Integration

1. Normal type maths:

(i) $\int \sin mx \, dx$

$= -\dfrac{\cos mx}{m} + c$

(ii) $\int e^{mx} dx$

$= \dfrac{e^{mx}}{m} + c$

(iii) $\int x(x^2+x-1)dx$

$= \int (x^3+x^2-x)dx$

$= \dfrac{x^4}{4} + \dfrac{x^3}{3} - \dfrac{x^2}{2} + c$

(iv) $\int (e^x - 5a^x + 2)dx$

$= e^x - 5 \cdot \dfrac{a^x}{\ln a} + 2x + c$

(v) $\int \dfrac{x+2\sqrt{x}+7}{\sqrt{x}} dx$

$= \int \left(\dfrac{x}{\sqrt{x}} + \dfrac{2\sqrt{x}}{\sqrt{x}} + \dfrac{7}{\sqrt{x}}\right) dx$

$= \int (\sqrt{x} + 2 + 7x^{-\frac{1}{2}}) dx$

$= \int (x^{\frac{1}{2}} + 7x^{-\frac{1}{2}} + 2) dx$

$= \dfrac{x^{3/2}}{3/2} + 7 \cdot \dfrac{x^{1/2}}{1/2} + 2x + c$

$= \dfrac{2}{3}x^{3/2} + 14x^{1/2} + 2x + c$

(vi) $\int (\sin 2x + \cos 2x) dx$

$= \left\{ -\dfrac{\cos 2x}{2} + \dfrac{\sin 2x}{2} \right\} + c$

$= \dfrac{1}{2}(\sin 2x - \cos 2x) + c$

Ans.

(vii) $\int \dfrac{3+4x^2+5x^4}{\sqrt[3]{x}} dx$

$= \int \left(\dfrac{3}{\sqrt[3]{x}} + \dfrac{4x^2}{\sqrt[3]{x}} + \dfrac{5x^4}{\sqrt[3]{x}}\right) dx$

$= \int (3x^{-\frac{1}{3}} + 4x^{2-\frac{1}{3}} + 5x^{4-\frac{1}{3}}) dx$

$= \int (3x^{-\frac{1}{3}} + 4x^{-5/3} + 5x^{\frac{11}{3}}) dx$

$= 3\dfrac{x^{2/3}}{2/3} + 4\dfrac{x^{-2/3}}{-2/3} + 5\dfrac{x^{\frac{14}{3}}}{14/3} + c$

$= \dfrac{9}{2}x^{2/3} - 6x^{-2/3} + \dfrac{15}{14}x^{14/3} + c$

Ans!

2. Solve after modifying:

(i) $\int \sin^2 x\, dx$

$= \frac{1}{2}\int 2\sin^2 x\, dx$

$= \frac{1}{2}\int (1-\cos 2x)\, dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

(ii) $\int \cos^4 x\, dx$

$= \frac{1}{4}\int (2\cos^2 x)^2\, dx$

$= \frac{1}{4}\int (1+\cos 2x)^2\, dx$

$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\, dx$

$= \frac{1}{4}\left(x + 2\cdot\frac{\sin 2x}{2}\right) + \frac{1}{4}\int \cos^2 2x\, dx + c'$

$= \frac{1}{4}(x + \sin 2x) + \frac{1}{8}\int 2\cos^2 2x\, dx + c'$

$= \frac{1}{4}(x+\sin 2x) + \frac{1}{8}\int (1+\cos 4x)\, dx + c'$

$= \frac{1}{4}(x+\sin 2x) + \frac{1}{8}\left(x + \frac{\sin 4x}{4}\right) + c$

$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$

Do yourself

(i) $\int \cos^2 x\, dx$

(ii) $\int \sin^4 x\, dx$

(iii) $\int (\sin^4 x - \cos^4 x)\, dx$

(iii) $\int \sin^3 x \, dx$

$= \int \sin^2 x \cdot \sin x \, dx$

$= \int (1-\cos^2 x) \sin x \, dx$

Let, $\cos x = z$

$\therefore -\sin x \, dx = dz$

So, we get

$= \int (1-z^2) \cdot (-dz)$

$= -\int (1-z^2) dz$

$= -\left[z + \frac{z^2}{2}\right] + c$

$= -\cos x + \frac{\cos^2 x}{2} + c$

(iv) $\int \sin^4 x \cos^3 x \, dx$

$\cancel{= \int (1-\cos^2 x)^2}$

$= \int \sin^4 x \cdot (1-\sin^2 x) \cos x \, dx$

Let $\sin x = z$

$\therefore \cos x \, dx = dz$

$\therefore$ We get.

