Radicals and Exponents

Shirin Sultana
Senior Lecturer
Department of GED
Daffodil International University

Definition

Exponent

If a is any real number and n is a positive integer, then the product of n numbers is defined as

$$\underbrace{a \cdot a \cdot a \cdots a}_{n\ factors} = a^n$$

Where n is the index or exponent or power and a is the base.

Base (real number)

Exponent (integer)

$$a^n$$

Radical

An expression containing the radical symbol $(\sqrt{\ })$ is called a radical.

The general form of a radical is $\sqrt[n]{a}$, where n is the index and a is the radicand.

Index

Radical Sign

$$\sqrt[n]{a}$$

Radicand

Note:

If $n = 2$ omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$.

Properties of Exponents

Let a and b be real numbers, variables, or algebraic expressions and let m and n be integers (All denominators and bases are nonzero).

Property	***Example***
1. $a^0 = 1, a^1 = a$	$(x^2+1)^0 = 1, (x^2+1)^1 = x^2+1$
2. $a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$	$3^{-4} = \frac{1}{3^4} = \left(\frac{1}{3}\right)^4$
3. $a^m a^n = a^{m+n}$	$3^2 3^4 = 3^{2+4} = 3^6 = 729$
4. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{3^2}{3^4} = 3^{2-4} = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
5. $(a^m)^n = a^{mn}$	$(y^3)^{-4} = y^{3(-4)} = y^{-12} = \frac{1}{y^{12}}$
6. $(ab)^m = a^m b^m$	$(5x)^3 = 5^3 x^3 = 125x^3$
7. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{2}{x}\right)^3 = \frac{2^3}{x^3} = \frac{8}{x^3}$

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Properties of Radicals

Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive integers.

Property	***Example***
1. $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$	$\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$
2. For n even, $\sqrt[n]{a^n} = \lvert a \rvert$	$\sqrt[4]{(-12)^4} = \lvert -12 \rvert = 12$
3. For n odd, $\sqrt[n]{a^n} = a$	$\sqrt[3]{(-12)^3} = -12$
4. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	$\sqrt{35} = \sqrt{5 \cdot 7} = \sqrt{5} \cdot \sqrt{7}$
5. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}, \qquad b \neq 0$	$\sqrt[3]{\dfrac{27}{8}} = \dfrac{\sqrt[3]{27}}{\sqrt[3]{8}} = \dfrac{\sqrt[3]{3^3}}{\sqrt[3]{2^3}} = \dfrac{3}{2}$
6. $\sqrt[\mathrm{m}]{\sqrt[n]{a}} = \sqrt[\mathrm{mn}]{\mathrm{a}}$	$\sqrt[3]{\sqrt[5]{10}} = \sqrt[(3\times5)]{10} = \sqrt[15]{10}$

Simplification of Radicals

An expression involving radicals can be simplified by,

1. by removing the perfect nth powers of the radicand.
2. by reducing the index of the radical
3. by rationalizing of the denominator of the radicand.

Find the simplest form of the followings:

a. $\left(\sqrt[3]{6}\right)^3$ b. $\sqrt[4]{6480}$ c. $\sqrt[5]{\frac{5}{32}}$ d. $\sqrt[3]{(27)^4}$ e. $\sqrt[3]{\sqrt{5}}$

a. We have $\left(\sqrt[3]{6}\right)^3$ Property 3

$$= (6)^{\frac{1}{3}.3}$$
$$= 6$$

b. We have $\sqrt[4]{6480}$ Property 4

$$= \sqrt[4]{3^4 \cdot 2^4 \cdot 5}$$
$$= \sqrt[4]{3^4} \cdot \sqrt[4]{2^4} \cdot \sqrt[4]{5}$$
$$= 3 \cdot 2 \cdot \sqrt[4]{5}$$
$$= 6\sqrt[4]{5}$$

c. We have $\sqrt[5]{\frac{5}{32}}$ Property 5

$$= \frac{\sqrt[5]{5}}{\sqrt[5]{32}}$$
$$= \frac{\sqrt[5]{5}}{\sqrt[5]{2^5}}$$
$$= \frac{\sqrt[5]{5}}{2}$$

