Pascal’s Triangle

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The triangle array of the number proposed by famous mathematician Balise Pascal is called Pascal’s Triangle. This triangle begins with 1 and in the next line the starting and the end numbers are fixed to be 1 then the middle number is generated by taking the sum of the above two numbers.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Consequently,

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^3 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Pascal’s Triangle Formula

$${}^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k}$$

Example: Find the third element in the third row of Pascal's triangle.

Solution:

We have to find the 3rd element in the 3rd row of Pascal's triangle.

Pascal Triangle Formula is,

$^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k}$

where $^{n}C_{k}$ represent $(k+1)^{th}$ element in n^{th} row.

Thus, 3rd element in the 3rd row is,

$^{3}C_{2} = {}^{2}C_{1} + {}^{2}C_{2}$

$\Rightarrow {}^{3}C_{2} = 2 + 1$

$\Rightarrow {}^{3}C_{2} = 3$

Thus, the third element in the third row of Pascal's triangle is 3.

Pascal's Triangle Binomial Expansion

We can easily find the coefficient of the binomial expansion using Pascal's Triangle. The elements in the (n+1)th row of the Pascal triangle represent the coefficient of the expanded expression of the polynomial $(a + b)^n$.

$$(a + b)^n = {}^{n}C_0 a^n b^0 + {}^{n}C_1 a^{n-1} b + {}^{n}C_2 a^{n-2} b^2 + {}^{n}C_3 a^{n-3} b^3 + \ldots\ldots + {}^{n}C_r a^{n-r} b^r + {}^{n}C_n a^0 b^n$$

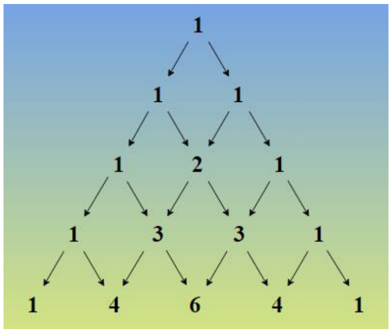
Here, (r+1)th term $= {}^{n}C_r a^{n-r} b^r$

Pascal's Triangle Examples

Example 1: Find the fifth row of Pascal's triangle.

Solution:

The Pascal triangle with 5 row is shown in the image below,



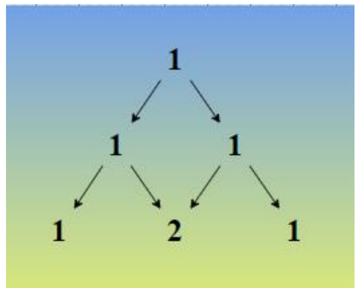
Example 2: Expand using Pascal Triangle $(a + b)^2$.

Solution:

First write the generic expressions without the coefficients.

$$(a + b)^2 = c_0a^2b^0 + c_1a^1b^1 + c_2a^0b^2$$

Now let's build a Pascal's triangle for 3 rows to find out the coefficients.



The values of the last row give us the value of coefficients.

$c_0 = 1$, $c_1 = 2$, $c_2 = 1$

$$(a + b)^2 = a^2b^0 + 2a^1b^1 + a^0b^2$$

Thus verified.

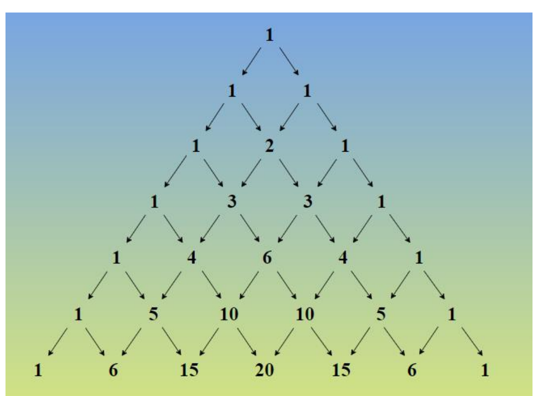
Example 3: Expand using Pascal Triangle $(a + b)^6$.

