FORMULA

1. $\int \sqrt{x^2+a^2}dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})$ x=a$\tan\theta$,   a$\cot\theta$
8. $\int \sqrt{x^2-a^2}dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}$ $\ln(x+\sqrt{x^2-a^2})$ x=a$\sec\theta$, a$\operatorname{cosec}\theta$
9. $\int \sqrt{a^2-x^2}dx = \frac{x\sqrt{a^2-x^2}}{2} - \frac{a^2}{2}\sin^{-1}\frac{x}{a}$ x=a$\sin\theta$,   a$\cos\theta$

The results can be obtained by two methods

Integration by substitution and Integration by parts

$$I=\int \sqrt{x^2+a^2}\,1dx = \sqrt{x^2+a^2}.x - \frac{1}{2}\int \frac{2x}{\sqrt{x^2+a^2}}.x\,dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}dx \quad = x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$$

$$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\,dx + a^2\int \frac{1}{\sqrt{x^2+a^2}}dx$$

$$2I = x\sqrt{x^2+a^2} + a^2\int \frac{1}{\sqrt{x^2+a^2}}dx$$

$$I = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2}) + c$$

Type 13 $\int \sqrt{ax^2+bx+c}$ dx

Procedure $\int \sqrt{ax^2+bx+c}$ dx= $\sqrt{a}\int \sqrt{(x+\frac{b}{2a})^2+\frac{4ac-b^2}{4a^2}}\,dx$

Example: Workout $\int \sqrt{2x^2+3x+1}dx$

Solution: $I = \int \sqrt{2x^2+3x+1}dx = \int \sqrt{2(x^2+\frac{3}{2}x+\frac{1}{2})}dx$

$$= \sqrt{2}\int \sqrt{(x+\frac{3}{4})^2+\frac{1}{2}-\frac{9}{16}}\,dx$$

$= \sqrt{2}\int\sqrt{(x+\frac{3}{4})^2-(\frac{1}{4})^2}dx$

$=\sqrt{2}\left[\frac{\left(x+\frac{3}{4}\right)\sqrt{(x+\frac{3}{4})^2-(\frac{1}{4})^2}}{2}+\frac{1}{32}ln\left(x+\frac{3}{4}+\sqrt{(x+\frac{3}{4})^2-(\frac{1}{4})^2}\right)\right]$+c

Type 14 $\int(px+q)\sqrt{ax^2+bx+c}\,dx$

Procedure $\int(px+q)\sqrt{ax^2+bx+c}\,dx$

$= \int\left[\frac{p}{2a}(2ax+b)+\frac{2aq-pb}{2a}\right]\sqrt{ax^2+bx+c}\,dx$

Example: Workout $\int(3x+4)\sqrt{2x^2+3x+1}\,dx$

Solution :$I = \int(3x+4)\sqrt{2x^2+3x+1}\,dx$

$= \int\left(\frac{3}{4}(4x+3)+4-\frac{9}{4}\right)\sqrt{2x^2+3x+1}\,dx$

$= \frac{3}{4}\int(4x+3)\sqrt{2x^2+3x+1}\,dx$ $+\frac{7}{4}\int\sqrt{2x^2+3x+1}dx$

$=\frac{3}{4}\int\sqrt{z}dz$$+\frac{7}{4}\int\sqrt{2x^2+3x+1}dx$

$$=\frac{1}{2}z\sqrt{z}+\frac{7\sqrt{2}}{4}\int\sqrt{(x+\frac{3}{4})^2-\frac{1}{16}}\ dx$$

$$=\ \frac{1}{2}(4x+3)^{3/2}$$

$$+\frac{7\sqrt{2}}{4}\left[\frac{\left(x+\frac{3}{4}\right)\sqrt{(x+\frac{3}{4})^2-(\frac{1}{4})^2}}{2}+\frac{1}{32}\ln\left(x+\frac{3}{4}+\sqrt{(x+\frac{3}{4})^2-(\frac{1}{4})^2}\right)\right]+c$$

Type15 $\int\frac{dx}{a+b\sin x+c\cos x}$, $\int\frac{dx}{a+b\sin x}$, $\int\frac{dx}{a+b\cos x}$

