Gamma function and Beta function

Gamma function

The integral $\int_0^\infty x^{n-1}e^{-x}dx,\ \ n > 0$ knows as the second

Eulerian , is called the gamma function of n and is denoted by $\Gamma(n)$

i, e $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$

Properties

1) $\Gamma(1) = \int_0^\infty e^{-x}dx = [-e^{-x}]_0^\infty = 1$

2) $\Gamma(n+1) = \int_0^\infty x^n e^{-x}dx = [-x^n e^{-x}]_0^\infty + \int_0^\infty n x^{n-1}e^{-x}dx$

$$= n\int_0^\infty x^{n-1}e^{-x}dx = n\Gamma n$$

3) $\Gamma(n+1) = n(n-1)\Gamma(n-1)$

$$= n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots\ldots\ldots\ldots 3.2.1\Gamma(1)$$

$= n!$, if n is an integer

4) $\Gamma(n)\Gamma(n-1) = \frac{\pi}{sinn\pi}$

Different forms of $\boldsymbol{\Gamma(n)}$

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

(1) put $x = ay$ then

$$\Gamma(n) = a^n \int_0^\infty y^{n-1} e^{-ay} dy = a^n \int_0^\infty x^{n-1} e^{-ax} dx$$

(2) put $x = y^2$ then

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1} dy = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$$

(3) put $x = -(m+1) lny$ then

$$\Gamma(n) = (m+1)^n \int_0^\infty y^m \left(ln\frac{1}{y}\right)^{n-1} dy$$

$$= (m+1)^n \int_0^\infty x^m \left(ln\frac{1}{x}\right)^{n-1} dx$$

Prove that $\boldsymbol{\Gamma(\frac{1}{2}) = \sqrt{\pi}}$ **and** $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Proof. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ put $x = y^2$ and $n = \frac{1}{2}$

$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-y^2} dy$ Let $I = \int_0^\infty e^{-y^2} dy$

$I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$ ----(1)

put $x = r cos\theta, y = r sin\theta \quad , we\ get \ \ dxdy = rdrd\theta$

$and \quad as\ x\ and\ y \ \ varies\ from \ \ 0 \ \ to \ \ \infty$ then r varies from

$0 \ \ to \ \ \infty$ and θ varies from $0 \ \ to \ \ \frac{\pi}{2}$. so (1) becomes

$$I^2 = \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} r dr\, d\theta = \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2}\ e^{-r^2}\right]_0^{\infty} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ d\theta = \frac{\pi}{4} \qquad \therefore \quad I = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\therefore \quad \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

<u>Beta function</u>

The integral $\int_0^1 x^{m-1} e(1-x)^{n-1} dx, \ \ n > 0$ knows as the first Eulerian integral , is called the Beta function of m and n and is denoted by $B(m,n) \ \ or \quad \beta(m,n)$

i, e $\quad B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Property

$$B(m,n) = \int_0^1 (1-x)^{m-1} x^{n-1}\, dx = \int_0^1 x^{n-1}(1-x)^{m-1}\, dx$$

$$= B(n,m)$$

Different forms of $B(m,n)$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

(1) put $x = \frac{y}{1+y}$ then

$$B(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

(2) put $x = sin^2\theta$ then

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta\sin^{2n-1}\theta\,d\theta$$

(3) put $x = \frac{y}{a}$ then

$$B(m,n) = \frac{1}{a^{m+n-1}}\int_0^a y^{m-1}(a-y)^{n-1}\,dy$$

$$= \frac{1}{a^{m+n-1}}\int_0^a x^{m-1}(a-x)^{n-1}\,dy$$

Relation between Gamma and Beta functions

$$\boldsymbol{B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

Proof :

We know that

$\Gamma(m) = 2\int_0^\infty e^{-x^2}x^{2m-1}dx$ and $\Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2n-1}dx$