Solution:

First write the generic expressions without the coefficients.

$$(a + b)^6 = c_0a^6b^0 + c_1a^5b^1 + c_2a^4b^2 + c_3a^3b^3 + c_4a^2b^4 + c_5a^1b^5 + c_6a^0b^6$$

Now let's build a Pascal's triangle for 7 rows to find out the coefficients.



The values of the last row give us the value of coefficients.

$c_0 = 1$, $c_1 = 6$, $c_2 = 15$, $c_3 = 20$, $c_4 = 15$, $c_5 = 6$ and $c_6 = 1$.
$(a + b)^6 = 1a^6b^0 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + 1a^0b^6$

Example 4: Find the second element in the third row of Pascal's triangle.

Solution:

We have to find the 2nd element in the 3rd row of Pascal's triangle.

We know that the nth row of Pascal's triangle is ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ${}^nC_3$…
The Pascal Triangle Formula is,

$${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$$

where ${}^nC_k$ represent $(k+1)^{th}$ element in n^{th} row.

Thus, 2nd element in the 3rd row is,

$${}^3C_1 = {}^2C_0 + {}^2C_1$$
$$= 1 + 2$$
$$= 3$$

Thus, the second element in the third row of Pascal's triangle is 3.

Binomial

Binomial Theorem Statement

The binomial theorem is the method of expanding an expression that has been raised to any finite power. A binomial theorem is a powerful tool of expansion which has applications in Algebra, probability, etc

Binomial Expansion

Important points to remember

- The total number of terms in the expansion of $(x+y)^n$ is $(n+1)$
- The sum of exponents of x and y is always n.
- nC_0, nC_1, nC_2, … .., nC_n are called binomial coefficients and also represented by C_0, C_1, C_2, ….., C_n

- The binomial coefficients, which are equidistant from the beginning and from the ending, are equal, i.e., $nC_0 = nC_n$, $nC_1 = nC_{n-1}$, $nC_2 = nC_{n-2}$,….. etc.

Binomial Expansion Formula: Let $n \in N, x, y, \in R$

Then $(x + y)^n = {}^n\Sigma_{r=0}\ nC_r\ x^{n-r} \cdot y^r$

where,

$$nC_r = \frac{n!}{(n-r)!r!}$$

Some other useful expansions:

- $(x + y)^n + (x-y)^n = 2[C_0\ x^n + C_2\ x^{n-1}\ y^2 + C_4\ x^{n-4}\ y^4 + \ldots]$
- $(x + y)^n - (x-y)^n = 2[C_1\ x^{n-1}\ y + C_3\ x^{n-3}\ y^3 + C_5\ x^{n-5}\ y^5 + \ldots]$
- $(1 + x)^n = {}^n\Sigma_{r-0}\ nC_r\ .\ x^r = [C_0 + C_1\ x + C_2\ x^2 + \ldots\ C_n\ x_n]$
- $(1+x)^n + (1-x)^n = 2[C_0 + C_2\ x^2 + C_4\ x^4 + \ldots]$
- $(1+x)^n - (1-x)^n = 2[C_1\ x + C_3\ x^3 + C_5\ x^5 + \ldots]$

Binomial Theorem for Negative Index

If the rational number and $-1 < x < 1$, then,

- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots\ \infty$
- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ (-1)^r\ x^r + \ldots\ \infty$
- $(1 - x)^{-2} = 1 + 2x + 3x^2 - 4x^3 + \ldots + (r + 1)x^r + \ldots\ \infty$
- $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots + (-1)^r\ (r + 1)x^r + \ldots\ \infty$

- $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots. + \frac{n(n-1)(n-2)\ldots..(n-4+1)}{r!}\ \ldots\ldots\infty$
- $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \ldots. + (-1)'\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}x^r + \ldots..\infty$
- $(1-x)^n = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots. + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots\infty$
- $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 \ldots. \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + (-1)^r\frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots\infty$
- $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots + \frac{(r+1)(r+2)}{r!} + \ldots\infty$
- $(1+x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots + (-1)^r\frac{(r+1)(r+2)}{r!} + \ldots\infty$

