Definite Integral

Integration as the limit of a sum

Let $f(x)$ be a bounded continuous function defined in the interval (a, b) , a and b being finite quantity and $b > a$ and let the interval (a, b) be divided into n equal sub-intervals each of length $h = \frac{b-a}{n}$, by the points

$$a = a, a + h, a + 2h - - - a + nh = b$$

then $\lim_{h \to 0} h[f(a) + f(a + h) + f(a + 2h) - - + f(a + (n - 1))$

$$= \operatorname*{Lt}_{h \to 0} h \sum_{r=0}^{n-1} f(a + rh)$$

is defined as the definite integral of $f(x)$ with respect to x between the limit a and b and is denoted by symbol

$\int_a^b f(x)\,dx$ i,e

$$\int_a^b f(x)\,dx = \operatorname*{Lt}_{h \to 0} h \sum_{r=0}^{n-1} f(a + rh)$$

where a is called lower limit and b is called upper limit.

Note: $\int_a^b f(x)\,dx$ is also represented as

$$\operatorname*{Lt}_{h \to 0} h \sum_{r=1}^{n} f(a + rh)$$

Note: If $a = 0$ & $b = 1$, $h = \frac{1}{n}$ $\therefore$ $f(a + rh) = f\left(\frac{r}{n}\right)$

$$\operatorname*{Lt}_{h \to 0} h \sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \operatorname*{Lt}_{h \to 0} h \sum_{r=1}^{n} f(rh) = \int_0^1 f(x)dx$$

Geometrical meaning

The definition of definite integral $\int_a^b f(x)dx$ in gemetrical approach is based on the concept of area under the curve y=f(x) and above the x-axis from *x=a* to *x=b*

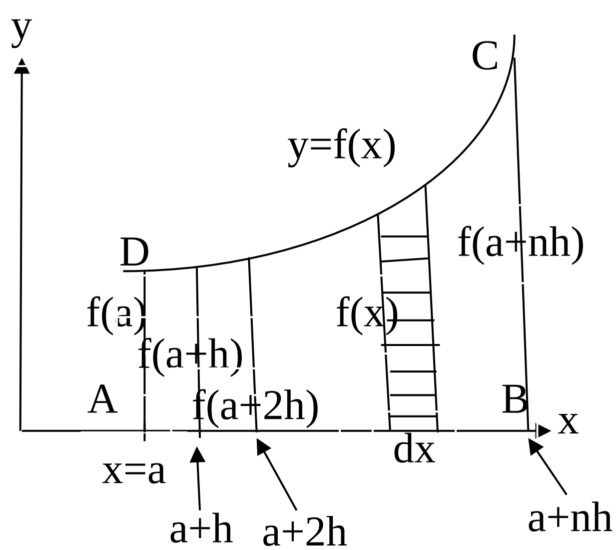

Fig 1

$$\underset{h\to 0}{\mathrm{Lt}}\, h[f(a) + f(a+h) + f(a+2h) -- + f(a+(n-1))$$

$$= \underset{h\to 0}{\mathrm{Lt}}\, h\sum_{r=1}^{n} f(a+rh) = area\ ABCD = \int_a^b f(x)dx$$

Evaluate $\int_0^1 x^3 dx$ using summation

Soln:

$$\int_0^1 x^3 dx = \underset{h\to 0}{Lt} \sum_{r=1}^{n} f(a+rh)h \quad , \frac{1-0}{n} = h \quad \text{nh=1}$$

Here $f(x) = x^3$

$$f(a+rh) = f(rh) = (rh)^3$$

$$\therefore \int_0^1 x^3\, dx = \underset{h\to 0}{Lt} \sum_{r=1}^{n} (rh)^3 h = \underset{h\to 0}{Lt}\, h^4 \sum_{r=1}^{n} r^3$$

$$= \underset{h\to 0}{Lt}\, h^4(1^3 + 2^3 + 3^3 + -----n^3)$$

$$= \underset{h\to 0}{Lt}\, h^4 \left\{\frac{n(n+1)}{2}\right\}^2 = \underset{h\to 0}{Lt} \left\{\frac{nh(nh+h)}{2}\right\}^2$$

$$= \underset{h\to 0}{Lt} \left\{\frac{1(1+h)}{2}\right\}^2 = \frac{1}{4}$$

Evaluate $\int_a^b \sin x\, dx$ *using summation*

Soln

$\int_a^b \sin x\, dx = \underset{h\to 0}{Lt} \sum_{r=1}^{n} f(a+rh)h$, $\frac{b-a}{n} = h,$ *nh=b-a*

