Integral Calculus

Def: 1. **Integration is the inverse of differentiation**

2. **Integration is summation**

Two types of integral

1. **Indefinite Integral**
2. **Definite Integral**

Indefinite Integral

FUNDAMENTAL FORMULAS

1. $\frac{dx^n}{dx} = nx^{n-1}, \quad \int x^n\, dx = \frac{x^{n+1}}{n+1} + c$

2. $\frac{de^x}{dx} = e^x, \quad \int e^x\, dx = e^x + c$

3. $\frac{d}{dx} lnx = \frac{1}{x}, \quad \int \frac{1}{x}\, dx = lnx + c$

4. $\frac{d}{dx} sinx = cosx \quad , \int sinx\, dx = -cosx + c$

5. $\frac{d}{dx} cosx = -sinx \quad , \int cosx\, dx = sinx + c$

6. $\frac{d}{dx} secx = secx\, tanx \quad , \int secx\, tanx\, dx = secx + c$

7. $\frac{d}{dx} cosecx = -cosecx\, cotx \quad , \int cosecx\, cotx\, dx = -cosecx + c$

8. $\frac{d}{dx} tanx = sec^2x \quad , \int sec^2x\, dx = tanx + c$

9. $\frac{d}{dx}\boldsymbol{cotx = -cosec^2x}$, $\int \boldsymbol{cosec^2}x\,dx = -cotx + c$

10. $\frac{d}{dx}\boldsymbol{sinhx = coshx}$, $\int sinhx\,dx = coshx + c$

11. $\frac{d}{dx}\boldsymbol{coshx = sinhx}$, $\int cosxh\,dx = sinhx + c$

12. $\frac{d}{dx}\boldsymbol{tanhx = sech^2x}$, $\int \boldsymbol{sech^2x}\,dx = tanhx + c$

Methods of Integration

1. Integration by substitution
2. Integration by parts
3. Integration by partial fraction
4. Integration by successive reduction

Integration by substitution

1Example: Workout $\int \frac{lnx}{x\sqrt{1+lnx}}dx$

Solution: $\int \frac{lnx}{x\sqrt{1+lnx}}dx$

put $1+lnx=z^2 => lnx=z^2-1 => \frac{dx}{x}=2zdz$

$$\int \frac{lnx}{x\sqrt{1+lnx}}dx=\int \frac{2z(z^2-1)}{z}dz=2\int (z^2-1)dz$$

$$=\frac{2}{3}z^3-2z+c=(\frac{2}{3}z^2-2)z+c$$

$$=\left[\frac{2}{3}(1+lnx)-2\right]\sqrt{1+lnx}+c$$

2Example: Workout $\int \frac{1}{a^2\sin^2 x+b^2\cos^2 x}dx$

Solution: $\int \frac{1}{a^2\sin^2 x+b^2\cos^2 x}dx=\frac{1}{a^2}\int \frac{\sec^2 x}{tan^2 x+\frac{b^2}{a^2}}dx$

Put $tanx=z => \sec^2 x\,dx=dz$

$$=\frac{1}{a^2}\int \frac{1}{z^2+\frac{b^2}{a^2}}dz=\frac{1}{a^2}\frac{a}{b}\tan^{-1}\frac{az}{b}+\boldsymbol{c}$$

$$=\frac{1}{ab}\tan^{-1}(\frac{a}{b}\boldsymbol{tanx})+\boldsymbol{c}$$

Example: Workout $\int \frac{x^{\frac{1}{2}}}{1+x^{\frac{1}{3}}}dx$

Solution: $I = \int \frac{x^{\frac{1}{2}}}{1+x^{\frac{1}{3}}}dx$ put $x = z^6,\ dx = 6z^5dz$

$$=\int \frac{z^3 6z^5 dz}{1+z^2} = 6\int \frac{z^8 dz}{1+z^2}$$

$$=6\int \frac{z^6(z^2+1)-z^4(z^2+1)+z^2(z^2+1)-(z^2+1)+1}{z^2+1}dz$$

$$=6\int (z^6 - z^4 + z^2 - 1)dz + 6\int \frac{1}{z^2+1}dz$$

$$=6\left(\frac{z^7}{7} - \frac{z^5}{5} + \frac{z^3}{3} - z\right) + 6\tan^{-1} z + c \quad \text{where } z = x^{\frac{1}{6}}$$

