Introduction to Cost-Benefit Analysis

Lecture 3

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CBA: Lecture 3

Valuing time.

- Assume now that crossing the river with the bridge saves 15 min compared to crossing with the ferry.
- This is an additional benefit to building the bridge which should be taken into account.
- But how much exactly? How to put a price on time? In other words, how much money would an individual be willing to pay to gain 15 min of time?
- ***Problem***: there is no market for time that we could use to infer the individuals' willingness to pay!

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Valuing time.

- Easy solution would be to use each individual's hourly wage to evaluate the gain in monetary terms. But everyone might not end up working 15 min more.
- Shorter commuting time means individuals could either work more or spend more time on leisure.
- Time is a special type of good because it is not valued in itself (it does not enter into the utility function directly), but for what it allows the individual to do: time has ***instrumental*** value.

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Valuing time.

- To simplify, assume individuals care only about two things: how much leisure time they get to enjoy and how much consumption they get to spend.
- The value individuals put on the bridge (excluding time saving) will be translated into consumption (money).
- Formally, individual i's welfare can be represented by a utility function $u_i = u_i(l_i, c_i)$, where $\mathbf{l_i}$ is i's leisure time and $\mathbf{c_i}$ is i's consumption (in \$).
- The utility function is increasing in both leisure time and consumption, but marginal utility with respect to both is decreasing:

$$\frac{\partial u_i}{\partial l_i} > 0, \quad \frac{\partial u_i}{\partial c_i} > 0, \quad \frac{\partial^2 u_i}{\partial l_i^2} < 0, \quad \frac{\partial^2 u_i}{\partial c_i^2} < 0$$

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Valuing time.

- Because time is constrained (there are only 24 hours in a day), the more leisure time the individual chooses to spend, the less time she can work, and therefore the less money she makes.
- Formally, if the individual's hourly wage is w, and her total available time in the day is t (accounting for sleeping and commuting), then her (daily) income is equal to:

$$w(t - l_i)$$

- After paying for her (daily) compulsory expenses e, the individual is left with $w(t - l_i) - e$ for her consumption.
- Therefore, we get the following budget constraint:

$$c_i \leq w(t - l_i) - e \quad \Leftrightarrow \quad c_i + wl_i \leq wt - e$$

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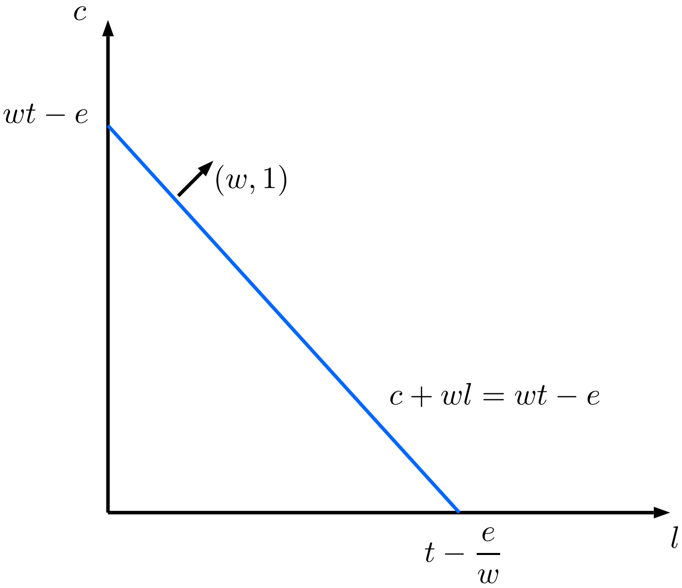
Valuing time.



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Valuing time.

- So we now get to write what we call the ***consumer's problem***: faced with a budget constraint, the individual needs to choose how much to spend on leisure time (or conversely how much to work) and how much to consume.
- Formally, this writes as a constrained maximization program:

$$\max_{l,c} u(l,c) \quad \text{subject to} \quad c \leq w(t - l_i) - e$$

- Assume that the utility function is of the ***Cobb-Douglas*** type:

$$u(c,l) = c^{\alpha} l^{1-\alpha}$$

- Mostly for tractability, but very common assumption in economics. Parameter α reflects how much the individual values consumption vs leisure time.