Here $f(x) = \sin x$

$f(a+rh) = \sin(a+rh)$

$\therefore \int_a^b \sin x\, dx = \underset{h\to 0}{Lt} \sum_{r=1}^{n} \sin(a+rh)h$

$= \underset{h\to 0}{Lt}\, h[\sin(a+h) + \sin(a+2h) + \sin(a+3h)$

$+ \ldots\ldots\ldots\ldots\ldots\ldots \sin(a+nh)\ ]$

$= \underset{h\to 0}{Lt}\, h\, S$, where

$S = \sin(a+h) + \sin(a+2h) + \sin(a+3h)$

$+ \ldots\ldots\ldots\ldots\ldots\ldots \sin(a+nh)\ ]$

$=> 2\sin\frac{h}{2} S = 2\sin(a+h)\sin\frac{h}{2} + 2\sin(a+2h)\sin\frac{h}{2}$

$+2\sin(a+3h)\sin\frac{h}{2} + \ldots\ldots\ldots +2\sin(a+nh)\sin\frac{h}{2}$

$$= cos(\, a + \frac{h}{2}) - cos(\, a + \frac{3h}{2}) + cos(\, a + \frac{3h}{2})$$
$$- cos(\, a + \frac{5h}{2}) + \text{---------------} - cos(\, a + \frac{(2n-1)h}{2})$$

$$+ cos(\, a + \frac{(2n-1)h}{2}) - cos(\, a + \frac{(2n+1)h}{2})$$

$$=> S = \frac{cos(a+\frac{h}{2}) - cos(a+\frac{(2n+1)h}{2})}{2\, sin\frac{h}{2}}$$
$$\therefore \int_a^b sin\, x\, dx = \underset{h\to 0}{Lt}\, hS = \underset{h\to 0}{Lt}\, h\, \frac{cos(a+\frac{h}{2}) - cos(a+\frac{(2n+1)h}{2})}{2\, sin\frac{h}{2}}$$

$$= \underset{h\to 0}{Lt}\, \frac{\frac{h}{2}}{sin\frac{h}{2}}\ \underset{h\to 0}{Lt}\, cos(\, a + \frac{h}{2}) - cos(\, a + \frac{(2n+1)h}{2})$$

$$= 1\, \underset{h\to 0}{Lt}\left[cos\, a - cos(\, a + \frac{2(b-a)+h}{2})\right]$$

$$= cos\, a - cos\, b$$

Summation of series by definite integral

$$\int_a^b f(x)dx = \underset{n\to\infty}{Lt}\, h \sum_{r=1}^{n} f(a+rh) \quad , \quad h = \frac{b-a}{n}$$

$$\int_0^1 f(x)dx = \underset{n\to\infty}{Lt}\, h \sum_{r=1}^{n} f(rh) \, , \quad \frac{1-0}{n} = h \;=>\; h = \frac{1}{n}$$

Evaluate $\underset{n\to\infty}{Lt}\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots\dots\dots + \frac{n}{n^2+n^2}\right]$

Soln

$$\underset{n\to\infty}{Lt}\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots\dots\dots + \frac{n}{n^2+n^2}\right]$$

$$= \underset{n\to\infty}{Lt} \sum_{r=1}^{n} \frac{n}{n^2+r^2} = \sum_{r=1}^{n} \frac{1}{1+\left(\frac{r}{n}\right)^2} \frac{1}{n}$$

$$= \underset{n\to\infty}{Lt}\, h \sum_{r=1}^{n} \frac{1}{1+(rh)^2} \quad = \int_0^1 \frac{1}{1+x^2} dx$$

$$= [tan^{-1} x]_0^1 = \frac{\pi}{4}$$

Evaluate $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\ldots\ldots\left(1+\frac{n^2}{n^2}\right)\right]$

