Integration by successive reduction

1Exam: Find the reduction formula for $I_n = \int x^n e^{ax} dx$ hence find $\int x^3 e^{ax} dx$

Soln. $I_n = \int x^n e^{ax} dx$ $= x^n \frac{e^{ax}}{a} - \int n x^{n-1} \frac{e^{ax}}{a} dx$

$= x^n \frac{e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$ $\quad = x^n \frac{e^{ax}}{a} - \frac{n}{a} I_{n-1}$

$$I_n = x^n \frac{e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

2nd part

$I_3 = \int x^3 e^{ax} dx = x^3 \frac{e^{ax}}{a} - \frac{3}{a} I_2$

$= x^3 \frac{e^{ax}}{a} - \frac{3}{a}\left(x^2 \frac{e^{ax}}{a} - \frac{2}{a} I_1\right) = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^2} I_1$

$= \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^2}\left(x \frac{e^{ax}}{a} - \frac{1}{a} I_0\right)$

$= \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^3} x e^{ax} - \frac{6}{a^3} \int e^{ax} dx$

$= \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6x e^{ax}}{a^3} - \frac{6 e^{ax}}{a^4} + c$

2Example: Find the reduction formula for $I_n = \int tan^n x \, dx$ hence find $\int tan^6 x \, dx$

Soln.

$I_n = \int tan^n x \, dx = \int tan^{n-2} x \, tan^2 x \, dx$

$= \int tan^{n-2} x \, (sec^2 x - 1) dx$

$=\int tan^{n-2} x \, sec^2 x \, dx - \int tan^{n-2} x \, dx$

$=\int tan^{n-2} x \, d(tanx) = \frac{tan^{n-1} x}{n-1} - I_{n-2}$

$I_n = \frac{tan^{n-1} x}{n-1} - I_{n-2}$

2nd

$$I_6 = \int tan^6 x \, dx = \frac{tan^5 x}{5} - I_4$$

$= \frac{tan^5 x}{5} - \left[\frac{tan^3 x}{3} - I_2\right] = \frac{tan^5 x}{5} - \frac{tan^3 x}{3} + tan\, x - I_0$

$= \frac{tan^5 x}{5} - \frac{tan^3 x}{3} + tan\, x - x + c$

3Example: Find the reduction formula for $I_n = \int sin^n x \, dx$ hence find $\int sin^4 x \, dx$

Soln:

$I_n = \int sin^n x \, dx \quad = \int sin^{n-1} x \, sin\, x \, dx$

$=sin^{n-1} x \,(-\,cos\, x) - \int (n-1)\, sin^{n-2} x \, cos\, x \,(-\,cos\, x) dx$

$= -\,sin^{n-1} x \, cos\, x + (n-1) \int sin^{n-2} x \,(1 - sin^2 x) dx$

$= -\,sin^{n-1} x \, cos\, x + (n-1) \int sin^{n-2} x \, dx - (n-1) \int sin^n x \, dx$

$= -\,sin^{n-1} x \, cos\, x + (n-1) I_{n-2} - (n-1) I_n$

$=> I_n + (n-1) I_n = -\,sin^{n-1} x \, cos\, x + (n-1) I_{n-2}$

$$=> I_n = -\frac{sin^{n-1} x \, cos\, x}{n} + \frac{n-1}{n} I_{n-2}$$

2nd part

$$I_4 = -\frac{sin^3 x \, cos\, x}{4} + \frac{3}{4} I_2 = -\frac{sin^3 x \, cos\, x}{4} + \frac{3}{4}\int sin^2 x \, dx$$

$$= -\frac{sin^3 x \, cos\, x}{4} + \frac{3}{8}\int (1 - cos 2x)\, dx$$

$$= -\frac{sin^3 x \, cos\, x}{4} + \frac{3x}{8} - \frac{3cos2x}{16} + c$$

4Example: Find the reduction formula for $I_n = \int e^{ax} cos^n x \, dx$

$$I_n = \int e^{ax} \cos^n x dx$$

$$= \frac{\cos^n x e^{ax}}{a} + \frac{n}{a}\int e^{ax} \cos^n x \sin x dx$$

$$= \frac{\cos^n x e^{ax}}{a} + \frac{n}{a}$$

$$\left[\cos^{n-1} x \sin x \frac{e^{ax}}{a} - \int [(n-1)\cos^{n-2} x(-\sin x)\sin x + \cos^{n-1} x \cos x]\frac{e^{ax}}{a} dx\right]$$

$$= \frac{\cos^n x e^{ax}}{a} + \frac{n \cos^{n-1} x \sin x e^{ax}}{a^2} - \frac{n}{a^2}\int \left[e^{ax}[(n-1)\cos^{n-2} x[\cos^2 x - 1] + \cos^n x\right]dx$$

$$\frac{e^{ax}\cos^{n-1} x\{a\cos x + n\sin x\}}{a^2} - \frac{n}{a^2}\left[n\int e^{ax}\cos^n x dx - (n-1)\int e^{ax}\cos^{n-2} x dx\right]$$

$$\left(1 + \frac{n^2}{a^2}\right) I_n = \frac{e^{ax}\cos^{n-1} x\{a\cos x + n\sin x\}}{a^2} + \frac{n(n-1)}{a^2} I_{n-2}$$

$$I_n = \frac{e^{ax}\cos^{n-1} x\{a\cos x + n\sin x\}}{n^2 + a^2} + \frac{n(n-1)}{n^2 + a^2} I_{n-2}$$