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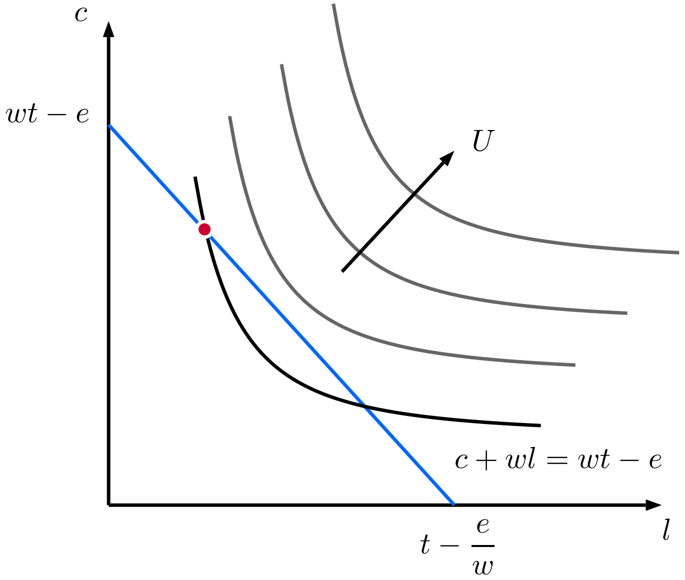
Valuing time.



- Here, it is easier to solve the maximization problem with $v = ln(u)$ as the objective function:

$$\max_{l,c} \ \alpha ln(c) + (1-\alpha) ln(l) \quad \text{subject to} \quad c \leq w(t - l_i) - e$$

- Since the utility function is increasing in both c and l, we know the budget constraint must be binding at the optimum. So we get:

$$\max_{l,c} \ \alpha ln(c) + (1-\alpha) ln(l) \quad \text{subject to} \quad c = w(t - l_i) - e$$

- We get a constrained maximization problem with equality.
- Before we actually solve this particular problem, let's review the basic technique to solve constrained optimization problems!

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Reminder on Constrained Optimization

- Consider the following constrained optimization problem:

$$\max_{x,y} f(x,y) \quad \text{subject to} \quad g(x,y) = 0$$

- In order to solve such a problem, we must first introduce a new function called the *Lagrangian*, defined as follows:

$$\mathscr{L} = f(x,y) - \lambda g(x,y)$$

.

- The new variable λ is called the Lagrange multiplier.
- If (x^*, y^*) is an interior solution to the maximization problem, and the partial derivatives of the constraint function g are not both equal to 0 at λ^*, then there exists λ^* such that (x^*, y^*, λ^*) is a ***stationary*** point of $\mathscr{L}$:

$$\nabla_{x,y}\mathscr{L} = \left(\frac{\partial \mathscr{L}}{\partial x}, \frac{\partial \mathscr{L}}{\partial y}, \frac{\partial \mathscr{L}}{\partial \lambda}\right)\bigg|_{x^*.y^*,\lambda^*} = (0,0,0)$$

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Reminder on Constrained Optimization:

- Finding the stationary points of the Lagrangian thus requires solving a system of three equations with three unknowns.
- Any (interior) solution to the constrained maximization problem is a critical point of the Lagragian, but not all critical points are solutions to the constrained maximization problem.
- In order to know whether a critical point is indeed a local maximizer (or minimizer) we need to check the second order conditions.
- However, in all examples we will consider FOC also sufficient for optimality because of concavity of the objective function.

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The value of time as a shadow price.