2Example: Workout $\int \boldsymbol{\cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)dx}$

Solution: $I = \int \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)dx$

Put $x = \cos 2z,\quad dx = -2\sin 2z\, dz$

$$I = -2\int \cos\left(2\cot^{-1}\sqrt{\frac{1-\cos 2z}{1+\cos 2z}}\right)\sin 2z\, dz$$

$$= -2\int \cos\left(2\cot^{-1}\sqrt{\tan^2 z}\right)\sin 2z\, dz$$

$$= -2\int cos(2cot^{-1}tanz)sin2z\,dz$$

$$= -2\int cos\left(2cot^{-1}\cot\left(\frac{\pi}{2}-z\right)\right)sin2z\,dz$$

$$= -2\int cos2\left(\frac{\pi}{2}-z\right)sin2z\,dz = -2\int cos(\pi-2z)sin2z\,dz$$

$$= 2\int cos2z\,sin2z\,dz = \int sin4z\,dz = -\frac{cos4z}{4}+c$$

$$= -\frac{2\cos^2 2z-1}{4} = -\frac{2x^2-1}{4}+c = -\frac{x^2}{2}+A$$

Type-1 $\int \frac{\boldsymbol{x^m\,dx}}{\boldsymbol{(a+bx)^n}}$, $\boldsymbol{m}$ + $\boldsymbol{ve}$ $\boldsymbol{integer}$

Procedure **Substitute** $\boldsymbol{a+bx = z}$

3Example: Workout $\int \frac{\boldsymbol{x^2\,dx}}{\boldsymbol{(2+3x)^3}}$

Solution: $I = \int \frac{\boldsymbol{x^2\,dx}}{\boldsymbol{(2+3x)^3}}$

put $2+3x = z \Rightarrow x = \frac{z-2}{3} \Rightarrow dx = \frac{1}{3}dz$

$$I = \frac{1}{9}\int \frac{(z-2)^2}{z^3}dz = \frac{1}{9}\int \left(\frac{1}{z}-\frac{4}{z^2}+\frac{4}{z^3}\right)dz$$

$$= lnz+\frac{4}{z}-\frac{2}{z^2}+c = \ln(2+3x)+\frac{4}{2+3x}-\frac{2}{(2+3x)^2}+c$$

Type-2 $\int \frac{dx}{x^m(a+bx)^n}$, $(m+n) > 1$ *and integer*

Procedure **Substitute** $\frac{a+bx}{x} = z$

4Example: Workout $\int \frac{dx}{x^{1/2}\,(3+2x)^{3/2}}$

Solution: $I = \int \frac{dx}{x^{1/2}\,(3+2x)^{3/2}}$, here $\frac{1}{2} + \frac{3}{2} = 2 > 1$

put $\frac{3+2x}{x} = z \Rightarrow x = \frac{3}{z-2} \Rightarrow dx = -\frac{3}{(z-2)^2}dz$

: $I = \int \frac{dx}{x^{1/2}\,(3+2x)^{3/2}} = \int \frac{1}{x^{1/2}\;x^{3/2}} \left(\frac{x}{3+2x}\right)^{3/2} dx$

$= -\int \frac{(z-2)^2}{9} \left(\frac{1}{z}\right)^{3/2} \frac{3}{(z-2)^2} dz = -\int \frac{1}{3} \left(\frac{1}{z}\right)^{3/2} dz$