d. We have $\sqrt[3]{(27)^4}$ Property 1

$$= \left(\sqrt[3]{3^3}\right)^4$$
$$= 3^4$$
$$= 81$$

e. We have $\sqrt[3]{\sqrt{5}}$ Property 6

$$= \sqrt[(3\cdot 2)]{5}$$
$$= \sqrt[6]{5}$$

Find the simplest form of the followings:

$$a.\ \sqrt[6]{81a^2} \quad b.\ \left(7\sqrt[3]{4ab}\right)^2 \quad c.\ \sqrt[3]{64x^7y^{-6}} \quad d.\ \sqrt[3]{\frac{(x+1)^3}{(y-2)^6}}$$

a. We have, $\sqrt[6]{81a^2}$

$$= \sqrt[6]{9^2a^2}$$

$$= \sqrt[6]{(9a)^2}$$

$$= \left((9a)^2\right)^{\frac{1}{6}}$$

$$= (9a)^{\frac{1}{3}} = \sqrt[3]{9a}$$

b. We have, $\left(7\sqrt[3]{4ab}\right)^2$

$$= 49 \cdot \left(\sqrt[3]{4ab}\right)^2$$

$$= 49 \cdot \sqrt[3]{(4ab)^2}$$

$$= 49 \cdot \sqrt[3]{2^3.2a^2b^2}$$

$$= 98 \cdot \sqrt[3]{2a^2b^2}$$

c. We have,

$$\sqrt[3]{64x^7y^{-6}}$$

$$= \sqrt[3]{4^3 \cdot x^6 \cdot x \cdot y^{-6}}$$

$$= \sqrt[3]{4^3} \cdot \sqrt[3]{(x^2)^3} \cdot \sqrt[3]{x} \cdot \sqrt[3]{y^{-6}}$$

$$= 4 \cdot x^2 \cdot \sqrt[3]{x} \cdot y^{-2}$$

$$= \frac{4x^2}{y^2}\sqrt[3]{x}$$

d. We have,

$$\sqrt[3]{\frac{(x+1)^3}{(y-2)^6}}$$

$$= \frac{\sqrt[3]{(x+1)^3}}{\sqrt[3]{(y-2)^6}}$$

$$= \frac{x+1}{(y-2)^2}$$

Calculate the followings:

$a.\ \sqrt{18}+\sqrt{50}-\sqrt{72}$ b. $2\sqrt{27}-4\sqrt{12}$ c. $\frac{112}{\sqrt{196}}\times\frac{\sqrt{576}}{12}\times\frac{\sqrt{256}}{8}$ $d.\ \sqrt{248+\sqrt{52+\sqrt{144}}}$

a.

$$\sqrt{18}+\sqrt{50}-\sqrt{72}$$
$$=\sqrt{2\cdot 3^2}+\sqrt{2\cdot 5^2}-\sqrt{2^3\cdot 3^2}$$
$$=3\sqrt{2}+5\sqrt{2}-3\sqrt{2^2\cdot 2}$$
$$=3\sqrt{2}+5\sqrt{2}-6\sqrt{2}$$
$$=2\sqrt{2}$$

b.

$$2\sqrt{27}-4\sqrt{12}$$
$$=2\sqrt{3^2\cdot 3}-4\sqrt{2^2\cdot 3}$$
$$=2\cdot 3\sqrt{3}-4\cdot 2\sqrt{3}$$
$$=6\sqrt{3}-8\sqrt{3}$$
$$=-2\sqrt{3}$$

c.

$$\frac{112}{\sqrt{196}}\times\frac{\sqrt{576}}{12}\times\frac{\sqrt{256}}{8}$$
$$=\frac{112}{\sqrt{2^2\cdot 7^2}}\times\frac{\sqrt{2^6\cdot 3^2}}{12}\times\frac{\sqrt{2^8}}{8}$$
$$=\frac{112}{2\cdot 7}\times\frac{2^3\cdot 3}{12}\times\frac{2^4}{8}$$
$$=\frac{112}{2\cdot 7}\times\frac{8\cdot 3}{12}\times\frac{16}{8}=32$$
$$=32$$

d.

$$\sqrt{248+\sqrt{52+\sqrt{144}}}$$
$$=\sqrt{248+\sqrt{52+\sqrt{2^4\cdot 3^2}}}$$
$$=\sqrt{248+\sqrt{52+2^2\cdot 3}}$$
$$=\sqrt{248+\sqrt{52+12}}$$
$$=\sqrt{248+\sqrt{64}}$$
$$=\sqrt{248+\sqrt{2^6}}$$
$$=\sqrt{248+2^3}$$
$$=\sqrt{248+8}$$
$$=\sqrt{2^8}=2^4=16$$