Soln

Let S= $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{\frac{1}{n}}$

$$=> ln\, S = \underset{n\to\infty}{Lt}\frac{1}{n}\ \left[\ ln\left(1+\frac{1^2}{n^2}\right)+\ ln\left(1+\frac{2^2}{n^2}\right)\right.$$

$$\left. + ln\left(1+\frac{3^2}{n^2}\right) + --------ln\left(1+\frac{n^2}{n^2}\right)\right.$$

$$= \underset{n\to\infty}{Lt}\ \frac{1}{n}\ \sum_{r=1}^{n}\ln\left(1+\frac{r^2}{n^2}\right) = \underset{n\to\infty}{Lt}\ h\ \sum_{r=1}^{n}\ln\left(1+r^2h^2\right)$$

$$= \int_0^1 ln(1+x^2)dx$$

$$=[x\, ln(1+x^2)]_0^1 - \int_0^1 \frac{2x}{1+x^2}\ x\, dx$$

$$=ln\, 2 - 2\int_0^1 \frac{1+x^2-1}{1+x^2}dx =ln\, 2 - 2\int_0^1 dx + 2\int_0^1 \frac{1}{1+x^2}dx$$

$$= \boldsymbol{ln2 - 2}[x - tan^{-1}x]_0^1$$

$$=> lnS = ln\, 2 - 2 + 2\frac{\pi}{4} = ln\, 2 - 2 + \frac{\pi}{2}$$

$$\therefore\ \ S = e^{ln\, 2-2+\frac{\pi}{2}} = e^{ln2}\ \ e^{\frac{\pi}{2}-2} = 2\, e^{\frac{\pi}{2}-2}$$

Evaluate $\int_0^1 xln(1+2x)dx$

Soln $I = \int_0^1 xln(1+2x)dx$

$$= \left[\frac{x^2}{2}\ \ln\,(1+2x)\right]_0^1 - \int_0^1 \frac{x^2}{2}\,\frac{2}{1+2x}dx$$

$$= \frac{1}{2}\ \ln 3 - \frac{1}{2}\int_0^1 \frac{2x^2+x-x}{1+2x}\ dx$$

$$= \frac{1}{2}\ \ln 3 - \frac{1}{2}\int_0^1 x\ dx + \frac{1}{4}\int_0^1 \frac{2x+1-1}{1+2x}\ dx$$

$$= \frac{1}{2}\ \ln 3 - \frac{1}{2}\int_0^1 x\ dx + \frac{1}{4}\int_0^1\ dx - \frac{1}{4}\int_0^1 \frac{1}{1+2x}\ dx$$

$$= \frac{1}{2}\ \ln 3 - \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{4}[x]_0^1 - \frac{1}{8}[\ln\,(1+2x)]_0^1$$

$$= \frac{1}{2}\ \ln 3 - \frac{1}{4} + \frac{1}{4} - \frac{1}{8}ln3 = \frac{3}{8}ln3$$

Evaluate $\int_0^1 x^2\sqrt{4-x^2}\,dx$

Soln $I = \int_0^1 x^2\sqrt{4-x^2}\,dx$

Put $x = 2sin\theta,\ \ dx = 2cos\theta\ d\theta$

When $x = 0\ \ then\ \ \theta = 0\ \ \ and\ x = 1\ \ then\ \ \ \theta = \pi/6$

$$I = \int_0^{\pi/6} 4sin^2\theta\ \ .2cos\theta.2cos\theta\ \ d\theta = 4\int_0^{\pi/6}(2sin\theta cos\theta\,)^2\ d\theta$$

$$= 4\int_0^{\pi/6} sin^2 2\theta \quad d\theta = 2\int_0^{\pi/6}(1 - cos4\theta) \quad d\theta$$

$$= 2\left[\theta - \frac{sin4\theta}{4}\right]_0^{\pi/6} = 2\left[\pi/6 - \frac{\sqrt{3}}{8}\right]$$

Evaluate $\int_0^{\pi/2} \frac{dx}{1+4\cot^2 x}$

Soln $I = \int_0^{\pi/2} \frac{dx}{1+4\cot^2 x} = \int_0^{\frac{\pi}{2}} \frac{cosec^2 x}{cosec^2 x(1+4\cot^2 x)} dx$

$$= \int_0^{\frac{\pi}{2}} \frac{cosec^2 x}{(1+\cot^2 x)(1+4\cot^2 x)} dx$$

Put $cotx = t, \ -cosec^2 x dx = dt$

When $x = 0 \quad then \quad t = \infty \quad and \ x = \frac{\pi}{2} \quad then \quad t = 0$