$= -\frac{2}{15} \left(\frac{1}{z}\right)^{\frac{5}{2}} + c = -\frac{2}{15} \left(\frac{x}{3x+2}\right)^{\frac{5}{2}} + c$

Type-3 $\int \frac{dx}{x^n\,\sqrt{a+bx^2}}$, n is +ve even integer

Procedure **Substitute** $x = \frac{1}{z}$

5Example: Workout $\int \frac{dx}{x^2\,\sqrt{x^2-4}}$

Solution: $I = \int \frac{\boldsymbol{dx}}{\boldsymbol{x^2 \sqrt{x^2-4}}}$

put $x = \frac{1}{z} = z \implies dx = -\frac{\mathbf{1}}{\boldsymbol{z^2}} dz$

: $I = -\int \frac{zdz}{\sqrt{1-4z^2}} = \frac{1}{8}\int \frac{d\left(1-4z^2\right)}{\sqrt{1-4z^2}} = \frac{1}{4}\sqrt{1-4z^2} + \boldsymbol{c}$

$$= \frac{1}{4}\sqrt{1-\frac{4}{x^2}} + c = \frac{1}{4x}\sqrt{x^2-4} + c$$

1111111111111111111

Standard integral formula

1. $\int \frac{dx}{x^2+a^2} = \frac{1}{a} tan^{-1} \frac{x}{a}$+c,

2. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} ln \frac{x-a}{x+a}$+c,

3. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} ln \frac{a+x}{a-x}$+c ,

Type-4 $\int \frac{\boldsymbol{dx}}{\boldsymbol{ax^2+bx+c}}$

Type equation here.

Procedure $\int \frac{\boldsymbol{dx}}{\boldsymbol{ax^2+bx+c}} = \frac{1}{\mathrm{a}} \int \frac{dx}{(x+\frac{b}{2a})^2+\frac{4ac-b^2}{4a^2}}$

Example: Workout $\int \frac{dx}{4x^2+4x+5}$

Solution: $I = \int \frac{dx}{4x^2+4x+5} = \int \frac{dx}{4\left(x^2+x+\frac{5}{4}\right)}$

$$= \frac{1}{4} \int \frac{dx}{(x+\frac{1}{2})^2+\frac{5}{4}-\frac{1}{4}} == \frac{1}{4} \int \frac{dx}{(x+\frac{1}{2})^2+1}$$

$$= \frac{1}{4} tan^{-1} \frac{x+\frac{1}{2}}{1} + c = \frac{1}{4} tan^{-1}(x+\frac{1}{2}) + c$$

Example: Workout $\int \frac{dx}{2x^2+3x+1}$

Solution: $I = \int \frac{dx}{2x^2+3x+1} = \int \frac{dx}{2\left(x^2+\frac{3}{2}x+\frac{1}{2}\right)}$

$$= \frac{1}{2}\int \frac{dx}{(x+\frac{3}{4})^2+\frac{1}{2}-\frac{9}{16}} = \frac{1}{2}\int \frac{dx}{(x+\frac{3}{4})^2-(\frac{1}{4})^2}$$

$$= \frac{1}{2.\frac{1}{2}} ln\frac{x+\frac{3}{4}-\frac{1}{4}}{x+\frac{3}{4}+\frac{1}{4}} + c = ln\frac{4x+2}{4x+4} + c$$

Type-5 $\int \frac{px+q}{ax^2+bx+c}dx$

Procedure $\int \frac{px+q}{ax^2+bx+c}dx = \int \frac{\frac{p}{2a}\frac{d}{dx}(ax^2+bx+c)+(q-\frac{pb}{2a})}{ax^2+bx+c} \quad dx$

Example: Workout $\int \frac{3x+4}{2x^2+3x+1}dx$

Solution: $I = \int \frac{3x+4}{2x^2+3x+1}dx = \int \frac{\frac{3}{4}(4x+3)+4-\frac{9}{4}}{2x^2+3x+1}dx$