- We go back to our problem.
- We start by writing the lagrangian:

$$\mathscr{L} = \alpha ln(c) + (1-\alpha) ln(l) - \lambda(c - w(t-l) + e)$$

- Any solution to our constrained maximization problem must necessarily be a *stationary* point of the Lagrangian.
- Therefore, we look for the solution to: $\nabla_{l,c,\lambda}\mathscr{L} = 0$.
- We get the following system of equations:

$$\frac{\alpha}{c} - \lambda = 0, \quad \frac{1-\alpha}{l} - \lambda w = 0, \quad c = w(t-l) - e$$

- Which yields:

$$c^* = \alpha(wt - e), \quad l^* = (1-\alpha)(t - \frac{e}{w}), \quad \lambda^* = \frac{1}{wt-e}$$

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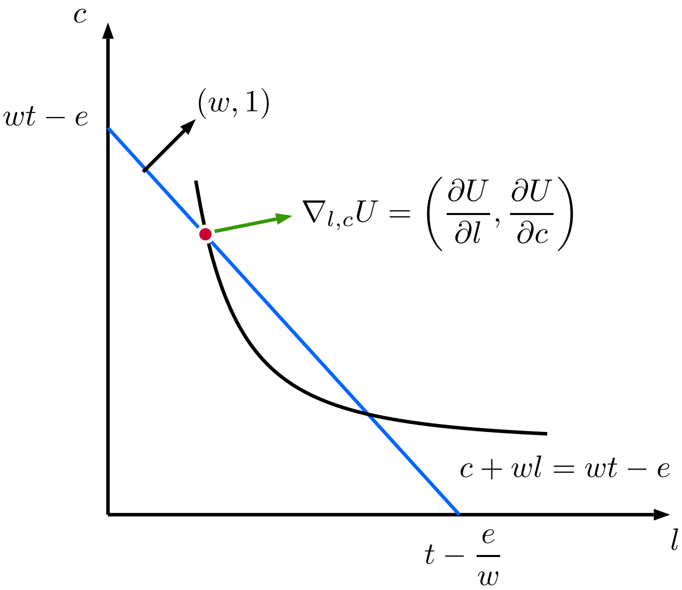
The value of time as a shadow price.



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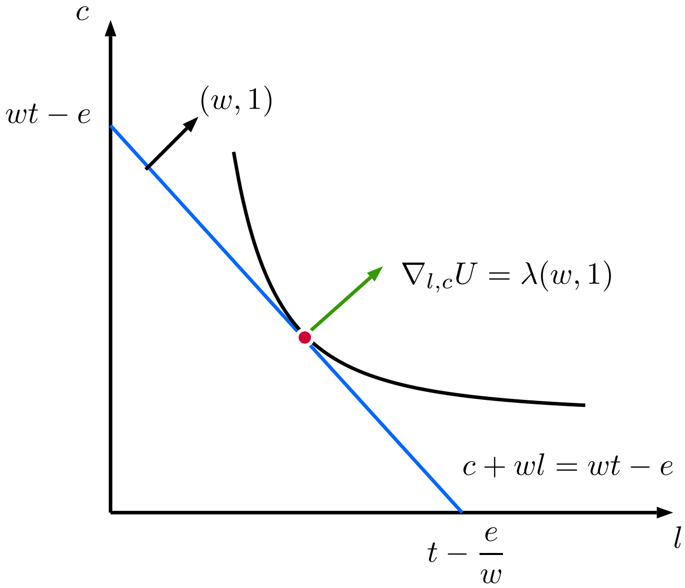
The value of time as a shadow price.



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The value of time as a shadow price.

- How to value an increase in available time t relative to a reduction in compulsory expenses or relative to a decrease in wage?
- What matters eventually is not the increase in time itself (or the reduction in expenses or wages) but how much the individual's utility increases as a consequence!
- ***Envelope Theorem***:

$$\frac{\partial u(c^*,l^*)}{\partial t}(t,e,w) = \frac{\partial \mathscr{L}}{\partial t}\Big|_{c^*,l^*,\lambda^*}$$
$$\frac{\partial u(c^*,l^*)}{\partial e}(t,e,w) = \frac{\partial \mathscr{L}}{\partial e}\Big|_{c^*,l^*,\lambda^*}$$
$$\frac{\partial u(c^*,l^*)}{\partial w}(t,e,w) = \frac{\partial \mathscr{L}}{\partial w}\Big|_{c^*,l^*,\lambda^*}$$

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The value of time as a shadow price.

