Introduction to Cost-Benefit Analysis

Lecture 2

Rafael Treibich

University of Southern Denmark

Today's lecture

- A first go at CBA: building a new bridge.
  1. Initial Setup
  2. The demand curve.
  3. Consumer Surplus and Social Welfare.
  4. Aggregating over time: net present value.
  5. CBA recommendation.

A first go at CBA

Building a new bridge

- Suppose there is a river which at present can only be crossed by ferry. The government considers building a bridge, which, being rather upstream, would take the traveler the same time to complete a crossing.
- The ferry is a privately owned monopoly and charges 0.20 per crossing, while its total costs per crossing are 0.15. It is used for $5,000$ crossings per year.
- The bridge would cost $30,000$ to build but would be open free of charge.
- It is expected that there will be $25,000$ crossings a year with the bridge and that the ferry would go out of business.

The government asks the cost-benefit analyst to advise them on whether to go ahead with the bridge.

A first go at CBA

Building a new bridge

Overall approach:

1. Specify alternatives.
2. Define standing: in this example there are four main categories of affected individuals: taxpayers, ferry owners, existing travelers and new travelers (who previously did not cross but would do so at the lower price).
3. List and categorize the costs and benefits associated to the project.
4. Predict the costs and benefits in each year of the project, first for each individual, then for society as a whole.
5. Monetize the costs and benefits so as to make them comparable.
6. Obtain an aggregate present value of the project by discounting costs and benefits in future years to make them commensurate with present costs and benefits, and then adding them up.

A first go at CBA
Costs

- The taxpayers lose $30,000$ by paying for the construction of the bridge.
- The ferry owners lose their excess profits of 250 [i.e., $0.05 \times 5,000$] in each future year forever.

A first go at CBA
Benefits

- For now, we assume there is no difference in crossing the river with a ferry as opposed to taking the bridge. Therefore, the benefit of building the bridge comes down to:
  - Existing travellers not having to pay the cost of the ferry anymore, and,
  - New travellers enjoying the benefits of crossing the river.
- The financial gain for existing travellers is equal to $1,000$ $[0.20 \times 5,000]$ in each future year, for the lifetime of the project.
- The value of the bridge to the new travellers depends on how many of them will use the bridge and how much benefit it will bring them? Need to estimate new travellers' **willingness to pay** for crossing the river because they only cross the river if the bridge is built.
- There is no need to estimate existing travellers' willingness to pay for crossing the river because they cross the river with and without the bridge.
- This information can be inferred from the **demand curve**.

Today's lecture

- A first go at CBA: building a new bridge.
  1. Initial Setup
  2. The demand curve.
  3. Consumer Surplus and Social Welfare.
  4. Aggregating over time: net present value.
  5. CBA recommendation.

A first go at CBA

The Demand Curve

- The demand curve tells us how many travellers are interested in crossing the river for any given price.
- For example, we know that if the price is 0.20, then there will be $5,000$ crossing per year, and that if the price is 0, then there will be $25,000$ crossing per year.
- As the price decreases, more and more travellers will be interested in crossing the bridge: demand curve is downward sloping.
- The demand curve (or function) tells us exactly how many.
- The demand curve is very useful in CBA. because it allows us to infer the total willingness to pay for the corresponding good.

A first go at CBA

The Demand Curve

- To simplify, assume individual's (daily) utility depends on her level of income x_i and whether or not she crosses the river, $B_i \in \{0,1\}$: $u_i(x_i, B_i)$.
- Then her willingness to pay for crossing the river is such that:

$$u_i(x_i - \Delta_i, 1) = u_i(x_i, 0)$$

- An individual wants to cross the river if her willingness to pay exceeds the cost of crossing.
- Since the demand was equal to $5,000$ before the bridge was built, it means that $5,000$ individuals have a willingness to pay larger or equal to the cost of using the ferry, 0.20.
- Similarly, since the demand is equal to $25,000$ after the bridge is built, it means that $25,000$ individuals have a willingness to pay larger or equal to 0 (the cost of using the bridge).

A first go at CBA

The Demand Curve

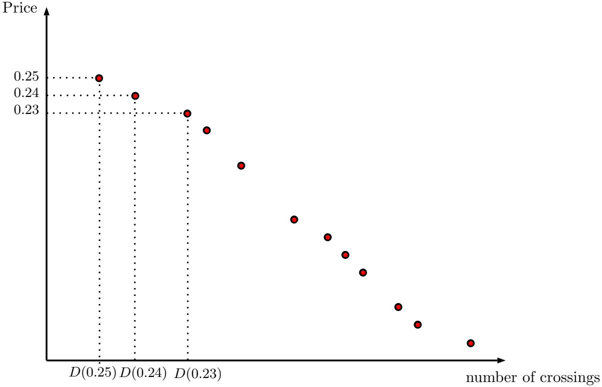
- So how to measure the total WTP from the demand curve? First we need to understand how to build the demand curve.
- To simplify, assume first that the WTP takes a finite number of values between 0 and 0.25:

$$\Delta_i \in \{0, 0.01, 0.02, \ldots, 0.25\}$$

- The demand curve gives us the quantity of good demanded (here number of crossings) for each level of price.

Lecture 2

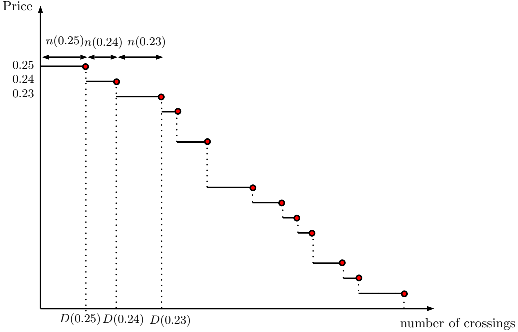
Demand curve and WTP



- Here prices take discrete values: 0.25, 0.24, 0.23, etc. More generally, the demand curve $D(.)$ gives the quantity demanded for any level of price.

Lecture 2

Demand curve and WTP



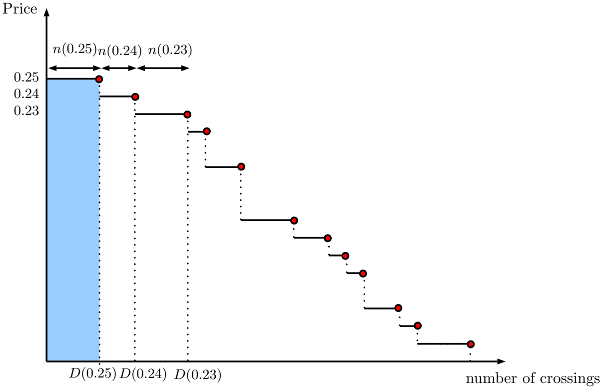
- How many individuals have a WTP equal to 0.25? How many have a WTP equal to 0.24?
- Number of individuals with a WTP of δ is equal to:

$$n(\delta) = \#\{i \in N \mid \Delta_i \geq \delta\} - \#\{i \in N \mid \Delta_i \geq \delta + 0.01\} = D(\delta) - D(\delta + 0.01)$$

- Demand curve allows to recover the number of individuals with each level of WTP δ.