Procedure Substitute $sinx=2\sin\frac{x}{2}\cos\frac{x}{2}$, $\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}$

& $1=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}$

Examples $\int\frac{dx}{5+4\sin x}$, $\int\frac{dx}{3+5\cos x}$, $\int\frac{dx}{2\sin x+3\cos x}$, $\int\frac{dx}{6+3\sin x+4\cos x}$

Example: Workout $\int\frac{dx}{5+4sinx}$

Solution $I=\int\frac{dx}{5+4sinx}$

$$I=\int\frac{dx}{5(\cos^2\frac{x}{2}+\sin^2\frac{x}{2})+8\sin\frac{x}{2}\cos\frac{x}{2}}$$

$=\int\frac{sec^2\frac{x}{2}dx}{5(1+tan^2\frac{x}{2})+8\tan\frac{x}{2}}$, put $z=tan\frac{x}{2}$, $dz=\frac{1}{2}\sec^2\frac{x}{2}dx$

$$=\int \frac{2dz}{5(1+z^2)+8z)}=\int \frac{2dz}{5z^2+8z+5} = \frac{2}{5}\int \frac{dz}{z^2+\frac{8}{5}z+1}$$

$$= \frac{2}{5}\int \frac{dz}{(z+\frac{4}{5})^2+1-\frac{16}{25}-} \quad =\int \frac{dz}{(z+\frac{4}{5})^2+\left(\frac{3}{5}\right)^2} \quad = \quad \frac{2}{3}tan^{-1}\frac{z+\frac{4}{5}}{\frac{4}{5}} + c$$

$$= \quad \frac{2}{3}tan^{-1}\frac{5\,tan\frac{x}{2}+4}{4} + c$$

Example: Workout $\int \frac{dx}{6+3\sin x+4\cos x}$

Solution $\quad I = \int \frac{dx}{6+3\,sin\,x+4\,cos\,x}$

Put $\quad sinx = 2\sin\frac{x}{2}\cos\frac{x}{2}$ & $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$$I = \int \frac{dx}{6(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}) + 3.2\,sin\frac{x}{2}\,cos\frac{x}{2} + 4(cos^2\frac{x}{2} - sin^2\frac{x}{2})}$$

$$= \quad \int \frac{sec^2\frac{x}{2}dx}{6(1+tan^2\frac{x}{2})+6\,tan\frac{x}{2}+4(1-tan^2\frac{x}{2})}$$

put $z = tan\frac{x}{2}$, $dz = \frac{1}{2}\sec^2\frac{x}{2}dx$

$$=\int \frac{2dz}{6(1+z^2)+6z+4(1-z^2)}=\int \frac{2dz}{2z^2+6z+10}=\int \frac{dz}{z^2+3z+5}$$

$$= \int \frac{dz}{(z+\frac{3}{2})^2+5-\frac{9}{4}} \quad =\int \frac{dz}{(z+\frac{3}{2})^2+\left(\frac{\sqrt{11}}{2}\right)^2} \quad = \quad \frac{2}{\sqrt{11}}tan^{-1}\frac{z+\frac{3}{2}}{\frac{\sqrt{11}}{2}} + c$$

$$= \quad \frac{2}{\sqrt{11}}tan^{-1}\frac{2\,tan\frac{x}{2}+3}{\sqrt{11}} + c$$

Type16 $\int \frac{\sin x}{a\sin x+b\cos x}dx, \int \frac{\cos x}{a\sin x+b\cos x}dx,$

$\int \frac{a\sin x+b\cos x}{c\sin x+do s\, x}dx$, $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}dx$

Procedure Substitute

$$a\sin x+b\cos x+c = l(d\sin x+e\cos x+f)$$

$$+m\frac{d}{dx}(d\sin x+e\cos x+f)+n$$

find the value of l , m & n

Examples Workout $\int \frac{\sin x}{\sin x+\cos x}dx, \int \frac{\cos x}{\sin x+\cos x}dx,$

$\int \frac{2\sin x+3\cos x}{4\sin x+\cos x}dx, \quad \int \frac{2\sin x+3\cos x+5}{\sin x+2\cos x+6}dx$

