part of the original problem, a modeling "trick" is needed to force them to zero value by the time the optimum iteration is reached (assuming the problem has a feasible solution). The desired goal is achieved by assigning a **penalty** defined as:

$$\text{Artificial variable objective function coefficient} = \begin{cases} -M, \text{ in maximization problems} \\ M, \text{ in minimization problems} \end{cases}$$

M is a sufficiently large positive value (mathematically, $M \rightarrow \infty$).

3-37. Consider the following set of constraints:

$$-2x_1 + 3x_2 = 3 \quad (1)$$
$$4x_1 + 5x_2 \geq 10 \quad (2)$$
$$x_1 + 2x_2 \leq 5 \quad (3)$$
$$6x_1 + 7x_2 \leq 3 \quad (4)$$
$$4x_1 + 8x_2 \geq 5 \quad (5)$$
$$x_1, x_2 \geq 0$$

For each of the following problems, develop the z-row after substituting out the artificial variables:

(a) Maximize $z = 5x_1 + 6x_2$ subject to (1), (3), and (4).

(b) Maximize $z = 2x_1 - 7x_2$ subject to (1), (2), (4), and (5).

(c) Minimize $z = 3x_1 + 6x_2$ subject to (3), (4), and (5).

(d) Minimize $z = 4x_1 + 6x_2$ subject to (1), (2), and (5).

(e) Minimize $z = 3x_1 + 2x_2$ subject to (1) and (5).

So we have the following problem:

Minimize $z = 3x_1 + 6x_2$ (Objective Function)

subject to

$-2x_1 + 3x_2 = 3$ (Constraint 1)

$4x_1 + 5x_2 \geq 10$ (Constraint 2)

$4x_1 + 8x_2 \geq 5$ (Constraint 5)

$x_1, x_2 \geq 0$ (Non-negativity constraint)

Solution:

Problem 3-37 d:

objective Eqn is

minimize: $Z - 4x_1 - 6x_2 - MR_1 - MR_2 - MR_3 = 0$

s.t

$$-2x_1 + 3x_2 + R_1 = 3$$

$$4x_1 + 5x_2 - S_1 + R_2 = 10$$

$$4x_1 + 8x_2 - S_2 + R_3 = 5$$

Using $M = 10$, the starting tableau is.

Basic	x_1	x_2	S_1	S_2	R_1	R_2	R_3	R.H.V
Z	-4	-6	0	0	-10	-10	-10	0
R_1	-2	3	0	0	1	0	0	3
R_2	4	5	-1	0	0	1	0	10
R_3	4	8	0	-1	0	0	1	5

Iteration 1

R_1-row + 10(R_2 + R_3 + R_4)

Row2 Row3 Row4

Basic	x_1	x_2	S_1	S_2	R_1	R_2	R_3	R.H.V	Ratio
Z	56	154	-10	-10	0	0	0	180	
R_1	-2	3	0	0	1	0	0	3	1
R_2	4	5	-1	0	0	1	0	10	2
R_3	4	8	0	-1	0	0	1	5	0.625

Row1 $- 154\left(\frac{\text{Row4}}{8}\right)$, Row2 $- 3\left(\frac{\text{Row4}}{8}\right)$, Row3 $- 5\left(\frac{\text{Row4}}{8}\right)$

$\frac{\text{Row4}}{8}$

Basic	x_1	x_2	S_1	S_2	R_1	R_2	R_3	R.H.V	Ratio
Z	-21	0	-10	$\frac{37}{4}$	0	0	$-\frac{77}{4}$	$\frac{335}{4}$	
R_1	$-7/2$	0	0	$3/8$	1	0	$-3/8$	$9/8$	3
R_2	$3/2$	0	-1	$5/8$	0	1	$-5/8$	$55/8$	11
x_2	$1/2$	1	0	$-1/8$	0	0	$1/8$	$5/8$	nA

$S_2 \to$ Incoming, $R_1 \to$ Leaving.