$$I = -\int_\infty^0 \frac{1}{(1+t^2)(1+4t^2)} dt = -\frac{1}{3}\int_\infty^0 \left[\frac{4}{(1+4t^2)} - \frac{1}{(1+t^2)}\right] dt$$

$$= -\frac{1}{3}\int_\infty^0 \left[\frac{1}{(t^2+\frac{1}{4})} - \frac{1}{(1+t^2)}\right] dt$$

$$= -\frac{1}{3}[2\tan^{-1} 2t - \tan^{-1} t]_\infty^0$$

$$= -\frac{1}{3}[2\tan^{-1} 0 - \tan^{-1} 0] + \frac{1}{3}[2\tan^{-1}\infty - \tan^{-1}\infty]$$

$$= \frac{1}{3}[2\,\frac{\pi}{2} - \frac{\pi}{2}] = \frac{\pi}{6}$$

Evaluate $\int_0^1 \cot^{-1}(1-x+x^2)$

Soln

$$I = \int_0^1 \cot^{-1}(1-x+x^2)\ dx$$

$$= \int_0^1 \tan^{-1}\frac{1}{1-x+x^2}\,dx = \int_0^1 \tan^{-1}\frac{x-(x-1)}{1+x(x-1)}\,dx$$

$$= \int_0^1 \tan^{-1}x\ dx - \int_0^1 \tan^{-1}(x-1)\ dx$$

$$= \int_0^1 \tan^{-1}x\ dx - \int_{-1}^0 \tan^{-1}z\ dz \quad ,$$

where $x-1=z$

$$= [x\tan^{-1}x]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$-[z\tan^{-1}z]_{-1}^0 + \int_{-1}^0 \frac{z}{1+z^2}\,dz$$

$$= \frac{\pi}{4} - \frac{1}{2}[\ln(1+x^2)]_0^1 + \frac{\pi}{4} + \frac{1}{2}[\ln(1+z^2)]_{-1}^0$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2 + \frac{\pi}{4} - \frac{1}{2}\ln 2 = \frac{\pi}{2} - \ln 2$$

Properties of definite integral

1. $\int_a^b f(x)dx = \int_a^b f(z)dz = \int_a^b f(t)dt$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$= \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx + \ldots\ldots.. + \int_{c_n}^{b} f(x)dx$

4. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ and

$\int_a^b f(x)dx = \int_0^a f(a-x)dx$

Proof : Let x=a+b-z dx=-dz when x=a, z=b when x=b, z=a

$\int_a^b f(x)dx = -\int_b^a f(a+b-z)dz = \int_a^b f(a+b-z)dz$

$= \int_a^b f(a+b-x)dx$ **proved**

If a=0 b=1 then $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Evaluate $\int_0^{\frac{\pi}{2}} sin^2\,\theta\, d\theta$

Soln

Let I=$\int_0^{\frac{\pi}{2}} sin^2\,\theta\, d\theta = \int_0^{\frac{\pi}{2}} sin^2(\frac{\pi}{2} - \theta) d\theta = \int_0^{\frac{\pi}{2}} cos^2\,\theta\, d\theta$ =I

$$\therefore\;\; 2I = I + I = \int_0^{\frac{\pi}{2}} sin^2\,\theta\, d\theta + \int_0^{\frac{\pi}{2}} cos^2\,\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (sin^2\,\theta + cos^2\,\theta) d\theta = \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}$$

$$2I = \frac{\pi}{2} => I = \frac{\pi}{4}$$

Evaluate $\int_0^{\frac{\pi}{2}} \frac{x}{sinx+cosx} dx$

Soln

$$I = \int_0^{\frac{\pi}{2}} \frac{x}{sinx+cosx} dx = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)} dx$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{1}{sinx+cosx} dx - \int_0^{\frac{\pi}{2}} \frac{x}{sinx+cosx} dx$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{1}{sinx+cosx} dx - \mathrm{I}$$

$$=> 2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{sinx + cosx} dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2sin\frac{x}{2}cos\frac{x}{2}+\cos^2(\frac{x}{2}) - sin^2(\frac{x}{2})} dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2(\frac{x}{2})}{1+2tan(\frac{x}{2}) - tan^2(\frac{x}{2})} dx$$

Put $tan\left(\frac{x}{2}\right) = z => \frac{1}{2}\sec^2(\frac{x}{2})\, dx = dz$

