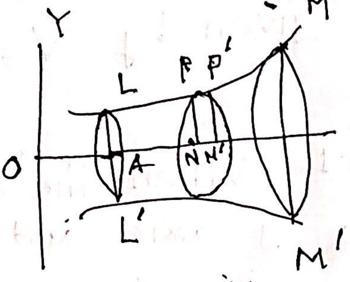
Volume $b-t$

Formation



Let a curve LM whose cartesian eqn is given by $y = f(x)$ be rotated about the x-axis say, so as to form a solid of revolution and let us consider the portion LL'MM' of this solid of revolution bounded by $x = x_1$ and $x = x_2$. We can imagine this solid to be divided into an infinite number of infinitely thin circular slices by planes perpendicular to the axis of revolution $\overrightarrow{OX}$. If $\overline{PN}$ and $\overline{P'N'}$ be two adjacent ordinates of the curve, where the coordinates of P and P' are (x, y) and $(x+\Delta x, y+\Delta y)$ respectively; the volume of the corresponding slice, which has its thickness Δx, is ultimately equal to $\pi y^2 \Delta x$.

Hence the total volume of the solid bounded by $x = x_1$ and $x = x_2$

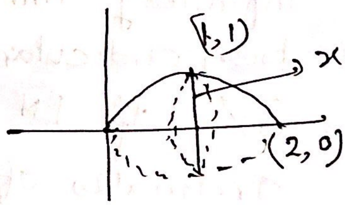
is given by $V = \underset{\Delta x \to 0}{Lt} \sum \pi y^2 \Delta x = \pi \int_{x_1}^{x_2} y^2 dx$

Cor I. When the axis of revolution is the y-axis and we consider the solid bounded by $y = y_1$ & $y = y_2$ respectively

$$V = \pi \int_{y_1}^{y_2} x^2 dy$$

Ex. 1 The area bounded by the x-axis and $y = 2x - x^2$ is rotated about x-axis. Find the volume of the solid of revolution

Soln



The curve intersects at $x = 0$ and $x = 2$

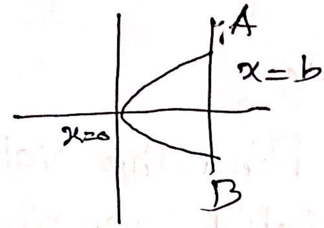
$$\therefore V = \pi \int_0^2 y^2 dx = \pi \int_0^2 (2x - x^2) dx$$

$$= \pi \left[\frac{2x^2}{2} - \frac{x^3}{3} \right]_0^2 = \frac{16}{15} \pi \text{ cubic unit}$$

Ex. 2: Find the volume of the solid generated by revolving about x-axis the area formed by $y^2 = 4ax$ and part of x-axis between $x = 0$, $x = b$

p-3

Required Volume

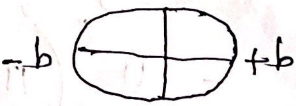


$$V = \pi \int_0^b y^2 dx$$

$$= \pi \int_a^b 4ax\, dx = \frac{4a\pi}{2}\left[x^2\right]_0^b = 2\pi a b^2 \text{ cu.unit}$$

Ex3 Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ about minor axis

Soln



$$\cancel{\pi \int_{-b}^{b} x^2 dy = 2\int_0^b x^2 dy}$$

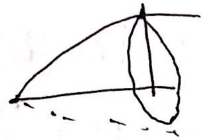
$$V = \pi \int_{-b}^{b} x^2 dy = 2\pi \int_0^b x^2 dy$$

$$= 2\pi \int_0^b \frac{a^2}{b^2}(b^2 - y^2)\, dy$$

$$= \frac{2\pi a^2}{b^2}\left[b^2 y - \frac{y^3}{3}\right]_0^b$$

$$= \frac{2\pi a^2}{b^2}\left(b^3 - \frac{b^3}{3}\right) = \frac{4}{3}\pi a^2 b \text{ cubic unit}$$