$\frac{8}{3}$ Row2 , Row1 $- \frac{37}{4}\left(\frac{8}{3}\text{Row}_2\right)$, Row3 $- \frac{5}{8}\left(\frac{8}{3}\text{Row2}\right)$, Row4 $+ \frac{1}{8}\left(\frac{8}{3}\text{Row2}\right)$

Z	$196/3$	0	-10	0	$-74/3$	0	-10	56
S_2	$-28/3$	0	0	1	$8/3$	0	-1	3
R_2	$22/3$	0	-1	0	$-5/3$	1	0	5
~~R_2~~ x_2	$-2/3$	1	0	0	$1/3$	0	0	1

$\frac{3}{22}$ Row3 , Row1 $- \frac{196}{3} \times \frac{3}{22}$ Row3 , Row2 $+ \frac{28}{3} \times \frac{3}{22}$ Row3, Row4 $+ \frac{1}{11}$ Row3

Z	0	0	$-12/11$	0	$-108/11$	$-98/11$	-10	$\boxed{126/11}$ optimal.
S_2	0	0	$-14/11$	1	$6/11$	$14/11$	-1	$103/11$
x_1	1	0	$-3/22$	0	$-5/22$	$3/22$	0	$15/22$
x_2	0	1	$-1/11$	0	$2/11$	$1/11$	0	$16/11$

3-38. Consider the following set of constraints:

$$x_1 + x_2 + x_3 = 7$$

$$2x_1 - 5x_2 + x_3 \geq 10$$

$$x_1, x_2, x_3 \geq 0$$

Solve the problem for each of the following objective functions:

(a) Maximize $z = 2x_1 + 3x_2 - 5x_3$.

(b) Minimize $z = 2x_1 + 3x_2 - 5x_3$.

(c) Maximize $z = x_1 + 2x_2 + x_3$.

(d) Minimize $z = 4x_1 - 8x_2 + 3x_3$.

Solution:

Problem 3-38(a)

Soln Max $Z = 2x_1 + 3x_2 - 5x_3 - MS_1 - MS_2$

s.t

$$x_1 + x_2 + x_3 + S_1 = 7$$

$$2x_1 - 5x_2 + x_3 - x_4 + S_2 = 10$$

$$x_i \geq 0, \; S_1, S_2 \geq 0 \quad \because i = 1,2,3,4.$$

The tableau form is By Taking $M = 50$

Basic	x_1	x_2	x_3	x_4	S_1	S_2	R.H.V
Z	-2	-3	5	0	M=50	M=50	0
S_1	1	1	1	0	1	0	7
S_2	2	-5	1	-1	0	1	10

$R_1 - 50(R_2 + R_3)$

Basic	x_1	x_2	x_3	x_4	S_1	S_2	R.H.V	Ratio.
Z	-152	197	-95	50	0	0	-850	
S_1	1	1	1	0	1	0	7	7
S_2	2	-5	1	-1	0	1	10	5

$x_1 \rightarrow$ Incoming variable, $S_2 \rightarrow$ outgoing

$\frac{R_3}{2}$, $R_1 + 152\left(\frac{R_3}{2}\right)$, $R_2 - \frac{R_3}{2}$

Basic	x_1	x_2	x_3	x_4	S_1	S_2	R.H.V
Z	0	$\cancel{-178}$ -183	$\cancel{-20}$ -19	-26	0	76	-90
S_1	0	7/2	1/2	1/2	1	-1/2	2
x_1	1	-5/2	1/2	-1/2	0	1/2	5

$x_2 \rightarrow$ Incoming , $S_1 \rightarrow$ outgoing

$\frac{2}{7}R_2$, $R_1 + 183(\frac{2}{7}R_2)$, $R_3 + \frac{5}{2}(\frac{2}{7}R_2)$

Basic	x_1	x_2	x_3	x_4	S_1	S_2	R.H.V
Z	0	0	$50/7$	$1/7$	$366/7$	$349/7$	$\boxed{102/7}$ optimal
x_2	0	1	$1/7$	$1/7$	$2/7$	$-1/7$	$4/7$
x_1	1	0	$6/7$	$-1/7$	$5/7$	$1/7$	$45/7$