Lecture 2

Demand curve and WTP

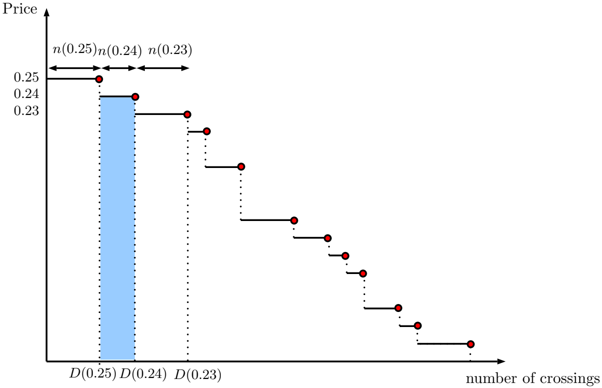


- Total WTP of individuals who have a WTP equal to 0.25 is equal to the blue area:

$$n(0.25) \times 0.25 = D(0.25) \times 0.25$$

Lecture 2

Demand curve and WTP

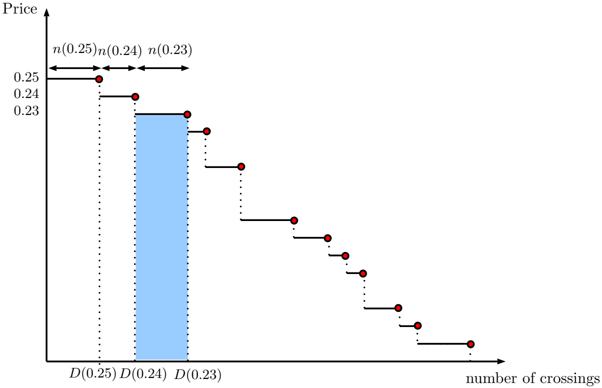


- Total WTP of individuals who have a WTP equal to 0.24 is equal to the blue area:

$$n(0.24) \times 0.24 = [D(0.24) - D(0.25)] \times 0.24$$

Lecture 2

Demand curve and WTP

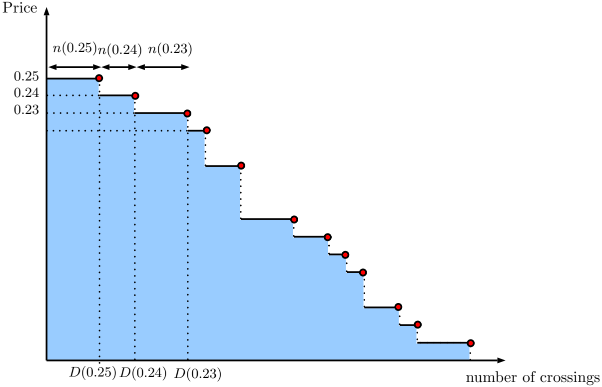


- Total WTP of individuals who have a WTP equal to 0.23 is equal to the blue area:

$$n(0.23) \times 0.23 = [D(0.23) - D(0.24)] \times 0.23$$

Lecture 2

Demand curve and WTP



- Total WTP for all individuals obtained by taking the sum of all these blue columns:

$$\text{Total WTP} = \sum_{i \in N} \Delta_i = \sum_{\delta=0}^{0.25} n(\delta)\delta$$

Lecture 2

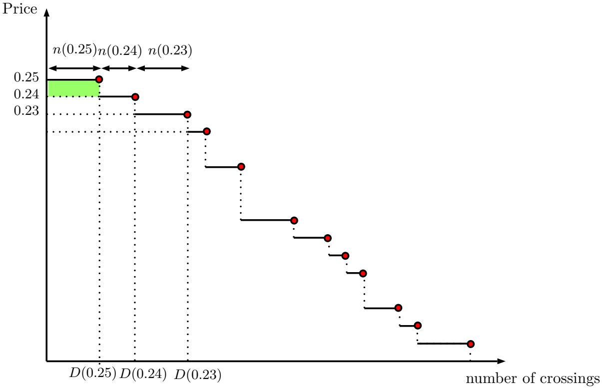
Demand curve and WTP

- Alternative way to compute the total WTP:

$$
\begin{aligned}
\sum_{i \in N} \Delta_i &= \sum_{\delta=0}^{0.25} n(\delta)\delta \\
&= \sum_{\delta=0}^{0.25} \left(D(\delta) - D(\delta+0.01)\right)\delta \\
&= \sum_{\delta=0}^{0.25} D(\delta)\delta - \sum_{\delta=0.01}^{0.26} D(\delta)(\delta - 0.01) \\
&= \sum_{\delta=0}^{0.25} D(\delta)0.01 - D(0.26)\times 0.25 + D(0)\times 0 \\
&= \sum_{\delta=0}^{0.25} D(\delta)0.01
\end{aligned}
$$

Lecture 2

Demand curve and WTP

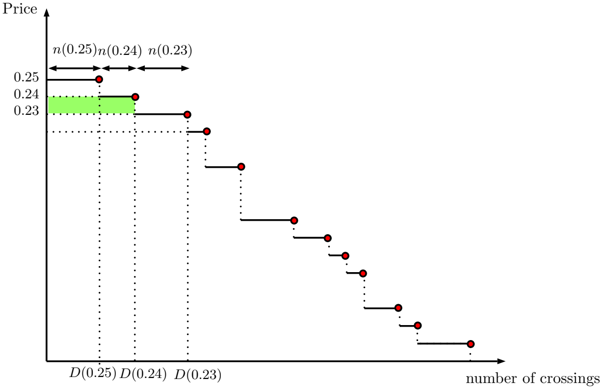


- Green area corresponds to:

$$D(0.25) \times 0.01$$

Lecture 2

Demand curve and WTP

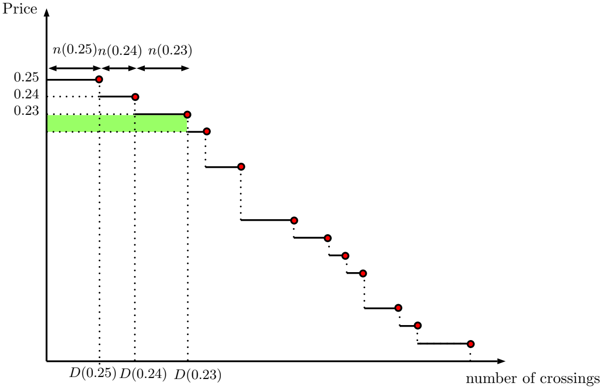


- Green area corresponds to:

$$D(0.24) \times 0.01$$

Lecture 2

Demand curve and WTP

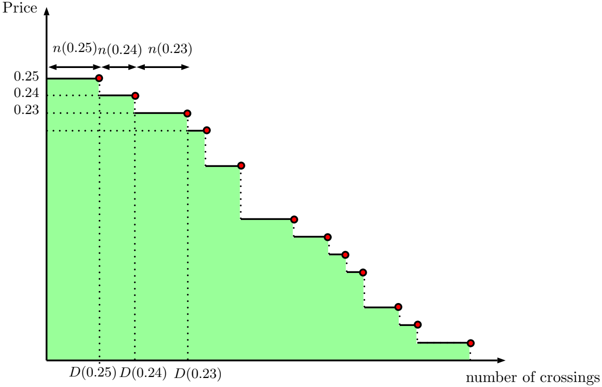


- Green area corresponds to:

$$D(0.23) \times 0.01$$

Lecture 2

Demand curve and WTP



- Total WTP for all individuals obtained by taking the sum of all these green rows:

$$\text{Total WTP} = \sum_{i \in N} \Delta_i = \sum_{\delta=0}^{0.25} D(\delta)0.01$$

A first go at CBA
Consumer Surplus

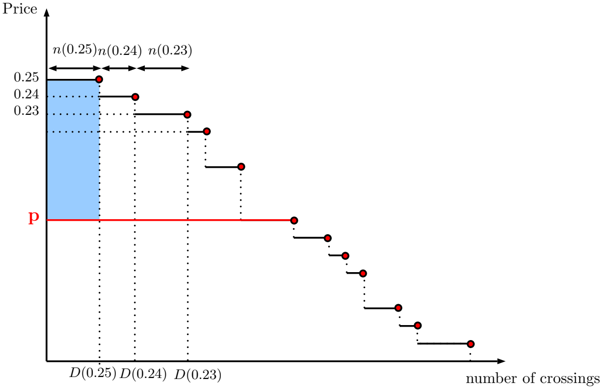
- We define the consumer surplus as the total sum of each individual's difference between her willingness to pay and the price (for consumers who actually cross the river).

$$
\begin{aligned}
CS &= \sum_{i \in N | \Delta_i \geq p} (\Delta_i - p) \\
&= \sum_{\delta = p}^{0.25} n(\delta)(\delta - p)
\end{aligned}
$$

- Consumer surplus reflects the welfare of consumers who buy the good at price p.
- Social welfare = consumer surplus + firm profits.

