Differential Equations

Differential equation

Definition. An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a *differential equation*.

$$dy = (x + \sin x)\, dx,$$

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t,$$

$$y = \sqrt{x}\,\frac{dy}{dx} + \frac{k}{dy/dx},$$

$$k\,(d^2y/dx^2) = \{1 + (dy/dx)^2\}^{3/2}$$

$$\partial^2 v/\partial t^2 = k\,(\partial^3 v/\partial x^3)^2$$

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0$$

Ordinary differential equation

Definition. A differential equation involving derivatives with respect to a single independent variable is called an *ordinary differential equation*.

Partial differential equation

Definition. A differential equation involving partial derivatives with respect to more than one independent variables is called a *partial differential equation*.

Order of a differential equation

Definition. The order of the highest order derivative involved in a differential equation is called the *order of the differential equation.*

Degree of a differential equation

Definition. The *degree of a differential equation* is the degree of the highest derivative which occurs in it, after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned. Note that the above definition of degree does not require variables x, t, u etc. to be free from radicals and fractions.

Linear and non-linear differential equations

Definition. A differential equation is called *linear* if (*i*) every dependent variable and every derivative involved occurs in the first degree only, and (*ii*) no products of dependent variables and/or derivatives occur. A differential equation which is not linear is called a *non-linear differential equation.*

Example

linear.

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0,$$

$$\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = xe^x.$$

nonlinear

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0,$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0.$$

$$\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0.$$

Formation of differential equations

Step I. Write the equation of the given family of curves.

Step II. Differentiate the equation of step I, n times so as to get n additional equations containing the n arbitrary constants and derivatives.

Step III. Eliminate n arbitrary constants from the $(n + 1)$ equations obtained in steps I and II. Thus, we obtain the required differential equation involving a derivative of nth order.

Ex. 1. *Find the differential equation of the family of curves* $y = e^{mx}$, *where m is an arbitrary constant.*

Ex. 2. (*a*) *Find the differential equation of all straight lines passing through the origin.*

(*b*) *Find the differential equation of all the straight lines in the xy-plane.*

Sol. (*a*) Equation of any straight line passing through the origin is

$$y = mx, \; m \text{ being arbitrary constant.} \qquad \ldots (1)$$

Differentiating (1) w.r.t. 'x', $\quad dy/dx = m. \qquad \ldots (2)$

Eliminating m from (1) and (2), we get $\quad y = x\,(dy/dx).$

(*b*) We know that equation of any straight line in the *xy*-plane is given by

$$y = mx + c, \; m \text{ and } c \text{ being arbitrary constants.} \qquad \ldots (1)$$

Differentiating (1) w.r.t. 'x', we get $\quad dy/dx = m. \qquad \ldots (2)$

Differentiating (2) w.r.t. 'x', we get $\quad d^2y/dx^2 = 0, \qquad \ldots (3)$

Ex. 3. (*a*) *Obtain a differential equation satisfied by family of circles* $x^2 + y^2 = a^2$, *a being an arbitrary constant.*

(*b*) *Obtain a differential equation satisfied by the family of concentric circles.*

Sol. (*a*) Given $x^2 + y^2 = a^2$. ... (1)

Differentiating (1) w.r.t. 'x', we get $2x + 2y\,(dy/dx) = 0$ or $x + y\,(dy/dx) = 0$,

which is the required differential equation.

(*b*) Let the centre of the given family of concentric circles be (0, 0). Then we know that the equation of the family of concentric circles is given by $x^2 + y^2 = a^2$, a being arbitrary constant.

Now proceed as in part (*a*). **Ans.** $x + y\,(dy/dx) = 0$.

Ex. 6. *Find the differential equation which has* $y = a \cos (mx + b)$ *for its integral, a and b being arbitrary constants and m being a fixed constant.*

Sol. Given that $y = a \cos (mx + b)$. ... (1)

Differentiating (1) w.r.t. 'x', we get $dy/dx = -\,am \sin (mx + b)$. ... (2)

Differentiating (2) w.r.t. 'x', we get $d^2y/dx^2 = -\,am^2 \cos (mx + b)$. ... (3)

or $d^2y/dx^2 = -\,m^2 y$, using (1)

Thus, the required differential equation is $d^2y/dx^2 + m^2 y = 0$.