General Term in Binomial Expansion:

We have $(x + y)^n = nC_0\, x^n + nC_1\, x^{n-1} . y + nC_2\, x^{n-2} . y^2 + \ldots + nC_n\, y^n$

General Term = $T_{r+1} = nC_r\, x^{n-r} . y^r$

- General Term in $(1 + x)^n$ is $nC_r\, x^r$
- In the binomial expansion of $(x + y)^n$, the r^{th} term from the end is $(n – r + 2)^{th}$

Example 1: Expand $(x/3 + 2/y)^4$

Sol:

$$\left(\frac{x}{3} + \frac{2}{y}\right)^4 = 4c_o \left(\frac{x}{3}\right)^4 + 4c_1 \left(\frac{x}{3}\right)^3 \left(\frac{2}{y}\right) + 4c_2 \left(\frac{x}{3}\right)^2 \left(\frac{2}{y}\right)^2 + 4c_3 \left(\frac{x}{3}\right) \left(\frac{2}{y}\right)^3 + 4c_4 \left(\frac{2}{y}\right)^4$$

$$\Rightarrow \frac{x^4}{81} + \frac{8x^3}{27y} + \frac{8x^2}{3y^2} + \frac{32x}{3y^3} + \frac{16}{y^4}$$

Example 2: Find the number of terms in $(1 + 2x + x^2)^{50}$

Sol:

$(1 + 2x + x^2)^{50} = [(1 + x)^2]^{50} = (1 + x)^{100}$

The number of terms = (100 + 1) = 101

Example 3: Find the fourth term from the end in the expansion of $(2x – 1/x^2)^{10}$

Sol:

Required term $= T_{10 – 4 + 2} = T_8 = 10C_7\, (2x)^3\, (−1/x^2)^7 = −960x^{-11}$

Middle Term(s) in the Expansion of $(x+y)^n$

- If n is even, then (n/2 + 1) Term is the middle term.
- If n is odd then, $[(n+1)/2]^{th}$ and $[(n+3)/2)^{th}$ terms are the middle terms.

Example 4: Find the middle term of $(1 - 3x + 3x^2 - x^3)^{2n}$

Sol:

$(1 - 3x + 3x^2 - x^3)^{2n} = [(1 - x)^3]^{2n} = (1 - x)^{6n}$

Middle Term = $[(6n/2) + 1]$ term = $6nC_{3n} (-x)^{3n}$

Find the Independent term of x

The term Independent of in the expansion of $[ax^p + (b/x^q)]^n$ is

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$, where r = (np/p+q) (integer)

Example 5: Find the independent term of x in $(x+1/x)^6$

Sol:

r = [6(1)/1+1] = 3

The independent term is $6C_3 = 20$

Applications of Binomial Theorem

The binomial theorem has a wide range of applications in Mathematics, like finding the remainder, finding the digits of a number, etc. The most common binomial theorem applications are as follows:

Finding Remainder Using Binomial Theorem

Example 7: Find the remainder when 7^{103} is divided by 25.

Sol:

$(7^{103} / 25) = [7(49)^{51} / 25)] = [7(50 - 1)^{51} / 25]$

$= [7(25K - 1) / 25] = [(175K - 25 + 25 - 7) / 25]$

$= [(25(7K - 1) + 18) / 25]$

$\therefore$ The remainder = 18

Example 8: If the fractional part of the number $(2^{403} / 15)$ is $(K/15)$, then find K.

Sol:

$(2^{403} / 15) = [2^3 (2^4)^{100} / 15]$

$= 8/15 (15 + 1)^{100} = 8/ 15 (15\lambda + 1) = 8\lambda + 8/15$

$\because 8\lambda$ is an integer, fractional part = 8/15

So, K = 8.

