Department of Mathematics and Natural Science

MAT110: Differential Calculus and Co–ordinate Geometry

Section ______, Quiz ______(Fall'22)

Name (PRINT): ______________________________	ID: __________________

Time: 25 minutes **Total Marks: 20**

Use following expressions if needed

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1,\qquad \lim_{x\to 0}\frac{1-\cos(x)}{x}=0,\qquad \lim_{x\to 0}\frac{\tan(x)}{x}=1\qquad \lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^{x}=\mathrm{e}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)]=f(x)\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]+g(x)\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]\qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]-f(x)\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]}{(g(x))^2}$$

Problem 1

Let

$$f(x)=\begin{cases}\dfrac{\tan(x)}{|x|}, & x\neq 0\\[2ex] 1, & x=0\end{cases}$$

(a) **Mathematically** show that the function, $f(x)$ is **NOT continuous** at $x=0$

(b) What can you say about the **differentiability** of the function at $x=0$ from (a). [3+1]

Solution:

(a) $f(0)=1$, so $f(x)$ is defined at $x=0$,

But,

When $x<0$,

$$\lim_{x\to 0^-}\frac{\tan(x)}{|x|}=\lim_{x\to 0^-}\frac{\tan(x)}{-x}=-1\times\underbrace{\lim_{x\to 0^-}\frac{\tan(x)}{x}}_{:=1}=-1$$

When $x > 0$,

$$\lim_{x \to 0^+} \frac{\tan(x)}{|x|} = \underbrace{\lim_{x \to 0^+} \frac{\tan(x)}{x}}_{:=1} = 1$$

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

As the limit does not exist at $x = 0$, so the function $f(x)$ is not continuous at $x = 0$

(b) As the function is not continuous at $x = 0$, so the function cannot be differentiable at $x = 0$

Problem 2

Find the derivative of $y = \dfrac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}$, using **logarithmic differentiation** [6]

Solution:

$$\begin{aligned}\ln y &= \ln\left(\frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\right)\\ &= \ln x^2 + \ln(7x-14)^{1/3} - \ln(1+x^2)^4\\ &= 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)\end{aligned}$$

Differentiating with respect to x

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\ln y) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)\right)\\ &= \frac{2}{x} + \frac{7}{3(7x-14)} - \frac{4\cdot 2x}{1+x^2}\\ &= \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\cdot\left(\frac{2}{x} + \frac{1}{3(x-2)} - \frac{8x}{1+x^2}\right)\end{aligned}$$

Problem 3

$$\text{Let } f(x) = |x - 1|$$

(a) **By finding limit and functional value**, show that $f(x)$ is **continuous** at $x = 1$.

(b) **By finding limiting value** of $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, **show that** $f(x)$ is **NOT differentiable** at $x = 1$,

(c) **Sketch the graph** of $f(x)$

[3 + 2 + 1 = 6]

Solution:

(a) For checking continuity,

At first, clearly, $f(1) = |1 - 1| = 0$, so $f(x)$ is defined at $x = 1$

Then $\lim_{x \to 1} f(x) = |1 - 1| = 0$ for both $x \to 1^-$ and $x \to 1^+$, so limit exists at $x = 1$

finally, $f(1) = \lim_{x \to 1} f(x) = 0$, or limiting value = functional value.

(b) For checking differentiability,

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{|x+h| - |x|}{h}$ at $x = 1$,

$f'(x) = \lim_{h \to 0} \dfrac{|1 + h - 1| - |1 - 1|}{h} = \lim_{h \to 0} \dfrac{|h|}{h}$

As $|h|$ changes sign on both sides of 0, so we need to check the one sided limits

When $h \to 0^-$, $h < 0$, so, $|h| = -h$

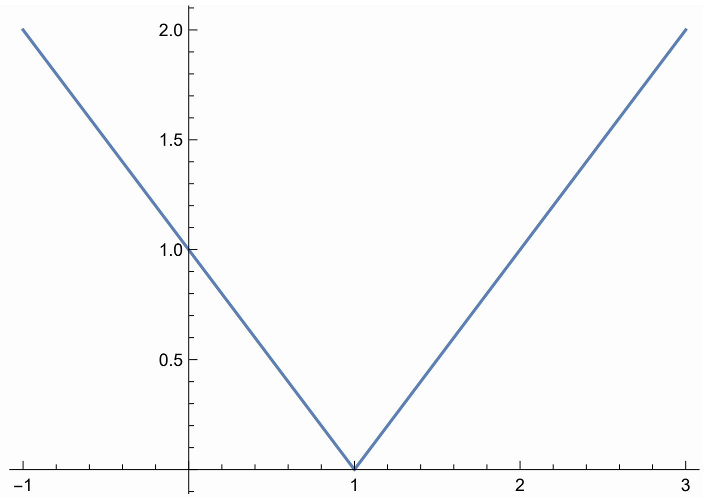
$$\lim_{h \to 0-} \frac{|h|}{h} = \frac{-h}{h} = -1$$

When $h \to 0^+$, $h > 0$, so, $|h| = h$

$$\lim_{h \to 0+} \frac{|h|}{h} = \frac{h}{h} = 1$$

As, $\lim_{h \to 0^-} f'(x) \neq \lim_{h \to 0^+} f'(x)$, so the function is not differentiable.

(c)



2.0
1.5
1.0
0.5
−1
1
2
3

Problem 4

Let

$$f(x) = \begin{cases} \dfrac{\sin(mx)}{-x}, & x < 0 \\ 2, & x = 0 \\ \dfrac{\tan(x)}{kx}, & x > 0 \end{cases}$$

Find values of the constants k and m, that will make the function $f(x)$ **CONTINUOUS** at $x = 0$.

[4]

Solution:

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= f(0) = \lim_{x\to 0^+} f(x) \\ \text{or, } \lim_{x\to 0^-} \frac{\sin mx}{-x} &= 2 = \lim_{x\to 0^-} \frac{\tan x}{kx} \\ \text{or, } \lim_{x\to 0^-} \frac{\sin mx}{mx} \times (-m) &= 2 = \lim_{x\to 0^-} \frac{1}{k} \times \frac{\tan x}{x} \\ \text{or, } \lim_{x\to 0^-} \frac{\sin mx}{mx} \times (-m) &= 2 = \lim_{x\to 0^-} \frac{1}{k} \times \frac{\tan x}{x} \\ \text{or, } -m &= 2 = \frac{1}{k} \end{aligned}$$

$$\implies m = -2 \text{ and } k = 1/2$$