5Example: Find the reduction formula for $I_n = \int (a^2 + x^2)^n \, dx$

Soln:

$$I_n = \int (a^2 + x^2)^n \, dx = x(a^2 + x^2)^n - n \int (a^2 + x^2)^{n-1} \, 2x^2 dx$$

$$= x(a^2 + x^2)^n - 2n \int (a^2 + x^2)^{n-1} \, (a^2 + x^2 - a^2) dx$$

$$= x(a^2 + x^2)^n - 2n \int (a^2 + x^2)^n \, dx + 2na^2 \int (a^2 + x^2)^{n-1} \, dx$$

$$= x(a^2 + x^2)^n - 2nI_n + 2na^2 I_{n-1}$$

$$=> (2n+1)I_n = x(a^2 + x^2)^n + 2na^2 I_{n-1}$$

$$\therefore \quad I_n = \frac{x\left(a^2+x^2\right)^n}{2n+1} x + \frac{2na^2}{2n+1} I_{n-1}$$

6Example: Find the reduction formula for $I_{m,n} = \int \frac{dx}{x^m (a+bx)^n}$

and hence evaluate $\int \frac{dx}{x^4 (a+bx)^3}$

Soln:

$$I_{m,n} = \int \frac{dx}{x^m (a+bx)^n}$$

$$= \frac{1}{-(m-1)x^{m-1}(a+bx)^n} - \int \frac{-nbdx}{-(m-1)x^{m-1}(a+bx)^{n+1}}$$

$$= -\frac{1}{(m-1)x^{m-1}(a+bx)^n} - \frac{n}{(m-1)} \int \frac{bx \, dx}{x^m (a+bx)^{n+1}}$$

$$= -\frac{1}{(m-1)x^{m-1}(a+bx)^n} - \frac{n}{(m-1)} \int \frac{a+bx-a}{x^m (a+bx)^{n+1}} \, dx$$

$$= -\frac{1}{(m-1)x^{m-1}(a+bx)^n} - \frac{n}{(m-1)}\int \frac{a+bx}{x^m(a+bx)^{n+1}}\,dx$$

$$+\frac{an}{(m-1)}\int \frac{1}{x^m(a+bx)^{n+1}}\,dx$$

$$= -\frac{1}{(m-1)x^{m-1}(a+bx)^n} - \frac{n}{(m-1)}\int \frac{1}{x^m(a+bx)^{n}}\,dx$$

$$+\frac{an}{(m-1)}\int \frac{1}{x^m(a+bx)^{n+1}}\,dx$$

$$= -\frac{1}{(m-1)x^{m-1}(a+bx)^n} - \frac{n}{(m-1)}I_{m,n} + \frac{an}{(m-1)}I_{m,n+1}$$

$$=> \frac{an}{(m-1)}I_{m,n+1} = \frac{1}{(m-1)x^{m-1}(a+bx)^n} + (1+\frac{n}{(m-1)})I_{m,n}$$

Changing n to n-1 0n both sides ,

$$\frac{a(n-1)}{(m-1)}I_{m,n} = \frac{1}{(m-1)x^{m-1}(a+bx)^{n-1}} + (1+\frac{n-1}{(m-1)})I_{m,n-1}$$

$$I_{m,n} = \frac{1}{a(n-1)x^{m-1}(a+bx)^{n-1}} + \frac{m+n-2}{a(n-1)}I_{m,n-1}$$

2nd part

$$I_{4,3} = \int \frac{dx}{x^4(a+bx)^3}$$

7Example: Find the reduction formula for $I_{m,n} = \int x^m (ln\,x)^n\,dx$

Soln:

$$I_{m,n} = \int x^m (ln\,x)^n\,dx = (ln\,x)^n \frac{x^{m+1}}{m+1} - \int n(ln\,x)^{n-1}\frac{1}{x}\frac{x^{m+1}}{m+1}dx$$

$$= (ln\,x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1}\int x^m (ln\,x)^{n-1}\,dx$$

$$= (ln\,x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}$$

$$I_{m,n} = (ln\,x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}$$

8Example: Find the reduction formula for

$I_{m,n} = \int cos^m x \, cos\, n\, x\, dx$ hence find $\int cos^3 x\, cos\, 3\, x\, dx$

