Multiple Integration

Integration of multi-variable is called multiple Integration. Repeated definite and indefinite of form $\int_a^b \int_{y_1}^{y_2} f(x,y)dydx \ \ and \ \ \int\int f(x,y)dydx$ for two variables are called iterated integral. Repeated definite and indefinite of form

$\int_a^b \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x,y,z)\,dzdydx \ \ and \ \ \int\int\int f(x,y,z)dzd\,ydx$ for three variables

Evaluation of double integral

$\int_a^b \int_{y_1}^{y_2} f(x,y)dydx$, $f(x,y)$ is first integrated with respect to y treating x as constant between the limits y_1 and y_2 and then the result is integrated with respect to x between the limits a and b .

Example1 $\int_0^3 \int_x^{\sqrt{25-x^2}} x\,dydx$

Sol. $\int_0^3 \int_x^{\sqrt{25-x^2}} x\,dydx \ = \int_0^3 x[y]_x^{\sqrt{25-x^2}}dx = \int_0^3 x\left[\sqrt{25-x^2} - x\right]dx$

$= -\frac{1}{2}\int_0^3 \sqrt{25-x^2}\ d(25-x^2) - \int_0^3 x^2dx$

$= -\frac{1}{3}\left[(25-x^2)^{\frac{3}{2}} + \boldsymbol{x^3}\right]_{\mathbf{0}}^{\mathbf{2}} = \frac{34}{3}$

Evaluate: $\int\int_R \ 3x + 2y)dydx \ \ , R: -1 \leq x \leq 1, x^2 \leq y \leq 1 + x$

Soln. $\int\int_R (3x + 2y)dydx = \int_{-1}^{1}\int_{x^2}^{x+1}(3x + 2y)dydx$

$= \int_{-1}^{1}[3xy + y^2]_{x^2}^{x+1}\mathrm{dx}$

$$= \int_{-1}^{1}(3x(x+1) + (x+1)^2 - 3x^3 - x^4)dx$$

$$= \int_{-1}^{1}(3x^2 + 3x + x^2 + 2x + 1 - 3x^3 - x^4)dx$$

$$= \int_{-1}^{1}(4x^2 + 5x + 1 - 3x^3 - x^4)dx$$

$= 2\int_0^1(4x^2 - x^4)dx + \int_{-1}^{1} dx$

$= 2\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^1 + [x]_{-1}^{1}$

$= 2(\frac{4}{3} - \frac{1}{5}) + 2 = 2\left(\frac{20-3+30}{15}\right) = \frac{94}{15}$

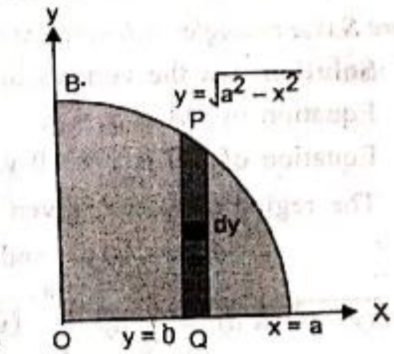
Example 4. *Evaluate* $\iint_R x\,y\,dx\,dy$

where R is the quadrant of the circle $x^2+y^2 = a^2$ *where* $x \geq 0$ *and* $y \geq 0$.

(Madras 1996, A.M.I.E.T.E., Summer 1999)

Solution. Let the region of integration be the first quadrant of the circle OAB.

$$\iint_R x\,y\,dx\,dy \quad (x^2+y^2 = a^2,\ y = \sqrt{a^2-x^2})$$



First we integrate w.r.t. y and then w.r.t. x.

