Indian Institute of Technology Bhubaneswar

School of Infrastructure

Session: Spring 2026

Finite Element Method (CE6L303)

Notes on Beltrami Identity in the Calculus of Variations

Course Instructor: Dr. Mohammad Masiur Rahaman

1. Introduction

In the calculus of variations, the fundamental problem is to determine a function $y(x)$ that extremizes a functional of the form

$$\mathcal{J}[y] = \int_{x_1}^{x_2} F(x, y, y')\, dx, \tag{1}$$

where $y' = \dfrac{dy}{dx}$ and F is the integrand (or Lagrangian).

The necessary condition for an extremum is given by the *Euler–Lagrange equation*. In special cases, this equation admits a first integral known as the *Beltrami identity*.

2. Variation of a Functional

Let $y(x)$ be an admissible function satisfying prescribed boundary conditions

$$y(x_1) = y_1, \qquad y(x_2) = y_2. \tag{2}$$

We introduce a family of varied functions

$$y_\varepsilon(x) = y(x) + \varepsilon\eta(x), \tag{3}$$

where:

- ε is a small parameter,
- $\eta(x)$ is an arbitrary smooth function,
- $\eta(x_1) = \eta(x_2) = 0$ to preserve the boundary conditions.

The functional becomes a function of ε:

$$\Phi(\varepsilon) = \mathcal{J}[y_\varepsilon] = \int_{x_1}^{x_2} F\big(x, y + \varepsilon\eta, y' + \varepsilon\eta'\big)\, dx. \tag{4}$$

For $y(x)$ to be an extremum, a necessary condition is

$$\left.\frac{d\Phi}{d\varepsilon}\right|_{\varepsilon=0} = 0. \tag{5}$$

3. Derivation of the Euler–Lagrange Equation

Differentiate $\Phi(\varepsilon)$ with respect to ε:

$$\frac{d\Phi}{d\varepsilon} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \right] dx. \tag{6}$$

Evaluating at $\varepsilon = 0$,

$$\delta\mathcal{J} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \right] dx. \tag{7}$$

The second term contains η', which we eliminate using integration by parts:

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\eta'\, dx = \left[\frac{\partial F}{\partial y'}\eta \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \eta\, dx. \tag{8}$$

Since $\eta(x_1) = \eta(x_2) = 0$, the boundary term vanishes. Hence,

$$\delta\mathcal{J} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right] \eta\, dx. \tag{9}$$

Because $\eta(x)$ is arbitrary, the integrand must vanish identically:

$$\boxed{\frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0.} \tag{10}$$

This is the **Euler–Lagrange equation**.

4. Statement of the Beltrami Identity

If the integrand does not explicitly depend on the independent variable x, i.e.,

$$F = F(y, y'), \tag{11}$$

then the Euler–Lagrange equation admits a first integral:

$$\boxed{F - y'\frac{\partial F}{\partial y'} = C,} \tag{12}$$

known as the **Beltrami identity**.

5. Derivation of the Beltrami Identity

Since $F = F(y, y')$, its total derivative is

$$\frac{dF}{dx} = \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y''. \tag{13}$$

Consider

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'} \right). \tag{14}$$

Using the product rule,

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'} \right) = \frac{dF}{dx} - \left[y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right]. \tag{15}$$

Substituting for $\frac{dF}{dx}$ and simplifying,

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = y'\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]. \tag{16}$$

Using the Euler–Lagrange equation, the term in brackets vanishes, yielding

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0. \tag{17}$$

Therefore,

$$\boxed{F - y'\frac{\partial F}{\partial y'} = C.} \tag{18}$$

6. Physical Interpretation

The Beltrami identity arises from invariance of the functional under translations in the independent variable. This reflects a conserved quantity and is a special case of *Noether's theorem.*

In classical mechanics, it corresponds to conservation of total energy.

7. Example: Shortest Path Between Two Points

Minimize

$$\mathcal{J}[y] = \int \sqrt{1 + y'^2}\, dx. \tag{19}$$

Applying the Beltrami identity,

$$\frac{1}{\sqrt{1 + y'^2}} = C \quad \Rightarrow \quad y' = \text{constant}. \tag{20}$$

Hence,

$$y = mx + b, \tag{21}$$

showing that the shortest path is a straight line.

8. Relevance to Engineering and FEM

- Euler–Lagrange equations form the basis of weak formulations in FEM.
- Energy minimization principles in mechanics directly lead to variational formulations.
- Beltrami-type identities are useful for analytical benchmarks and verification of numerical solutions.

9. Summary

- Euler–Lagrange equations are obtained from the first variation of a functional.
- The Beltrami identity is a first integral valid when the integrand is independent of the independent variable.
- Both play a central role in mechanics, physics, and finite element methods.