Sum of devisors & Number of devisors
$\sigma(n)$ $(\tau(n))$ ← tau n

Number of divisors

for, $n = 8$

$d = \{1, 2, 4, 8\}$

$\tau(8) = 4$

$\sigma(n)$ = sum of +ve divisor of n

$\sigma(8) = 1+2+4+8$

$= 15$

Remark:

1,2,3,6

- $\tau(1) = 1$ $\tau(2) = 2$ $\tau(3) = 2$ $\tau(4) = 3$ $\tau(5) = 2$ $\tau(6) = 4$
- $\sigma(1) = 1$ $\sigma(2) = 3$ $\sigma(3) = 4$ $\sigma(4) = 7$ $\sigma(5) = 6$ $\sigma(6) = 12$
- $\tau(n) = 2$ n is prime
- $\sigma(n) = n+1$ n is prime
- $\sum_{d/n} f(d)$; $\tau(n) = \sum_{d/n} 1$; $\sigma(n) = \sum_{d/n} d$

$n = 10$

$\rightarrow d : \{1, 2, 5, 10\}$

$\tau(10) = \sum_{d/10} 1$

$= (1+1+1+1)$

$= 4$

$\sigma(10) = \sum_{d/10} (d)$

$= 1+2+5+10$

$= 18$

Example $n=180$

divisons of $180 = \{1,2,3,4,5,\ldots\}$ ----- >

$n = 180 = 2^2 \cdot 3^2 \cdot 5^1$

$d = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3}$, $0 \le a_1 \le 2$

$0 \le a_2 \le 2$

$0 \le a_3 \le 1$

$d_1 = 2^0 \cdot 3^0 \cdot 5^0 = 1$

$d_2 = 2^0 3^0 5^1 = 5$

$d_3 = 2^0 3^1 5^0 = 3$

$d_4 = 2^1 3^0 5^0 = 2$

$d_5 = 2^0 3^1 5^1 = 15$

⋮

$d_{17} = 2^1 \cdot 3^2 \cdot 5^1 = 90$

$d_{18} = 2^2 3^2 5^1 = 180$

set of divisons of 180 are :

$\{1,2,3,4,5,6,9,10,12,15,18,20,30,36,45,60,90,180\}$

$\tau(180) = 18$

$\sigma(180) = ?$

Theorem: If $n = p_1^{k} p_2^{k_2} \cdots p_n^{k_n}$ is the prime factorization

(a) $\tau(n) = (k_1+1)(k_2+1)\cdots(k_n+1)$

(b) $\sigma(n) = \dfrac{p_1^{k_1+1}-1}{p_1-1} \cdot \dfrac{p_1^{k_2+1}}{p_2-1} \cdots \dfrac{p_n^{k_2+1}}{p_n-1}$

Lets proof: The division of n takes the form $d = p_1^{a_1} p_2^{a_2} \cdots a_n^{n}$

$0 \le a_i \le k_i$ $1 \le i \le n$

(a) a_1: take k_1+1 $\quad \tau(n) = (k_1+1)\cdot(k_2+1)\cdots(k_n+1)$

a_2: takes k_2+1

a_n: takes k_n+1

The number of division $\tau(n) = (k_1+1)(k_2+1)\cdots(k_n+1)$

(b)

$$\underbrace{\left(1+p_1+p_1^2+p_1^3+\cdots+p_{1_k}^{n}\right)}_{GP}\left(1+p_2+p_2^2+\cdots p_{2_k}^{n}\right)\cdots\left(1+p_n+p_n^2+\cdots p_n^{a_n}\right)$$

Sum of devism $= \left(\frac{p^{k_1+1}-1}{p_1-1}\right)\times\left(\frac{p_2^{k_2+1}-1}{p_2-1}\right)\cdots\left(\frac{p_n^{k_n+1}}{p_n-1}\right)$

Lets take an example:

$n = 180 = 2^2\, 3^2\, 5$

$\tau(180) = (2+1)(2+1)(1+1) = 3\times 3\times 2 = 18$

$$\sigma(180) = \frac{2^{2+1}-1}{2-1}\cdot\frac{3^{2+1}-1}{3-1}\cdot\frac{5^{1+1}-1}{5-1}$$

$$= \frac{7}{1}\cdot\frac{26}{2}\cdot\frac{24}{4}$$

$$= 546$$

Multidicative Function

- multiplicative function
- $\tau(n)$, $\sigma(n)$ multiplicative fn
- $F(n) = \sum_{d|n} f(d)$; f is multiplicative then F is also multiplicative.