The limits for y are 0 and $\sqrt{a^2-x^2}$ and for x, 0 to a.

$$= \int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} y\,dy = \int_0^a x\,dx \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}$$

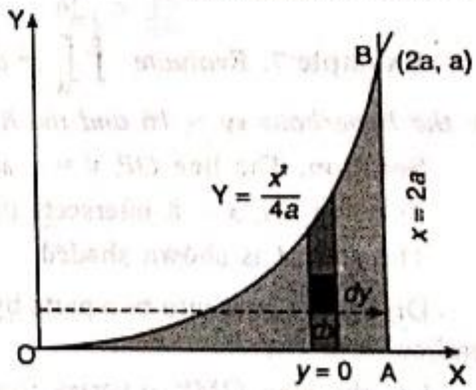
$$= \frac{1}{2}\int_0^a x\,(a^2-x^2)\,dx = \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{a^4}{8} \quad \textbf{Ans.}$$

Example 5. *Evaluate* $\iint_A xy\,dx\,dy$

where A is the domain bounded by x axis, ordinate $x = 2a$ *and the curve* $x^2 = 4ay$.

(A.M.I.E., Winter, 1996)

Solution. $\iint_A xy\,dx\,dy$



The area of integration is OAB. Here we integrate first w.r.t. y and then w.r.t. x.

The limits of integration are 0 to $2a$ for x and 0 to $\frac{x^2}{4a}$ for y.

Thus $$\iint_A x\,y\,dx\,dy = \int_0^{2a}\int_0^{x^2/4a} x\,y\,dx\,dy$$

$$= \int_0^{2a} x\,dx \int_0^{x^2/4a} y\,dy = \int_0^{2a} x\,dx \left[\frac{y^2}{2}\right]_0^{x^2/4a} = \int_0^{2a} x\,\frac{x^4}{32a^2}\,dx = \frac{1}{32\,a^2}\int_0^{2a} x^5\,dx$$

$$= \frac{1}{32\,a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{1}{192\,a^2}\,[(2a)^6 - 0] = \frac{64\,a^6}{192\,a^2} = \frac{a^4}{3} \quad \textbf{Ans.}$$

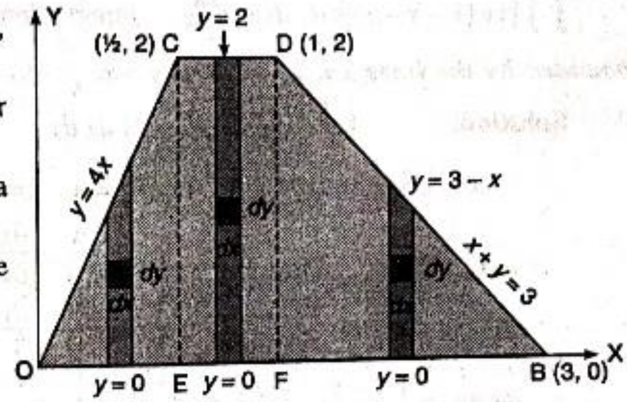
Example 8. *Evaluate* $\iint (x^2+y^2)\, dx\, dy$ *throughout the area enclosed by the curves* $y = 4x$, $x + y = 3$, $y = 0$ *and* $y = 2$.

Solution. Let OC represent $y = 4x$; BD, $x + y = 3$; OB, $y = 0$, and CD, $y = 2$.

The given integral is to be evaluated over the area A of the trapezium $OCDB$.

Area $OCDB$ consists of area OCE, area $ECDF$ and area FDB.

The co-ordinates of C, D and B are $\left(\frac{1}{2}, 2\right)$ (1, 2) and (3, 0) respectively.



$$\therefore \quad \iint_A (x^2+y^2)\, dy\, dx$$

$$= \iint_{OCE} (x^2+y^2)\, dy\, dx + \iint_{ECDF} (x^2+y^2)\, dy\, dx + \iint_{FDB} (x^2+y^2)\, dy\, dx$$

$$= \underbrace{\int_0^{1/2} dx \int_0^{4x} (x^2+y^2)\, dy}_{I_1} + \underbrace{\int_{1/2}^{1} dx \int_0^{2} (x^2+y^2)\, dy}_{I_2} + \underbrace{\int_1^3 dx \int_0^{3-x} (x^2+y^2)\, dy}_{I_3}$$

