Department of Mathematics and Natural Science

MAT110: Differential Calculus and Co–ordinate Geometry

Section ______, Quiz ______(Fall'22)

Name (PRINT): ______________________________	**ID:**__________________

Time: 25 minutes **Total Marks: 20**

Use following expressions if needed

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1, \qquad \lim_{x\to 0}\frac{1-\cos(x)}{x}=0, \qquad \lim_{x\to 0}\frac{\tan(x)}{x}=1 \qquad \lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^{x}=\mathrm{e}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)]=f(x)\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]+g(x)\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]-f(x)\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]}{(g(x))^2}$$

Problem 1

Find the values of x at which f is **NOT continuous**, and **by calculating limits determine** the **type of each discontinuity**.

$$f(x)=\frac{-x-2}{|x|-2}$$

[5]

Solution:

As the denominator cannot be equal to zero, $|x|-2\neq 0$.

So $|x|\neq 2$ or $x\neq\pm 2$

[2]

When $x\to -2$,

$x<0$, so $|x|=-x$,

$$\lim_{x\to -2}f(x)=\frac{-x-2}{-x-2}=1$$

So there is an removable discontinuity at $x=-2$

[1]

When $x\to 2$,

$x > 0$, so $|x| = x$, $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{-x-2}{x-2} = \lim_{x\to 2} \frac{-(x+2)}{x-2}$

as $(x-2)$ can change sign on both sides of $x = 2$,

we need to consider one sided limits

So, $\lim_{x\to 2^-} f(x) = +\infty$ [as the signs are $\frac{(-)}{(-)}$]

And, $\lim_{x\to 2^+} f(x) = -\infty$ [as the signs are $\frac{(-)}{(+)}$]

So there is an infinite discontinuity at $x = 2$

[2]

Problem 2

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ (by any method) :

$$y = (2\sqrt{x}+1)\cdot\left(\frac{2-x}{x^2+3x}\right) \qquad [5]$$

Solution:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \left(2\sqrt{x}+1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2-x}{x^2+3x}\right) + \left(\frac{2-x}{x^2+3x}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(2\sqrt{x}+1\right) \\
&= \left(2\sqrt{x}+1\right)\frac{\left(x^2+3x\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(2-x\right) - \left(x-2\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+3x\right)}{\left(x^2+3x\right)^2} + \left(\frac{2-x}{x^2+3x}\right)\left(\frac{2}{2\sqrt{x}}\right) \\
&= \left(2\sqrt{x}+1\right)\frac{\left(x^2+3x\right)\left(-1\right) - \left(x-2\right)\left(2x+3\right)}{\left(x^2+3x\right)^2} + \left(\frac{2-x}{x^2+3x}\right)\left(\frac{1}{\sqrt{x}}\right) \\
&= -\left(2\sqrt{x}+1\right)\frac{\left(x^2+3x\right) + \left(x-2\right)\left(2x+3\right)}{\left(x^2+3x\right)^2} + \left(\frac{2-x}{x^2+3x}\right)\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Problem 3

Calculate limits(**NOT** using L'Hôpitals rule):

$$\lim_{h\to 0}\frac{1-\cos\left(2h\right)}{\cos^2\left(5h\right)-1} \qquad [5]$$

Solution:

$$\begin{aligned}
\lim_{h\to 0} \frac{1-\cos(2h)}{\cos^2(5h)-1} &= \lim_{h\to 0} \frac{2\sin^2(2h/2)}{-\sin^2(5h)} \\
&= -2 \times \lim_{h\to 0} \left(\frac{\sin(h)}{h}\right)^2 \times (h)^2 \times \left(\frac{5h}{\sin(5h)}\right)^2 \times \frac{1}{(5h)^2} \\
&= -2 \times (1)^2 \times (1)^2 \times (1)^2 \times \frac{1}{(5)^2} = -\frac{2}{25}
\end{aligned}$$

Problem 4

(a) **Sketch the graph** of $f(x) = \sqrt{x}$.

(b) For $f(x) = \sqrt{x}$ **find the limit** of

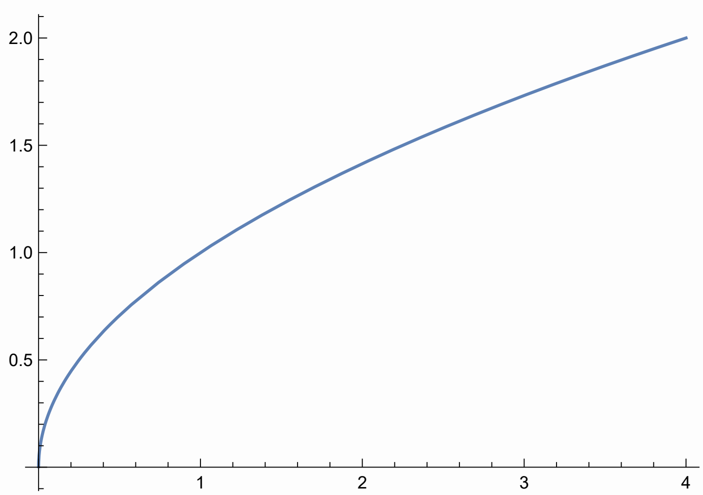
$$f'(x) = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h} \qquad \text{at } x = 0$$

to show that it has **vertical tangency** at $x = 0$ and hence **comment on the differentiability**.

[1 + 4 = 5]

Solution:

(a)



[1]

(b)

$$f'(x) = \lim_{h \to 0^+} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

at $x = 0$

$$\begin{aligned} f'(x) &= \lim_{h \to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} \\ &= \lim_{h \to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} \\ &= \lim_{h \to 0^+} \frac{\sqrt{h}}{h} \\ &= \lim_{h \to 0^+} \frac{1}{\sqrt{h}} = +\infty \end{aligned}$$

So we see that the slope of the straight line will be $+\infty$ at $x = 0$ or the function $f(x)$ will have a vertical tangent at $x = 0$.

As the limit does not exist(or not a finite value), so we can say the function is not differentiable at $x = 0$.

[4]