Research Based On
Personal Understanding of Quantitative Finance

Quantitative Finance

An in-depth analysis

Author: Dr. Acharya, Dr. Khan

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This document is created by Dr. Acharya and Dr. Khan while they were studying quantitative finance.

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Chapter 1

Introduction

"I am the worst TA."
– Dr. Acharya

chapter.1

We will be introducing the basic concepts of quantitative finance in this chapter.

1.1 σ-Algebra

Let X be some set, and let $\mathcal{P}(X)$ represent its power set. Then a subset $\Sigma \subseteq \mathcal{P}(X)$ is called a σ-algebra if it satisfies the following three properties:

1. X is in Σ.

2. Σ is closed under complementation.
3. Σ is closed under countable unions (also closed under countable intersections by applying De Morgan's laws).

1.2 Measurable Space

Consider a set X and a σ-algebra Σ on X. Then the tuple (X, Σ) is called a measurable space.

1.3 Measurable function

Let (X, Σ_X) and (Y, Σ_Y) be measurable spaces. A function $f : X \to Y$ is said to be measurable if for every $E \in \Sigma_Y$ the pre-image of E under f is in Σ_X.

1.4 Measure

Let X be a set and Σ a σ-algebra over X. A function $\mu : \Sigma \to \overline{\mathbb{R}}$ is called a measure on (X, Σ) if it satisfies the following properties:

1. Non-negativity: For all E in Σ, we have$\mu(E) \geq 0$.
2. Null empty set: $\mu(\phi) = 0$.
3. Countable additivity (or σ-additivity): For all countable collections $\{E_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in Σ,
$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

1.5 Measure space

A measure space is a triple (X, Σ, μ) where,

1. X is a set.
2. $\mathcal{A}$ is a σ-algebra on the set X.
3. μ is a measure on (X, Σ).

1.6 Probability space

A probability space is a measure space such that the measure of the whole space is equal to one. The expanded definition is the following: a probability space is a triple $(\Omega, \mathcal{F}, P)$ consisting of:

1. The sample space Ω - an arbitrary non-empty set (a set of possible outcomes).
2. The σ-algebra $\mathcal{F} \subseteq 2^{\Omega}$ (also called σ-field) on Ω, called events.

3. The probability measure $P : \mathcal{F} \to [0,1]$ - a function on $\mathcal{F}$ such that the measure of entire sample space is equal to one: $P(\Omega) = 1$.

1.7 Random Variable

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ a measurable space. An $(E, \mathcal{E})$-valued random variable is a measurable function $X : \Omega \to E$, which means that, for every subset $B \in \mathcal{E}$, its preimage $X^{-1}(B) \in \mathcal{F}$ where $X^{-1}(B) = \{\omega : X(\omega) \in B\}$. This definition enables us to measure any subset $B \in \mathcal{E}$ in the target space by looking at its preimage, which by assumption is measurable.

1.8 Stochastic or random process

A stochastic or random process can be defined as a collection of random variables that is indexed by some mathematical set, meaning that each random variable of the stochastic process is uniquely associated with an element in the set. A stochastic process can be denoted, among other ways, by $\{X(t)\}_{t \in T}$, $\{X_t\}_{t \in T}$, $\{X_t\}$, $\{X(t)\}$ or simply as $X(t)$, although $X(t)$ is regarded as an abuse of function notation.

A stochastic process is defined as a collection of random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$, where Ω is a sample space, $\mathcal{F}$ is a σ -algebra, and P is a probability measure; and the random variables, indexed by some set T, all take values in the same mathematical space S, which must be measurable with respect to some σ -algebra Σ.

1.9 Random field

If the random variables are indexed by the Cartesian plane or some higher-dimensional Euclidean space, then the collection of random variables is usually called a random field instead. The values of a stochastic process are not always numbers and can be vectors or other mathematical objects.

1.10 State space

The mathematical space S of a stochastic process is called its state space. The state space S is the real line or the natural numbers, but it can be n-dimensional Euclidean space or more abstract spaces such as Banach spaces.

1.11 Discrete-time and continuous-time stochastic processes

When interpreted as time, if the index set of a stochastic process has a finite or countable number of elements, such as a finite set of numbers, the set of integers, or the natural numbers, then the stochastic process is said to be in discrete time. If the index set is some interval of the real line, then time is said to be continuous. The two types of stochastic processes are respectively referred to as discrete-time and continuous-time stochastic processes.

1.12 Increment

An increment of a stochastic process is the difference between two random variables of the same stochastic process.

1.13 Stochastic processes

Applications and the study of phenomena have in turn inspired the proposal of new stochastic processes. Examples of such stochastic processes include the Wiener process or Brownian motion process, used by Louis Bachelier to study price changes on the Paris Bourse, and the Poisson process, used by A. K. Erlang to study the number of phone calls occurring in a certain period of time. These two stochastic processes are considered the most important and central in the theory of stochastic processes, and were discovered repeatedly and independently, both before and after Bachelier and Erlang, in different settings and countries. Stochastic processes can be grouped into various categories, which include random walks, martingales, Markov processes, Levy processes, Gaussian processes, random fields, renewal processes, and branching processes.

Appendix A

All Definitions

Appendix B

Bibliography