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Show that $3+\frac{1}{\sqrt{3}}+\frac{1}{3+\sqrt{3}}-\frac{1}{3-\sqrt{3}}=3$

$$\text{L.H.S} = 3+\frac{1}{\sqrt{3}}+\frac{1}{3+\sqrt{3}}-\frac{1}{3-\sqrt{3}}$$

$$=3+\frac{\sqrt{3}}{(\sqrt{3})(\sqrt{3})}+\frac{(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})}-\frac{(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})}$$

$$=3+\frac{\sqrt{3}}{3}+\frac{(3-\sqrt{3})}{3^2-(\sqrt{3})^2}-\frac{(3+\sqrt{3})}{3^2-(\sqrt{3})^2}$$

$$=3+\frac{\sqrt{3}}{3}+\frac{(3-\sqrt{3})}{9-3}-\frac{(3+\sqrt{3})}{9-3}$$

$$=3+\frac{\sqrt{3}}{3}+\frac{(3-\sqrt{3})}{6}-\frac{(3+\sqrt{3})}{6}$$

$$\text{L.H.S.}=\frac{18+2\sqrt{3}+3-\sqrt{3}-3-\sqrt{3}}{6}$$

$$=\frac{18}{6}=3$$

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Find the value of a & b if $a+b\sqrt{6}=\frac{7\sqrt{3}+5\sqrt{2}}{\sqrt{48}-\sqrt{18}}$

Given that,

$$a+b\sqrt{6}=\frac{7\sqrt{3}+5\sqrt{2}}{\sqrt{48}-\sqrt{18}}$$

$$=\frac{7\sqrt{3}+5\sqrt{2}}{4\sqrt{3}-3\sqrt{2}}$$

$$=\frac{(7\sqrt{3}+5\sqrt{2})(4\sqrt{3}+3\sqrt{2})}{(4\sqrt{3}-3\sqrt{2})(4\sqrt{3}+3\sqrt{2})}$$

$$=\frac{28\times 3+21\sqrt{2}\sqrt{3}+20\sqrt{2}\sqrt{3}+15\times 2}{48-18}$$

$$=\frac{84+21\sqrt{6}+20\sqrt{6}+30}{30}$$

$$=\frac{114+41\sqrt{6}}{30}$$

$$or, a+b\sqrt{6}=\frac{114}{30}+\frac{41}{30}\sqrt{6}$$

$$\therefore a=\frac{114}{30}=\frac{19}{5} \quad \text{and} \quad b=\frac{41}{30}$$

Express $\frac{(4-\sqrt{3})^2}{2+\sqrt{3}}$ in the form of $a + b\sqrt{3}$ where a and b are integers.

Given that,

$$\frac{(4-\sqrt{3})^2}{2+\sqrt{3}}$$

$$= \frac{16 - 8\sqrt{3} + 3}{2+\sqrt{3}}$$

$$= \frac{19 - 8\sqrt{3}}{2+\sqrt{3}}$$

$$= \frac{(19 - 8\sqrt{3})(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}$$

$$= \frac{38 - 19\sqrt{3} - 16\sqrt{3} + 24}{4-3}$$

$$= \frac{62 - 35\sqrt{3}}{1}$$

$$= 62 + (-35)\sqrt{3},$$

which has been expressd in the form of $a + b\sqrt{3}$, where $a = 62$ and $b = -35$.

What will be come in the place of question mark $\sqrt{86.49}+\sqrt{5+(?)}=12.3$?

Let the required value is x

According to the question we can write,

$$\sqrt{86.49}+\sqrt{5+(x)}=12.3$$

$$\text{or, } \sqrt{5+(x)}=12.3-\sqrt{86.49}$$

$$\text{or, } 5+x=\left(\frac{123}{10}-\sqrt{\frac{8649}{100}}\right)^2$$

$$\text{or, } x=\left(\frac{123}{10}-\frac{\sqrt{8649}}{\sqrt{100}}\right)^2-5$$

$$\text{or, } x=\left(\frac{123}{10}-\frac{93}{10}\right)^2-5$$

$$\text{or, } x=\left(\frac{30}{10}\right)^2-5$$

$$\text{or, } x=(3)^2-5$$

$$\text{or, } x=9-5$$

$$\text{or, } x=4$$

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What will be come in the place of question mark $(?)^{\frac{1}{4}} = \frac{48}{(?)^{\frac{3}{4}}}$?