Now, $$I_1 = \int_0^{1/2} dx \int_0^{4x} (x^2+y^2)\, dy = \int_0^{1/2} \left[x^2y + \frac{y^3}{3}\right]_0^{4x} dx = \int_0^{1/2} \frac{76}{3} x^3\, dx$$

$$= \frac{76}{3}\int_0^{1/2} x^3\, dx = \frac{76}{3}\left[\frac{x^4}{4}\right]_0^{1/2} = \frac{76}{3}\left[\frac{1}{4}\cdot\frac{1}{16}\right] = \frac{19}{48}$$

$$I_2 = \int_{1/2}^{1} dx \int_0^{2} (x^2+y^2)\, dy = \int_{1/2}^{1} \left[x^2 y + \frac{y^3}{3}\right]_0^{2} dx = \int_{1/2}^{1} \left(2x^2 + \frac{8}{3}\right) dx$$

$$= \left[\frac{2x^3}{3} + \frac{8}{3}x\right]_{1/2}^{1} = \left[\left(\frac{2}{3}+\frac{8}{3}\right) - \left(\frac{2}{3}\cdot\frac{1}{8} + \frac{8}{3}\cdot\frac{1}{2}\right)\right] = \frac{23}{12}$$

$$I_3 = \int_1^3 dx \int_0^{3-x} (x^2+y^2)\, dy$$

$$= \int_1^3 \left[x^2y + \frac{y^3}{3}\right]_0^{3-x} dx = \int_1^3 \left[x^2(3-x) + \frac{(3-x)^3}{3}\right] dx$$

$$= \int_1^3 \left[3x^2 - x^3 + \frac{(3-x)^3}{3}\right] dx = \left[x^3 - \frac{x^4}{4} - \frac{(3-x)^4}{12}\right]_1^3$$

$$= \left[27 - \frac{81}{4} - 0 - 1 + \frac{1}{4} + \frac{16}{12}\right] = \frac{22}{3}$$

$$\therefore \quad \iint_A (x^2+y^2)\, dy\, dx = I_1 + I_2 + I_3 = \frac{19}{48} + \frac{23}{12} + \frac{22}{3} = \frac{463}{48} = 9\frac{31}{48}. \qquad \textbf{Ans.}$$

Triple Integrals

Evaluate: $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \; dz\, dy\, dx$

Soln. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \; dz\, dy\, dx$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dydx = \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2)\, dydx$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} (xy - x^3y - xy^3)\, dydx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{8}\int_0^1 [2x(1-x^2) - 2x^3(1-x^2) - x(1-x^2)^2]\, dx$$

$$= \frac{1}{8}\int_0^1 [2x - 2x^3 - 2x^3 + 2x^5 - x + 2x^3 - x^5]\, dx$$

$$= \frac{1}{8}\int_0^1 [x - 2x^3 + x^5]\, dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6}\right]_0^1 = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{48}$$

Evaluate:

$\iiint_R (x-2y+z)dxdydz\,,\ R\!:\ 0 \le x \le 1, 0 \le y \le x^2\ , 0 \le z \le x+y$

Soln. $\iiint_R (x-2y+z)dxdydz$

$$= \int_0^1 \int_0^{x^2} \int_0^{x+y} (x-2y+z)dzdydx$$

$$= \int_0^1 \int_0^{x^2} [xz - 2yz + z^2]_0^{x+y} \text{dydx}$$

$$= \int_0^1 \int_0^{x^2} \left(x^2 + xy - 2xy - 2y^2 + \frac{(x+y)^2}{2}\right) dydx$$

$$= \frac{3}{2}\int_0^1 \int_0^{x^2} (x^2 - y^2)dydx$$

$$= \frac{3}{2}\int_0^1 \left[x^2 y - \frac{y^3}{3}\right]_0^{x^2}$$

$$= \frac{3}{2}\int_0^1 (x^4 - \frac{x^6}{3})\text{dx} = \frac{3}{2}\left[\frac{x^5}{5} - \frac{y^7}{21}\right]_0^1 = \frac{8}{35}$$

Change of variable

Sometime double , triple integral easily be evaluated by changing the dependent variables by suitable transformations .