$= \frac{3}{4}\int \frac{4x+3}{2x^2+3x+1}dx + \frac{7}{4}\int \frac{dx}{2x^2+3x+1}$

$= \frac{3}{4}\int \frac{dz}{z} + \frac{7}{4}\int \frac{dx}{2x^2+3x+1} = \frac{3}{2}\ln z + \frac{7}{4}\int \frac{dx}{2x^2+3x+1} + C$

FORMULA

4. $\int tanx \;\; dx =$ ln(secx)+c

5. $\int cotx \;\; dx =$ ln(sinx)+c

6. $\int secx \;\; dx =$ ln (secx+tan x)+c

7. $\int cosecx \;\; dx =$ ln tan(x/2)+c

8. $\int \frac{dx}{\sqrt{x^2+a^2}}$ =ln(x+$\sqrt{x^2+a^2}) + c = \sinh^{-1}\frac{x}{a}$ +c, x=atanθ,

9. $\int \frac{dx}{\sqrt{x^2-a^2}}$ =ln(x+$\sqrt{x^2-a^2}$)+c $= cos\text{h}^{-1}\frac{x}{a} + c$, x=asecθ,

10. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$ +c x=asinθ,

$\int \frac{dx}{\sqrt{x^2+a^2}}$ x=atanθ dx=$a\sec^2\theta\,$dθ

$\int \frac{a\sec^2\theta d\theta}{\sqrt{a\tan^2\theta+a^2}} = \int \sec\theta d\theta$=ln(sec$\theta$+tan$\theta$)+c= ln(x+$\sqrt{x^2+a^2}$)+c

Type6 $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Procedure : $\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}}$

Example: Workout $\int \frac{dx}{\sqrt{2x^2+3x+1}}$

Solution: $I = \int \frac{dx}{\sqrt{2x^2+3x+1}} = \int \frac{dx}{\sqrt{2(x^2+\frac{3}{2}x+\frac{1}{2})}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x+\frac{3}{4})^2+\frac{1}{2}-\frac{9}{16}}}$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x+\frac{3}{4})^2-(\frac{1}{4})^2}} = \frac{1}{\sqrt{2}} \ln\left(x+\frac{3}{4}+\sqrt{(x+\frac{3}{4})^2-(\frac{1}{4})^2}\right)+c$$

Example: Workout $\int \frac{dx}{\sqrt{6+11x-10x^2}}$

Solution: $I = \int \frac{dx}{\sqrt{6+11x-10x^2}} = \frac{1}{\sqrt{10}} \int \frac{dx}{\sqrt{\frac{3}{5}+\frac{11}{10}x-x^2}}$

$$= \frac{1}{\sqrt{10}} \int \frac{dx}{\sqrt{\frac{3}{5}-(x-\frac{11}{20})^2+\frac{121}{400}}}$$

$$= \frac{1}{\sqrt{10}} \int \frac{dx}{\sqrt{\left(\sqrt{\frac{19}{20}}\right)^2-(x-\frac{11}{20})^2}} = \frac{1}{\sqrt{10}} \sin^{-1} \frac{x-\frac{11}{20}}{\sqrt{\frac{19}{20}}} + C$$

Type-7 $\int \frac{dx}{\sqrt{(x-a)(x-b)}}$ **Procedure** substitute $\boldsymbol{x-b=z^2}$

Example: Workout $\int \frac{dx}{\sqrt{(x-3)(x-4)}}$

Solution: $I = \int \frac{dx}{\sqrt{(x-3)(x-4)}}$ put $x-4=z^2$, dx=2zdz

$=\int \frac{2zdz}{\sqrt{(z^2+1^2)z^2}} = 2\int \frac{dz}{\sqrt{(z^2+1^2)}} = 2\ln(z+\sqrt{z^2+1})+c$

$=2\ln(\sqrt{x-4}+\sqrt{x-4+1})+c$

$=2\ln(\sqrt{x-4}+\sqrt{x-3})+c$

Type-8 $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Procedure $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx = \int \frac{\frac{p}{2a}(2ax+b)+\frac{2aq-pb}{2a}}{\sqrt{ax^2+bx+c}}dx$