Let the required value is x

According to the question we can write,

$$(x)^{\frac{1}{4}} = \frac{48}{(x)^{\frac{3}{4}}}$$

$$\text{or, } (x)^{\frac{1}{4}}(x)^{\frac{3}{4}} = 48$$

$$\text{or, } (x)^{\frac{1}{4}+\frac{3}{4}} = 48$$

$$\text{or, } (x)^{\frac{4}{4}} = 48$$

$$\therefore\ x = 48$$

If $\sqrt{841} = 29$ then find the value of $\sqrt{841} + \sqrt{8.41} + \sqrt{0.0841} + \sqrt{0.000841}$

Given that, $\sqrt{841} = 29$

Now,

$$\sqrt{841} + \sqrt{8.41} + \sqrt{0.0841} + \sqrt{0.000841}$$

$$= \sqrt{841} + \sqrt{\frac{841}{100}} + \sqrt{\frac{841}{10000}} + \sqrt{\frac{841}{1000000}}$$

$$= \sqrt{841} + \frac{\sqrt{841}}{\sqrt{100}} + \frac{\sqrt{841}}{\sqrt{10000}} + \frac{\sqrt{841}}{\sqrt{1000000}}$$

$$= 29 + \frac{29}{10} + \frac{29}{100} + \frac{29}{1000}$$

$$= \frac{29000 + 2900 + 290 + 29}{1000}$$

$$= \frac{32219}{1000}$$

$$= 32.219$$

If $\sqrt{1+\frac{x}{144}}=\frac{13}{12}$, then find the value of x

We have,

$$\sqrt{1+\frac{x}{144}}=\frac{13}{12}$$

$$\text{or, } 1+\frac{x}{144}=\left(\frac{13}{12}\right)^2$$

$$\text{or, } \frac{x}{144}=\frac{169}{144}-1$$

$$\text{or, } \frac{x}{144}=\frac{169-144}{144}$$

$$\text{or, } \frac{x}{144}=\frac{25}{144}$$

$$\therefore \quad x=25$$

Find the value of $\frac{(243)^{\frac{n}{5}} \times 3^{2n+1}}{9^n \times 3^{n-1}}$

We have, $$\frac{(243)^{\frac{n}{5}} \times 3^{2n+1}}{9^n \times 3^{n-1}}$$

$$= \frac{(3)^{5 \cdot \frac{n}{5}} \times 3^{2n+1}}{3^{2n} \times 3^{n-1}} = \frac{3^n \times 3^{2n+1}}{3^{2n+n-1}}$$

$$= \frac{3^{n+2n+1}}{3^{2n+n-1}} \quad = \frac{3^{3n+1}}{3^{3n-1}}$$

$$= 3^{3n+1-3n+1}$$

$$= 3^2$$

$$= 9$$

If $2^x + 2^{1-x} = 3$, then find the value of x

Given that, $2^x + 2^{1-x} = 3$

or, $2^x + 2^1 \cdot 2^{-x} = 3$

or, $2^x + 2 \cdot \frac{1}{2^x} = 3$

or, $a + 2 \cdot \frac{1}{a} = 3$ Let $2^x = a$

or, $a^2 + 2 = 3a$

or, $a^2 - 3a + 2 = 0$

or, $a^2 - 2a - a + 2 = 0$

or, $a(a-2) - 1(a-2) = 0$

or, $(a-1)(a-2) = 0$

Therefore $a - 1 = 0$ and $a - 2 = 0$

$\Rightarrow a = 1$ $\Rightarrow a = 2$

$\Rightarrow 2^x = 2^0$ $\Rightarrow 2^x = 2^1$

$\Rightarrow x = 0$ $\Rightarrow x = 1$

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If $8^x.2^y = 512$ and $3^{3x+2y} = 9^6$, then what is the value of x and y?