Here

$x_2 = \frac{4}{7}$, $x_1 = \frac{45}{7}$, $S_1 = 0$, $S_2 = 0$, $x_3 = 0$, $x_4 = 0$

$Z = \frac{102}{7}$

***3-39.** Consider the problem

$$\text{Maximize } z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

subject to

$$x_1 + x_2 + x_3 \qquad = 4$$

$$x_1 + 4x_2 \qquad + x_4 = 8$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solve the problem with x_3 and x_4 as the starting basic variables and *without using any artificial variables*. (*Hint*: x_3 and x_4 play the role of slack variables. The main difference is that they have nonzero objective coefficients.)

Remarks. The use of the penalty M will not force an artificial variable to zero in the final simplex iteration if the LP does not have a feasible solution (i.e., the constraints cannot be satisfied simultaneously). In this case, the final simplex iteration will include at least one artificial variable with a positive value. Section 3.5.4 explains this situation.

3.4.2 Two-Phase Method

In the M-method, the use of the penalty, M, can result in computer roundoff error. The two-phase method eliminates the use of the constant M altogether. As the name suggests, the method solves the LP in two phases: Phase I attempts to find a starting basic feasible solution, and, if one is found, Phase II is invoked to solve the original problem.

Summary of the Two-Phase Method

Phase I. Put the problem in equation form, and add the necessary artificial variables to the constraints (exactly as in the M-method) to secure a starting basic solution. Next, find a basic solution of the resulting equations that *always* minimizes the sum of the artificial variables, regardless of whether the LP is maximization or minimization. If the minimum value of the sum is positive, the LP problem has no feasible solution. Otherwise, proceed to Phase II.

Phase II. Use the feasible solution from Phase I as a starting basic feasible solution for the *original* problem.

Problem 3-48 Two phase method:

Soln:

Phase 1: Min $z' = S_1 + S_2 + S_3$

S.t

$$2x_1 + x_2 + x_3 + S_1 = 4$$

$$x_1 + 3x_2 + x_3 + S_2 = 12$$

$$3x_1 + 4x_2 + 2x_3 + S_3 = 16$$

The starting tableau is,

Basic	x_1	x_2	x_3	S_1	S_2	S_3	R.H.V
z'	0	0	0	-1	-1	-1	0
S_1	2	1	1	1	0	0	4
S_2	1	3	1	0	1	0	12
S_3	3	4	2	0	0	1	16

Here, S_1, S_2, S_3 are Basic Soln and we try to vanish them in z'-row.
For this,
$R_1 + (R_2 + R_3 + R_4)$

								Ratio
z'	6	8	4	0	0	0	32	
S_1	2	1	1	1	0	0	4	4
S_2	1	3	1	0	1	0	12	4
S_3	3	4	2	0	0	1	16	4

→ Pick Arbitrary

$x_2 \rightarrow$ Entering variable, $S_1 \rightarrow$ outgoing.

$R_1 - 8R_2$, $R_3 - 3R_2$, $R_4 - 4R_2$

Basic	x_1	x_2	x_3	S_1	S_2	S_3	R.H.V
Z'	-10	0	-4	-8	0	0	0
x_2	2	1	1	1	0	0	4
S_2	-5	0	-2	-3	1	0	0
S_3	-6	0	-2	-4	0	1	0

Here we have $x_2 = 4$, $Z' = 0$, $S_2 = 0$, $S_3 = 0$, $S_1 = 0$. $x_1 = 0$

Phase II

Basic	x_1	x_2	x_3	S_1	S_2	S_3	R.H.V
Z	-3	-2	-3	0	0	0	0
x_2	2	1	1	1	0	0	4
S_2	-5	0	-2	-3	1	0	0
S_3	-6	0	-2	-4	0	1	0

$x_3 \rightarrow$ Entering, $S_2 \rightarrow$ Leaving.

$R_1 + 3\left(\frac{R_3}{-2}\right)$, $R_2 - 1\left(\frac{R_3}{-2}\right)$, $\frac{R_3}{-2}$, $R_4 - R_3$

DIY