Lecture 2

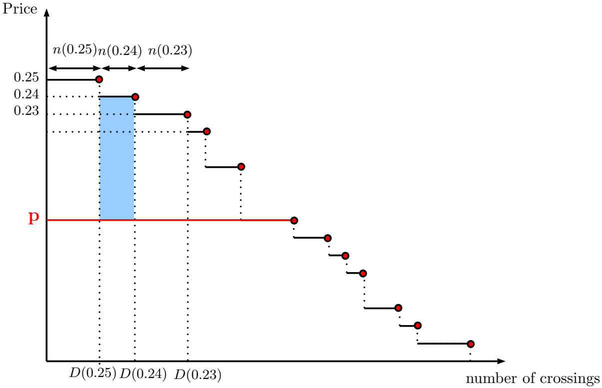
Demand curve and WTP



- Blue area corresponds to:

$$n(0.25) \times (0.25 - p)$$

Lecture 2
Demand curve and WTP

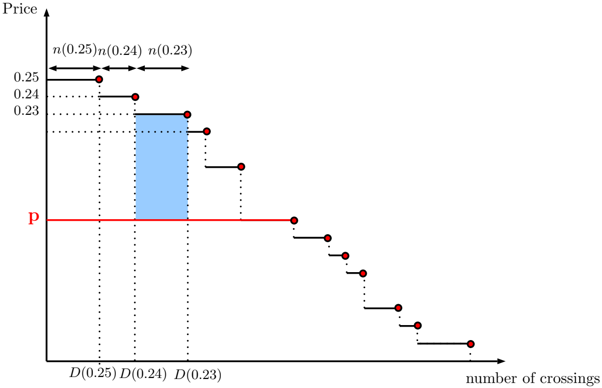


- Blue area corresponds to:

$$n(0.24) \times (0.24 - p)$$

Lecture 2

Demand curve and WTP

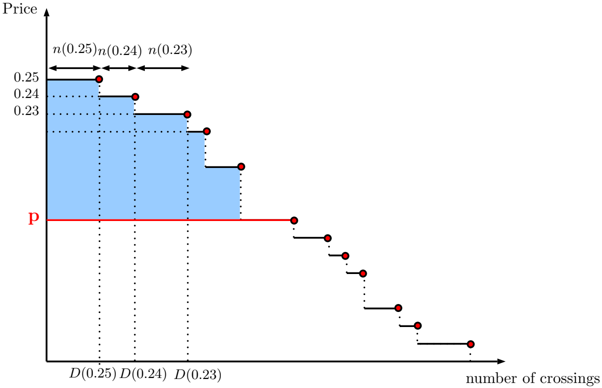


- Blue area corresponds to:

$$n(0.23) \times (0.23 - p)$$

Lecture 2

Demand curve and WTP



- Consumer surplus obtained by summing up the blue columns for WTP above p.

$$CS = \sum_{\delta=p}^{0.25} n(\delta)(\delta - p)$$

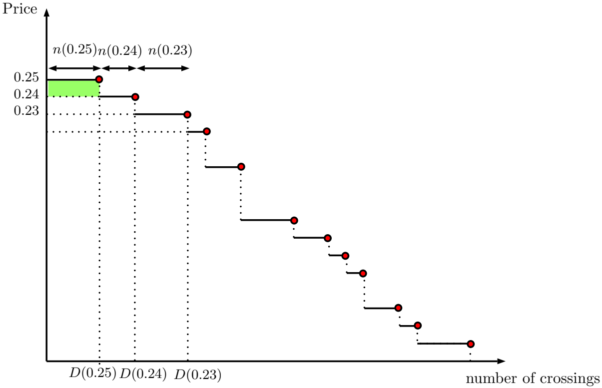
A first go at CBA
Consumer Surplus

- Alternative way to compute the CS:

$$
\begin{aligned}
CS &= \sum_{i \in N | \Delta_i \geq p} (\Delta_i - p) \\
&= \sum_{\delta = p}^{0.25} n(\delta)\delta - \#\{i \in N \mid \Delta_i \geq p\}p \\
&= \sum_{\delta = p}^{0.25} D(\delta)0.01 - D(0.26)0.25 + D(p)p - D(p)p \\
&= \sum_{\delta = p}^{0.25} D(\delta)0.01
\end{aligned}
$$

Lecture 2

Demand curve and WTP

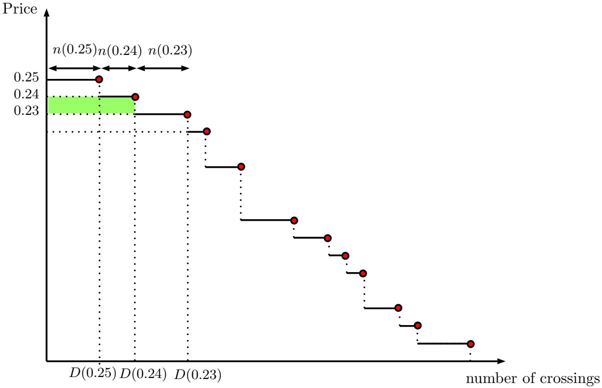


- Green area corresponds to:

$$D(0.25) \times 0.01$$

Lecture 2

Demand curve and WTP

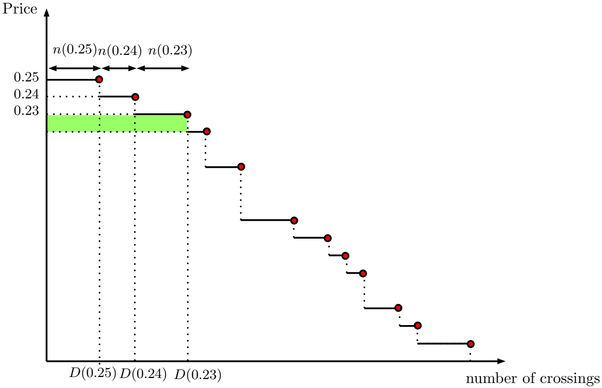


- Green area corresponds to:

$$D(0.24) \times 0.01$$

Lecture 2

Demand curve and WTP

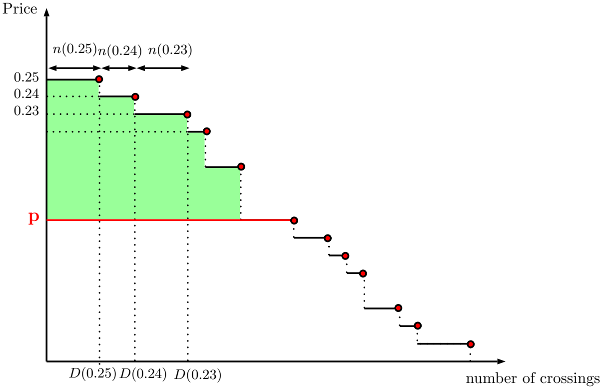


- Green area corresponds to:

$$D(0.23) \times 0.01$$

Lecture 2

Demand curve and WTP



- Consumer surplus obtained by summing up the green rows for WTP above p.