Finding Digits of a Number

Example 9: Find the last two digits of the number $(13)^{10}$

Sol:

$(13)^{10} = (169)^5 = (170 - 1)^5$

$= 5C_0 (170)^5 - 5C_1 (170)^4 + 5C_2 (170)^3 - 5C_3 (170)^2 + 5C_4 (170) - 5C_5$

$= 5C_0 (170)^5 - 5C_1 (170)^4 + 5C_2 (170)^3 - 5C_3 (170)^2 + 5(170) - 1$

A multiple of 100 + 5(170) – 1 = 100K + 849

$\therefore$ The last two digits are 49.

Relation between Two Numbers

Example 10: Find the larger of $99^{50} + 100^{50}$ and 101^{50}

Sol:

$101^{50} = (100 + 1)^{50} = 100^{50} + 50 . 100^{49} + 25 . 49 . 100^{48} + \ldots$

$\Rightarrow 99^{50} = (100 - 1)^{50} = 100^{50} - 50 . 100^{49} + 25 . 49 . 100^{48} - \ldots.$

$\Rightarrow 101^{50} – 99^{50} = 2[50 . 100^{49} + 25(49) (16) 100^{47} + …]$

$= 100^{50} + 50 . 49 . 16 . 100^{47} + … >100^{50}$

$\therefore 101^{50} – 99^{50} > 100^{50}$

$\Rightarrow 101^{50} > 100^{50} + 99^{50}$

Divisibility Test

Example 11: Show that $11^9 + 9^{11}$ is divisible by 10.

Sol:

$11^9 + 9^{11} = (10 + 1)^9 + (10 − 1)^{11}$

$= (9C_0 . 10^9 + 9C_1 . 10^8 + … 9C_9) + (11C_0 . 10^{11} − 11C_1 . 10^{10} + … −11C_{11})$

$= 9C_0 . 10^9 + 9C_1 . 10^8 + … + 9C_8 . 10 + 1 + 10^{11} − 11C_1 . 10^{10} + … + 11C_{10} . 10−1$

$= 10[9C_0 . 10^8 + 9C_1 . 10^7 + … + 9C_8 + 11C_0 . 10^{10} − 11C_1 . 10^9 + … + 11C_{10}]$

= 10K, which is divisible by 10.

Problems on Binomial Theorem

Question 1: If the third term in the binomial expansion of $\left(1 + x^{\log\, \frac{x}{2}}\right)^5$ equals 2560, find x.

Solution:

$$T_3 = 5C_2. \left(x^{\log_2^x}\right)^2 = 2560 \Rightarrow 10.\, x^{2\log_2^x} = 2560 \Rightarrow x^{2\log_2^x} = 256$$

$\Rightarrow (\log_2{}^x)^2 = 4$

$\Rightarrow \log_2{}^x = 2$ or -2

$\Rightarrow$ x = 4 or 1/4.

Question 2: Find the positive value of λ for which the coefficient of x^2 in the expression $x^2[\sqrt{x} + (\lambda/x^2)]^{10}$ is 720.

Solution:

$\Rightarrow x^2\,[{}^{10}C_r \,.\, (\sqrt{x})^{10-r} \,.\, (\lambda/x^2)^r] = x^2\,[{}^{10}C_r \,.\, \lambda^r \,.\, x^{(10-r)/2} \,.\, x^{-2r}]$

$= x^2\,[{}^{10}C_r \,.\, \lambda^r \,.\, x^{(10-5r)/2}]$

Therefore, r = 2

Hence, ${}^{10}C_2 \,.\, \lambda^2 = 720$

$\Rightarrow \lambda^2 = 16$

$\Rightarrow \lambda = \pm 4$.

Question 3: The sum of the real values of x for which the middle term in the binomial expansion of $(x^3/3 + 3/x)^8$ equals 5670 is?

