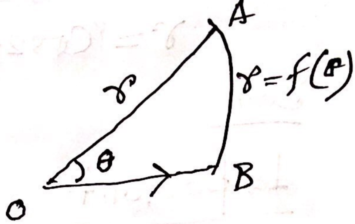
13/9/20 P-1

Area for polar curve



a) Equation $r = f(\theta)$

b) $OA = r$ is radius vector

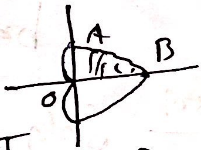
c) θ is vectorial angle

d) OB is initial Line

e) O is pole

Area for polar curve is defined by Area $= \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$

Ex.1 Find the area of the cardioide $r = a(1-\cos\theta)$



Soln

$$A = 2 \cdot \frac{1}{2}\int r^2 \, d\theta$$

$$= \int_{\theta_1}^{\theta_2} a^2 (1-\cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^{\pi} (1-\cos\theta)^2 \, d\theta = a^2 \int_0^{\pi} (1-\cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^{\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= a^2 \int_0^{\pi} \left[1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right] d\theta$$

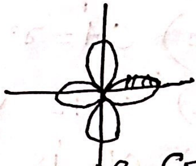
$$= a^2 \left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= a^2 \left[\frac{3\pi}{2} - 2\sin\pi + \frac{\pi}{2} + \frac{\sin 2\pi}{4}\right]$$

$$= \frac{3\pi a^2}{2} \text{ sq. unit Ans.}$$

Ex.2, Find the area of 4 loops of the curve

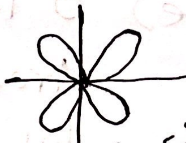
$r = \cos 2\theta$


$r = \cos 2\theta$

Soln

Important

If $r = \sin^n\theta$
or $r = \cos^n\theta$


$r = \sin 2\theta$

There will be $2n$ loops if n is even and n loops if n is odd.

If $n = 1$ the $r = \sin\theta$ or $r = \cos\theta$ will be a circle

Now Soln Here n is even $= 2$

$\therefore$ There will be for loops

Area of a loop $= 2 \cdot \frac{1}{2}\int r^2 d\theta$

$= \int_0^{\pi/4} r^2 d\theta = \int_0^{\pi/4} \cos^2 2\theta\, d\theta$

$= \frac{1}{2}\int_0^{\pi/4}(1+\cos 4\theta)\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/4}$

$= \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}$ sq unit

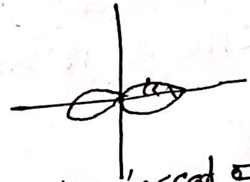
$\therefore$ Total area $= 4 \cdot \frac{\pi}{8} = \frac{\pi}{2}$ sq unit

Ans

P-3

Ex.3. Find the area of a loop of the curve $r^2 = a^2 \cos 2\theta$.

Soln.



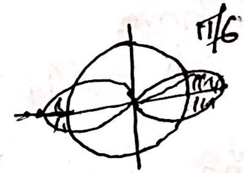
Leminscat of Bernoulli

$$A = 2 \cdot \frac{1}{2} \int r^2 \, d\theta$$

$$= \int_0^{\pi/4} a^2 \cos 2\theta \, d\theta$$

$$= \frac{a^2}{2} \left[\sin 2\theta \right]_0^{\pi/4}$$

$$= \frac{a^2}{2} \text{ sq. unit Ans.}$$



Ex.4

Find the area within the Leminscate $r^2 = 2a^2 \cos 2\theta$ and outside the circle $r = a$

$x = r\cos\theta$

$y = r\sin\theta$

Soln

$$2a^2 \cos 2\theta = a^2$$

$$\therefore \cos 2\theta = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\therefore \theta = \pi/6$$

$$A = 4 \cdot \frac{1}{2} \int_0^{\pi/6} (r_1^2 - r_2^2) \, d\theta$$

$$= 2 \int_0^{\pi/6} (2a^2 \cos 2\theta - a^2) \, d\theta$$

$$= 2a^2 \int_0^{\pi/6} (2\cos 2\theta - 1) \, d\theta$$

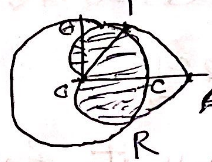
$$= 2a^2 \left[\frac{2 \sin 2\theta}{2} - \theta \right]_0^{\pi/6}$$

$$= 2a^2 \left(\sin\frac{\pi}{3} - \pi/6 \right) = 2a^2 \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) \text{ sq. unit}$$

Ex.5. Find the area common to the cardioide $r = a(1+\cos\theta)$ and the circle $r = \frac{3a}{2}$

Soln

At the common point P of the two curves we have



$$\frac{3\cancel{a}}{2} = \cancel{a}(1+\cos\theta) \therefore \cos\theta = \frac{1}{2} = \cos\frac{\pi}{3} \therefore \theta = \frac{\pi}{3}$$

The required area is easily seen to be

~~2[OCPQ~~ 2[Area OCP + Area PQO]

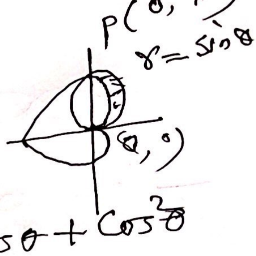
$$= 2\left[\frac{1}{2}\int_0^{\pi/3}\left(\frac{3a}{2}\right)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} a^2(1+\cos\theta)^2 d\theta\right]$$

$$= \frac{9a^2}{4}$$

$$= \frac{9a^2}{4}[\theta]_0^{\pi/3} + a^2\int_{\pi/3}^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$\left(\frac{7\pi}{4} - \frac{9\sqrt{3}}{8}\right)a^2 \text{ sq unit}$$

Ex. 6. Find the area within the circle $r = \sin\theta$ and outside the cardioide $r = 1 - \cos\theta$



Soln

$\sin\theta = 1 - \cos\theta$

$\cancel{\sin^2\theta} = 1 - 2\cos\theta + \cos^2\theta$

$= \cancel{\sin^2\theta} + \cos^2\theta - 2\cos\theta + \cos^2\theta$

$2\cos^2\theta - 2\cos\theta = 0$

$\cos\theta(\cos\theta - 1) = 0 \quad \therefore \theta = 0, \frac{\pi}{2}$

Required area $A = \frac{1}{2}\int_0^{\pi/2}\left[\sin^2\theta - (1-\cos\theta)^2\right] d\theta$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(2\cos\theta - 1 - \cos 2\theta\right) d\theta$$

$$= \int_0^{\pi/2}\left[\cos\theta - \frac{1+\cos 2\theta}{2}\right] d\theta$$

$$= \left[\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$= \left(1 - \frac{\pi}{4}\right)$ sq. unit

Ai