$$CS = \sum_{\delta=p}^{0.25} D(\delta)0.01$$

Lecture 2

Demand curve and WTP

- The definition of the CS generalizes to continuous demand curves.

$$CS = \int_{p}^{+\infty} D(t)dt \quad (\sim \sum_{\delta=p}^{0.25} D(\delta)0.01 \text{ in discrete case})$$

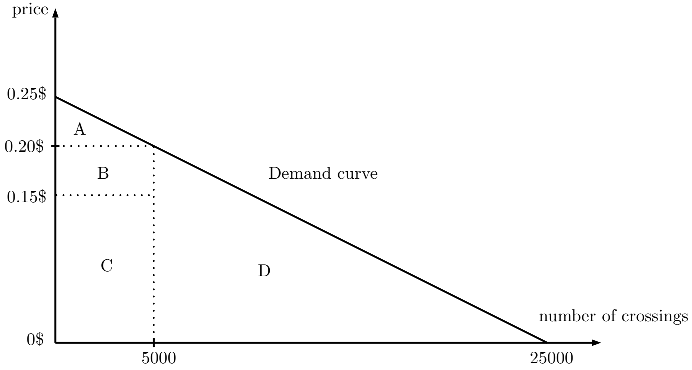
- The CS can also be expressed as a function of the inverse demand function $p(D)$ (price as a function of demand):

$$CS = \int_{0}^{D(p)} (p(D)-p)dD \quad (\sim \sum_{\delta=p}^{0.25} n(\delta)(\delta-p) \text{ in discrete case})$$

- Graphically, the CS is given by the area located between the demand curve and the price line.

A first go at CBA

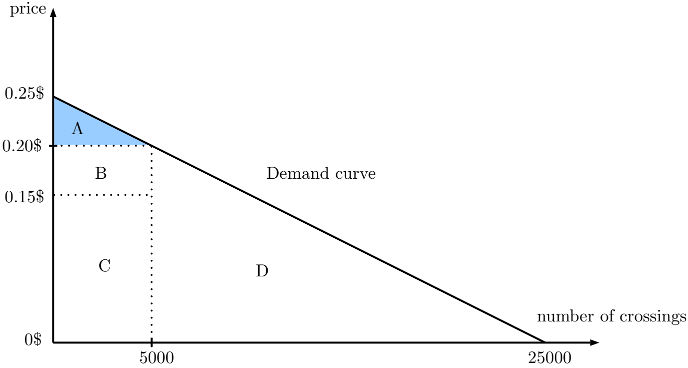
Consumer Surplus



- Assume a linear demand curve.
- From previous assumptions, the number of crossings is equal to $5,000$ when the price is 0.20 and to $25,000$ when the price is 0.
- Can we say by how much social welfare will increase (annually) if we build the bridge? (without accounting for its cost).

A first go at CBA

Consumer Surplus

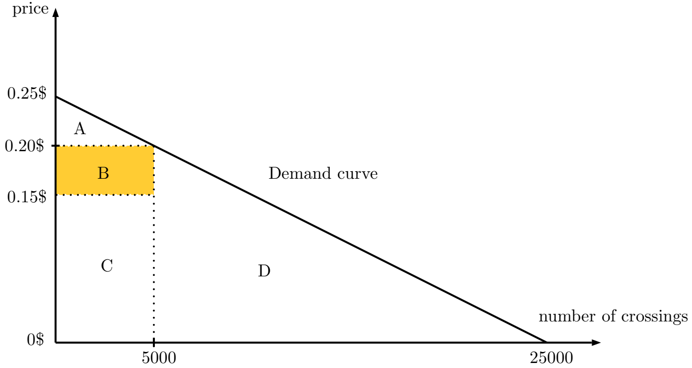


Before the bridge is built:

$$\text{Consumer Surplus} = \text{Area A}$$
$$= \frac{1}{2} \times 5,000 \times 0.05 = 125.$$

A first go at CBA

Consumer Surplus

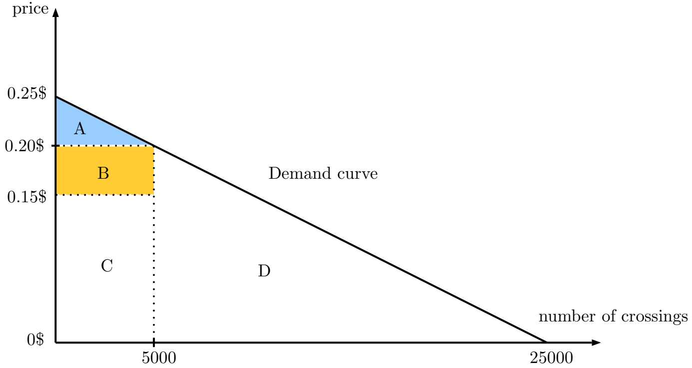


Before the bridge is built:

$$
\begin{aligned}
\text{Ferry's profit} &= \text{Revenue - Cost} \\
&= (5000 \times 0,20) - (5000 \times 0,15) = 250 = \text{Area B}.
\end{aligned}
$$

A first go at CBA

Consumer Surplus

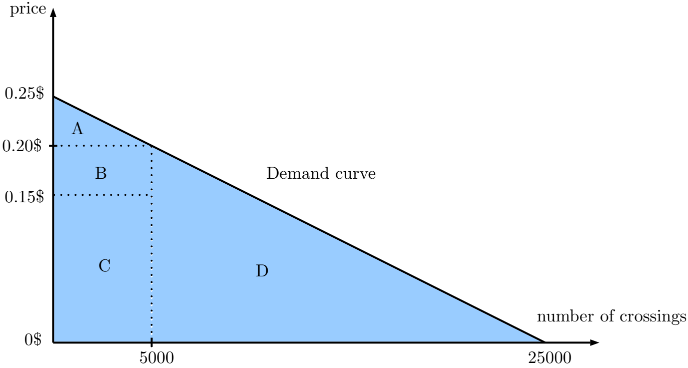


Before the bridge is built:

$$\begin{aligned}\text{Social Welfare} &= \text{Consumer Surplus} + \text{Profits}\\ &= 125 + 250 = 375.\end{aligned}$$

A first go at CBA

Consumer Surplus

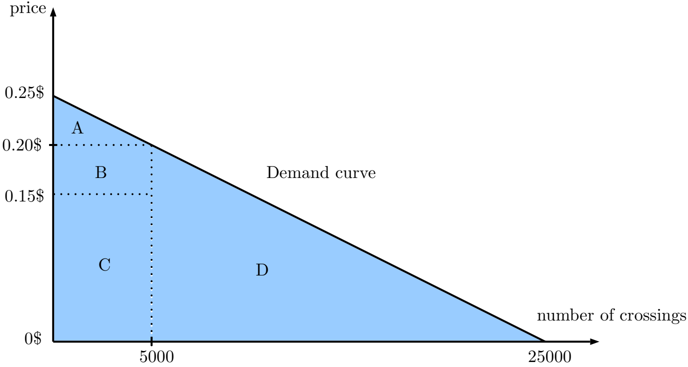


After the bridge is built:

$$\text{Consumer Surplus} = \text{A+B+C+D}$$
$$= \frac{1}{2} \times 25,000 \times 0.25 = 3,125.$$

A first go at CBA

Social Welfare



After the bridge is built:

$$\text{Social Welfare} = \text{Consumer Surplus} + \text{Profits}$$
$$= 3,125 + 0 = 3,125.$$

A first go at CBA

Social Welfare

- So the increase in social welfare due to the construction of the bridge is equal to:

$$\begin{aligned}\Delta SW &= \text{SW(after the bridge) - SW(before the bridge)} \\ &= (A+B+C+D)-(A+B) \\ &= 3,125-375 = 2,750\end{aligned}$$

- **Conclusion**: if the bridge is built, it will bring an additional $2,750$ in social welfare (=benefits for society) every year.