Solution:

$T_5 = {}^8C_4 \times (x^{12}/81) \times (81/x^4) = 5670$

$\Rightarrow 70\, x^8 = 5670$

$\Rightarrow x = \pm \sqrt{3}$.

Question 4: Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1x + a_2\, x_2 + \ldots\ldots + a_{50}\, x^{50}$ for all $x \in R$, then a_2/a_0 is equal to?

Solution:

$\Rightarrow (x + 10)^{50} + (x - 10)^{50}$:

$a_2 = 2 \times {}^{50}C_2 \times 10^{48}$

$a_0 = 2 \times 10^{50}$

$\Rightarrow a_2/a_0 = {}^{50}C_2/10^2 = 12.25$.

Question 5: Find the coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3) \ldots\ldots (1 + x^{100})$.

Solution:

x^9 can be formed in 8 ways.

i.e., $x^9\ x^{1+8}\ x^{2+7}\ x^{3+6}\ x^{4+5},\ x^{1+3+5},\ x^{2+3+4}$

∴ The coefficient of $x^9 = 1 + 1 + 1 + \ldots + 8$ times $= 8$.

Question 6: The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5:10:14. Find n.

Solution:

Let T_{r-1}, T_r, T_{r+1} are three consecutive terms of $(1 + x)^{n+5}$

$\Rightarrow T_{r-1} = (n+5)\, C_{r-2} \,.\, x^{r-2}$

$\Rightarrow T_r = (n+5)\, C_{r-1} \,.\, x^{r-1}$

$\Rightarrow T_{r+1} = (n+5)\, C_r \,.\, x^r$

Given

$(n+5)\, C_{r-2} : (n+5)\, C_{r-1} : (n+5)\, C_r = 5 : 10 : 14$

Therefore, $[(n+5)\, C_{r-2}]/5 = [(n+5)\, C_{r-1}]/10 = (n+5)\, C_r/14$

Comparing first two results we have $n - 3r = -9$ . . . . . . (1)

Comparing last two results we have $5n - 12r = -30$ . . . . . . (2)

From equations (1) and (2), $n = 6$

Question 7: The digit in the units place of the number $183! + 3^{183}$.

Solution:

$\Rightarrow 3^{183} = (3^4)^{45}.3^3$

$\Rightarrow$ unit digit = 7 and 183! ends with 0

$\therefore$ The units digit of 183! + 3^{183} is 7.

Question 8: Find the total number of terms in the expansion of $(x + a)^{100} + (x - a)^{100}$.

Solution:

$\Rightarrow$ $\mathbf{(x + a)^{100} + (x - a)^{100}} = 2[{}^{100}C_0\, x^{100}.\ {}^{100}C_2\ x^{98} . a^2 + ... +\ {}^{100}C_{100}\, a^{100}]$

$\therefore$ Total Terms = 51

Question 9: Find the coefficient of t^4 in the expansion of $[(1-t^6)/(1 - t)]$.

Solution:

$\Rightarrow$ $\mathbf{[(1-t^6)/(1 - t)]} =$ $(1 - t^{18} - 3t^6 + 3t^{12})\ (1 - t)^{-3}$

Coefficient of t in $(1 - t)^{-3} = 3 + 4 - 1$

$C_4 = {}^6C_2 = 15$

The coefficient of x^r in $\mathbf{(1 - x)^{-n}} = (r + n - 1)\ C_r$

Question 10: Find the ratio of the 5th term from the beginning to the 5th term from the end in the binomial expansion of $[2^{1/3} + 1/\{2.(3)^{1/3}\}]^{10}$**.**

Solution:

$$\frac{T_5}{T_5^1} = \frac{10C_4\left(2^{1/3}\right)^{10-4}\left[\frac{1}{2(3)^{1/3}}\right]^4}{10C_4\left(\frac{1}{2(3^{1/3})}\right)^{10-4}.\left(2^{1/3}\right)^4} = 4.(36)^{1/3}$$

Question 11: Find the coefficient of $a^3b^2c^4d$ **in the expansion of $(a\text{-}b\text{-}c\text{+}d)^{10}$.**

Solution:

Expand $\mathbf{(a - b - c + d)^{10}}$ using the multinomial theorem, and by using the coefficient property, we can obtain the required result.