When $x = 0\,,\ then\ \ z = 0\ and\ x = \frac{\pi}{2}\ ,\ then\ \ z = 1$

$$\therefore \quad I = \frac{\pi}{2}\int_0^1 \frac{1}{1+2z - z^2} dx = \frac{\pi}{2}\int_0^1 \frac{1}{2 - (z-1)^2} dx$$

$$= \frac{\pi}{2}\frac{1}{2\sqrt{2}}\left[ln\frac{\sqrt{2}+x}{\sqrt{2}-x}\right]_0^1 = \frac{\pi}{2}\frac{1}{2\sqrt{2}} ln\frac{\sqrt{2}+1}{\sqrt{2}-1} - 0$$

$$= \frac{\pi}{2}\frac{1}{2\sqrt{2}} ln\frac{\left(\sqrt{2}+1\right)^2}{2-1} = \frac{\pi}{2\sqrt{2}} \ln\,(\sqrt{2}+1) \text{ Ans.}$$

Evaluate $\int_0^{\frac{\pi}{2}} ln\sin\theta\, d\theta,\qquad \int_0^{\frac{\pi}{2}} ln\cos\theta\, d\theta$

Soln

$$I = \int_0^{\frac{\pi}{2}} ln\sin\theta\, d\theta = \int_0^{\frac{\pi}{2}} ln\sin(\frac{\pi}{2}-\theta)d\theta = \int_0^{\frac{\pi}{2}} ln\cos\theta\, d\theta = I$$

$$2I = \int_0^{\frac{\pi}{2}} ln\sin\theta\, d\theta + \int_0^{\frac{\pi}{2}} ln\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} ln\sin\theta\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} ln(\frac{1}{2}\sin 2\,\theta)d\theta$$

$$= \int_0^{\frac{\pi}{2}} ln(\frac{1}{2})d\theta + \int_0^{\frac{\pi}{2}} ln\sin 2\,\theta d\theta$$

$= \frac{\pi}{2} ln(\frac{1}{2}) + I_1$, where

$I_1 = \int_0^{\frac{\pi}{2}} ln\sin 2\,\theta d\theta$ put $2\theta = z$ $2d\theta = dz$

$$= \frac{1}{2}\int_0^{\pi} ln\sin z\, dz = \frac{1}{2}\int_0^{2\frac{\pi}{2}} ln\sin z\, dz$$

$= \frac{1}{2}.2\int_0^{\frac{\pi}{2}} ln\sin z\, dz = \int_0^{\frac{\pi}{2}} ln\sin z\, dz = \int_0^{\frac{\pi}{2}} ln\sin x\, dx = I$

$\therefore 2I = \frac{\pi}{2} ln\frac{1}{2} + I \qquad => I = \frac{\pi}{2} ln\frac{1}{2}$

Evaluate $\int_0^{\pi} x\, ln\sin x\, dx$

Soln

$$I = \int_0^{\pi} x\, ln\sin x\, dx = \int_0^{\pi} (\pi - x)\, ln\sin(\pi - x) dx$$

$$= \int_0^{\pi} (\pi - x)\, ln\sin x\, dx$$

$$=\pi \int_0^{\pi} ln\sin x\, dx - \int_0^{\pi} x\, ln\sin x\, dx$$

$$= \pi \int_0^{\pi} ln\sin x\, dx - I$$

$$\therefore \quad 2I = \pi \int_0^{\pi} ln\sin x\, dx = 2\pi \int_0^{\frac{\pi}{2}} ln\sin x\, dx$$

$$=> I = \pi \int_0^{\frac{\pi}{2}} ln\sin x\, dx = \frac{\pi^2}{2} ln\frac{1}{2}$$

Evaluate $\int_0^{\frac{\pi}{4}} ln(1 + \tan\theta) d\theta$

Soln

$$I = \int_0^{\frac{\pi}{4}} ln(1 + \tan\theta) d\theta = \int_0^{\frac{\pi}{4}} ln(1 + \tan(\frac{\pi}{4} - \theta) d\theta$$

$$= \int_0^{\frac{\pi}{4}} ln\left(1 + \frac{\tan\frac{\pi}{4} - \tan\theta}{\tan\frac{\pi}{4} + \tan\theta}\right) d\theta = \int_0^{\frac{\pi}{4}} ln\left(1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} ln\left(\frac{2}{1+tan\,\theta}\right) d\theta = \int_0^{\frac{\pi}{4}} \{ln\,2 - ln(1 + tan\,\theta)\} d\theta$$

$$= \frac{\pi}{4} ln\,2 - I \quad => \quad 2I = \frac{\pi}{4} ln\,2 \quad => \quad I = \frac{\pi}{8} ln\,2$$