Example: Workout $\int \frac{3x+4}{\sqrt{2x^2+3x+1}}dx$

Solution: $I = \int \frac{3x+4}{\sqrt{2x^2+3x+1}}dx = \int \frac{\frac{3}{4}(4x+3)+4-\frac{9}{4}}{\sqrt{2x^2+3x+1}}dx$

$= \frac{3}{4}\int \frac{4x+3}{\sqrt{2x^2+3x+1}}dx + \frac{7}{4}\int \frac{dx}{\sqrt{2x^2+3x+1}} = I_1 + I_2$

$I_1 = \frac{3}{4}\int \frac{4x+3}{\sqrt{2x^2+3x+1}}dx,\quad \frac{7}{4}\int \frac{dx}{\sqrt{2x^2+3x+1}} = I_2$

$I_1 = \frac{3}{4}\int \frac{4x+3}{\sqrt{2x^2+3x+1}}dx$ put z= $2x^2+3x+1 \Rightarrow dz = (4x+3)dx$

$$= \frac{3}{4}\int\frac{dz}{\sqrt{z}} = \frac{3}{2}\sqrt{z} = \frac{3}{2}\sqrt{2x^2+3x+1} + c_1$$

$$I_2 = \frac{7}{4}\int\frac{dx}{\sqrt{2x^2+3x+1}} = \int\frac{dx}{\sqrt{2(x^2+\frac{3}{2}x+\frac{1}{2})}} = \frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{(x+\frac{3}{4})^2+\frac{1}{2}-\frac{9}{16}}}$$

$$= \frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{(x+\frac{3}{4})^2-(\frac{1}{4})^2}} = \frac{1}{\sqrt{2}}\ln\left(x+\frac{3}{4}+\sqrt{(x+\frac{3}{4})^2-(\frac{1}{4})^2}\right) = c_2$$

Type-9 $\int\sqrt{\frac{(ax+b)}{cx+d}}\,dx$

Process $\int\sqrt{\frac{(ax+b)(ax+b)}{(cx+d)(ax+b)}}\,dx$

Example: Workout $\int\sqrt{\frac{2x+4}{3x+3}}\,dx$

Solution: $I = \int\sqrt{\frac{2x+4}{3x+3}}\,dx = \int\frac{2x+4}{\sqrt{(3x+3)(2x+4)}}\,dx = \int\frac{2x+4}{\sqrt{6x^2+18x+12}}\,dx$

Type- 10 $\int\frac{dx}{(ax+b)\sqrt{cx+d}}$ **Procedure** Substitute $cx+d = z^2$

Example: Workout $\int\frac{dx}{(1-x)\sqrt{x}}$

Solution $I = \int\frac{dx}{(1-x)\sqrt{x}}$ put $x = z^2$ dx=2zdz

$$= \int\frac{2zdz}{(1-z^2)z} = \int\frac{2dz}{1-z^2} = \frac{1}{2}\,ln\,\frac{1+z}{1-z} + c = \frac{1}{2}\,ln\,\frac{1+x^2}{1-x^2} + c$$

Type11 $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$

Procedure Substitute $ax+b=\frac{1}{z}$

Example: Workout $\int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$

Solution: $I=\int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$

put $1+x=\frac{1}{z}$, $x=\frac{1}{z}-1$, $dx=-\frac{1}{z^2}dz$

$$1+x-x^2=\frac{1}{z}-\left(\frac{1}{z}-1\right)^2=\frac{1}{z}-\frac{1}{z^2}+\frac{2}{z}-1=\frac{3}{z}-\frac{1}{z^2}-1=\frac{3z-z^2-1}{z^2}$$

$$I=\int \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{\frac{3z-z^2-1}{z^2}}}=-\int \frac{dz}{\sqrt{3z-z^2-1}}$$