$= \int z^4 (1-z^2) dz$

$= (z^4 - z^6) dz$

$= \frac{z^5}{5} - \frac{z^7}{7} + c$

$= \frac{1}{5} \sin^4 x - \frac{1}{7} \sin^7 x + c$

(v) $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx$

$= \int \frac{(\sin x + \cos x) \, dx}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x \cdot \cos x}}$

$= \int \frac{\sin x + \cos x}{\sin x + \cos x} dx$

$= \int 1 \, dx$

$= x + c$

3. UV formula: $\int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx$

Follow: LIATE where L = logarithmic function
I = Inverse "
A = Algebraic "
T = Trigonometric "
E = Exponential "

① $\int e^{ax}\sin bx\,dx$

Let, $I = \int e^{ax}\sin bx\,dx$

$= \sin bx\int e^{ax}dx - \int\left\{\frac{d}{dx}(\sin bx)\int e^{ax}dx\right\}dx$

$= \sin bx\cdot\frac{e^{ax}}{a} - \int b\cos bx\cdot\frac{e^{ax}}{a}dx + c$

$= \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx + c$

$= \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\left[\cos bx\int e^{ax}dx - \int\left\{\frac{d}{dx}(\cos bx)\int e^{ax}dx\right\}dx\right] + c$

$= \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\left[\cos bx\cdot\frac{e^{ax}}{a} - \int\frac{e^{ax}}{a}\cdot(-b\sin bx)dx\right] + c$

$= \frac{e^{ax}}{a}\sin bx - \frac{be^{ax}}{a^2}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx + c$

বা, $I + \frac{b^2}{a^2}I = \frac{e^{ax}}{a^2}[a\sin bx - b\cos bx] + c$

বা, $I\left(1+\frac{b^2}{a^2}\right) = \frac{e^{ax}}{a^2}(a\sin bx - b\cos bx) + c$

$\Rightarrow I = \frac{e^{ax}\cdot a^2}{(a^2+b^2)a^2}(a\sin bx - b\cos bx) + c$

$\therefore \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

Do yourself: ~~① $\int e^{ax}\sin$~~ ① $\int e^{ax}\cos bx\,dx$

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(iii) $\int x^2 \sin 2x \, dx$

$= \int x^2 \sin 2x \, dx$

$= x^2 \int \sin 2x \, dx - \int \left\{ \frac{d}{dx}(x^2) \int \sin 2x \, dx \right\} dx$

$= x^2 \cdot \frac{-\cos 2x}{2} - \int 2x \cdot \frac{-\cos 2x}{2} dx + c$

$= -\frac{x^2}{2} \cos 2x + \int x \cos 2x \, dx + c$

$= -\frac{x^2}{2} \cos 2x + x \int \cos 2x - \int \left\{ \frac{d}{dx}(x) \int \cos 2x \, dx \right\} dx + c$

$= -\frac{x^2}{2} \cos 2x + x \cdot \frac{\sin 2x}{2} - \int 1 \cdot \frac{\sin 2x}{2} dx + c$

$= -\frac{x^2}{2} \cos 2x + \frac{x}{2} \sin 2x - \frac{1}{2} \int \sin 2x \, dx + c$

