Integral Calculus

Def: 1. **Integration is the inverse of differentiation**

2. **Integration is summation**

Two types of integral

1. **Indefinite Integral**
2. **Definite Integral**

Indefinite Integral

FUNDAMENTAL FORMULAS

1. $\frac{dx^n}{dx} = nx^{n-1}, \quad \int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

2. $\frac{de^x}{dx} = e^x, \quad \int e^x \, dx = e^x + c$

3. $\frac{d}{dx} lnx = \frac{1}{x}, \quad \int \frac{1}{x} \, dx = lnx + c$

4. $\frac{d}{dx} sinx = cosx \quad , \int sinx \, dx = -cosx + c$

5. $\frac{d}{dx} cosx = -sinx \quad , \int cosx \, dx = sinx + c$

6. $\frac{d}{dx} secx = secx \, tanx \quad , \int secx \, tanx \, dx = secx + c$

7. $\frac{d}{dx} cosecx = -cosecx \, cotx \quad , \int cosecx \, cotx \, dx = -cosecx + c$

8. $\frac{d}{dx} tanx = sec^2x \quad , \int sec^2x \, dx = tanx + c$

9. $\frac{d}{dx}\boldsymbol{cotx} = \boldsymbol{-cosec^2x}$, $\int \boldsymbol{cosec^2}x\,dx = -cotx + c$

10. $\frac{d}{dx}\boldsymbol{sinhx} = \boldsymbol{coshx}$, $\int sinhx\,dx = coshx + c$

11. $\frac{d}{dx}\boldsymbol{coshx} = \boldsymbol{sinhx}$, $\int cosxh\,dx = sinhx + c$

12. $\frac{d}{dx}\boldsymbol{tanhx} = \boldsymbol{sech^2x}$, $\int \boldsymbol{sech^2x}\,dx = tanhx + c$

$$\boldsymbol{sinx = \frac{e^{ix} - e^{-ix}}{2i}}, \qquad \boldsymbol{cosx = \frac{e^{ix} + e^{-ix}}{2}}$$

$$\boldsymbol{sinhx = \frac{e^{x} - e^{-x}}{2}}, \qquad \boldsymbol{coshx = \frac{e^{x} + e^{-x}}{2}}$$

Methods of Integration

1. Integration by substitution
2. Integration by parts
3. Integration by partial fraction
4. Integration by successive reduction

Integration by substitution

1Example: Workout $\int \frac{lnx}{x\sqrt{1+lnx}} dx$

Solution: $I = \int \frac{lnx}{x\sqrt{1+lnx}} dx$

put $1 + lnx = z^2 => lnx = z^2 - 1 => \frac{dx}{x} = 2zdz$

$$\int \frac{lnx}{x\sqrt{1+lnx}} dx = \int \frac{2z(z^2 - 1\,)}{z} dz = 2\int (z^2 - 1\,)dz$$

$$= \frac{2}{3}z^3 - 2z + c = (\frac{2}{3}z^2 - 2)z + c$$

$$= \left[\frac{2}{3}(1 + lnx) - 2\right]\sqrt{1 + lnx} + c$$

2Example: Workout $\int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx$

Solution: $I = \int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx = \frac{1}{a^2} \int \frac{\sec^2 x}{tan^2 x + \frac{b^2}{a^2}} dx$

Put $tanx = z => \sec^2 x\, dx = dz$

$$= \frac{1}{a^2} \int \frac{1}{z^2 + \frac{b^2}{a^2}} dz = \frac{1}{a^2} \frac{a}{b} \tan^{-1} \frac{az}{b} + \boldsymbol{c}$$

$$= \frac{1}{ab} \tan^{-1}(\frac{a}{b} \boldsymbol{tanx}) + \boldsymbol{c}$$

Example: Workout $\int \frac{x^{\frac{1}{2}}}{1+x^{\frac{1}{3}}} dx$

Solution: $I = \int \frac{x^{\frac{1}{2}}}{1+x^{\frac{1}{3}}} dx$ put $x = z^6,\ dx = 6z^5 dz$

$$=\int \frac{z^3 6z^5 dz}{1+z^2} = 6\int \frac{z^8 dz}{1+z^2}$$

$$=6\int \frac{z^6(z^2+1)-z^4(z^2+1)+z^2(z^2+1)-(z^2+1)+1}{z^2+1} dz$$

$$=6\int (z^6 - z^4 + z^2 - 1)dz+6\int \frac{1}{z^2+1} dz$$

$$=6(\frac{z^7}{7} - \frac{z^5}{5} + \frac{z^3}{3} - z) + 6\tan^{-1} z + c \quad \text{where } z = x^{\frac{1}{6}}$$

2Example: Workout $\int \boldsymbol{cos}\left(\boldsymbol{2cot^{-1}}\sqrt{\frac{1-x}{1+x}}\right) \boldsymbol{dx}$