Using the multinomial theorem, we have

$$(a-b-c+d)^{10} = \sum_{r_1+r_2+r_3+r_4=10} \frac{(10)!}{r_1!r_2!r_3!r_4!}(a)^{r_1}(-b)^{r2}(-c)^{r3}(d)^{r_4}$$

We want to get coefficient of $a^3b^2c^4d$ this implies that $r_1 = 3$, $r_2 = 2$, $r_3 = 4$, $r_4 = 1$,

∴ The coefficient of $a^3b^2c^4d$ is $[(10)!/(3!.2!.4)]\ (-1)^2\ (-1)^{-4} = 12600$

Question 12: Find the coefficient of in the expansion of $(1 + x + x^2 + x^3)^{11}$.

Solution:

By expanding the given equation using the expansion formula, we can get the coefficient x^4

i.e. $1 + x + x^2 + x^3 = (1 + x) + x^2 (1 + x) = (1 + x) (1 + x^2)$

$\Rightarrow (1 + x + x^2 + x^3)\ x^{11} = (1+x)^{11} (1+x^2)^{11}$

$= 1+ {}^{11}C_1 x^2 + {}^{11}C_2 x^2 + {}^{11}C_3 x^3 + {}^{11}C_4 x^4 \ldots\ldots.$

$= 1 + {}^{11}C_1 x^2 + {}^{11}C_2 x^4 + \ldots\ldots$

To find the term in from the product of two brackets on the right-hand-side, consider the following products terms as

$= 1 \times {}^{11}C_2 x^4 + {}^{11}C_2 x^2 \times {}^{11}C_1 x^2 + {}^{11}C_4 x^4$

$= {}^{11}C_2 + {}^{11}C_2 \times {}^{11}C_1 + {}^{11}C_4]\ x^4$

$\Rightarrow [55 + 605 + 330]\ x^4 = 990x^4$

∴ The coefficient of x^4 is 990.

Question 13: Find the number of terms free from the radical sign in the expansion of $(\sqrt{5} + \sqrt[4]{n})^{100}$.

Solution:

$T_{r+1} = {}^{100}C_r \,.\, 5^{(100 - r)/2}\ n^{r/4}$

Where r = 0, 1, 2, ... , 100

r must be 0, 4, 8, … 100

The number of rational terms = 26

Question 14: Find the degree of the polynomial $[x + \{\sqrt{(3^{(3-1)})}\}^{1/2}]^5 + [x + \{\sqrt{(3^{(3-1)})}\}^{1/2}]^5$.

Solution:

$[x + \{ \sqrt{(3^{(3-1)})} \}^{1/2}]^5$:

$= 2 [^5C_0\, x^5 + {}^5C_2\, x^5 (x^3 – 1) + {}^5C_4 . x . (x^3 – 1)^2]$

Therefore, the highest power = 7.

Question 15: Find the last three digits of 27^{26}.

Solution:

By reducing 27^{26} into the form $(730 – 1)^n$ and using simple binomial expansion, we will get the required digits.

We have **$27^2 = 729$**

Now $27^{26} = (729)^{13} = (730 – 1)^{13}$

$= {}^{13}C_0 (730)^{13} – {}^{13}C_1 (730)^{12} + {}^{13}C_2 (730)^{11} – . . . . . – {}^{13}C_{10} (730)^3 + {}^{13}C_{11}(730)^2 – {}^{13}C_{12} (730) + 1$

$= 1000m + [(13 \times 12)]/2] \times (14)^2 – (13) \times (730) + 1$

Where ‘m’ is a positive integer

$= 1000m + 15288 – 9490 = 1000m + 5799$

Thus, the last three digits of 17^{256} are 799