Ex. 11. *Find the differential equation of all circles of radius a.*

Sol. The equation of all circles of radius a is given by

$$(x-h)^2+(y-k)^2=a^2, \qquad \dots (1)$$

where h and k, are to be taken as arbitrary constants.

Diff. (1) w.r.t. 'x', we get $\quad (x-h)+(y-k)\,y'=0. \qquad \dots (2)$

Diff. (2), $\quad 1+(y')^2+(y-k)\,y''=0 \quad$ or $\quad y-k=-\{1+(y')^2\}/y''. \qquad \dots (3)$

Putting this value of $y-k$ in (2), we get

$$x-h=-(y-k)\,y'=\{1+(y')^2\}\times(y'/y''). \qquad \dots (4)$$

Using (3) and (4), (1) gives the required equation as

$$\frac{\{1+(y')^2\}^2\,(y')^2}{(y'')^2}+\frac{\{1+(y')^2\}^2}{(y'')^2}=a^2 \qquad \text{or} \qquad \{1+(y')^2\}^3=a^2\,(y'')^2.$$

1. Form the differential equations for the following:

(*a*) $y = Ae^{2x} + Be^{-2x}$, A, B being arbitrary constants.

(*b*) $y = k \sin^{-1} x$, k parameter

(*c*) $y = \alpha x + \beta x^2$, α, β parameters

(*d*) $y = A \cos nt + B \sin nt$, ($A$, B parameters)

(*e*) $xy = ae^x + be^{-x}$, (a, b parameters)

2. Find the differential equation of the family of curves $y = Ae^{3x} + Be^{5x}$; for different values of A and B.

Ans. $y'' - 8y' + 15y = 0$

3. Find the differential equation of all circles passing through origin and having their centres on the x-axis.

Ans. $2xy' = y^2 - x^2$

4. Show that $v = B + A/r$ is a solution of $(d^2v/dr^2) + (2/r) \times (dv/dr) = 0$.

5. Find a differential equation with the following solution: $y = ae^x + be^{-x} + c \cos x + d \sin x$, where a, b, c and d are parameters.

Ans. $d^4y/dx^4 - y = 0$

6. Classify the following equations as linear and non-linear equations and write down their orders

(*a*) $\dfrac{a^3y}{dx^3} + \dfrac{d^2y}{dx^2} \cdot \dfrac{dy}{dx} + y = x.$ (*b*) $x \dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = e^x.$ (*c*) $\dfrac{dy}{dx} + y^2 = x^2.$

Ans. (*a*) Non-linear; 3 (*b*) Linear, 4 (*c*) Non-linear, 1

7. Write down the order and degree of $x^2 (d^2y/dx^2)^3 + y (dy/dx)^4 + y^4 = 0$. How many constants does the general solution of the differential equation must contain. **Ans.** 2, 3, 2

8. Find the differential equation of the family of parabolas $y^2 = 4ax$. **Ans.** $y = 2x (dy/dx)$

9. Show that the differential equation of the family of circles of fixed radius r with centre on y-axis is $(x^2 - r^2) (dy/dx)^2 + x^2 = 0$.

10. Find the differential equation of all

(*a*) parabolas of latusrectum $4a$ and axis parallel to y-axis.

(*b*) tangent lines to the parabola $y = x^2$.

(*c*) ellipses centered at the origin.

(*d*) circles through the origin.

(*e*) circles tangent to y-axis.

(*f*) parabolas with axis parallel to the axis of y.

(*g*) parabolas with foci at the origin and axis along x-axis.

(*h*) all conics whose axes coincide with axes of co-ordinates.

Ans. (*a*) $2ay_2 - 1 = 0$ (*b*) $4 (y - xy_1) + (y_1)^2 = 0$

(*c*) $xyy_2 + x (y_1)^2 - yy_1 = 0$ (*d*) $(x^2 + y^2) y_2 = 2 (xy_1 - y) (1 + y_1^2)$