Ex-4



Find the volume of the solid generated by revolving about x-axis formed by the curve $y = \sin x$ and part of the x-axis between $x=0$; $x=\pi$

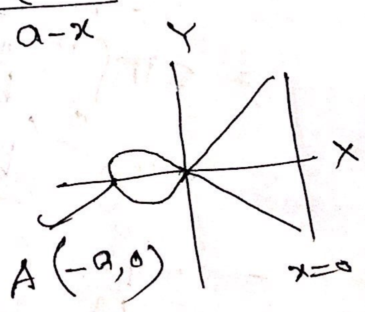
Soln

$$V = \pi \int_0^{\pi} y^2 dx = \pi \int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) dx = \frac{\pi}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi}$$

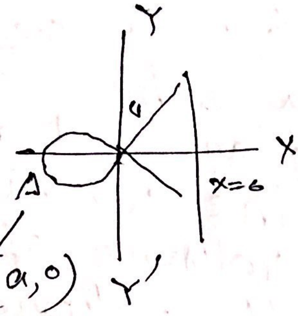
$$= \frac{\pi}{2} (\pi) = \frac{\pi^2}{2} \text{ cubic unit}$$

Ex.5. Find the volume of the solid formed by the revolution about the x-axis of the loop of the curve $y^2 = \frac{x^2(a+x)}{a-x}$



Soln

For the upper half of the loop x varies from $-a$ to 0.



$\therefore$ Required volume

$$V = \pi\int_{-a}^{0} \pi y^2\,dx = \pi\int_{-a}^{0} \frac{x^2(a+x)}{a-x}\,dx$$

$$= \pi\int_{-a}^{0} \frac{ax^2+x^3}{a-x}\,dx$$

$$= \pi\int_{-a}^{0} \left(-x^2-2ax-2a^2+\frac{2a^3}{a-x}\right)dx$$

Dividing numerator by denominate

$$= \pi\left[-\frac{x^3}{3}-ax^2-2a^2x-2a^3\log(a-x)\right]_{-a}^{0}$$

$$= \pi\left[-2a^3\log a-\left(\frac{a^3}{3}-a^3+2a^3-2a^3\log 2a\right)\right]$$

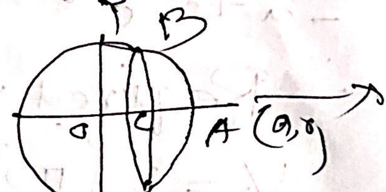
$$= \pi\left[-\frac{4a^3}{3}+2a^3(\log 2a-\log a)\right]$$

$$= \cancel{2\pi a^2} = 2\pi a^3\left(\log 2-\frac{2}{3}\right)$$

cubic unit

P-6

Ex-6. A segment is cut off from a sphere of radius a by a plane at a distance $\frac{a}{2}$ from the centre. Show that the volume of the segment $\frac{5}{32}$ of the volume of the sphere



Soln

$OC = \frac{a}{2}$

The area ABCA is rotated x-axis and for BA the limits of x are $\frac{a}{2}$ to a

$$V = \pi\int_{a/2}^{a} y^2\,dx = \pi\int_{a/2}^{a} (a^2 - x^2)\,dx$$

$x^2 + y^2 = a^2$

$y^2 = a^2 - x^2$

$$= \pi\left[a^2 x - \frac{x^3}{3}\right]_{a/2}^{a}$$

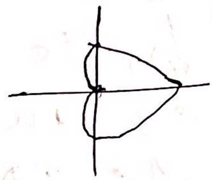
$$= \frac{5}{32} \cdot \frac{4\pi a^3}{3}$$

$= \frac{5}{32} \times$ volume of sphere of radius a

Proved

Ex. 7 Find the volume of the solid generated by revolving the cardioide $r = a(1-\cos\theta)$ about the initial line