A first go at CBA

Social Welfare

- Which assumption have we been making implicitly here?
- As is usual in CBA, we've been adding up individual WTPs.
- We've therefore ignored whether some individuals deserved to be given a larger weight in the computation of social welfare.
- This is acceptable if we believe that there is no systematic relationship between one's willingness to pay to cross the river and one's income. Otherwise, it could be problematic.
- In this case, we have no particular reason to believe the bridge would affect individuals of different wealth differently.

A first go at CBA

Social Welfare

- If not, there are only two alternatives: to use some system of distributional weights or simply to show the net benefits to each party and let the policy maker apply his own evaluation.
- If distributional weights are used they need not be unique: it may be that the weights can take a wide range of alternative values and yet provide an unambiguous verdict on a project.
- Can sometimes be interesting to look for the most extreme weighting for which the project is still considered worthwhile.

A first go at CBA

Social Welfare

- Notice that both areas A and B cancel out in the calculation of ΔSW.
- In the case of area A, the reason is that existing travelers are still crossing the river, but for a lower price (0 instead of 0.20).
- In the case of area B, the reason is that it was a transfer payment (monopoly rent) rather than a payment for real goods and services; and if everybody's benefit is equally valuable, transfers cannot change social welfare.
- Consumers used to pay this rent and now they do not, but there is no saving in resource cost as a result of the non-payment after the bridge is built.

A first go at CBA

Social Welfare

- The economic cost-saving from the demise of the ferry comes from the liberation of resources worth C for production elsewhere in the economy.
- The only other economic change is the value of the additional consumption D.
- Thus, if one had wanted to take a short cut to estimating future net benefits (granted all dollars equally valuable) one could have straightaway identified only the changes of real economic significance, i.e., the cost-saving on the ferry (C) and the consumers' surplus on the generated traffic (D).
- ΔSW gives the yearly increase in social welfare due to the bridge. How to aggregate these increases over time?

Today's lecture

- A first go at CBA: building a new bridge.
    1. Initial Setup
    2. The demand curve.
    3. Consumer Surplus and Social Welfare.
    4. Aggregating over time: net present value.
    5. CBA recommendation.

A first go at CBA
Discounting for time

- Assume the bridge has a lifetime of 50 years before it needs to be rebuilt. For simplicity ignore any maintenance cost. Therefore, we need only consider the cost of building the bridge today.
- Assume that the increase in social welfare ΔSW computed previously applies to each of the next 50 years.
- Before we can decide whether the bridge is worth building, we need to aggregate these benefits over time so as to come up with a measure of the overall (intertemporal) social benefit of the bridge, that we will then be able to compare with the cost of building the bridge.
- Should we simply multiply ΔSW by 50? Why getting a $ tomorrow is not the same as getting a $ today?
- Because if I receive a $ today, I can invest it and get more than a $ tomorrow. So a $ today is more valuable than a $ tomorrow.

A first go at CBA

Discounting for time

- So we need to know the value of next year's consumption relative to this year's, and so on for all future years.
- This relative value is expressed in terms of a discount rate. To simplify (for now), think of the discount rate as the interest rate at which you can invest money.
- If this rate is 5% per year, this means that 1\$ of next year's consumption is worth only about 0.95\$ of this year's.
- Indeed, if I invest 0.95\$ today, I will have $0.95 + 0.95 \times 0.05 = 1$\$ next year.
- More generally, if r is the discount rate, we value each 1\$ of next year's consumption the same as $1/(1+r)$\$ of this year's, since:

$$\frac{1}{1+r} + \frac{1}{1+r} \times r = 1$$

A first go at CBA

Discounting for time

- The same logic applies to consumption (benefits or costs) which happen later in the future.
- If I invest $1/(1+r)^t$\$ today, my investment will be worth exactly 1\$ after t years, since:

$$\underbrace{\left[\underbrace{\left[\frac{1}{(1+r)t}\times(1+r)\right]}_{\text{investment after one year}}\times(1+r)\right]}_{\text{investment after two years}}\times\cdots\times(1+r)=1$$

- Therefore 1\$ in t years is worth $1/(1+r)^t$\$ from today's point of view.

A first go at CBA

Present value

- We call the *net present value* the discounted sum of net benefits over the lifetime of the project. In our bridge example it is equal to:

$$NPV = \sum_{t=0}^{t=49} \frac{\Delta SW}{(1+r)^t} - C$$

  because we assume the bridge has a lifetime of 50 years and the cost is borne at the start of the project.
- Note that $t = 0$ refers to the current year while $t = 1$ refers next year.

A first go at CBA

Present value

- Remember the formula for the sum of the first n terms of a geometric series ($p \neq 1$):

$$\sum_{k=0}^{n} p^k = \frac{1-p^{n+1}}{1-p}$$

- Here, we get:

$$\begin{aligned} NPV &= \sum_{t=0}^{t=49} \frac{\Delta SW}{(1+r)^t} \; - \; C \\ &= \Delta SW \sum_{t=0}^{t=49} \frac{1}{(1+r)^t} \; - \; C \\ &= \Delta SW \left(\frac{1-\left(\frac{1}{1+r}\right)^{50}}{1-\frac{1}{1+r}} \right) - \; C \; \approx \; \Delta SW \left(\frac{1+r}{r} \right) \; - \; C \end{aligned}$$

- The approximation is valid as long as r is not too close to 0 and the project's lifetime is long enough.

A first go at CBA

Present value

- For $r = 10\%$, we get:

$$NPV = 2,750\left(\frac{1+0.1}{0.1}\right) - 30,000 = 250 > 0$$

- Here, the net present value is positive: we should build the bridge.
- More generally, if the lifetime of a project goes from $t = 0$ to $t = T$, and each year generates net benefits $NB_t = B_t - C_t$, then the NPV is given by:

$$NPV = \sum_{t=0}^{T} \frac{NB_t}{(1+r)^t}$$

A first go at CBA

Discount Rate

- But how do we choose the discount rate?
- This is an issue of considerable practical importance.
- How far is it legitimate to depress present levels of living for the sake of the future?
- The horizon for a road scheme might well extend to forty years into the future. A benefit valued at 100$ in forty years' time will have a present value of 14.2$ at a discount rate of 5%, 4.6$ at 8% and only 2.2$ at 10%!
- A high rate of discount will count against projects which take a long time to gestate.

A first go at CBA

Exercise

- A highway department is considering building a temporary bridge to cut travel time during the three years it will take to build a permanent bridge. The temporary bridge can be put up in a few weeks at a cost of $900,000$.
- At the end of three years, it would be removed and the steel would be sold for scrap for an estimated value of $81,000$.
- Based on estimated time savings and wage rates, fuel savings, and reductions in risks of accidents, department analysts predict that the benefits in real dollars would be $275,000$ during the first year, $295,000$ during the second year, and $315,000$ during the third year. Departmental regulations require use of a real discount rate of 4%.