Exam Workout $\int \frac{\cos x}{\sin x+\cos x}dx$

Solution: $\boldsymbol{I} = \int \frac{\cos x}{\sin x+\cos x}dx$

Let $\cos x = l(\sin x+\cos x)+m\frac{d}{dx}(\sin x+\cos x)$

Equating the coefficient of $\cos x$ & $\sin x$, we get
$l+m=1$, $l-m=0$, from this eqution we get $l=m=\frac{1}{2}$,

$$I = \int \frac{\cos x}{\sin x+\cos x}dx$$

$$=\frac{1}{2}\int \frac{\cos x+\sin x}{\sin x+\cos x}dx+\frac{1}{2}\int \frac{d(\sin x+\cos x)}{\sin x+\cos x}$$

$$=\frac{1}{2}\int dx+\frac{1}{2}\int \frac{d(\sin x+\cos x)}{\sin x+\cos x}$$

$$=\frac{1}{2}x+\frac{1}{2}\ln((\sin x+\cos x)+c$$

Exam Workout $\int \frac{2\sin x+3\cos x+5}{\sin x+2\cos x+6}dx$

Solution: $\boldsymbol{I}=\int \frac{2\sin x+3\cos x+5}{\sin x+2\cos x+6}dx$

Let $2\sin x+3\cos x+5$

$$=l(\sin x+2\cos x+6)+m\frac{d}{dx}(\sin x+2\cos x+6)+n$$
$$=l(\sin x+2\cos x+6)+m(\cos x-2\sin x)\text{+n}$$

Equating the coefficient of $\cos x$ & $\sin x$, we get

$l-m=2\ ----(1)$
$2\,l+m=3\ ---\ (2)$
$6l+n=5---\ (3)$

from this eqution we get $l=\frac{5}{3}$, $m=-\frac{1}{3}$ and $n=-5$

$$\therefore\ \boldsymbol{I}=\frac{5}{3}\int \frac{\sin x+2\cos x+6}{\sin x+2\cos x+6}dx-\frac{1}{3}\int \frac{d(\sin x+2\cos x+6)}{\sin x+2\cos x+6}dx$$

$$-5\int \frac{1}{\sin x+2\cos x+6}dx$$

$$= \frac{5}{3}x - \frac{1}{3}\ln(\sin x + 2\cos x + 6) - 5I_1$$

$$I_1 = \int \frac{1}{\sin x + 2\cos x + 6}\,dx$$

Put $\quad sinx = 2\sin\frac{x}{2}\cos\frac{x}{2}$ & $cosx = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$$I = \int \frac{dx}{6(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}) + \sin\frac{x}{2}\cos\frac{x}{2} + 2(\cos^2\frac{x}{2} - \sin^2\frac{x}{2})}$$

$$= \int \frac{\sec^2\frac{x}{2}dx}{6(1+\tan^2\frac{x}{2})+\tan\frac{x}{2}+2(1-\tan^2\frac{x}{2})}$$

put $z = \tan\frac{x}{2}$, $dz = \frac{1}{2}\sec^2\frac{x}{2}dx$

$$= \int \frac{2dz}{6(1+z^2)+z+2(1-z^2)} = \int \frac{2dz}{4z^2+z+8}$$

$$= \frac{1}{2}\int \frac{dz}{(z+\frac{1}{8})^2+2-\frac{1}{64}} = \int \frac{dz}{(z+\frac{3}{2})^2+\left(\frac{\sqrt{127}}{8}\right)^2} = \frac{4}{\sqrt{127}}\tan^{-1}\frac{z+\frac{1}{8}}{\frac{\sqrt{127}}{8}} + c$$

$$= \frac{4}{\sqrt{127}}\tan^{-1}\frac{8\tan\frac{x}{2}+1}{\sqrt{127}} + c$$

$$\therefore \quad I = \frac{5}{3}x - \frac{1}{3}\ln(\sin x + 2\cos x + 6) - 5\frac{4}{\sqrt{127}}\tan^{-1}\frac{8\tan\frac{x}{2}+1}{\sqrt{127}} + c$$