Let $x = f_1(u,v)$, $y = f_2(u,v)$, the double integral

$$\iint\limits_R f(x,y)dxdy = \iint\limits_{R^/} f(f_1(u,v), f_2(u,v)) \quad |J|dudv$$

where J= Jacobian $=\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

similarly for triple integral can be written as

$$\iiint_R f(x,y,z)dxdydz$$

$$= \iiint_{R^/} f\big(f_1(u,v,w), f_2(u,v,w), f_3(u,v,w)\big)|J|dudvdw$$

and $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$ and so on.

Example: Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)dxdy$ by transformations.

Soln. put $x = rcos\theta \quad , y = rsin\theta$

$$\frac{\partial x}{\partial r} = cos\theta, \quad \frac{\partial x}{\partial \theta} = -rsin\theta, \qquad \frac{\partial y}{\partial r} = sin\theta, \quad \frac{\partial y}{\partial \theta} = rcos\theta,$$

$$dxdy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} drd\theta = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} drd\theta$$

$$= r(\cos^2\theta + r\sin^2\theta)drd\theta = rdrd\theta$$

Lower limit of x is 0 and upper limit of $x = \sqrt{a^2-y^2}$

$$x^2+y^2 = a^2 => r^2 = a^2 => r = a$$

Lower limit of $y = 0 => rsin\theta = 0 => \theta = 0$ and upper limit of

$y = a => rsin\theta = a => asin\theta = a => \theta = \frac{\pi}{2}$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) dxdy = \int_0^{\frac{\pi}{2}} \int_0^a r^3 dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^a d\theta = \frac{a^4}{4} [\theta]_0^{\frac{\pi}{2}} = \frac{\pi a^4}{8}$$

Evaluate $\int_0^{\sqrt{2}} \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2 + y^2 + z^2}\, dzdxdy$ by transformations

Soln. put $x = rsin\theta cos\varphi$, $y = rsin\theta sin\varphi$, $z = rcos\theta$

$$\frac{\partial x}{\partial r} = sin\theta cos\varphi, \quad \frac{\partial x}{\partial \theta} = rcos\theta cos\varphi, \quad \frac{\partial x}{\partial \varphi} = -rsin\theta sin\varphi$$

$$\frac{\partial y}{\partial r} = sin\theta sin\varphi, \quad \frac{\partial y}{\partial \theta} = rcos\theta sin\varphi, \quad \frac{\partial y}{\partial \varphi} = rsin\theta cos\varphi$$

$$\frac{\partial z}{\partial r} = cos\theta, \quad \frac{\partial z}{\partial \theta} = -rsin\theta, \quad \frac{\partial x}{\partial \varphi} = 0$$

$$dzdxdy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} drd\theta d\varphi$$

$$= \begin{vmatrix} sin\theta cos\varphi, & rcos\theta cos\varphi & -rsin\theta sin\varphi \\ sin\theta sin\varphi & rcos\theta sin\varphi & rsin\theta cos\varphi \\ cos\theta & -rsin\theta & 0 \end{vmatrix} drd\theta d\varphi$$

$$= r^2 sin\theta drd\theta d\varphi$$

Lower limit of z is 0 and upper limit of $z = \sqrt{4 - x^2 - y^2}$

$$x^2 + y^2 + z^2 = 4 => r^2 = 4 => r = 2$$

Lower limit of $x = 0 => rsin\theta cos\varphi = 0 => \theta = 0$ and upper limit of $x = \sqrt{4 - y^2}$

$$=> r^2 \sin^2\theta \cos^2\varphi + r^2 \sin^2\theta \cos^2\varphi = 4$$

$$=> r^2 \sin^2\theta = 4 => rsin\theta = 2 => sin\theta = \frac{2}{r} = \frac{2}{2} = 1 => \theta = \frac{\pi}{2}$$

Lower limit of $y = 0 => rsin\theta sin\varphi = 0 => \theta = 0$ and upper limit of $y = \sqrt{2} => rsin\theta sin\varphi = \sqrt{2} => 2sin\varphi = \sqrt{2} => \varphi = \frac{\pi}{4}$

$$\therefore \int_0^{\sqrt{2}} \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2 + y^2 + z^2}\, dzdxdy$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^{2} r\, r^2 sin\theta\ drd\theta x d\varphi = \pi$$