A first go at CBA

Exercise

Questions:

1. Calculate the present value of net benefits assuming that the benefits are realized at the end of each of the three years.
2. Calculate the present value of net benefits assuming that the benefits are realized at the beginning of each of the three years.
3. Calculate the present value of net benefits assuming that the benefits are realized in the middle of each of the three years.
4. Calculate the present value of net benefits assuming that half of each year's benefits are realized at the beginning of the year and the other half at the end of the year.
5. Should the highway department build the temporary bridge?

A first go at CBA

Exercise

- If the benefits are realized at the end of each year:

$$\begin{aligned} NPV &= -900,000 + \frac{275,000}{1.04} + \frac{295,000}{1.04^2} + \frac{315,000 + 81,000}{1.04^3} \\ &= -10,790 < 0 \end{aligned}$$

- If the benefits are realized at the beginning of each year:

$$\begin{aligned} NPV &= -900,000 + 275,000 + \frac{295,000}{1.04} + \frac{315,000 + 81,000}{1.04^2} \\ &= 24,778 > 0 \end{aligned}$$

A first go at CBA

Exercise

- If the benefits are realized in the middle of each year:

$$
\begin{aligned}
NPV &= -900,000 + \frac{275,000}{1.04^{0.5}} + \frac{295,000}{1.04^{1.5}} + \frac{315,000}{1.04^{2.5}} + \frac{81,000}{1.04^{3}} \\
&= 5,393 > 0
\end{aligned}
$$

- If half the benefits are realized at the beginning and half of the benefits are realized at the end of each year:

$$
\begin{aligned}
NPV &= -900,000 + 0.5 \times 275,000 + \frac{0.5 \times 275,000 + 0.5 \times 295,000}{1.04} \\
&+ \frac{0.5 \times 295,000 + 0.5 \times 315,000}{1.04^{2}} + \frac{0.5 \times 315,000 + 81,000}{1.04^{3}} \\
&= 5,554 > 0
\end{aligned}
$$

- NPV is positive under 3/4 scenarios, but there does not seem to be a great case in favour of building the temporary bridge.

CBA: Lecture 2

Internal Rate of Return

- So far, we have taken the discount rate as given and used it to determine the NPV of our considered projects.
- For example, in our bridge example, we had:

$$NPV = \Delta SW \left(\frac{1 - \left(\frac{1}{1+r}\right)^{50}}{1 - \frac{1}{1+r}} \right) - C \approx \Delta SW \left(\frac{1+r}{r} \right) - C$$

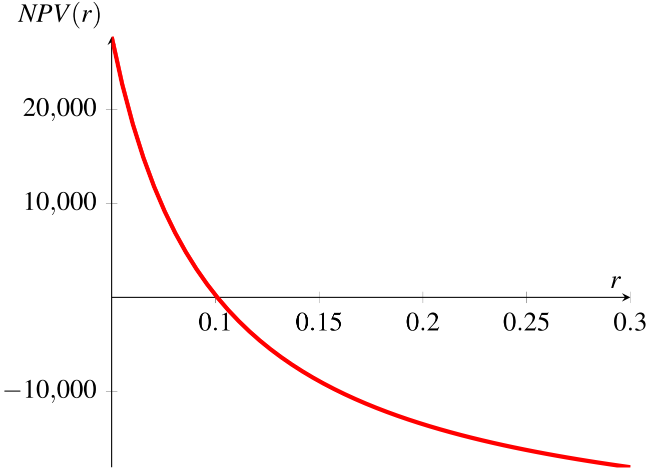
- We chose $r = 10\%$, and obtained:

$$NPV = 2,750 \left(\frac{1+0.1}{0.1} \right) - 30,000 = 250 > 0$$

- What if we vary the discount rate?

CBA: Lecture 2

Internal Rate of Return



CBA: Lecture 2

Internal Rate of Return

- In this example NPV is a decreasing function of the discount rate r.
- In many projects, costs realize early, while benefits realize late. As the discount rate increases, benefits are discounted relatively more than costs.
- When this is the case, the NPV is decreasing with the discount rate; positive for small values of r, negative for large values of r.
- **Internal rate of return** (IRR) $=$ discount rate for which the NPV is equal to zero.

CBA: Lecture 2

Internal Rate of Return

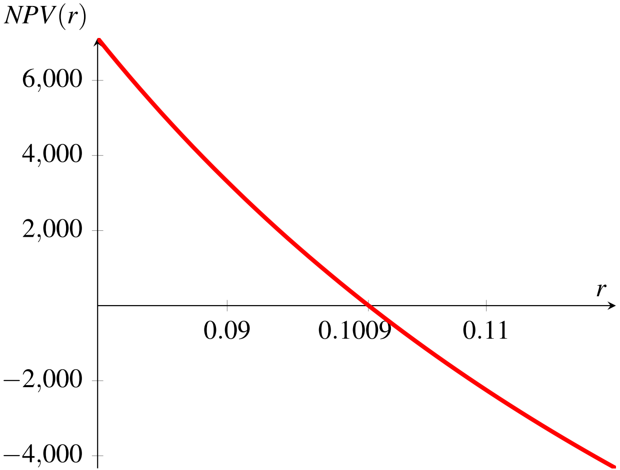
- In our bridge example, we can solve analytically for the (IRR).

$$NPV(IRR) = 2,750\left(\frac{1+IRR}{IRR}\right) - 30,000 = 0$$
$$\frac{2,750}{IRR} - 27,250 = 0$$
$$IRR = \frac{2,750}{27,250} = 10.09\%$$

- Not surprisingly, the IRR is very close (but larger) than 10% since we obtained a NPV almost equal to zero (250) when used $r = 10\%$.

CBA: Lecture 2

Internal Rate of Return



CBA: Lecture 2

Internal Rate of Return

- When NPV decreasing with discount rate, and only one project is under consideration, equivalent criteria for deciding whether to adopt a project or not: the NPV is positive if and only if the actual discount rate is lower than the IRR.
- Beware, however, that this is not a general rule, and depends on how costs and benefits are distributed over time (since it requires the NPV to decrease with the discount rate).
- IRR interesting for sensitivity analysis but always think in terms of NPV.

CBA: Lecture 2

Exercises: Computing the NPV + Sensitivity

- A city is considering building a recreation center. The estimated construction cost is 12 million with annual staffing and maintenance costs of $750,000$ over the 30 year life of the project.
- Analysts estimate the first-year benefits (accruing at the beginning of the first year) to be 1.2 million. They expect the annual benefit to grow in real terms due to increases in population and income. Their prediction is a growth rate of 4%.
- Analysts estimate the real discount rate to be 6%.

CBA: Lecture 2

Exercises: Computing the NPV + Sensitivity

Questions:

- Calculate the net present value for the project using the analysts' predictions.
- Sensitivity analysis: find the maximum maintenance cost for which the project becomes beneficial (for a growth rate of 4% and a discount rate of 6%).
- Would you recommend building the recreation center?