$$=-\int \frac{dz}{\sqrt{-1+3z-z^2}}=-\int \frac{dz}{\sqrt{-1-(z-\frac{3}{2})^2+\frac{9}{4}}}$$

$$=-\int \frac{dz}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-(z-\frac{3}{2})^2}}=\sin^{-1}\left(\frac{z-\frac{3}{2}}{\frac{\sqrt{5}}{2}}\right)+c$$

$$=\sin^{-1}\left(\frac{\frac{1}{x+1}-\frac{3}{2}}{\frac{\sqrt{5}}{2}}\right)+c$$

Type12 $\int \frac{dx}{x(a+x^n)}$, $\int \frac{dx}{x\sqrt{a+bx^n}}$

Procedure Substitute $x^n = \frac{1}{z^2}$, $lnx^n = ln\frac{1}{z^2}$

$$n\frac{dx}{x} = -\frac{2}{z}dz => \frac{dx}{x} = -\frac{2}{nz}dz$$

Example: Workout $\int \frac{dx}{x\sqrt{1+x^5}}$

Solution: $I = \int \frac{dx}{x\sqrt{1+x^5}}$

put $x^5 = \frac{1}{z^2}$, $lnx^5 = lnz^{-2} => 5lnx = -2lnz$

$$\frac{dx}{x} = -\frac{2}{5z}dz$$

$$I = -\frac{2}{5}\int \frac{1}{\frac{1}{z}\sqrt{z^2+1}} \quad \frac{1}{z}dz = -\frac{2}{5}\int \frac{1}{\sqrt{z^2+1}} \quad dz$$

$$= -\frac{2}{5}\ln\left(z+\sqrt{z^2+1}\right)+c$$

$$= -\frac{2}{5}\ln\left(\frac{1}{\sqrt{x^5}}+\sqrt{\frac{1+x^5}{x^5}}\right)+c$$

$$= -\frac{2}{5}\ln\left(\frac{1+\sqrt{1+x^5}}{\sqrt{x^5}}\right)+c$$

2222222222222

FORMULA

11. $\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})$ x=atanθ, acotθ

8. $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}$ $\ln(x+\sqrt{x^2-a^2})$x=asecθ, acosecθ

9. $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} - \frac{a^2}{2}\sin^{-1}\frac{x}{a}$ x=asinθ, acosθ

The results can be obtained by two methods

Integration by substitution and Integration by parts

I= $\int \sqrt{x^2+a^2}\,1dx = \sqrt{x^2+a^2}\,.x - \frac{1}{2}\int \frac{2x}{\sqrt{x^2+a^2}}.x\,dx$

$= x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}dx \quad = x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$

$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\,dx + a^2\int \frac{1}{\sqrt{x^2+a^2}}dx$

2I= $x\sqrt{x^2+a^2} + a^2\int \frac{1}{\sqrt{x^2+a^2}}dx$

I= $\frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}$ $\ln(x+\sqrt{x^2+a^2})$+c

Type 13 $\int \sqrt{ax^2+bx+c}$ dx

Procedure $\int \sqrt{ax^2+bx+c}$ dx= $\sqrt{a}\int \sqrt{(x+\frac{b}{2a})^2 + \frac{4ac-b^2}{4a^2}}\,dx$

Example: Workout $\int \sqrt{2x^2+3x+1}\,dx$

Solution: $I = \int \sqrt{2x^2+3x+1}\,dx = \int \sqrt{2(x^2+\frac{3}{2}x+\frac{1}{2})}\,dx$

$= \sqrt{2}\int \sqrt{(x+\frac{3}{4})^2 + \frac{1}{2} - \frac{9}{16}}\,dx$

$$= \sqrt{2}\int\sqrt{(x+\tfrac{3}{4})^2-(\tfrac{1}{4})^2}\,dx$$

$$=\sqrt{2}\left[\frac{\left(x+\frac{3}{4}\right)\sqrt{(x+\frac{3}{4})^2-(\frac{1}{4})^2}}{2}+\frac{1}{32}ln\left(x+\frac{3}{4}+\sqrt{(x+\tfrac{3}{4})^2-(\tfrac{1}{4})^2}\right)\right]+c$$