- We get:

$$\frac{\partial u(c^*,l^*)}{\partial t}(t,e,w) = w\lambda^* = \frac{w}{wt-e}$$
$$\frac{\partial u(c^*,l^*)}{\partial e}(t,e,w) = -\lambda^* = -\frac{1}{wt-e}$$
$$\frac{\partial u(c^*,l^*)}{\partial w}(t,e,w) = \lambda^*(t-l^*) = \frac{\alpha t + (1-\alpha)\frac{e}{w}}{wt-e}$$

- We call these derivatives ***shadow prices***.
- Saving 1 unit of time is equivalent to reducing the compulsory expenses by w\$ (w is the salary per unit of time worked):

$$\frac{\partial u^*}{\partial t} = (-w)\frac{\partial u^*}{\partial e}$$

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The value of time as a shadow price.

- Similarly, increasing the wage by 1$ is equivalent to saving $(t-l^*)/w$ units of time.

$$\frac{\partial u^*}{\partial w} = \lambda^*(t-l^*) = \left(\frac{t-l^*}{w}\right)\frac{\partial u^*}{\partial t}$$

- The more I work at the optimum, $t-l^*$, the more valuable is an increase in my wage.
- Equivalently, the larger weight I put on consumption relative to leisure (as reflected by parameter α), the more valuable is an increase in my wage.

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The value of time as a shadow price.

- So, according to this model, would the bridge be a worthwhile project or not?
- Assume the bridge is funded by imposing a daily tax of τ to each individual during the project's lifetime (independent of individuals' income). In order for the bridge to be funded, we must have:

$$\sum_{t=0}^{40} \frac{365N\tau}{(1+r)^t} \approx 365N\tau\left(\frac{1+r}{r}\right) = C \;\Leftrightarrow\; \tau = \frac{Cr}{365N(1+r)}$$

- Here, because the net benefits of the project are constant over time, the project is beneficial if the increase in each individual's daily utility Δu^* is positive.

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The value of time as a shadow price.

- Taking a first order approximation, we have:

$$\Delta u^* \approx \frac{\partial u^*}{\partial t}\Delta t + \frac{\partial u^*}{\partial e}\Delta e + \frac{\partial u^*}{\partial w}\Delta w$$

$$\approx \frac{w}{wt-e}\Delta t - \frac{1}{wt-e}\Delta e + \frac{\alpha t + (1-\alpha)\frac{e}{w}}{wt-e}\Delta w$$

- So the first step is to define the costs and benefits of the bridge and determine the values of t, e, w, Δt, Δe and Δw.

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The value of time as a shadow price.

- ***Time Saving***: the bridge would reduce commuting time by 0.25h, so that $\Delta t = 0.25$.
- ***Increased labor competition***: more people are now able to reach jobs thanks to the bridge increasing labor supply and reducing equilibrium wage on the job market. The wage (per hour) is expected to decrease from 20$ to 19$, so that $\Delta w = -0.1$.
- ***Additional benefit from new travellers crossing the river***: the bridge now makes it possible for new travellers (those with a WTP below .20) to cross the river. This is measured to be equivalent to receiving an additional consumption of W $ every day.
- Therefore, building the bridge increases the compulsory expenses by $\Delta e = \tau - W$.

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The value of time as a shadow price.

Additional parameters:

- Total work day is $t = 12h$.
- Compulsory expenses e is equal to 72.
- Finally assume that the total population in the standing is $N = 3.10^6$, that the the total cost of construction is $C = 3.10^9$ and that the interest rate is $r = 4\%$.
- As a result, the daily lump sum tax is equal to:

$$\tau = \frac{Cr}{365N(1+r)} = 1.05.$$

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The value of time as a shadow price.

