Department of Mathematics and Natural Science

MAT110: Differential Calculus and Co–ordinate Geometry

Section ______, Quiz ______(Fall'22)

Name (PRINT): ________________________________	**ID:**__________________

Time: 25 minutes **Total Marks: 20**

Use following theorem and expressions if needed.
Any function f is said to be **continuous everywhere** if the following conditions are satisfied for **every value** of $x = c \in (-\infty, \infty)$

- $f(c)$ is defined.
- $\lim_{x \to c} f(x)$ exists.
- $\lim_{x \to c} f(x) = f(c)$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1, \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0, \qquad \lim_{x\to 0} \frac{\tan(x)}{x} = 1$$

Problem 1 Complete the statement for **The Squeezing Theorem**

Solution: Let $f(x), g(x),$ and $h(x)$ be functions satisfying

$$\boxed{g(x) \leq f(x) \leq h(x)}$$

for **all** x in some **open interval containing the number** c, with the possible exception that the inequalities need not hold at c. If limit of $g(x)$ and $h(x)$ as x approaches c, is given by,

$$\boxed{\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L}$$

then $f(x)$ also has this limit as x approaches c, that is,

$$\boxed{\lim_{x\to c} f(x) = L}$$

[2+1+1 = 4]

Problem 2 Given that, $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, prove **using the squeezing theorem** that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$$

[4]

Solution: Given, $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

for $x > 0$, $-x \leq \sin\left(\frac{1}{x}\right) \leq x$

for $x < 0$, $-x \geq \sin\left(\frac{1}{x}\right) \geq x$

combining both, $-|x| \leq \sin\left(\frac{1}{x}\right) \leq |x|$

Now, as $\lim_{x\to 0} |x| = \lim_{x\to 0} -|x| = 0$

So, by squeezing theorem, $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$

Problem 3 Carefully read the instructions:

(a) **Find a value** of the **constant** k, if possible, that will make the function **continuous everywhere** and **then show the function in a graph**:

(i) $f(x) = \begin{cases} 4 - x^2, & x \geq -2 \\ k/x^2, & x < -2 \end{cases}$ (ii) $g(x) = \begin{cases} 2x + 1, & x \leq -1 \\ kx^2 & x > -1 \end{cases}$

(b) Find a **non-zero value** for the constant k that makes $h(x) = \begin{cases} \dfrac{\sin(kx)}{x}, & x < 0 \\ 3x + 2k^2 & x \geq 0 \end{cases}$ continuous at $x = 0$

[(2 + 2) + (2 + 2) + 4 = 12]

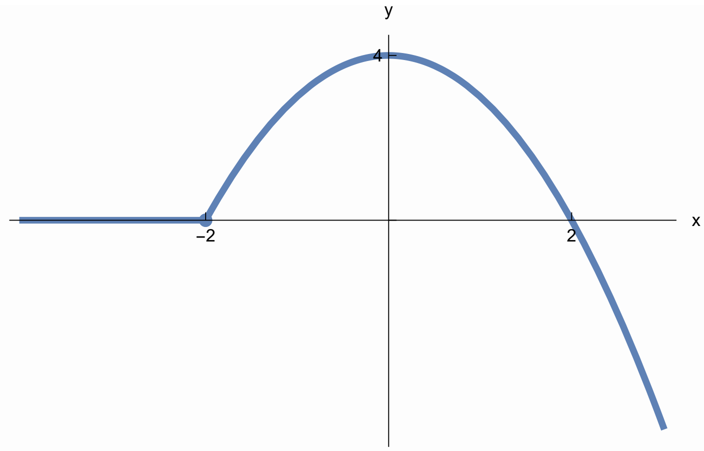
Solution: (a) (i) For continuity, we need

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) = f(-2) = 4 - (-2)^2 = 0$$

$$\text{Or, } 0 = \frac{k}{(-3)^2}$$

$$\implies k = 0$$

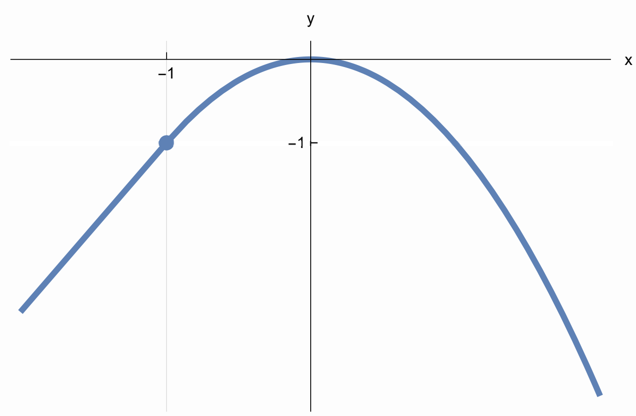
So, $f(x) = \begin{cases} 4 - x^2, & x \geq -2 \\ 0, & x < -2 \end{cases}$



(ii) For continuity, we need

$$\lim_{x\to -1^-} g(x) = \lim_{x\to -1^+} g(x) = g(-1) = 2(-1)+1 = -1$$
$$\text{Or, } -1 = k(-1)^2$$
$$\implies k = -1$$

So, $g(x) = \begin{cases} 2x+1, & x \le -1 \\ -x^2, & x > -1 \end{cases}$



(b) For $h(x)$ to be continuous,

$$\lim_{x\to 0^-} h(x) = \lim_{x\to 0^+} h(x) = h(0) = 3(0) + 2k^2$$
$$\text{Or, } \lim_{x\to 0^-} \frac{\sin kx}{x} = 2k^2$$
$$\text{Or, } \lim_{x\to 0^-} \frac{k\sin kx}{kx} = 2k^2$$
$$\text{Or, } k = 2k^2 \qquad \left(\text{as } \lim_{x\to 0^-} \frac{\sin kx}{kx} = 1\right)$$
$$\text{Or, } 1 = 2k \qquad (\text{as } k \neq 0)$$
$$\implies k = 1/2$$