Given that, $8^x.2^y = 512$

or, $(2^3)^x.2^y = 2^9$

or, $2^{3x}.2^y = 2^9$

or, $2^{3x+y} = 2^9$

$\therefore\ 3x + y = 9 \ldots\ldots\ldots (i)$

Again, $3^{3x+2y} = 9^6$

or, $3^{3x+2y} = (3^2)^6$

or, $3^{3x+2y} = 3^{12}$

$\therefore\ 3x + 2y = 12 \ldots\ldots\ldots (ii)$

Subtracting equation (i) from equation (ii) we get,

$$y = 3$$

Putting the value of y in equation (i) we get,

$$3x + 3 = 9$$

or, $3x = 9 - 3 = 6$

$\therefore\ x = 2$

Therefore, $x = 2$ & $y = 3$

Solve the equation $\frac{5^{4y-1}}{25^y} = \frac{125^{y+3}}{25^{2-y}}$.

Given that,

$$\frac{5^{4y-1}}{25^y} = \frac{125^{y+3}}{25^{2-y}}$$

$$\text{or, } \frac{5^{4y-1}}{(5^2)^y} = \frac{(5^3)^{y+3}}{(5^2)^{2-y}}$$

$$\text{or, } \frac{5^{4y-1}}{5^{2y}} = \frac{5^{3y+9}}{5^{4-2y}}$$

$$\text{or, } 5^{4y-1-2y} = 5^{3y+9-4+2y}$$

$$\text{or, } 5^{2y-1} = 5^{5y+5}$$

$$\text{or, } 2y - 1 = 5y + 5$$

$$\text{or, } -3y = 6$$

$$\text{or, } y = -\frac{6}{3}$$

$$\therefore\ y = -2$$

Express $\frac{54\sqrt[3]{27^{2x}}}{9^{x+1}\times 216(3^{2x-1})}$ as a fraction in its simplest form.

Given that,

$$\frac{54\sqrt[3]{27^{2x}}}{9^{x+1}\times 216(3^{2x-1})}$$

$$=\frac{\left(\sqrt[3]{27}\right)^{2x}}{(3^2)^{x+1}\times 4\times 3^{2x-1}}$$

$$=\frac{\left(\sqrt[3]{3^3}\right)^{2x}}{3^{2x+2}\times 4\times 3^{2x-1}}$$

$$=\frac{3^{2x}}{3^{2x+2+2x-1}\times 4}$$

$$=\frac{3^{2x}}{3^{4x+1}\times 4}$$

$$=\frac{3^{2x-4x-1}}{4}$$

$$=\frac{3^{-2x-1}}{4}$$

$$=\frac{1}{4\times 3^{2x+1}}$$

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Find the values of $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$.

Let, $x = \sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$

Squaring both sides we get,

$$x^2 = 6 + \sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$$

$$x^2 = 6 + x$$

$$x^2 - x - 6 = 0$$

$$x^2 - 3x + 2x - 6 = 0$$

$$x(x-3) + 2(x-3) = 0$$

$$(x-3)(x+2) = 0$$

$$\therefore\ x = 3, -2$$

Since a negative number cannot be under root, hence $x = 3$.

Exercise

- Rationalize these surds, writing them in the form of $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are integers.

$$(i)\ \frac{1}{1-\sqrt{3}}\quad (ii)\ \frac{1}{3+\sqrt{2}}\quad (iii)\ \frac{1}{3-\sqrt{7}}\quad (iv)\ \frac{1+3\sqrt{3}}{2-2\sqrt{3}}\quad (v)\ \frac{3-\sqrt{2}}{2+3\sqrt{2}}\quad (vi)\ \frac{2-\sqrt{7}}{3+\sqrt{7}}$$

- Show that $\sqrt{5+2\sqrt{6}} = \sqrt{3}+\sqrt{2}$.
- Find the cube root of 2744.
- If $x = 1+\sqrt{2}$ and $y = 1-\sqrt{2}$ then find the value of x^2+y^2.
- If $2^{x-1}+2^{x+1} = 1280$ then find the values of x.
- If $\frac{5+2\sqrt{3}}{7+4\sqrt{3}} = a+b\sqrt{3}$ then what are the values of a & b?
- Find the value of $\frac{\sqrt{12-\sqrt{12-\sqrt{12-\cdots}}}}{\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}}$ is a rational number.
- If $t = 2+\sqrt[3]{4}+\sqrt[3]{2}$, determine the value of t^3-6t^2+6t-2.