To summarize, we have:

- $\Delta t = 0.25$, $\Delta w = -0.1$, and $\Delta e = 1.05 - W$.
- $w = 20$, $t = 12h$, and $e = 72$.

Furthermore, from our previous computations we have:

$$\frac{\partial u^*}{\partial t} = \frac{w}{wt-e} = 0.12$$
$$\frac{\partial u^*}{\partial e} = -\frac{1}{wt-e} = -0.006$$
$$\frac{\partial u^*}{\partial w} = \frac{\alpha t + (1-\alpha)\frac{e}{w}}{wt-e} = 0.006\,(8.4\alpha + 3.6)$$

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The value of time as a shadow price.

- So that:

$$\begin{aligned}
\Delta u^* &\approx \frac{\partial u^*}{\partial t}\Delta t + \frac{\partial u^*}{\partial e}\Delta e + \frac{\partial u^*}{\partial w}\Delta w \\
&\approx \frac{w}{wt-e}\Delta t - \frac{1}{wt-e}\Delta e + \frac{\alpha t + (1-\alpha)\frac{e}{w}}{wt-e}\Delta w \\
&\approx 0.12 \times 0.25 - 0.006(1.05 - W) - 0.003\,(8.4\alpha + 3.6) \\
&\approx 0.0129 + 0.006W - 0.025\alpha
\end{aligned}$$

- The project is worthwhile if:

$$\Delta u^* = 0.0129 + 0.006W - 0.025\alpha > 0 \quad \Leftrightarrow W > 4.2\alpha - 2.15$$

- What are sensible values for α?

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The value of time as a shadow price.

- Say we would like the optimal number of hours worked to fall between 6 and 8h. Then we must have:

$$6 \leq \alpha t + (1-\alpha)\frac{e}{w} \leq 8 \quad \Leftrightarrow \quad 0.29 \leq \alpha \leq 0.52$$

- If $\alpha = 0.29$, meaning that $t - l^* = 6$h, then:

$$W \geq 4.2 \times 0.29 - 2.15 = -0.93$$

- The project is worthwhile even if individuals do not get any additional benefit from the bridge!
- If $\alpha = 0.52$, meaning that $t - l^* = 8$h, then:

$$W \geq 4.2 \times 0.29 - 3.16 = 0.03$$

- Individuals need to get at least some additional benefit for the project to be worthwhile.

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The value of time as a shadow price.

- When comparing the impact of several variables (available time, income, etc...) on the individual's welfare, one should always try to use shadow prices.
- Shadow prices reflect the (marginal) effect of an increase of a given variable on the individual's welfare (at the optimum).
- We care about the effect of the project on the individual's welfare, not on the variable itself.
- The problem is that it might be difficult to estimate shadow prices directly. Our computations require estimating the individual's utility function, which is not easy.

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The value of time as a shadow price.

- In some cases, shadow prices actually coincide with market prices. In that case the project is worthwhile if and only if it is predicted that the money values of the revenues will exceed the money values of the costs.
- This explains why market prices are often used as a basis for estimating the overall impact of a policy.
- However, in many cases they do not coincide and it is therefore important to be aware of such limitations.

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Exercises: income tax.

- Assume now that the cost of the bridge is financed by imposing a tax rate k on each individual's income (instead of a lump sum tax) from today until the end of the project in 40 years.
- The available income of each individual (after taxes) is now equal to $w(1-k)t$ instead of wt, but there is no lump sum tax τ anymore.

Question: find the value k^* so that the project is fully paid out at the end of the project's life.

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Exercises: income tax.

- In order to evaluate how much revenue the government will raise for a given tax rate k, we must first determine how much individuals will be willing to work if their income is taxed at a tax rate k.
- From our previous computations, we know that:

$$t - l^* = \alpha t + (1-\alpha)\frac{e}{w(1-k)}$$

- So the total revenue raised by the government in a year is equal to:

$$YTR = 365N(t-l^*)k = 365Nkw\left(\alpha t + (1-\alpha)\frac{e}{w(1-k)}\right)$$

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Exercises: income tax.

