Improper Integral

Def: If either a**ny** one of (or both) the limit of the integral is (or are) infinite or $f(x)$ is infinitely discontinuous or indeterminate at a (lower limit) or b (upper limit) or both or at any one or more points between a and b , then the integral is called an improper integral or infinity integral. The improper integral can be classified into three kinds

1) **Infinite range** : The limit of the integral either lower or upper or both limits are infinite. This type of integrals can be written as

   (i) $\int_a^\infty f(x)dx = \underset{\in\to\infty}{\text{Lt}} \int_a^\in f(x)dx$ provided $f(x)$ is integrable in (a,b) and this limit exists.

   (ii) $\int_{-\infty}^b f(x)dx = \underset{\in\to-\infty}{\text{Lt}} \int_\in^b f(x)dx$ provided $f(x)$ is integrable in (a,b) and this limit exists.

   (iii) $\int_{-\infty}^{+\infty} f(x)dx =$
   $\underset{\in\to-\infty}{\text{Lt}} \int_\epsilon^c f(x)dx + \underset{\in\to\infty}{\text{Lt}} \int_c^\epsilon f(x)dx, \quad -\infty < c < \infty$ and c arbitrary , provided $f(x)$ is integrable in (a,b) and this limit exists

$$\int_a^b f$$

Evaluate1 $\int_0^{\infty} \frac{xdx}{(1+x)(1+x^2)}$

Soln Here upper limit is infinity

$$\therefore \quad I = \int_0^{\infty} \frac{xdx}{(1+x)(1+x^2)} = \underset{\in\to\infty}{\text{Lt}} \int_0^{\in} \frac{xdx}{(1+x)(1+x^2)}$$

$$= \underset{\in\to\infty}{\text{Lt}} \ \frac{1}{2} \int_0^{\in} \left[\frac{x+1}{1+x^2} - \frac{1}{1+x}\right] dx$$

$$= \underset{\in\to\infty}{\text{Lt}} \ \frac{1}{2} \int_0^{\in} \left[\frac{x}{1+x^2} + \frac{1}{1+x^2} - \frac{1}{1+x}\right] dx$$

$$= \frac{1}{2} \underset{\in\to\infty}{\text{Lt}} \left[\frac{1}{2} \ln(1+x^2) + \tan^{-1} x - \ln(1+x)\right]_0^{\in}$$

$$= \frac{1}{2} \underset{\in\to\infty}{\text{Lt}} \left[\quad ln \frac{\left(\sqrt{1+\epsilon^2}\right)}{(1+\epsilon)} + \tan^{-1} \epsilon \right] - 0$$

$$= \frac{1}{2} \underset{\in\to\infty}{\text{Lt}} \, ln \left[\frac{\sqrt{\frac{1}{\epsilon^2}+1}}{\frac{1}{\epsilon}+1}\right] + \frac{1}{2} \tan^{-1} \infty$$

$$= 0 + \frac{\pi}{4}$$

Evaluate2 $\int_{-\infty}^{-1} \frac{xdx}{(1+x)^2}$

Soln Here lower limit is infinity . So this

integral can be written as

$$\int_{-\infty}^{-1} \frac{xdx}{(1+x^2)^2} = \underset{\in\to-\infty}{\mathrm{Lt}} \int_{\in}^{-1} \frac{xdx}{(1+x^2)^2}$$

$$= \frac{1}{2} \underset{\in\to-\infty}{\mathrm{Lt}} \int_{\in}^{-1} \frac{d(1+x^2)}{(1+x^2)^2} = \frac{1}{2} \underset{\in\to-\infty}{\mathrm{Lt}} \left[-\frac{1}{1+x^2}\right]_{\in}^{-1}$$

$$= \frac{1}{2} \underset{\in\to-\infty}{\mathrm{Lt}} \left[-\frac{1}{2} + \frac{1}{1+\in^2}\right] = -\frac{1}{4}$$

Evaluate3 $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

Soln Here upper and lower limit both are infinity . So this

integral can be written as

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \underset{\in\to-\infty}{\mathrm{Lt}} \int_{\epsilon}^{c} \frac{dx}{x^2+2x+2} + \underset{\in\to\infty}{\mathrm{Lt}} \int_{c}^{\epsilon} \frac{dx}{x^2+2x+2}$$

$$= \underset{\in\to-\infty}{\mathrm{Lt}} \int_{\epsilon}^{c} \frac{d(x+1)}{(x+1)^2+1} + \underset{\in\to\infty}{\mathrm{Lt}} \int_{c}^{\epsilon} \frac{dx}{(x+1)^2+1},$$

$$= \underset{\in\to-\infty}{\mathrm{Lt}} [\tan^{-1}(x+1)]_{\in}^{c} + \underset{\in\to\infty}{\mathrm{Lt}} [\tan^{-1}(x+1)]_{c}^{\in}$$

