Equations of First Order And First Degree

There are two standard forms of differential equations of first order and first degree, namely,

(*i*) $dy/dx = f(x, y)$ (*ii*) $M(x, y)\, dx + N(x, y)\, dy = 0$.

Separation of variables

If in an equation, it is possible to get all the functions of x and dx to one side and all the functions of y and dy to the other, the variables are said to be separable.

(*a*) *Solve* $dy/dx = e^{x-y} + x^2e^{-y}$ (*b*) *Solve* $dy/dx = e^{x+y} + x^2e^{y}$

$$e^y dy = (x^2 + e^x)\, dx.$$

$e^y = x^3/3 + e^x + c$, c being an arbitrary constant.

2. *Find the curves passing through* (0, 1) *and satisfying sin* $(dy/dx) = c$.

$dy/dx = \sin^{-1} c$ or $dy = (\sin^{-1} c)\, dx$.

Integrating, $y = x \sin^{-1} c + c'$, c' being arbitrary constant. ... (1)

Since, (1) must pass through (0, 1), we put $x = 0$ and $y = 1$ in (1) and obtain $c' = 1$. Hence, (1)

$y = x \sin^{-1} c + 1$ or $(y - 1)/x = \sin^{-1} c$

$\sin \{(y - 1)/x\} = c$, which gives the desired curves.

Solve (dy/dx) *tan* $y =$ *sin* $(x + y) +$ *sin* $(x - y)$.

$(\tan y)\,(dy/dx) = 2 \sin x \cos y$ $\quad \sin C + \sin D = 2 \sin \{(C + D)/2\} \cos \{(C - D)/2\}$

$\sec y \tan y\, dy = 2 \sin x\, dx$.

$\sec y = -2 \cos x + c$, c being an arbitrary constant.

(*i*) $\dfrac{dy}{dx} = \dfrac{\sin x + x \cos x}{y\,(2 \log y + 1)}$ (*ii*) $\dfrac{dy}{dx} = \dfrac{x\,(2 \log x + 1)}{\sin y + y \cos y}$.

Solve $y - x\,(dy/dx) = a\,(y^2 + dy/dx)$.

The given equation can be re-written as

$$(a + x)\,\frac{dy}{dx} = y - ay$$

$$\frac{dx}{x + a} = \frac{dy}{y\,(1 - ay)}$$

$$\frac{dx}{x + a} = \left[\frac{a}{1 - ay} + \frac{1}{y}\right] dy.$$

$$x + a = \frac{cy}{1 - ay} \quad \text{or}$$

$(x + a)\,(1 - ay) = cy$, which is the required solution.

Solve $\sqrt{(1+x^2+y^2+x^2y^2)} + xy\,(dy/dx) = 0.$

$$\sqrt{[(1+x^2)\,(1+y^2)]} + xy\,(dy/dx) = 0$$

$$\sqrt{(1+x^2)}\,\sqrt{(1+y^2)} + xy\,(dy/dx) = 0$$

$$\frac{(1+x^2)\,dx}{x\sqrt{(1+x^2)}} + \frac{y\,dy}{\sqrt{(1+y^2)}} = 0.$$

$$\int \frac{dx}{x(1+x^2)^{1/2}} + \int \frac{x\,dx}{(1+x^2)^{1/2}} + \int \frac{y\,dy}{(1+y^2)^{1\backslash 2}} = \text{C}.$$

$$\int \frac{dx}{x(1+x^2)^{1/2}} = \int \frac{(-1/t^2)\,dt}{(1/t)\sqrt{1+(1/t)^2}}, \text{ putting } x = \frac{1}{t}$$

Solve $dy/dx = e^{x+y} + x^2 e^{x^3+y}.$

$$e^{-y}\,dy = (e^x + x^2 e^{x^3})\,dx.$$

Transformation of some equations in the form in which variables are separable Equations of the form

$dy/dx = f(ax + by + c)$ substitution $ax + by + c = v$

Ex. 1. (*a*) *Solve* $dy/dx = (4x + y + 1)^2$.

(*b*) $dy/dx = (4x + y + 1)^2$ *if* $y(0) = 1$.

Let $4x + y + 1 = v$.

Differentiating (1) with respect to x,

$4 + (dy/dx) = dv/dx$

$dy/dx = (dv/dx) - 4$

Now,

$(dv/dx) - 4 = v^2$

$dx = (dv) / (4 + v^2)$

$$4x + y + 1 = 2 \tan(2x + c),$$

(*b*) Putting $x = 0$, $y = 1$ in (2), we get $\tan c = 1$, so that $c = \pi/4$.

$\therefore$ Required solution is $4x + y + 1 = 2 \tan(2x + \pi/4)$.

Ex. 2. *Solve* $(x + y)^2\,(dy/dx) = a^2$.

Let $x + y = v$.

Ex. 3. *Solve* $dy/dx = \sec\,(x + y)$

Ex. 4. *Solve* $dy/dx = \sin\,(x + y) + \cos\,(x + y)$.

Ex. 5. *Solve* $(x + y)\,(dx - dy) = dx + dy$.

$(x + y - 1)\,dx = (x + y + 1)\,dy$

$$\frac{dy}{dx} = \frac{x + y - 1}{x + y + 1}.$$

$x + y = v$.

$1 + dy/dx = dv/dx$

$dy/dx = (dv/dx) - 1$.

$$\frac{dv}{dx} - 1 = \frac{v - 1}{v + 1}$$

$$2dx = \left(1 + \frac{1}{v}\right) dv.$$

Solve $dy/dx = (4x + 6y + 5) / (3y + 2x + 4)$

Solve $(x + 2y - 1)\, dx = (x + 2y + 1)\, dy$

Homogeneous equation Definition. A differential equation of first order and first degree is said to be homogeneous if it can be put in the form $dy/dx = f(y/x)$

Working rule $dy/dx = f(y/x)$.

$y = vx$. $dy/dx = v + x\,(dv/dx)$.

$$v + x\frac{dv}{dx} = f(v)$$

$$\frac{dx}{x} = \frac{dv}{f(v) - v}$$

1. *Solve* $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0.$

$$\frac{dy}{dx} = -\frac{x^3 + 3xy^2}{y^3 + 3x^2y} = -\frac{1 + 3\,(y/x)^2}{(y/x)^3 + 3\,(y/x)}$$

Take $y/x = v$, $\quad y = vx.$ $\qquad dy/dx = v + x\,(dv/dx).$

$$v + x\,\frac{dv}{dx} = -\frac{1 + 3v^2}{v^3 + 3v}$$

$$4\,\frac{dx}{x} = -\frac{4v^3 + 12v}{v^4 + 6v^2 + 1}\,dv$$

Solve: $x\,dy - y\,dx = (x^2 + y^2)^{1/2}\,dx$

Solve $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$.