- For $\alpha = 0.4$, $t = 12$, $w = 20$, $e = 72$, and $N = 3.10^5$, we get:

$$YTR = 1.095k\left(96 + \frac{2.16}{(1-k)}\right)10^8$$

- The net present value of the yearly tax revenue over the next 40 years must be equal to C:

$$\sum_{t=0}^{39} \frac{YTR}{(1+r)^t} = 1.095k\left(96 + \frac{2.16}{(1-k)}\right)10^8\left(\frac{1+r}{r}\right) = C$$

- For $C = 3.10^9$ and $r = 6\%$, this is equivalent to:

$$-96k^2 + 93.8k - 1.55 = 0 \quad \Rightarrow \quad k = 0.017$$

- In order for the project to be fully funded, the government must impose a tax of 1.7%.

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Exercises: quasilinear utility.

- Let's revisit our analysis of saving time with a different utility function to see how the results differ.
- Consider the following *quasilinear* utility function:

$$u(c,l) = \beta c + \sqrt{l}$$

- The larger β, the more the individual values consumption relative to leisure.
- The maximization problem becomes:

$$\max_{l,c} \beta c + \sqrt{l} \quad \text{subject to} \quad c = w(t-l) - e$$

Question: Find the shadow prices and under which condition on *WTP* and β the bridge is worthwhile.

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Exercises: quasilinear utility.

- We first write the Lagrangian:

$$\mathscr{L} = \beta c + \sqrt{l} - \lambda \left(c - w(t-l) + e \right)$$

- The first order conditions yield:

$$\frac{\partial \mathscr{L}}{\partial c} = \beta - \lambda = 0$$
$$\frac{\partial \mathscr{L}}{\partial l} = \frac{1}{2\sqrt{l}} - \lambda w = 0$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} = w(t - l_i) - e - c = 0$$

- From the first two equations, we get:

$$\lambda = \beta \quad \text{and} \quad l^* = \frac{1}{4\beta^2 w^2}$$

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Exercises: quasilinear utility.

- Using the budget constraint we recover c^*:

$$c^* = w\left(t - \frac{1}{4\beta^2 w^2}\right) - e$$

- Here the shadow prices are equal to:

$$\begin{aligned}
\frac{\partial u(c^*, l^*)}{\partial t}(t, e, w) &= w\lambda^* = w\beta \\
\frac{\partial u(c^*, l^*)}{\partial e}(t, e, w) &= -\lambda^* = -\beta \\
\frac{\partial u(c^*, l^*)}{\partial w}(t, e, w) &= \lambda^*(t - l^*) = \beta\left(t - \frac{1}{4\beta^2 w^2}\right)
\end{aligned}$$

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Exercises: quasilinear utility.

- We can now plug the values of all parameters to find out under which conditions the project is worthwhile.
- Note that in order to get reasonable values for l^* (say between 6 and 8h per day), we must choose $\beta \in [0.0102, 0.0125]$.
- Here we get:

$$\begin{aligned}
\Delta u^* &\approx w\beta\Delta t - \beta\Delta e + \beta\left(t - \frac{1}{4\beta^2 w^2}\right)\Delta w \\
&\approx 20\beta \times 0.25 - \beta \times (1.05 - W) + \beta\left(12 - \frac{1}{4\beta^2 20^2}\right)(-0.5) \\
&\approx 3.95\beta + \beta \times W - \frac{1}{2}\beta\left(12 - \frac{1}{1600\beta^2}\right)
\end{aligned}$$

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Exercises: quasilinear utility.

- The project is worthwhile if:

$$\Delta u^* \approx 3.95\beta + \beta \times W - \frac{1}{2}\beta\left(12 - \frac{1}{1600\beta^2}\right) > 0$$

- Which is equivalent to:

$$W > 2.05 - \frac{1}{3200\beta^2}$$

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Exercises: quasilinear utility.

