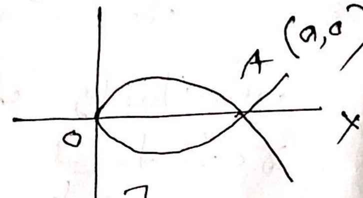
Surface Area

[redacted]

$$S = 2\pi \int_{-1}^{1} 2\,dx = 4\pi \int_{-1}^{1} dx = 4\pi \Big[x\Big]_{-1}^{1} = 8\pi \text{ sq}$$

Ex 2 Find the area of the surface generated by revolving about x-axis the loop of the curve $3ay^2 = x(x-a)^2$

Soln



$$3ay^2 = x(x-a)^2$$

$$\therefore\ y = \frac{1}{\sqrt{3a}}\, x^{\frac{1}{2}}(x-a)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{3a}}\left[x^{\frac{1}{2}}\cdot 1 + (x-a)\,\frac{1}{2}x^{-\frac{1}{2}}\right]$$

$$= \frac{1}{\sqrt{3a}}\left(\sqrt{x} + \frac{x-a}{2\sqrt{x}}\right)$$

$$= \frac{1}{\sqrt{3a}} \cdot \frac{2x+x-a}{2\sqrt{x}} = \frac{3x-a}{2\sqrt{3ax}}$$

$$\therefore\ 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(3x-a)^2}{12ax} = \frac{(3x+a)^2}{12ax}$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{(3x+a)^2}{12ax}} = \frac{3x+a}{2\sqrt{3ax}}$$

$$S = 2\pi \int_0^a \frac{1}{\sqrt{3a}}\, x^{\frac{1}{2}}(x-a)\, \frac{3x+a}{2\sqrt{3ax}}\, dx$$

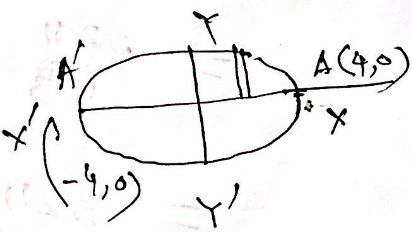
$$S = \frac{\pi}{3a}\int_0^a (x-a)(3x+a)\,dx$$

$$= \frac{\pi}{3a}\int_0^a \left(3x^2 - 2ax - a^2\right) dx$$

$$= \frac{\pi}{3a}\left[x^3 - ax^2 - a^2x\right]_0^a = -\frac{\pi a^2}{3}$$

$$\therefore = \frac{\pi a^2}{3}$$ sq. unit neglecting − sign

Ex-3 Find the surface area of the solid generated by the revolution of the ellipse $x^2+4y^2=16$ about its major axis.



Soln

$$S = 2\pi\int y\,ds$$

$$= 2\pi\int_{-4}^{4} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\int_{-4}^{4} y\sqrt{1+\frac{x^2}{16y^2}}\,dx$$

$$= \frac{2\pi}{4}\int_{-4}^{4}\sqrt{16y^2+x^2}\,dx$$

P-4

$$S = \frac{\pi}{2}\int_{-4}^{4} \sqrt{16y^2 + x^2}\, dx$$

$$= \frac{\pi}{2}\int_{-4}^{4} \sqrt{64 - 4x^2 + x^2}\, dx$$

$$= \frac{\pi}{2}\int_{-4}^{4} \sqrt{64 - 3x^2}$$ Put $\sqrt{3}x = 8\sin\theta$

$\sqrt{3}\,dx = 8\cos\theta\, d\theta$

$$= \frac{\pi}{2}\int_{-\frac{\pi}{3}}^{\pi/3} \sqrt{64 - 64\sin^2\theta}\, \frac{8}{\sqrt{3}}\cos\theta\, d\theta$$

$$= \frac{\pi}{2} \cdot \frac{8}{\sqrt{3}} \cdot 8 \int_{-\frac{\pi}{3}}^{\pi/3} \cos\theta \cos\theta\, d\theta$$

$$= \frac{32\pi}{\sqrt{3}} = \frac{16\pi}{\sqrt{3}} \cdot 2\int_{0}^{\pi/3} (1 + \cos 2\theta)\, d\theta$$

$$= \frac{32\pi}{\sqrt{3}} \left[\theta + \frac{\sin 2\theta}{2}\right]_{0}^{\pi/3}$$

$$= \frac{32\pi}{\sqrt{3}} \left[\pi/3 + \frac{\sqrt{3}}{4}\right]$$

$$= 8\pi\left(\frac{4\pi}{3\sqrt{3}} + 1\right) \text{ Sq. unit}$$

Ex-04 Find the surface area of the solid obtained by rotating $y = \sqrt[3]{x}$, $1 \le y \le 2$ about y-axis

Soln $S = 2\pi \int_1^2 \cancel{y}\, x\, ds$

$= 2\pi \int_1^2 y^3 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$= 2\pi \int_1^2 y^3 \sqrt{1 + 9y^4}\, dy$

$= \frac{2\pi}{36} \int_1^2 36\, y^3 \sqrt{1 + 9y^4}\, dy$

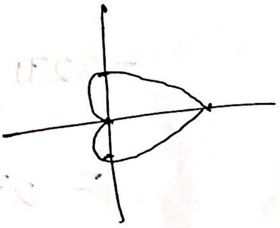
$= \cancel{\frac{2\pi}{36}}\ \frac{\pi}{27} \left[(1 + 9y^4)^{3/2} \right]_{+1}^{2}$

$= \frac{\pi}{27} \left(145^{3/2} - 10^{3/2} \right)$ sq unit

~~Ex-05~~ Polar curve

Ex. 05 The curve $r = a(1-\cos\theta)$ ~~revo~~ revolves about the initial line. Find the area of the surface generated.