$\Gamma(m)\Gamma(n) = 2\int_0^\infty e^{-x^2}x^{2m-1}dx\, 2\int_0^\infty e^{-y^2}y^{2n-1}dy$

$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2m-1}y^{2n-1}dxdy$ --------------(1)

put $x = rcos\theta, y = rsin\theta$ $, we\ get\ dxdy = rdrd\theta$

$and\ \ as\ x\ and\ y\ varies\ from\ \ 0\ to\ \ \infty$ then r varies from

$0\ to\ \ \infty$ and θ varies from $0\ to\ \frac{\pi}{2}$. so (1) becomes

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2}r^{2m-1}\cos^{2m-1}\theta\, r^{2n-1}\sin^{2n-1}\theta\, rdrd\theta$$

$$= 2\int_0^\infty e^{-r^2}r^{2(m+n)-1}dr.\ \ 2\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\, d\theta$$

$$=\Gamma(m+n)B(m,n)$$

$$=> \ \ B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Prove that $\frac{B(p,q+1)}{q} = \frac{B(p+1,q)}{p} = \frac{B(p,q)}{p+q}$

Proof:

$$B(p,q+1) = \frac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)} = \frac{q\Gamma(p)\Gamma(q)}{(p+q)\Gamma(p+q)}$$

$$=> \frac{B(p,q+1)}{q} = \frac{\Gamma(p)\Gamma(q)}{(p+q)\Gamma(p+q)} = \frac{B(p,q)}{(p+q)} \text{ -------(1)}$$

Again

$$B(p+1,q) = \frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+q+1)} = \frac{p\Gamma(p)\Gamma(q)}{(p+q)\Gamma(p+q)}$$

$$=> \frac{B(p+1,q)}{p} = \frac{\Gamma(p)\Gamma(q)}{(p+q)\Gamma(p+q)} = \frac{B(p,q)}{(p+q)} \text{ ---------(2)}$$

From (1) and (2) we get

$$\frac{B(p,q+1)}{q} = \frac{B(p+1,q)}{p} = \frac{B(p,q)}{p+q} \text{ proved}$$

Prove that $\int_0^{\frac{\pi}{2}} sin^p\,\theta\, cos^q\,\theta\, d\theta = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})}$

Proof:

We have

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$=> \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\ B(m,n) = \frac{\Gamma(m)\Gamma(n)}{2\,\Gamma(m+n)}$$

Put $2m - 1 = p \quad and \quad 2n - 1 = q$

$\therefore \quad m = \frac{p+1}{2} \quad and \quad n = \frac{q+1}{2}$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

Example Evaluate: $\int_0^{\frac{\pi}{2}} sin^4 \theta \, cos^5 \theta \, d\theta$

Soln we know that $\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$

Put $p = 4 \quad and \quad q = 5$, we get

$$\int_0^{\frac{\pi}{2}} \sin^4 \theta \cos^5 \theta \, d\theta = \frac{\Gamma\left(\frac{4+1}{2}\right) \Gamma\left(\frac{5+1}{2}\right)}{2\Gamma\left(\frac{4+5+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{5}{2}\right) \Gamma(3)}{2\Gamma\left(\frac{11}{2}\right)} = \frac{\Gamma\left(\frac{5}{2}\right) 2}{2 \; \frac{9}{2} \frac{7}{2} \frac{5}{2} \Gamma\left(\frac{5}{2}\right)} = \frac{8}{9.7.5} = \frac{8}{315}$$

Example Evaluate: $\int_0^{\frac{\pi}{2}} sin^5 \theta \, d\theta$

Soln we know that $\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$

Put $p = 5 \quad and \quad q = 0$, we get

$$\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta = \frac{\Gamma(3)\Gamma(\frac{1}{2})}{2\Gamma(\frac{7}{2})} = \frac{2\Gamma(\frac{1}{2})}{2\,\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\,\Gamma(\frac{1}{2})} = \frac{8}{15}$$