Soln



Required volume

$$V = \pi\int y^2\,dx \qquad x = r\cos\theta,\quad y = r\sin\theta$$

$$= \pi\int_{\pi}^{0} r^2\sin^2\theta\, d(r\cos\theta)$$

$$= \pi a^2\int_{\pi}^{0} \cancel{1-\cos^2}\,(1-\cos\theta)^2\sin^2\theta\, d[r\cos\theta]$$

$$\cancel{= \pi a^2\int (1-\cos\theta)^2(1-\cos^2\theta)}$$

$$= \pi a^2\int_{\pi}^{0} (1-\cos\theta)^2\sin^2\theta\, d[(1-\cos\theta)\cos\theta]\,d\theta$$

$$= \pi a^2\int_{\pi}^{0} (1-\cos\theta)^2(1-\cos^2\theta)\, d[(1-\cos\theta)\cos\theta]\,d\theta$$

$$\cos\theta = z$$

$$= \pi a^3\int_{-1}^{1} (1-z)^2(1-z^2)(1-2z)\,dz$$

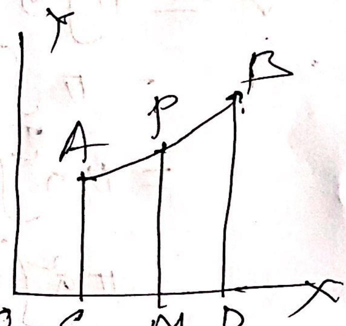
$$= \frac{8}{3}\pi a^3 \text{ cubic unit}$$

~~Q~~ p-8

Revolution about any line

The volume of the solid by the revolution about any line CD of the area bounded by the curve AB, the line CD and the perpendicular AC, BD on the axis is

$$\int_{OC}^{OD} \pi (PM)^2 \, d(OM)$$

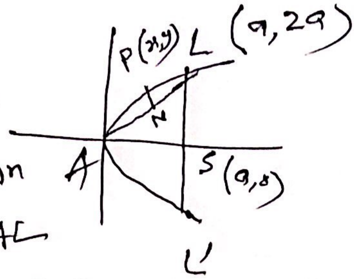


Ex-8

A chord is drawn joining the vertex of the parabola $y^2 = 4ax$ to one end of the latus rectum. If the intersected part is rotated around the chord find the volume of the solid generated.

P-9

Soln



P (x, y) is a point on AL, PN is drawn perpendicular to AL

$\therefore$ Equ^n of AL is $y = \frac{2a}{a}x \Rightarrow y - 2x = 0$

$$PN = \frac{y-2x}{\sqrt{5}} = \frac{2\sqrt{ax} - 2x}{\sqrt{5}} = \frac{2}{\sqrt{5}}\left(\sqrt{ax} - x\right)$$

$$AN^2 = AP^2 - PN^2 = x^2 + y^2 - \frac{y^2 - 4xy}{5}$$

$$= \frac{(x+2y)^2}{5}$$

$$\therefore AN = \frac{x+2y}{\sqrt{5}} = \frac{x}{\sqrt{5}} + \frac{2}{\sqrt{5}} \cdot 2\sqrt{ax}.$$

$$\cancel{d(AN) = \frac{1}{\sqrt{5}} + \frac{4\sqrt{a}}{\sqrt{5}} \cdot \frac{1}{2\sqrt{x}}}$$

$$= \frac{1}{\sqrt{5}} + \frac{2\sqrt{a}}{\sqrt{5x}}$$

$$V = \int_0^{AC} \pi PN^2 \, d(AN)$$

$$= \pi \int_0^a \frac{4}{5}\left(\sqrt{ax} - x\right)^2 \cdot \frac{1}{\sqrt{5}}\left(1 + 2\sqrt{\frac{a}{x}}\right) dx$$

$$= \frac{4\pi}{5\sqrt{5}} \int_0^a \left(ax + x^2 - 2\sqrt{a}\,x^{3/2}\right)\left(1 + 2\sqrt{\frac{a}{x}}\right) dx$$

$$= \frac{2\pi a^3}{15\sqrt{5}} \text{ cubic units}$$