Exercises: Computing the NPV + Sensitivity

- For a growth rate of g and a discount rate of r, we get:

$$NPV = -12 * 10^6 - \sum_{t=0}^{29} \frac{75.10^4}{(1+r)^t} + \sum_{t=0}^{29} \frac{12.10^5 (1+g)^t}{(1+r)^t}$$

$$= -12.10^6 - 75.10^4 \left(\frac{1 - \left(\frac{1}{1+r}\right)^{30}}{1 - \left(\frac{1}{1+r}\right)} \right)$$

$$+ 12.10^5 \left(\frac{1 - \left(\frac{1+g}{1+r}\right)^{30}}{1 - \left(\frac{1+g}{1+r}\right)} \right)$$

CBA: Lecture 2

Exercises: Computing the NPV + Sensitivity

- Note here that we cannot assume $\left(\frac{1+g}{1+r}\right)^{30} \approx 0$ because $\frac{1+g}{1+r}$ is too close to 1.
- For $g = 4\%$ and $r = 6\%$, we find:

$$
\begin{aligned}
NPV = & -12.10^6 - 75.10^4 \left(\frac{1 - \left(\frac{1}{1.06}\right)^{30}}{1 - \left(\frac{1}{1.06}\right)} \right) \\
& + 12.10^5 \left(\frac{1 - \left(\frac{1.04}{1.06}\right)^{30}}{1 - \left(\frac{1.04}{1.06}\right)} \right) \\
& = 4.74 \times 10^6.
\end{aligned}
$$

CBA: Lecture 2

Exercises: Computing the NPV + Sensitivity

- Let M be the yearly maintenance cost. The NPV writes:

$$\begin{aligned} NPV &= -12.10^6 - M \times \left(\frac{1-\left(\frac{1}{1.06}\right)^{30}}{1-\left(\frac{1}{1.06}\right)} \right) + 12.10^5 \left(\frac{1-\left(\frac{1.04}{1.06}\right)^{30}}{1-\left(\frac{1.04}{1.06}\right)} \right) \\ &= -12.10^6 - 14.59 \times M + 2.77 \times 10^7 \end{aligned}$$

- Therefore the project is profitable if:

$$NPV > 0 \quad \Leftrightarrow \quad M < \frac{-12.10^6 + 2.77 \times 10^7}{14.59} = 1.07 \times 10^6.$$

- This means that the maintenance costs would have to be 43% bigger than the estimated $750,000$ for the project not to be profitable. Therefore, we can be fairly confident that the project would be worthwhile even if our estimate of the maintenance costs are not completely accurate.

CBA: Lecture 2

Exercises: More on the IRR

Two projects under considerations.

- Project 1 costs C today and yields benefit B next year.
- Project 2 costs C today and yields yearly benefit D for 50 years.

Questions:

1. Write the NPV for both projects as a function of the discount rate r.
2. Find the internal rate of return for both projects. When is project 1's IRR larger than project 2's?
3. Show why project 1 might have a larger IRR, yet generate a smaller NPV for certain values of the discount rate?
4. What should we conclude?

CBA: Lecture 2

Exercises: More on the IRR

- Project 1:

$$NPV_1 = -C + \frac{B}{1+r}$$

- Project 2:

$$NPV_2 = -C + \sum_{t=0}^{t=49} \frac{D}{(1+r)^t} = -C + D\left(\frac{1-\left(\frac{1}{1+r}\right)^{50}}{1-\frac{1}{1+r}}\right)$$

$$\approx -C + D\left(\frac{1+r}{r}\right)$$

Exercises: More on the IRR

- IRR for project 1:

$$NPV_1 = -C + \frac{B}{1+r_1^*} = 0 \quad \Leftrightarrow \quad r_1^* = \frac{B}{C} - 1$$

- IRR for project 2:

$$NPV_2 = -C + D\left(\frac{1+r_2^*}{r_2^*}\right) = 0 \quad \Leftrightarrow \quad r_2^* = \frac{D}{C-D}$$

- Therefore, project 1's IRR larger than project 2's if:

$$\frac{B}{C} - 1 > \frac{D}{C-D} \quad \Leftrightarrow \quad B > \frac{C^2}{C-D}$$

CBA: Lecture 2

Exercises: More on the IRR

- Assume previous condition holds, so that $r_1^* > r_2^*$. Anytime (for any discount rate) project 2 is profitable, project 1 is profitable as well. One could think that project 1 is better than project 2.
- However, for $r = 0$, we get:

$$NPV_1(r=0) = -C + \frac{B}{1+0} = -C + B$$

while,

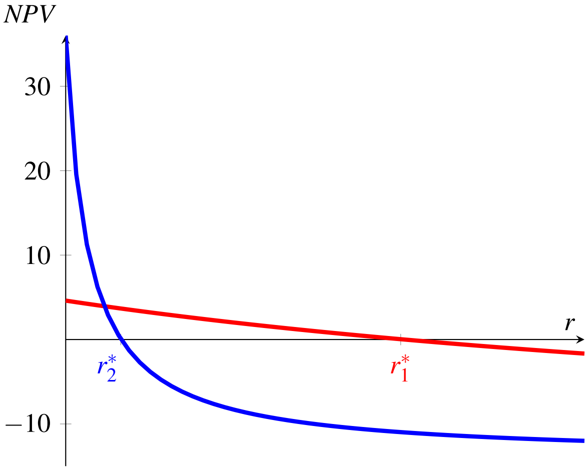
$$NPV_2(r=0) = -C + 50D > NPV_1(r=0) \quad \text{if } B < 50D$$

- For small values of the discount rate, project 2 yields (a possibly much) larger NPV than project 1.
- Conclusion: do not use IRRs to compare the profitability of different projects!

CBA: Lecture 2

Exercises: More on the IRR

NPV of both projects (red for 1, blue for 2) for $C = 15$, $B = 30$, and $D = 1$.



Lecture 2 recap

- We learned two important methods used in CBA.
- **Method 1**: how to evaluate consumers' willingness to pay for marketed goods. Necessary for Step 4 of the CBA analysis.
- What you should be able to do: compute the consumer surplus (CS).
- **Method 2**: how to aggregate net benefits over the lifetime of the project. Necessary for Step 7 of the CBA analysis.
- What you should be able to do: compute the Net Present Value (NPV) of the project.

CBA: Lecture 2

Lecture 2 recap

Method 1: how to evaluate consumers' willingness to pay for marketed goods.

- CBA requires evaluating the population's willingness to pay for specific goods: going to the Odense stadium, using the highway between Odense and Middlefart, crossing the great belt...
- If that good (or an equivalent good) is exchanged on a market, then possible to use the demand curve to infer the population's (aggregate) WTP as the area located below the demand curve.
- For each individual, net benefit equal to WTP minus price.
- Consumer Surplus = sum of every individual's net benefit = difference between total WTP of individuals who buy the good - total price paid by those individuals.

CBA: Lecture 2

Lecture 2 recap

- If $D(p)$ is the demand function ($D(p)$ is the quantity demanded when the price is equal to p), and p^* is the actual price on the market, then the Consumer Surplus is equal to:

$$CS = \int_{p^*}^{+\infty} D(p)dp$$

- If $p(D)$ is the inverse demand function (price as a function of demand) and the D^* is the actual quantity demanded on the market, the CS is equal to:

$$CS = \int_{0}^{D^*} (p(D) - p(D^*))dD$$

- Consumer surplus corresponds to the area located between the demand curve and the horizontal line corresponding to the actual price p.

Method 2: how to aggregate net benefits over the lifetime of the project.

- Steps 1 to 6 of the CBA analysis yield aggregate net benefits of the project over its lifetime.
- However, net benefits which realize later should count less than net benefits which realize earlier in time. Why?
- More precisely, receiving/paying 1 \$ in t years is equivalent to receiving/paying $1/(1+r)^t$ \$ today.
- Net Present Value (NPV) expresses the aggregate net benefits of the project from today's point of view:

$$NPV = \sum_{t=0}^{T} \frac{NB_t}{(1+r)^t}$$

CBA: Lecture 2

Exercises: Computing the Consumer Surplus.