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{sin\,x}}{\sqrt{sin\,x}+\sqrt{cos\,x}} dx$

Soln

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{sin\,x}}{\sqrt{sin\,x}+\sqrt{cos\,x}} dx == \int_0^{\frac{\pi}{2}} \frac{\sqrt{sin(\frac{\pi}{2}-x)}}{\sqrt{sin(\frac{\pi}{2}-x)}+\sqrt{cos(\frac{\pi}{2}-x)}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{cos\,x}}{\sqrt{sin\,x}+\sqrt{cos\,x}} dx = I$$

$$=> 2I = I + I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{sin\,x}}{\sqrt{sin\,x}+\sqrt{cos\,x}} dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{cos\,x}}{\sqrt{sin\,x}+\sqrt{cos\,x}} dx$$

$$=> 2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{sin\,x}+\sqrt{cos\,x}}{\sqrt{sin\,x}+\sqrt{cos\,x}} dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{\pi}{4}$$

5. $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if \;\; f(-x) = f(x) \\ 0, & if \;\; f(-x) = -f(x) \end{cases}$

Proof : $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$

Now , $\int_{-a}^{0} f(x)dx$ put $x = -z \quad dx = -dz$

When $x = -a \;\; then \;\; z = a \;\; and \;\; x = 0 \; then \;\; z = 0$

$z = 0$

$\int_{-a}^{0} f(x)dx = -\int_a^0 f(-z)dz = \int_0^a f(-z)dz == \int_0^a f(-x)dx$

$\therefore \quad \int_{-a}^{a} f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx$

$= \int_0^a f(x)dx + \int_0^a f(x)dx$

$= 2\int_0^a f(x)dx$, if $f(-x) = f(x)$

Again

$\int_{-a}^{a} f(x)dx = \int_0^a f(-x)dx + \int_0^a f(x)dx$

$== -\int_0^a f(x)dx \int_0^a f(x)dx$

$=0$ if $f(-x) = -f(x)$

Hence $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if \;\; f \;\; (-x) = f(x) \\ 0 \;, & if \;\; f(-x) = -f(x) \end{cases}$

6. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx\,, & if \quad f(2a-x) = f(x) \\ 0, & if \quad f(2a-x) = -f(x) \end{cases}$

Proof: $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$

Now $\int_a^{2a} f(x)dx$ put $x = 2a - z \quad dx = -dz$

When $x = a \quad then\ z = a\ and\ x = 2a\ then\ z = 0$

$$z = 0$$

$$\int_a^{2a} f(x)dx = -\int_0^a f(2a-z)dz = \int_0^a f(2a-x)dx$$

$\therefore \quad \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

$= \int_0^a f(x)dx + \int_0^a f(x)dx$ if$f(2a-x) = f(x)$

$= 2\int_0^a f(x)dx$ if $f(2a-x) = f(x)$

Again

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

$= \int_0^a f(x)dx - \int_0^a f(x)dx$ if $f(2a-x) = -f(x)$

$= 0$

$\therefore \quad \int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx\,, & if\ \ f(2a-x) = f(x) \\ 0\ , & if \quad f(2a-x) = -f(x) \end{cases}$

Example

$$I = \int_0^{2\ \frac{\pi}{2}} sin^3 x\, dx$$

Here $f(x) = sin^3 x$

$$f(2\ \frac{\pi}{2} - x) = sin^3(\pi - x) = sin^3 x = f(x)$$

$$\int_0^{\pi} sin^3 x\, dx = 2\int_0^{\frac{\pi}{2}} sin^3 x\, dx$$

Example

$$I = \int_0^{\pi} cos^3 x\, dx$$

Here $f(x) = cos^3 x$

$$f(2\ \frac{\pi}{2} - x) = cos^3(\pi - x) = -cos^3 x = -f(x)$$

$$\int_0^{\pi} sin^3 x\, dx = 2\int_0^{\frac{\pi}{2}} sin^3 x\, dx$$

$$f(\pi - x) = cos^3(\pi - x) = -cos^3 x = -f(x)$$

$$\int_0^{\pi} cos^3 x\, dx = 0$$