⇒ A number theorilic function is multiplicative if

$$f(mn) = f(m) f(n) \quad ; \gcd(m,n) = 1$$

⇒ $\tau(m,n) = \tau(m)\tau(n)$

$\sigma(m,n) = \sigma(m)\sigma(n)$

Möbius function:

. Defination:

$$\mu(n) = \begin{cases} 1 & : n=1 \\ 0 & : p^2 | n ;\ P = \text{Prime} \\ (-1)^r & : n = P_1 P_2 \cdots P_r \ P \text{ is distint prime} \end{cases}$$

$\mu(1) = 1$

$\mu(2) = -1$

$\mu(3) = -1$

$\mu(4) = 2^2 = 0$

$\mu(5) = -1$

$\mu(6) = \mu(2\times3) = (-1)^2 = 1$

$\mu(7) = -1$

$\mu(8) = \mu(2^3) = 0$

$\mu(9) = \mu(3^2) = 0$

$\mu(10) = 1$

Theorem : For each positive integer $n \geq 1$

$$\sum_{d/n} \mu(d) = \begin{cases} 1 & ; n=1 \\ 0 & ; n>1 \end{cases}$$

where d runs through the positive divisors of n

Proof: Let $n = p_1^{k_1} . p_2^{k_2} . . . p_r^{k_r}$ and $F(n) = \sum_{d|n} \mu(d)$

$$F(n) = F(p_1^{k_1} \ldots p_n^{k_n})$$

$$= F(p_1^{k_1})\, f(p_2^{k_2}) \ldots F(p_n^{k_n})$$

$$= 0 . 0 . 0 . = 0$$

$$\therefore F(p_i^{k_i}) = \cancel{F(n)} \sum_{d/n} \mu(d)$$

$$= \mu(1) + \mu(p_i) + \mu(p^2) + \mu(p^3) + \ldots \mu(p^{k_i})$$

$$= 1 + (-1) + 0 + 0 . . . + 0$$

$$= 0$$

~~Suppose~~ $F(n) = 0$ if $n > 1$

Suppose

$n = 1$ $F(1) = \sum_{d/n} \mu(d)$

$$= \mu(1) = 1$$

Mobius Inverse:

Theorem: let F an f be two number-theoretic function related by the formula $F(n) = \sum_{d|n} f(d)$ — ① Then

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum \mu\left(\frac{n}{d}\right) F(d)$$

Proof

apply ①

$$\sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \sum_{c|n/d} f(c)$$

$$= \sum_{d|n} \sum_{c|n/d} \mu(d) f(c)$$

$$= \sum_{c|n} \sum_{d|n/c} \mu(d) f(c)$$

$n = d \cdot d'$

$\Rightarrow d' = \frac{d}{n}$

$d''c = d'$

$\Rightarrow \frac{d'}{c} = d''$

$\Rightarrow \frac{dd'}{c} = dd''$

$\Rightarrow \frac{n}{c} = dd''$

$$= \sum_{c|n} f(c) \underbrace{\sum_{d|n/c} \mu(d)}$$

$\mu(d) = 1$ if $n/c = 1$

$\mu(d) = 0$ if $n/c > 1$

take $n/c = 1$

$$= \sum_{c|n} f(c) \times 1$$

$$= \sum_{c|n} f(c)$$

Remark: If $F(n) = \sum_{d|n} f(d)$ then $f(n) = \sum_{d|n} \mu(d) F(\frac{n}{d})$

① $\tau(n) = \sum_{d|n} 1$; $F(n) = \tau(n)$; $1 = \sum_{d|n} \mu(d)\tau(n/d)$

$f(n) = 1$

$= \sum_{d|n} \mu(n/d)\,\mu(d)$

② $\sigma(n) = \sum_{d|n} d$; $F(n) = \sigma(n)$; $n = \sum_{d|n} \mu(d)\,\sigma(n/d)$

$f(n) = n$

$= \sum_{d|n} \mu(n/d)\,\sigma(d)$