- If $\beta = 0.102$, meaning that $t - l^* = 6$h, then the project is worthwhile if:

$$W \geq -0.95$$

- If $\beta = 0.125$, meaning that $t - l^* = 8$h, then the project is worthwhile if:

$$W \geq 0.05$$

- Note that even though the numerical results differ from what we obtained previously (the value of time depends on the individual's utility function), we get the same comparative statics with respect to the individual's taste for consumption.

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Exercises: shadow price.

- A planner in a developing country has to set the outputs of two commodities: manufactured goods and food.
- He values a unit of manufactured goods at 4$ and a unit of food at 7$.
- To produce a unit of manufactured goods requires two units of trained labour and one unit of capital equipment.
- To produce a unit of food requires three units of labour and one unit of capital equipment.
- Twelve units of labour and five units of capital are available.
- What output levels should be set in order to maximize the value of the plan?

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Exercises: shadow price.

- Denote the quantity of manufactured goods by M and the quantity of food by F. Then the problem can be described by the following constrained maximization program:

$$\max_{M \geq 0, F \geq 0} 4M + 7F \quad \text{subject to} \quad M + F \leq 5, \quad 2M + 3F \leq 12$$

- This is a linear programming problem. It is linear because the objective and the technical constraints all have coefficients which are independent of the volumes produced: doubling all outputs doubles the value of the plan and also doubles the input requirements.
- Solve the problem graphically.

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Exercises: shadow price.

- The first line $M + F = 5$ represents the capital constraint.
- The second line $2M + 3F = 12$ represents the labour constraint.
- Since the constraints must be observed simultaneously the feasible region is the area bounded by all the constraints.
- The diagram also shows lines corresponding to equation of the type $4M + 7F = K$.
- These are referred to as iso-revenue lines. If they were feasible then all points on these lines would yield the same value K (try drawing the ones corresponding to $K = 42$ and $K = 49$).

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Exercises: shadow price.

- The problem is solved by finding a point on a line parallel to these two lines, as far away to the north east of the origin as possible whilst being within the feasible region.
- This must occur at one of the three vertices and these are the only points we need to consider (classic result of linear programming).
- $M = 5$ and $F = 0$ yields a value of 20; $M = 3$ and $F = 2$ yields a value of 26; $M = 0$ and $F = 4$ yields a value of 28.
- As is obvious from the relative slopes of the lines in the diagrams, this last point is the solution to the problem: produce no units of manufactured goods, 4 units of food, giving a value of 28.

Exercises: shadow price.

- What are the shadow prices of the constraints?
- Remember, shadow prices tell us how valuable resources are in terms of the objective function. By how much does the objective increases (at the optimum) if the amount of a given resource (here either capital or labour) increases by one unit?
- Note here that the solution point lies inside the capital constraint. The capital constraint turns out to be non-binding, that is, irrelevant!
- Therefore the value of an extra unit of capital is zero - this is its shadow price.
- This is a general result. Any resource which is in excess supply at an optimum allocation of resources will have a social value of zero.

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Exercises: shadow price.

- In the context of our example note that this shadow price may be quite different from the international market price of capital.
- So we see that there may be a distinction between the market price of a resource and its true value measured relative to the objective.
- Finally, note that the shadow price is only meaningful in relation to the particular objective.
- For instance, if the relative valuations of the two products changed from the present ratio of 4 : 7 then the slope of the iso-revenue line would change. It might become sufficiently steep for the optimum point to shift to $M = 3$ and $F = 2$, or even to $M = 5$ and $F = 0$.
- In either case the capital constraint binds and the shadow price is definitely positive.

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Exercises: shadow price.

- What about labour? What is the shadow price of labour in this example?
- If the quantity of labour that is available increases from 12 to 13 then the intercept of the labour constraint on the vertical axis will increase from $12/3 = 4$ to $13/3$.
- The optimum allocation still coincides with the intercept of the labour constraint on the vertical axis.
- Therefore the output of food can increase by $1/3$. The value of this incremental output is $7/3$ and so that is the shadow price of the labour constraint.
- What if the available amount of labour is now equal to 15?