Solution: $I = \int cos\left(2cot^{-1}\sqrt{\frac{1-x}{1+x}}\right) dx$

Put $x = cos2z, \quad dx = -2sin2z\, dz$

$$I \quad = -2\int cos\left(2cot^{-1}\sqrt{\frac{1-cos2z}{1+cos2z}}\right) sin2z\, dz$$

$$= -2\int cos\left(2cot^{-1}\sqrt{\tan^2 z}\right)sin2z\, dz$$

$$= -2\int cos\left(2cot^{-1}tanz\right)sin2z\, dz$$

$$= -2\int cos\left(2cot^{-1}\cot\left(\frac{\pi}{2}-z\right)\right) sin2z\, dz$$

$$= -2\int cos2\left(\frac{\pi}{2}-z\right) sin2z\, dz = -2\int cos(\pi - 2z)sin2z\, dz$$

$$= 2\int cos2z\, sin2z\, dz = \int sin4z\, dz = -\frac{cos4z}{4} + c$$

$$= -\frac{2\cos^2 2z - 1}{4} = -\frac{2x^2-1}{4} + c = -\frac{x^2}{2} + A$$

Type-1 $\int \frac{\boldsymbol{x^m\, dx}}{\boldsymbol{(a+bx)^n}}$, $\boldsymbol{m + ve \quad integer}$

Process **Substitute** $\boldsymbol{a + bx = z}$

3Example: Workout $\int \frac{\boldsymbol{x^2\, dx}}{\boldsymbol{(2+3x)^3}}$

Solution: $I = \int \frac{\boldsymbol{x^2\, dx}}{\boldsymbol{(2+3x)^3}}$

put $2 + 3x = z \Rightarrow x = \frac{z-2}{3} \Rightarrow dx = \frac{1}{3}dz$

$$I=\frac{1}{27}\int\frac{(z-2)^2}{z^3}dz=\frac{1}{27}\int\left(\frac{1}{z}-\frac{4}{z^2}+\frac{4}{z^3}\right)dz$$

$$=\frac{1}{27}\left(lnz+\frac{4}{z}-\frac{2}{z^2}\right)+c$$

$$=\frac{1}{27}\left[\ln(3x+2)+\frac{4}{3x+2}-\frac{2}{(3x+2)^2}\right]+c$$

Type-2 $\int\frac{dx}{x^m(a+bx)^n}$, $(m+n)>1$ *and integer*

Process **Substitute** $\frac{a+bx}{x}=z$

4Example: Workout $\int\frac{dx}{x^{1/2}(3+2x)^{3/2}}$

Solution: $I=\int\frac{dx}{x^{1/2}(3+2x)^{3/2}}$, here $\frac{1}{2}+\frac{3}{2}=2>1$

put $\frac{3+2x}{x}=z \Rightarrow x=\frac{3}{z-2} \Rightarrow dx=-\frac{3}{(z-2)^2}dz$

: $I=\int\frac{dx}{x^{1/2}(3+2x)^{3/2}}=\int\frac{1}{x^{1/2}\ x^{3/2}}\left(\frac{x}{3+2x}\right)^{3/2}dx$

$$=-\int\frac{(z-2)^2}{9}\left(\frac{1}{z}\right)^{3/2}\frac{3}{(z-2)^2}dz=-\int\frac{1}{3}\left(\frac{1}{z}\right)^{3/2}dz$$

$$=\frac{2}{15}\left(\frac{1}{z}\right)^{\frac{5}{2}}+c=\frac{2}{15}\left(\frac{x}{3x+2}\right)^{\frac{5}{2}}+c$$

Type-3 $\int \frac{\boldsymbol{dx}}{\boldsymbol{x^n}\sqrt{\boldsymbol{a+bx^2}}}$, n is +ve even integer

Process **Substitute** $x = \frac{1}{z}$

5Example: Workout $\int \frac{\boldsymbol{dx}}{\boldsymbol{x^2}\sqrt{\boldsymbol{x^2-4}}}$

Solution: $I = \int \frac{\boldsymbol{dx}}{\boldsymbol{x^2}\sqrt{\boldsymbol{x^2-4}}}$

put $x = \frac{1}{z} = z \; => dx = -\frac{\boldsymbol{1}}{\boldsymbol{z^2}} dz$

: $I = -\int \frac{zdz}{\sqrt{1-4z^2}} = \frac{1}{8}\int \frac{d(1-4z^2)}{\sqrt{1-4z^2}} = \frac{1}{4}\sqrt{1-4z^2} + \boldsymbol{c}$

$$= \frac{1}{4}\sqrt{1-\frac{4}{x^2}} + c = \frac{1}{4x}\sqrt{x^2-4} + c$$