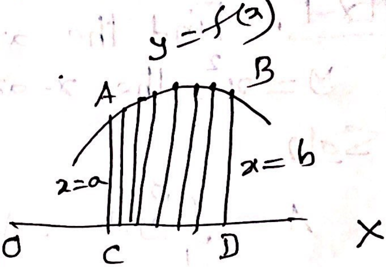
Area



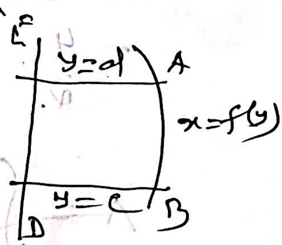
Let us find the area bounded by the curve $y = f(x)$, $x = a$, $x = b$ and x-axis. Let us divide the area bounded by the curve with heights $y_1, y_2, y_3, y_4 \ldots y_n$ and breath $\delta x_1, \delta x_2, \delta x_3, \delta x_4 \ldots \delta x_n$ respectively where $\delta x_1, \delta x_2, \delta x_3, \delta x_4 \ldots \delta x_n$ are infinitisely small and n is very large. Therefore the required area say A is given by

$$A = y_1 \delta x_1 + y_2 \delta x_2 + y_3 \delta x_3 + y_4 \delta x_4 + \quad + y_n \delta x_n$$

$$= \sum_{i=1}^{n} y_i \delta x_i$$

Now $A = \underset{\delta x_i \to 0}{Lt} \sum_{i=1}^{n} y_i \delta x_i = \int_a^b \boxed{y\,dx}$

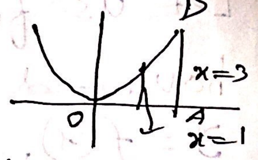
Similarly the area bounded by $x = f(y)$, $y = c$, $y = d$ and y-axis is given by



$$A = \int_c^d x\,dy.$$

Ex-1. Find the area bounded by the curve $y = x^2$, the x-axis and the lines $x=1$, $x=3$

Soln

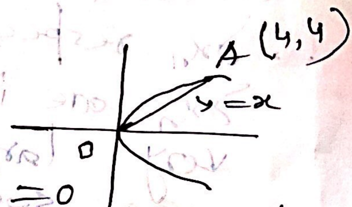


$$A = \int_1^3 y\,dx = \int_1^3 x^2\,dx = \frac{1}{3}\left[x^3\right]_1^3$$

$$= \frac{1}{3}(27-1) = \frac{26}{3} \text{ sq unit}$$

Ex-2. Find the area of the segment cut off from $y^2 = 4x$ by the line $y = x$

Soln



$y^2 = 4x$; $y = x$

$\therefore x^2 - 4x = 0$; $x(x-4) = 0$, $x = 0, 4$

$$\therefore A = \int_0^4 y\,dx = \int_0^4 2\sqrt{x}\,dx$$

$$= \frac{2\cdot 2}{3}\left[x^{3/2}\right]_0^4 = \frac{4}{3}\cdot 4^{3/2} = \frac{4}{3}\cdot 2^{2\cdot 3/2}$$

$$= \frac{4}{3}\cdot 2^3 = \frac{32}{3} \text{ sq unit}$$

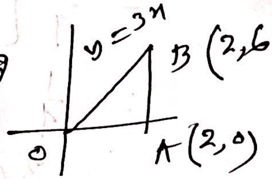
Second method

$$\cancel{A = \int x\,dy = \int_0^4 \frac{y^2}{4}\,dy}$$

$$\cancel{= \frac{1}{4}\cdot\frac{1}{3}\left[y^3\right]_0^4 =}$$

Ex-3. Find the area of the triangle bounded by the line $y = 3x$, the x-axis and the ordinate $x = 2$

Verify your result by finding the area as half of the product of the base and altitude



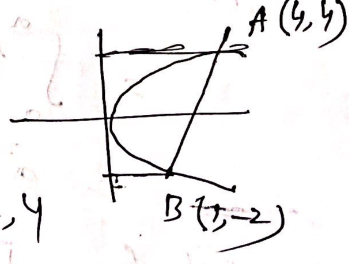
Soln

Area $A = \int_0^2 y\,dx = 6$ sq unit

Verification $A = \frac{1}{2} \cdot 2 \cdot 6 = 6$ sq un

Ex-4 Area bounded by 2 curves

Show that the area between the parabola $y^2 = 4x$ and the st. line $y = 2x - 4$ is 9 sq unit



Soln

Solving $y^2 = 4x$ and $y = 2x - 4$ we get

$x = 1, 4$ and $y = -2, 4$

$\therefore$ Required area

$$= \int_{-2}^{4} \left(\frac{y+4}{2} - \frac{y^2}{4} \right) dy$$

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p-4

$$A = \frac{1}{4}\int_{-2}^{4}(2y+8-y^2)\,dy$$

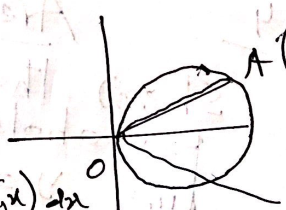
$$= \frac{1}{4}\left[2\frac{y^2}{2}+8y-\frac{y^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{4}\left[16+32-\frac{64}{3}-\left(4-16+\frac{8}{3}\right)\right]$$

$$= \frac{36}{4} = 9 \text{ sq units proved}$$

Ex-5 Find the area in the first quadrant enclosed by $y^2 = 2ax - x^2$ and $y^2 = ax$