Type 14 $\int(px+q)\sqrt{ax^2+bx+c}\,dx$

Procedure $\int(px+q)\sqrt{ax^2+bx+c}\,dx$

$$=\int\left[\frac{p}{2a}(2ax+b)+\frac{2aq-pb}{2a}\right]\sqrt{ax^2+bx+c}\,dx$$

Example: Workout $\int(3x+4)\sqrt{2x^2+3x+1}\,dx$

Solution :$I=\int(3x+4)\sqrt{2x^2+3x+1}\,dx$

$$=\int\left(\frac{3}{4}(4x+3)+4-\frac{9}{4}\right)\sqrt{2x^2+3x+1}\,dx$$

$$=\frac{3}{4}\int(4x+3)\sqrt{2x^2+3x+1}\,dx+\frac{7}{4}\int\sqrt{2x^2+3x+1}dx$$

$$=\frac{3}{4}\int\sqrt{z}dz+\frac{7}{4}\int\sqrt{2x^2+3x+1}dx$$

$$=\frac{1}{2}z\sqrt{z}+\frac{7\sqrt{2}}{4}\int\sqrt{(x+\frac{3}{4})^2-\frac{1}{16}}\,dx$$

$$=\ \frac{1}{2}(4x+3)^{3/2}$$

$$+\frac{7\sqrt{2}}{4}\left[\frac{\left(x+\frac{3}{4}\right)\sqrt{(x+\frac{3}{4})^2-(\frac{1}{4})^2}}{2}+\frac{1}{32}\ln\left(x+\frac{3}{4}+\sqrt{(x+\frac{3}{4})^2-(\frac{1}{4})^2}\right)\right]+c$$

Type15 $\int\frac{dx}{a+b\sin x+c\cos x}$, $\int\frac{dx}{a+b\sin x}$, $\int\frac{dx}{a+b\cos x}$

Procedure Substitute $sinx=2\sin\frac{x}{2}\cos\frac{x}{2}$, $\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}$

& $1=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}$

Examples $\int\frac{dx}{5+4\sin x}$, $\int\frac{dx}{3+5\cos x}$, $\int\frac{dx}{2\sin x+3\cos x}$, $\int\frac{dx}{6+3\sin x+4\cos x}$

Example: Workout $\int\frac{dx}{5+4sinx}$

Solution $I=\int\frac{dx}{5+4sinx}$

$$I=\int\frac{dx}{5(\cos^2\frac{x}{2}+\sin^2\frac{x}{2})+8\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\ \int\frac{sec^2\frac{x}{2}dx}{5(1+tan^2\frac{x}{2})+8\tan\frac{x}{2}}$$, put $z=tan\frac{x}{2}$, $dz=\frac{1}{2}\sec^2\frac{x}{2}dx$

$$=\int \frac{2dz}{5(1+z^2)+8z} = \int \frac{2dz}{5z^2+8z+5} = \frac{2}{5}\int \frac{dz}{z^2+\frac{8}{5}z+1}$$

$$= \frac{2}{5}\int \frac{dz}{(z+\frac{4}{5})^2+1-\frac{16}{25}} = \int \frac{dz}{(z+\frac{4}{5})^2+\left(\frac{3}{5}\right)^2} = \frac{2}{3}tan^{-1}\frac{z+\frac{4}{5}}{\frac{4}{5}} + c$$