$$= \tan^{-1}(c+1) - \underset{\in\to-\infty}{\mathrm{Lt}} \tan^{-1}(\epsilon+1)$$

$$+\underset{\in\to-\infty}{\mathrm{Lt}}\ \tan^{-1}(\epsilon+1)-\tan^{-1}(c+1)$$

$$= -\ \tan^{-1}(-\infty+1)\ +\ \tan^{-1}(\infty+1)$$

$$= \ \tan^{-1}(\infty-1)\ +\ \tan^{-1}(\infty+1)$$

$$=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

2) **Integrand Infinitely discontinuous at a point** :

$$\int_a^b f(x)dx$$

(i) If $f(x)$ is infinitely discontinuous at lower limit a,

i,e if $f(x)\to\infty$ as $x\to a$ then

$\int_a^b f(x)dx$ can be written as $\underset{\in\to 0}{\mathrm{Lt}} \int_{a+\in}^{b} f(x)dx$ provided $f(x)$ is integrable in (a,b) and this limit exists.

(ii) If $f(x)$ is infinitely discontinuous at upper limit b,

i,e if $f(x)\to\infty$ as $x\to b$ then

$\int_a^b f(x)dx$ can be written as $\underset{\in\to 0}{\mathrm{Lt}} \int_{a}^{b-\in} f(x)dx$ provided $f(x)$

is integrable in (a,b) and this limit exists.

(iii) If $f(x)$ is infinitely discontinuous at an internal point c, $((a < c < b)$ i,e if $f(x) \to \infty$ as $x \to c$

then $\int_a^b f(x)dx$ can be written as

$\underset{\in\to 0}{\text{Lt}} \int_a^{c-\in} f(x)dx + \underset{\partial\to 0}{\text{Lt}} \int_{c+\partial}^{b} f(x)dx$ provided

$f(x)$ is integrable in (a,b) and this limit exists

(iv) If $f(x)$ is infinitely discontinuous at both upper and lower limit a and b,

i,e if $f(x) \to \infty$ as $x \to a$ and $x \to b$ then

$\int_a^b f(x)dx$ can be written as $\int_a^c f(x)dx + \int_c^b f(x)dx,\ a < c < b$

and c arbitrary , where two integrals exists

Evaluate3 $\int_3^4 \frac{dx}{\sqrt{x-3}}$

Soln Here $f(x) = \frac{1}{\sqrt{x-3}} \to \infty$ as $x \to 3$ So this

integral can be written as

$$I = \underset{\in\to 0}{\text{Lt}} \int_{3+\in}^{4} \frac{dx}{\sqrt{x-3}} = \underset{\in\to 0}{\text{Lt}}\ 2\left[\sqrt{x-3}\right]_{3+\in}^{4}$$

$$=2\sqrt{4-3} - 2\underset{\in\to 0}{\text{Lt}} \sqrt{3+\in -3} = 2 - 0 = 2$$

Evaluate4 $\int_{-1}^{1} \sqrt{\frac{x+1}{1-x}}\, dx$

Soln Here $f(x) = \sqrt{\frac{x+1}{1-x}} \to \infty$ as $x \to 1$ So this integral can be written as

$$I = \underset{\in\to 0}{\mathrm{Lt}} \int_{-1}^{1-\in} \sqrt{\frac{x+1}{1-x}}\, dx = \underset{\in\to 0}{\mathrm{Lt}} \int_{-1}^{1-\in} \frac{1+x}{\sqrt{1-x^2}}\, dx$$

$$= \frac{1}{2} \underset{\in\to 0}{\mathrm{Lt}} \int_{-1}^{1-\in} \frac{2x}{\sqrt{1-x^2}}\, dx + \underset{\in\to 0}{\mathrm{Lt}} \int_{-1}^{1-\in} \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \underset{\in\to 0}{\mathrm{Lt}} \left[-\sqrt{1-x^2} + \sin^{-1} x\right]_{-1}^{1-\in}$$

$$= \underset{\in\to 0}{\mathrm{Lt}} \left[-\sqrt{1-(1-\in)^2} + \sin^{-1}(1-\in)\right] + 0 - \sin^{-1}(-1)$$

$$= \sin^{-1} 1 + \sin^{-1} 1 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Evaluate5 $\int_{3}^{4} \frac{dx}{x^2-2x}$