Soln



Area $A = \int_0^a (y_1 - y_2)\,dx$

$$= \int_0^a \left(\sqrt{2ax-x^2} - \sqrt{ax}\right)dx$$

$$= a^2\left(\frac{\pi}{4} - \frac{2}{3}\right) \text{ sq unit.}$$

Ex-6 Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

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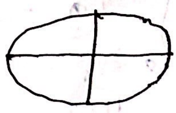
Soln



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\therefore\ y = b\sqrt{1-\frac{x^2}{a^2}}$$

$$\therefore\ \text{Total area} = \frac{4b}{a}\int_0^a (a^2-x^2)\,dx$$

put $x = a\sin\theta$

$dx = a\cos\theta\,d\theta$

$$\therefore\ A = \frac{4b}{\cancel{a}}\int_0^{\pi/2} \cancel{a}\cos\theta\, a\cos\theta\, d\theta$$

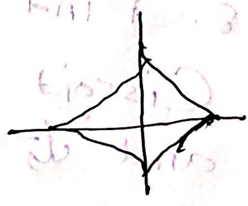
$$= 4ab\int_0^{\pi/2}\cos^2\theta\, d\theta = 4ab\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= 4ab \text{ sq units}$$

Ex-6

Find the area of the ~~hypocycloid~~ asteroid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$$



Soln $y^{\frac{2}{3}} = a^{\frac{2}{3}} - x^{\frac{2}{3}}$

$$y = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{3/2}$$

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$\therefore A = 4\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right) dx.$ Put $x = a\cos^3\theta$

$y = a\sin^3\theta$

$dx = -3a\cos^2\theta\sin\theta\, d\theta$

$= -4\int_0^{\pi/2} \left(a^{\frac{2}{3}} - a^{\frac{2}{3}}\cos^2\theta\right) \times 3a\cos^2\theta\sin\theta\, d\theta$

$= -12a^2\int_0^{\pi/2} \sin^4\theta\cos^2\theta\, d\theta$

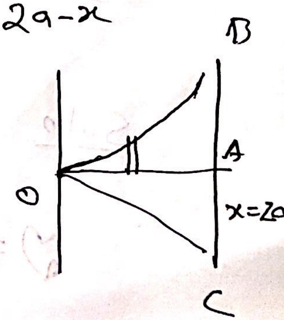
$= -12a^2 \cdot \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{6}\cdot\frac{\pi}{2}$

$= -\frac{3}{8}\pi a^2$ sq unit $= \frac{3}{8}\pi a^2$ sq unit

neglecting $-$ sign

Ex-7 Find the area of the circle $x^2+y^2=a^2$ Ans: πa^2 sq unit

Ex-8. Find the area between the Cissoid of Diocles $y^2=\frac{x^3}{2a-x}$ and its asymptote.



Soln

The asymptote is $x=2a$

$\therefore$ Required area

$A = 2\int_0^{2a} y\, dx$

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$\therefore A = 2\int_0^{2a} \frac{x^{3/2}}{\sqrt{2a-x}}\, dx$; Put $x = 2a\sin^2\theta$

$\therefore dx = 4a\sin\theta\cos\theta\, d\theta$

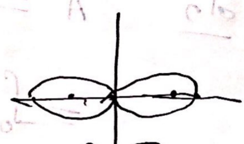
$= 2\int_0^{\pi/2} \frac{(2a\sin^2\theta)^{3/2} \cdot 4a\sin\theta\, \cancel{\cos\theta}\, d\theta}{\sqrt{2a}\, \cancel{\cos\theta}}$

$= 16a^2 \int_0^{\pi/2} \sin^4\theta\, d\theta = 16a^2 \cdot \frac{1.3}{2.4}\cdot\frac{\pi}{2}$

$= 3\pi a^2$ sq unit

Ex. 9. Find the area of the loop of the curve $a^2y^2 = x^2(a^2-x^2)$

Soln



There are two loops.

The curve is Leminscate of Bernoulli

Soln

$y = \frac{x}{a}\cdot\sqrt{a^2-x^2}$

$\therefore A = 2\int_0^a \frac{x}{a}\sqrt{a^2-x^2}\, dx$; Put $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

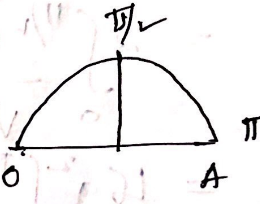
$= \frac{2}{a}\int_0^{\pi/2} a\sin\theta \cdot a\cos\theta \cdot a\cos\theta\, d\theta$

$= 2a^2 \int_0^{\pi/2} \sin\theta\cos^2\theta\, d\theta$

$= 2a^2 \frac{\Gamma\frac{1+1}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma\frac{1+2+2}{2}} = \frac{2a^2}{3}$ sq unit

Ex. 10

Find the area bounded by x-axis and one arc of sine curve. $y = \sin x$



Soln

$$\cancel{\text{Area } A = \int_0^{\pi/2} y\,dx + \int^{\pi} y\,dx}$$

$$\cancel{= \frac{1}{2}\left[y^2\right]_0^{\pi/2} + \frac{1}{2}\left[y^2\right]_{\pi/2}^{\pi}}$$

Soln $A = \int_0^{\pi/2} y\,dx + \int_{\pi/2}^{\pi} y\,dx$

$$= \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^{\pi} \sin x\,dx$$

$$= -\left[\cos x\right]_0^{\pi/2} - \left[\cos x\right]_{\pi/2}^{\pi} = 2 \text{ units}$$