Indian Institute of Technology Bhubaneswar
School of Infrastructure
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Finite Element Method (CE6L303)
Notes on The Brachistochrone Problem:
Variational Formulation and Connection with the Finite Element Method
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1. Introduction

The *Brachistochrone problem* is a classical problem in mechanics and calculus of variations. The term *brachistochrone* originates from the Greek words *brachistos* (shortest) and *chronos* (time).

> **Problem statement:** Determine the curve joining two points in a vertical plane along which a particle slides under gravity, starting from rest and without friction, in the *least possible time*.

This problem was posed by Johann Bernoulli in 1696 and played a central role in the development of the calculus of variations. Its mathematical structure closely resembles modern variational formulations used in the Finite Element Method (FEM).

2. Historical Motivation

Johann Bernoulli solved the Brachistochrone problem in 1696 using a powerful physical analogy with geometrical optics. At that time, neither the Euler–Lagrange equations nor modern variational calculus existed.

Bernoulli's guiding idea was:

> *A particle descending under gravity behaves like a ray of light traveling through a medium whose optical properties vary with depth.*

His derivation rests on Fermat's principle and Snell's law of refraction.

Precise Statement of the Problem

A particle slides without friction from point A to point B in a vertical plane under uniform gravity.
Assumptions:

- The particle starts from rest.
- Gravity is constant with acceleration g.
- No friction or air resistance.

We measure the vertical coordinate y *downward* from the starting point.

Speed of the Particle as a Function of Depth

From conservation of mechanical energy:

$$\frac{1}{2}mv^2 = mgy.$$

Canceling mass m, the velocity becomes:

$$\boxed{v(y) = \sqrt{2gy}.}$$

Thus, the particle moves faster as it descends.

3. Bernoulli's Layered Medium Idea

Bernoulli imagined the vertical space divided into a large number of *infinitesimally thin horizontal layers*. Within each layer:

- The vertical change in y is negligible,
- The speed $v(y)$ is approximately constant,
- The particle moves along a straight line segment.

At the boundary between two layers, the direction of motion changes, just as a light ray refracts at the interface between two optical media.

Analogy with Fermat's Principle

Fermat's principle in optics states:

Light chooses the path requiring the least time.

Bernoulli argued that:

- The falling particle also minimizes travel time,
- Therefore, it must obey the same refraction law as light.

Snell's Law for a Variable-Speed Medium

For a light ray crossing an interface between two media:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} = \text{constant},$$

where:

- θ is the angle with the vertical,
- v is the speed in the medium.

Bernoulli extended this idea to infinitely many layers, obtaining:

$$\boxed{\frac{\sin\theta}{v(y)} = C,}$$

where C is a constant along the curve.

Substituting the velocity expression:

$$\frac{\sin\theta}{\sqrt{2gy}} = C.$$

Geometric Interpretation of the Angle

Let the curve be described by $y = y(x)$.
The differential arc length is:

$$ds = \sqrt{dx^2 + dy^2}.$$

The angle θ between the tangent to the curve and the vertical direction satisfies:

$$\sin\theta = \frac{\text{horizontal component}}{\text{tangent length}} = \frac{dx}{ds}.$$

Hence:

$$\sin\theta = \frac{dx}{\sqrt{dx^2 + dy^2}} = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}.$$

Fundamental Differential Equation

Substitute this expression for $\sin\theta$ into Snell's law:

$$\frac{1}{\sqrt{2gy}\sqrt{1 + (y')^2}} = C.$$

Rearranging:

$$\sqrt{2gy}\sqrt{1 + (y')^2} = \frac{1}{C}.$$

Squaring both sides:

$$2gy\left(1 + (y')^2\right) = \text{constant}.$$

Absorbing constants:

$$\boxed{y\left(1 + (y')^2\right) = k,}$$

where k is a constant.

Reduction to a Parametric Form

Rewriting:

$$1 + (y')^2 = \frac{k}{y}.$$

Thus:

$$\left(\frac{dy}{dx}\right)^2 = \frac{k}{y} - 1.$$

Invert:

$$\frac{dx}{dy} = \sqrt{\frac{y}{k - y}}.$$

Trigonometric Substitution

Let:

$$y = k \sin^2\left(\frac{\phi}{2}\right) = \frac{k}{2}(1 - \cos\phi).$$

Then:

$$dy = \frac{k}{2} \sin\phi \, d\phi.$$

Substituting into the expression for dx:

$$dx = \sqrt{\frac{\frac{k}{2}(1 - \cos\phi)}{\frac{k}{2}(1 + \cos\phi)}} \cdot \frac{k}{2} \sin\phi \, d\phi.$$

Using trigonometric identities:

$$dx = \frac{k}{2}(1 - \cos\phi) \, d\phi.$$

Cycloidal Parametric Equations

Integrating:

$$x = \frac{k}{2} \int (1 - \cos\phi) \, d\phi = \frac{k}{2}(\phi - \sin\phi).$$

Recall:

$$y = \frac{k}{2}(1 - \cos\phi).$$

Thus, the curve is:

$$\boxed{\begin{aligned} x(\phi) &= a(\phi - \sin\phi), \\ y(\phi) &= a(1 - \cos\phi), \end{aligned}}$$

where $a = \frac{k}{2}$.

This is the *cycloid*.

Bernoulli's Physical Interpretation

Bernoulli emphasized:

- The cycloid descends steeply at first,
- Rapid early acceleration increases speed,
- The higher speed outweighs the longer distance.

Therefore, the fastest path is not the shortest path.