$$= \frac{2}{3}tan^{-1}\frac{5\,tan\frac{x}{2}+4}{4} + c$$

Example: Workout $\int \frac{dx}{6+3\sin x+4\cos x}$

Solution $I = \int \frac{dx}{6+3\,sin\,x+4\,cos\,x}$

Put $sinx = 2\sin\frac{x}{2}\cos\frac{x}{2}$ & $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$$I = \int \frac{dx}{6(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}) + 3.2\,sin\frac{x}{2}cos\frac{x}{2} + 4(cos^2\frac{x}{2} - sin^2\frac{x}{2})}$$

$$= \int \frac{sec^2\frac{x}{2}dx}{6(1+tan^2\frac{x}{2})+6\,tan\frac{x}{2}+4(1-tan^2\frac{x}{2})}$$, put $z = tan\frac{x}{2}$, $dz = \frac{1}{2}\sec^2\frac{x}{2}dx$

$$=\int \frac{2dz}{6(1+z^2)+6z+4(1-z^2)} = \int \frac{2dz}{2z^2+6z+10} = \int \frac{dz}{z^2+3z+5}$$

$$= \int \frac{dz}{(z+\frac{3}{2})^2+5-\frac{9}{4}} = \int \frac{dz}{(z+\frac{3}{2})^2+\left(\frac{\sqrt{11}}{2}\right)^2} = \frac{2}{\sqrt{11}}tan^{-1}\frac{z+\frac{3}{2}}{\frac{\sqrt{11}}{2}} + c$$

$$= \frac{2}{\sqrt{11}}tan^{-1}\frac{2\,tan\frac{x}{2}+3}{\sqrt{11}} + c$$

Type16 $\int \frac{\sin x}{a\sin x+b\cos x}dx,\ \int \frac{\cos x}{a\sin x+b\cos x}dx,$

$$\int \frac{a\sin x+b\cos x}{c\sin x+do s\,x}dx\ \ ,\qquad \int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}dx$$

Procedure Substitute

$$a\sin x+b\cos x+c=l(d\sin x+e\cos x+f)$$

$$+m\frac{d}{dx}(d\sin x+e\cos x+f)+n$$

find the value of l , m & n

Examples Workout $\int \frac{\sin x}{\sin x+\cos x}dx,\ \int \frac{\cos x}{\sin x+\cos x}dx,$

$\int \frac{2\sin x+3\cos x}{4\sin x+\cos x}dx,\quad \int \frac{2\sin x+3\cos x+5}{\sin x+2\cos x+6}dx$

Exam Workout $\int \frac{\cos x}{\sin x+\cos x}dx$

Solution: $\boldsymbol{I} = \int \frac{\cos x}{\sin x+\cos x}dx$

Let $cosx = l(\sin x+\cos x)+m\frac{d}{dx}(\sin x+\cos x)$

Equating the coefficient of $cosx$ & $sinx$, we get
$l+m=1\ ,\ l-m=0,$ from this eqution we get $l=m=\frac{1}{2},$

$$I=\int \frac{\cos x}{\sin x+\cos x}dx$$

$$= \frac{1}{2}\int \frac{\cos x + \sin x}{\sin x + \cos x} dx + \frac{1}{2}\int \frac{d(\sin x + \cos x)}{\sin x + \cos x}$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{d(\sin x + \cos x)}{\sin x + \cos x}$$

$$= \frac{1}{2}x + \frac{1}{2}\ln((\sin x + \cos x) + c$$

Exam Workout $\int \frac{2\sin x + 3\cos x + 5}{\sin x + 2\cos x + 6} dx$

Solution: $\boldsymbol{I} = \int \frac{2\sin x + 3\cos x + 5}{\sin x + 2\cos x + 6} dx$

Let $2\sin x + 3\cos x + 5$

$$= l(\sin x + 2\cos x + 6) + m\frac{d}{dx}(\sin x + 2\cos x + 6) + n$$
$$= l(\sin x + 2\cos x + 6) + m(\cos x - 2\sin x) + n$$

Equating the coefficient of *cosx* & *sinx*, we get

$l - m = 2$, $2l + m = 3$, $6l + n = 5$,

from this eqution we get