Soln



$r = a(1-\cos\theta)$

$\frac{dr}{d\theta} = a\sin\theta$

$S = 2\pi \int y\, ds = 2\pi \int_0^{\pi} y \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$= 2\pi \int_0^{\pi} r\sin\theta \sqrt{a^2(1-2\cos\theta+\cos^2\theta) + a^2\sin^2\theta}\, d\theta$

$= 2\pi \int_0^{\pi} r\sin\theta\, a\sqrt{2-2\cos\theta}\, d\theta$

$= 2\pi\sqrt{2}\, a \int_0^{\pi} r\sin\theta \sqrt{2}\sin\frac{\theta}{2}\, d\theta$

$= 4\pi a \int_0^{\pi} a \cdot 2\sin^2\frac{\theta}{2} \sin\theta \sin\frac{\theta}{2}\, d\theta$

$= 8\pi a^2 \int_0^{\pi} 2\sin^2\frac{\theta}{2} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \sin\frac{\theta}{2}\, d\theta$

$$S = 16\pi a^2 \int_0^{\pi} \sin^4\frac{\theta}{2} \cos\frac{\theta}{2}\, d\theta$$

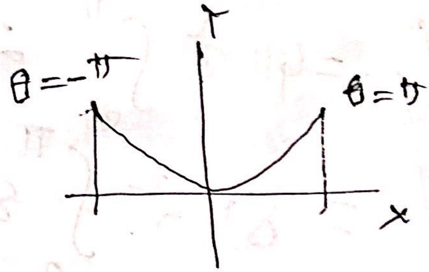
$\sin\frac{\theta}{2} = z$

$\cos\frac{\theta}{2}\, d\theta = 2dz$

$$= 16\pi a^2 \int_0^1 z^4 \cdot 2dz$$

$$= \frac{32\pi a^2}{5}\left[z^5\right]_0^1 = \frac{32\pi a^2}{5} \text{ sq unit}$$

Ex. 06. Find the area of the surface of the solid formed by the revolution of the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about x-axis.



Soln

$x = a(\theta + \sin\theta)$

$\frac{dx}{d\theta} = a(1 + \cos\theta)$

$y = a(1 - \cos\theta)$; $\frac{dy}{d\theta} = a\sin\theta$

$$\cancel{\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 =}$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= a^2\left(1+2\cos\theta+\cos^2\theta\right) + a^2\sin^2\theta$$

$$= a^2\left[2+2\cos\theta\right]$$

$$= 2a^2\left(1+\cos\theta\right)$$

$$= 4a^2\cos^2\frac{\theta}{2}$$

$$S = 2\pi\int_{-\pi}^{\pi} y\, ds$$

$$= 2\pi\int_{-\pi}^{\pi} a\left(1-\cos\theta\right)\cdot 2a\cos\frac{\theta}{2}\, d\theta$$

$$= 4\pi\int_{0}^{\pi} a^2\left(1-\cos\theta\right)\cos\frac{\theta}{2}\, d\theta$$

$$= 4\pi a^2\int_{0}^{\pi} 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta$$

$$= 8\pi a^2\int_{0}^{\pi} \sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta$$

$$= \frac{8\pi a^2\cdot 2}{3}\left[\sin^3\frac{\theta}{2}\right]_0^{\pi}$$

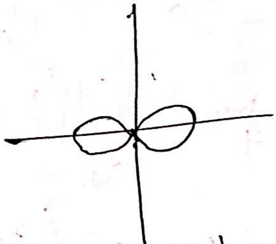
$$= \frac{32\pi a^2}{3} \text{ sq unit}$$

Ex. 07

~~The curve r=~~

The Leminscate $r^2 = a^2\cos 2\theta$ is rotated about initial line. Find the surface area generated

Sol



$$r^2 = a^2\cos 2\theta$$

$$\cancel{2}r\frac{dr}{d\theta} = -\cancel{2}a^2\sin 2\theta$$

$$\frac{dr}{d\theta} = -\frac{a^2\sin 2\theta}{a\cos^{\frac{1}{2}} 2\theta} = -\frac{a\sin 2\theta}{\cos^{\frac{1}{2}} 2\theta}$$

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{a^2\sin^2 2\theta}{\cos 2\theta}$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta}$$

$$= \frac{a^2}{\cos 2\theta}$$

$$\therefore ds = \frac{a}{\cos^{\frac{1}{2}} 2\theta}$$

p-10

$$S = 2\pi \int_0^{\pi/4} y\, ds$$

$$= 4\pi \int_0^{\pi/4} r \sin\theta \cdot \frac{a}{\cos^{\frac{1}{2}} 2\theta}\, d\theta$$

$$= 4\pi \int_0^{\pi/4} \frac{a \cos^{\frac{1}{2}} 2\theta \cdot \sin\theta \cdot a}{\cos^{\frac{1}{2}} 2\theta}\, d\theta$$

$$= 4\pi a^2 \int_0^{\pi/4} \sin\theta\, d\theta$$

$$= -4\pi a^2 \left[\cos\theta\right]_0^{\pi/4}$$

$$= -4\pi a^2 \left[\frac{1}{\sqrt{2}} - 1\right] \text{ sq. unit}$$

$$= 4\pi a^2 \left(\frac{1}{\sqrt{2}} - 1\right) \text{ sq unit}$$

$$= 4\pi a^2 \cdot \frac{1-\sqrt{2}}{\sqrt{2}} \text{ sq unit}$$

$$= 2\sqrt{2}\,\pi a^2 (1-\sqrt{2}) \text{ sq unit}$$

$$= 2\sqrt{2}\,\pi a^2 (\sqrt{2}-1) \text{ sq unit}$$

Ans.