Soln Here $f(x) = \frac{1}{x^2-2x} \to \infty$ as $x \to 2$ So this integral can be written as

$$I = \underset{\in\to 0}{\mathrm{Lt}} \int_{3}^{2-\in} \frac{1}{x^2-2x}\, dx + \underset{\partial\to 0}{\mathrm{Lt}} \int_{2+\partial}^{4} \frac{1}{x^2-2x}\, dx$$

$$= \frac{1}{2} \underset{\in\to 0}{\mathrm{Lt}} \int_{3}^{2-\in} \left[\frac{1}{x-2} - \frac{1}{x}\right] dx + \frac{1}{2} \underset{\partial\to 0}{\mathrm{Lt}} \int_{2+\partial}^{4} \left[\frac{1}{x-2} - \frac{1}{x}\right] dx$$

$$= \frac{1}{2} \underset{\in\to 0}{\mathrm{Lt}} [\ln(x-2) - lnx]_3^{2-\in} + \frac{1}{2} \underset{\partial\to 0}{\mathrm{Lt}} [\ln(x-2) - lnx]_{2+\partial}^4$$

$$= \frac{1}{2} \underset{\in\to 0}{\mathrm{Lt}} \left[\ln\left(\frac{x-2}{x}\right)\right]_3^{2-\in} + \frac{1}{2} \underset{\partial\to 0}{\mathrm{Lt}} \left[\ln\left(\frac{x-2}{x}\right)\right]_{2+\partial}^4$$

$$= \frac{1}{2} \underset{\in\to 0}{\mathrm{Lt}} \left[\ln\left(1-\frac{2}{x}\right)\right]_3^{2-\in} + \frac{1}{2} \underset{\partial\to 0}{\mathrm{Lt}} \left[\ln\left(1-\frac{2}{x}\right)\right]_{2+\partial}^4$$

$$= \frac{1}{2} \underset{\in\to 0}{\mathrm{Lt}} \ln\left(1-\frac{2}{2-\in}\right) - \frac{1}{2} \ln\left(\frac{1}{3}\right) + \frac{1}{2}\ln\left(\frac{1}{2}\right) - \frac{1}{2} \underset{\partial\to 0}{\mathrm{Lt}} \ln\left(1-\frac{2}{2+\partial}\right)$$

$$= \frac{1}{2}\ln\left(\frac{3}{2}\right)$$

Convergence and divergence of improper integral

If the value of $\int_a^\infty f(x)dx = \underset{\in\to\infty}{\mathrm{Lt}} \int_a^\in f(x)dx$ exists that is its value is finite then the integral $\int_a^\infty f(x)dx$ is convergent , otherwise it is divergent.

Example: Examine convergence of $\int_0^\infty \frac{x}{(1+x)^3} dx$

Soln. $\int_0^\infty \frac{x}{(1+x)^3} dx = \underset{\in\to\infty}{\mathrm{Lt}} \int_0^\in \frac{xdx}{(1+x)^3} = \underset{\in\to\infty}{\mathrm{Lt}} \int_0^\in \frac{1+x-1}{(1+x)^3} dx$

$$= \underset{\in\to\infty}{\mathrm{Lt}} \int_0^\in \frac{1+x-1}{(1+x)^3} dx$$

$$= \underset{\in\to\infty}{\mathrm{Lt}} \int_0^\in \frac{1}{(1+x)^2} dx - \underset{\in\to\infty}{\mathrm{Lt}} \int_0^\in \frac{1}{(1+x)^3} dx$$

$$= \underset{\in\to\infty}{\mathrm{Lt}} \left[-\frac{1}{1+x} + \frac{1}{2}\frac{1}{(1+x)^2}\right]_0^\in$$

$= \underset{\in\to\infty}{\text{Lt}} \left[-\frac{1}{1+\in} + \frac{1}{2}\frac{1}{(1+\in)^2}\right] - \left[-1 + \frac{1}{2}\right] = \frac{1}{2}$

Since $\int_0^\infty \frac{x}{(1+x)^3}\,dx$ is finite , so $\int_0^\infty \frac{x}{(1+x)^3}\,dx$ is convergent.

Example: Examine convergence of $\int_2^\infty \frac{1}{\sqrt{x^2-1}}\,dx$

Soln. $\int_2^\infty \frac{1}{\sqrt{x^2-1}}\,dx = \underset{\in\to\infty}{\text{Lt}} \int_0^\in \frac{1}{\sqrt{x^2-1}}\,dx$

$= \underset{\in\to\infty}{\text{Lt}} \left[\ln\left(x + \sqrt{x^2-1}\right)\right]_2^\in$

$= \underset{\in\to\infty}{\text{Lt}} \left[\ln\left(\in + \sqrt{\in^2 - 1}\right] - ln2 = ln \in - ln2 = \infty - ln2 = \infty$

Since $\int_2^\infty \frac{1}{\sqrt{x^2-1}}\,dx$ is not exist , so $\int_2^\infty \frac{1}{\sqrt{x^2-1}}\,dx$ is divergent.