$= -\frac{x^2}{2} \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x + C$

4. Using hard formula:

(i) $\int \frac{1}{\sqrt{16 - x^2}} dx$

$= \int \frac{1}{4\sqrt{1 - \left(\frac{x}{4}\right)^2}} dx$

$= \frac{1}{4} \int \frac{1}{\sqrt{1 - \left(\frac{x}{4}\right)^2}} dx$

$= \frac{1}{4} \sin^{-1} \frac{x/4}{1} + c$

$= \frac{1}{4} \sin^{-1} \frac{x}{4} + c$

(ii) $\int \sqrt{16 - x^2} \, dx$

$= \int \sqrt{4^2 - x^2} \, dx$

$= \frac{x\sqrt{4^2 - x^2}}{2} + \frac{4^2}{2} \sin^{-1} \frac{x}{4} + c$

$= \frac{x\sqrt{16 - x^2}}{2} + \frac{16}{2} \sin^{-1} \frac{x}{4} + c$

$= \frac{x}{2} \sqrt{16 - x^2} + 8 \sin^{-1} \frac{x}{4} + c$

Formula:

$\int \sqrt{a^2 - x^2} \, dx$

$= \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

5. Definite Integral:

(i) $\int_0^3 \frac{dx}{9+x^2}$

$= \int_0^3 \frac{dx}{3^2+x^2}$

$= \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$

$= \frac{1}{3}\left(\tan^{-1}\frac{3}{3} - \tan^{-1}\frac{0}{3}\right)$

$= \frac{1}{3}\left(\frac{\pi}{2} - 0\right)$

$= \frac{\pi}{6}$.

(ii) $\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$

Let, $\tan^{-1}x = z$

$\therefore \frac{1}{1+x^2}\,dx = dz$

x	1	0
z	$\frac{\pi}{2}$	0

$\therefore$ We get,

$= \int_0^{\pi/2} z\,dz$

$= \left[\frac{z^2}{2}\right]_0^{\pi/2}$

$= \frac{1}{2}\left(\frac{\pi^2}{4} - 0\right)$

$= \frac{\pi^2}{8}$,

(iii) $\int_0^1 x e^x\,dx$

Here, $I = \left[x\int e^x dx\right]_0^1 - \int_0^1 1\cdot e^x\,dx$

$= \left[xe^x - e^x\right]_0^1$

$= \left[(1\cdot e^1 - e^1) - (0 - e^0)\right]$

$= \left[(e-e) - (0-1)\right]$

$= 0+1$

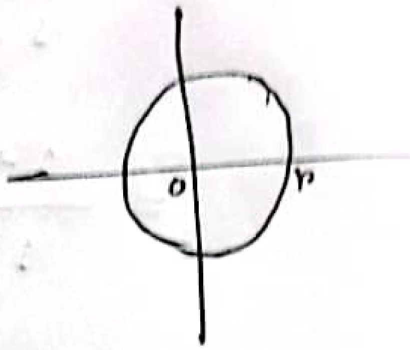
$= 1$.

6. Applications:

(i) Find the area of a circle $x^2+y^2=r^2$

Soln: Here, $x^2+y^2=r^2$

$\therefore \; y=\pm\sqrt{r^2-x^2}$



$\therefore$ Area $=4\int_0^r y\,dx$

$$=4\int_0^r \sqrt{r^2-x^2}\,dx$$

$$=4\left[\frac{x}{2}\sqrt{r^2-x^2}+\frac{r^2}{2}\sin^{-1}\frac{x}{r}\right]_0^r$$

$$=4\left[\left(\frac{r}{2}\times 0+\frac{r^2}{2}\sin^{-1}\frac{r}{r}\right)-(0+0)\right]$$

$$=4\times\frac{r^2}{2}\sin^{-1}(1)$$

$$=4\times\frac{r^2}{2}\cdot\frac{\pi}{2}$$

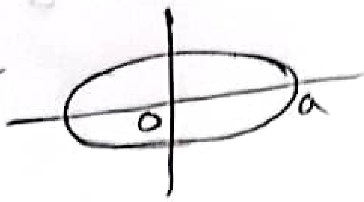
$$=\pi r^2$$

So, the area of the circle is πr^2.

(ii) Findd the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Soln: Here, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\Rightarrow y=\pm\frac{b^2}{a^2}\sqrt{a^2-x^2}$



$\therefore$ Area $=4\int_0^a y\,dx$

Do your self: Ans: πab