In our bridge example, assume the demand function is given by the following expression:

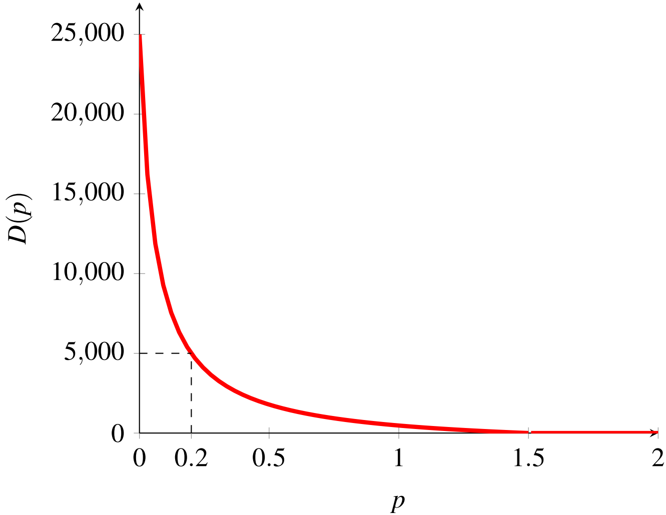
$$
\begin{aligned}
D(p) &= \frac{1560}{p+0.06} - 1000 && \text{if } p \leq 1.5 \\
&= 0 && \text{if } p > 1.5
\end{aligned}
$$

Questions:

- Check that this demand function is consistent with the data of the problem and represent it graphically.
- Find the inverse demand function $p(D)$.
- Compute the Consumer Surplus before the bridge is built and after the bridge is built.
- Find the CBA recommendation for $r = 10\%$.
- Find the Internal Rate of Return.

CBA: Lecture 2

Exercises: Computing the Consumer Surplus.



Exercises: Computing the Consumer Surplus.

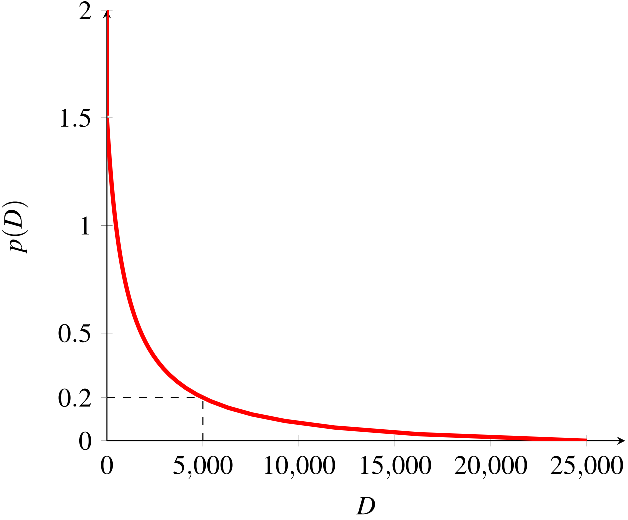
- We check that $D(0.2) = 5,000$ and $D(0) = 25,000$, consistent with the data of the problem.
- The inverse demand function is given by:

$$p(D) = \frac{1560}{D+1000} - 0.06 \qquad \text{if } D \leq 25,000$$

- No (positive) price generates a demand above $25,000$.

CBA: Lecture 2

Exercises: Computing the Consumer Surplus.



CBA: Lecture 2

Exercises: Computing the Consumer Surplus.

- The formula for computing the CS at level of price p^* using the demand function is:

$$CS = \int_{p^*}^{+\infty} D(p)dp$$

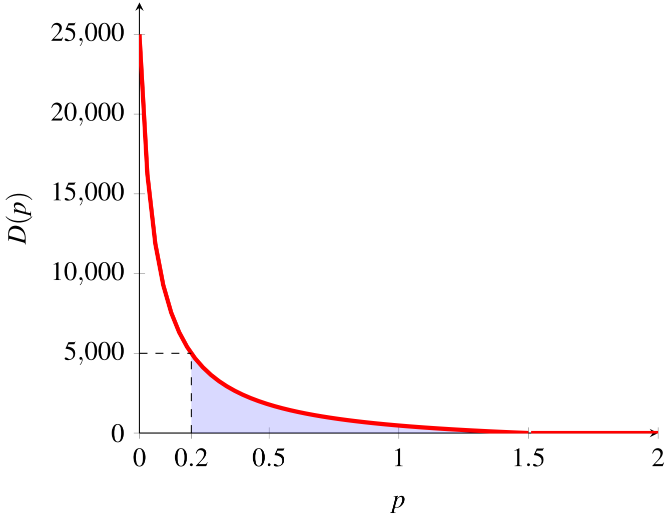
- Before the bridge is built the price is $p^* = 0.2$. Therefore the consumer surplus is equal to:

$$CS = \int_{0.2}^{+\infty} D(p)dp = \int_{0.2}^{1.5} \left(\frac{1560}{p + 0.06} - 1000 \right) dp$$

- Graphically we get.

CBA: Lecture 2

Exercises: Computing the Consumer Surplus.



CBA: Lecture 2

Exercises: Computing the Consumer Surplus.

- Analytically, we get:

$$
\begin{aligned}
CS &= \int_{0.2}^{1.5} \left(\frac{1560}{p+0.06} - 1000 \right) dp \\
&= \Big[1560ln(p+0.06) - 1000p \Big]_{0.2}^{1.5} \\
&= (1560ln(1.56) - 1000 \times 1.5) - (1560ln(0.26) - 1000 \times 0.2) \\
&= 1495
\end{aligned}
$$

CBA: Lecture 2

Exercises: Computing the Consumer Surplus.

- After the bridge is built, the price is equal to 0. Therefore, the consumer surplus is equal to the total WTP:

$$
\begin{aligned}
CS &= \int_0^{+\infty} D(p)dp = \int_0^{1.5} \left(\frac{1560}{p+0.06} - 1000 \right) dp \\
&= \Big[1560 \times ln(p+0.06) - 1000 \times p \Big]_0^{1.5} \\
&= (1560 \times ln(1.56) - 1000 \times 1.5) - (1560 \times ln(0.06) - 1000 \times 0) \\
&= 3582
\end{aligned}
$$

Exercises: Computing the Consumer Surplus.

- So the CS increases by $3582 - 1495 = 2087$.
- Social welfare is equal to the sum of the Consumer Surplus and the producer's profit. Before the bridge is built, the profit is equal to 5000×0.05 (since the price is 0.20 and the cost is 0.15). After the bridge is built there is no company and no profit.
- Therefore, the change in social welfare is equal to:

$$\Delta SW = \Delta CS - 0.05 \times 5000 = 2087 - 250 = 1837$$

- Remember that the NPV in this case is equal to:

$$NPV = \Delta SW \left(\frac{1 - \left(\frac{1}{1+r}\right)^{40}}{1 - \frac{1}{1+r}} \right) - C \approx \Delta SW \left(\frac{1+r}{r} \right) - C$$

CBA: Lecture 2

Exercises: Computing the Consumer Surplus.

- For $r = 10\%$, we get:

$$NPV = 1837\left(\frac{1+0.1}{0.1}\right) - 30,000 = -9793 < 0$$

- Therefore, it is not worthwhile to build the bridge.
- Internal Rate of Return r^* given by:

$$\Delta SW\left(\frac{1+r^*}{r^*}\right) - C = 0$$

which is equivalent to:

$$r^* = \frac{1}{\frac{C}{\Delta SW} - 1} = \frac{1}{\frac{30,000}{1837} - 1} = 6.5\%$$

- Project only worthwhile (i.e. $NPV > 0$) if discount rate smaller than $r^* = 6.5\%$.