Soln $I_{m,n} = \int cos^m x\, cos\, n\, x\, dx$

$$= cos^m x \frac{sin\, nx}{n} - \int m\, cos^{m-1} x\, (-\, sin\, x) \frac{sin\, nx}{n} dx$$

$$= cos^m x \frac{sin\, n\, x}{n} + \frac{m}{n} \int cos^{m-1} x\;\; sinx\; sinnx\; dx$$

$$\begin{bmatrix} \cos(n-1)\,x = cosnxcosx + sinnxsinx \\ sinnxsinx = \cos(n-1)\,x - cosnxcosx \end{bmatrix}$$

$$= cos^m x \frac{sin\, n\, x}{n} + \frac{m}{n} \int cos^{m-1} x\, [\cos(n-1)\,x - cosnx\; cosx]\, dx$$

$$= cos^m x \frac{sin\, nx}{n} + \frac{m}{n} \int cos^{m-1} x \cos(\text{n-1})\text{x} = \frac{m}{n} \int cos^m x \,\text{cosnx}\, dx$$

$$= cos^m x \frac{sin\, n\, x}{n} + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n}$$

$$=> \left(1 + \frac{m}{n}\right) I_{m,n} = \frac{cos^m x\, sin\, nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$=> \left(\frac{m+n}{n}\right) I_{m,n} = \frac{cos^m x\, sin\, n\, x}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$=> I_{m,n} = \frac{cos^m x \sin n\, x}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

2nd part

$$I_{3,3} = \frac{cos^3 x \sin 3\, x}{6} + \frac{3}{6} I_{2,2} = \frac{cos^3 x \sin 3\, x}{6} + \frac{3}{6}\left[\frac{cos^2 x \sin 2\, x}{4} + \frac{2}{4} I_{1,1}\right]$$

$$= \frac{\cos^3 x \sin 3x}{6} + \frac{\cos^2 x \sin 2x}{8} + \frac{1}{4} I_{1,1}$$

$$= \frac{cos^3 x \sin 3\, x}{6} + \frac{cos^2 x \sin 2\, x}{8} + \frac{1}{4}\left[\frac{cos\, x \sin x}{2} + \frac{1}{2} I_{0,0}\right]$$

$$= \frac{cos^3 x \sin 3\, x}{6} + \frac{cos^2 x \sin 2\, x}{8} + \frac{cos\, x \sin x}{8} + \frac{1}{8}\int dx$$

$$= \frac{cos^3 x \sin 3\, x}{6} + \frac{cos^2 x \sin 2\, x}{8} + \frac{cos\, x \sin x}{8} + \frac{x}{8} + c$$

9Example: Find the reduction formula for

$I_{m,n} = \int sin^m x\, cos^n x\ dx$, m, n positive integer is

$$I_{m,n} = \frac{\sin^{m+1} x\ \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

or

$$I_{m,n} = -\frac{\sin^{m-1} x\ \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$

hence find $\int sin^2 x\, cos^3 x\, dx$

Soln

$$I_{m,n} = \int sin^m x\, cos^n x \; dx$$

$$= \int cos^{n-1} x\,(cosx\, sin^m x \;)\, dx$$

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} - (n-1)\int \cos^{n-2} x\,(-sinx)\frac{\sin^{m+1} x}{m+1} dx$$

$$= \frac{\sin^{m+1} x\ \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x\,(1-\cos^2 x)\sin^m x\, dx$$

$$= \frac{\sin^{m+1} x\ \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\, dx$$

$$-\frac{n-1}{m+1}\int \cos^n x \sin^m x\, dx$$

$$= \frac{\sin^{m+1} x\ \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$=> \left[1 + \frac{n-1}{m+1}\right] I_{m,n} = \frac{\sin^{m+1} x\ \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$=> \left[\frac{m+n}{m+1}\right] I_{m,n} = \frac{\sin^{m+1} x\ \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$=> I_{m,n} = \frac{\sin^{m+1} x\ \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

Similarly

$$I_{m,n} = \int sin^{m-1} x\,(sinx\, cos^n x \;)\, dx$$

$$= -\frac{\sin^{m-1} x\ \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$

2nd part

$$I_{2,3} = \int sin^2 x\, cos^5 x \; dx$$

$$= \frac{\sin^3 x \ \cos^4 x}{7} + \frac{4}{7} I_{2,3}$$

$$= \frac{\sin^3 x \ \cos^4 x}{7} + \frac{4}{7}\left[\frac{\sin^3 x \ \cos^2 x}{5} + \frac{2}{5} I_{2,1}\right]$$

$$= \frac{\sin^3 x \ \cos^4 x}{7} + \frac{4\sin^3 x \ \cos^2 x}{35} + \frac{8}{15} I_{2,1}$$

$$= \frac{\sin^3 x \ \cos^4 x}{7} + \frac{4\sin^3 x \ \cos^2 x}{35} + \frac{8}{35}\int sin^2 x \cos x \ dx$$

$$= \frac{\sin^3 x \ \cos^4 x}{7} + \frac{4\sin^3 x \ \cos^2 x}{35} + \frac{8}{35}\int sin^2 x \, \mathrm{d}(\sin x)$$

$$= \frac{\sin^3 x \ \cos^4 x}{7} + \frac{4\sin^3 x \ \cos^2 x}{35} + \frac{8}{105} sin^3 x + c$$