Example Evaluate: $\int_0^{\pi} x \ cos^4 x\, dx$

Soln: Let

$$I = \int_0^{\pi} x \ cos^4 x\, dx = \int_0^{\pi} (\pi - x) \ cos^4(\pi - x)\, dx$$

$$= \int_0^{\pi} \pi\, cos^4 x\, dx - \int_0^{\pi} x \ cos^4 x\, dx$$

$$= \int_0^{\pi} \pi\, cos^4 x\, dx - I$$

$$=> 2I = \int_0^{\pi} \pi\, cos^4 x\, dx = \pi \frac{\Gamma\left(\frac{1}{2}\right)\ \Gamma(\frac{5}{2})}{2\ \Gamma(\frac{6}{2})} = \pi \frac{\frac{3}{2}\frac{1}{2}\ \Gamma(\frac{1}{2})\Gamma\left(\frac{1}{2}\right)}{2\ .\ \Gamma(3)}$$

$$\therefore \quad I = \frac{3\pi\sqrt{\pi}\sqrt{\pi}}{16} = \frac{3\pi^2}{16}$$

Evaluate $\int_0^1 x^3(1-x^2)^{\frac{5}{2}}dx$

Soln:

$I = \int_0^1 x^3(1-x^2)^{\frac{5}{2}}dx$

Put $\quad x = sin\,\theta \qquad$ dx=cosθdθ

When $x = 0\ \ then\ \ \theta = 0\ \ and\ \ \ x = 1\ \ then\ \ \theta = \frac{\pi}{2}$

$\therefore \quad I = \int_0^{\frac{\pi}{2}} sin^3\,\theta\, cos^6\,\theta\, d\theta = \frac{\Gamma(2)\,\Gamma(\frac{7}{2})}{2\,\Gamma(\frac{3+6+2}{2})} = \frac{2\ \Gamma(\frac{7}{2})}{2\ \frac{9}{2}\frac{7}{2}\Gamma(\frac{7}{2})} = \frac{4}{63}$

Evaluate $\int_0^1 \frac{xdx}{\sqrt{1-x^5}}$

Soln: $I = \int_0^1 \frac{xdx}{\sqrt{1-x^5}}$

Put $\quad x^5 = sin^2\,\theta\ \ x = sin^{\frac{2}{5}}\,\theta\ \ \ dx = \frac{2}{5}sin^{-\frac{3}{5}}\,\theta\, cos\,\theta\, d\theta$

$I = \frac{2}{5}\int_0^{\frac{\pi}{2}} \frac{sin^{\frac{2}{5}}\,\theta\, sin^{-\frac{3}{5}}\,\theta\, cos\,\theta d\theta}{cos\,\theta} = \frac{2}{5}\int_0^{\frac{\pi}{2}} sin^{-\frac{1}{5}}\,\theta\, d\theta$

$= \frac{1}{5}\ 2\int_0^{\frac{\pi}{2}} sin^{2\frac{2}{5}-1}\,\theta\, cos^{2\frac{1}{2}-1}\,\theta\, d\theta = \frac{1}{5}B(\frac{2}{5},\frac{1}{2})$

$= \frac{1}{5}\ \frac{\Gamma\left(\frac{2}{5}\right)\Gamma(\frac{1}{2})}{\Gamma(\frac{9}{10})}$

Evaluate $\int_0^1 \frac{x^4}{\sqrt{1-x^2}} dx, \quad x = \sin\theta$

Soln: $I = \int_0^1 \frac{x^4 dx}{\sqrt{1-x^2}}$

Put x=sin θ , $dx = \cos\theta \, d\theta$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^4\theta \cos\theta \, d\theta}{\cos\theta} = \int_0^{\frac{\pi}{2}} \sin^4\theta \; d\theta$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma(3)} = \frac{\frac{3}{2}.\frac{1}{2}\;\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma(3)} = \frac{3\pi}{16}$$