Relation to Modern Variational Calculus

Bernoulli's derivation:

- Avoids Euler–Lagrange equations,
- Uses physical laws and geometry,
- Arrives at the same condition as the Beltrami identity.

Modern calculus of variations later formalized this reasoning.

Concluding Remarks

Johann Bernoulli's solution of the Brachistochrone problem represents one of the earliest and most elegant applications of variational thinking. By using optics as an analogy, he anticipated the entire structure of modern analytical mechanics and numerical variational methods.

4. By Euler and Lagrange

Physical Assumptions and Geometry

The following assumptions are made:

- Motion occurs in a vertical plane.
- Gravitational acceleration g is constant.
- The particle starts from rest.
- No friction or energy loss is present.

Let the particle move from

$$A = (0,0) \quad \text{to} \quad B = (x_1, y_1),$$

where the vertical coordinate y is measured downward and $y_1 > 0$.

The curve is described by a function

$$y = y(x).$$

Kinematics and Energy Relation

4.1. Velocity of the Particle

Since the particle starts from rest, conservation of mechanical energy gives

$$\frac{1}{2}mv^2 = mgy.$$

Canceling the mass m, the velocity becomes

$$v(y) = \sqrt{2gy}.$$

4.2. Arc Length Element

The differential arc length along the curve is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

4.3. Differential Time Element

The time required to traverse the arc length ds is

$$dt = \frac{ds}{v}.$$

Substituting for ds and v,

$$dt = \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}}\, dx,$$

where $y' = \frac{dy}{dx}$.

Time as a Functional

The total time of travel from $x = 0$ to $x = x_1$ is

$$T[y] = \int_0^{x_1} \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}}\, dx.$$

This defines a *functional* of the unknown curve $y(x)$:

$$\boxed{T[y] = \int_0^{x_1} L(y, y')\, dx}$$

with the Lagrangian

$$L(y, y') = \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}}.$$

The Brachistochrone problem is therefore a variational problem of minimizing $T[y]$.

First Variation and Euler–Lagrange Equation

Consider a variation of the curve

$$y(x) \to y(x) + \varepsilon\eta(x),$$

where $\eta(x)$ is an admissible variation satisfying

$$\eta(0) = \eta(x_1) = 0.$$

The first variation of the functional is

$$\delta T = \frac{d}{d\varepsilon} T[y + \varepsilon\eta]\bigg|_{\varepsilon=0}.$$

Using standard variational arguments,

$$\delta T = \int_0^{x_1} \left(\frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial y'}\eta' \right) dx.$$

Integrating the second term by parts,

$$\delta T = \int_0^{x_1} \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) \right] \eta\, dx.$$

Since $\eta(x)$ is arbitrary, the integrand must vanish, yielding the *Euler–Lagrange equation*:

$$\boxed{\frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) - \frac{\partial L}{\partial y} = 0.}$$

Beltrami Identity

The Lagrangian $L(y, y')$ does not explicitly depend on x. Hence, the Beltrami identity applies:

$$L - y'\frac{\partial L}{\partial y'} = C,$$

where C is a constant.

Substituting the expression for L and simplifying, we obtain

$$\frac{1}{\sqrt{2gy}\sqrt{1 + (y')^2}} = C.$$

Rearranging,

$$y\left(1 + (y')^2\right) = \text{constant}.$$

Cycloidal Solution

The above differential equation can be solved parametrically by introducing a parameter θ. The solution is a *cycloid*:

$$\boxed{\begin{aligned} x(\theta) &= a(\theta - \sin\theta), \\ y(\theta) &= a(1 - \cos\theta), \end{aligned}}$$

where a is a constant determined from the boundary conditions.

Thus, the curve of fastest descent under gravity is a cycloid.

5. Variational Structure and Connection with FEM

5.1. Common Variational Philosophy

Both the Brachistochrone problem and FEM are based on variational principles:

- The Brachistochrone minimizes a time functional.
- FEM minimizes an energy functional or enforces a weak form.
- Governing equations arise from stationarity of a functional.

5.2. Weak Form Analogy

The condition

$$\delta T = 0$$

is directly analogous to the FEM weak form

$$\delta \Pi(u) = 0,$$

where Π is the total potential energy and u is the field variable.

5.3. Finite Element Discretization

In FEM spirit, the curve is approximated by

$$y_h(x) = \sum_{i=1}^{n} N_i(x) y_i,$$

where:

- $N_i(x)$ are shape functions,
- y_i are nodal values.

Substituting into the time functional,

$$T[y_h] = \int_0^{x_1} \frac{\sqrt{1 + (y_h')^2}}{\sqrt{2 g y_h}} \, dx.$$

The problem reduces to minimizing T with respect to the nodal values:

$$\boxed{\min_{\{y_i\}} T(y_1, y_2, \ldots, y_n).}$$

This leads to a nonlinear system of algebraic equations, typically solved using Newton–Raphson or gradient-based optimization methods.

6. Strong Form vs FEM Approximation

Variational Calculus	**Finite Element Method**
Euler–Lagrange equation	Governing PDE
Exact cycloidal solution	Approximate numerical solution
Infinite-dimensional space	Finite-dimensional subspace

7. Concluding Remarks

The Brachistochrone problem:

- Is a fundamental variational problem in mechanics,
- Naturally leads to the Euler–Lagrange equations,
- Demonstrates the power of functional minimization,
- Provides a clear conceptual bridge to FEM.

It serves as an excellent pedagogical example linking classical mechanics, calculus of variations, and modern computational methods.