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MATERIAL AND SPATIAL GAUGE THEORIES OF SOLIDS—III. DYNAMICS OF DISCLINATION FREE STATES

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Abstract—The dynamics of disclination free states of material bodies are analyzed by restricting the spatial and material gauge groups so that only the translation subgroups act locally. This eliminates the spatial and material curvature tensors for the rotation sectors and simplifies the problem. Field equations for the matter fields, χ^i, the spatial translation compensating fields, ϕ^i_a, and the material translation compensating fields, ψ^a_b, are obtained. All of the field equations are shown to be gauge covariant. Effects of local material translations are pervasive because they arise from local transformations of the base manifold. In the absence of material defects, the field equations for the χ^i and ϕ^i_a fields reduce to those previously reported, and the integrability conditions of the field equations for the ϕ's are the field equations for the χ's in all cases. The new field equations for the material translation compensating fields are shown to have sources that are linear in the components of the gauge field momentum-energy complex, and hence they are intrinsically nonlinear. The integrability conditions for the field equations for the ψ's are shown to be equations of balance for gauge momentum-energy of the matter fields with explicitly calculated source terms that vanish in the absence of defects.

1. INTRODUCTION

One of the aspects of gauge-theoretic models of solids that leads to difficulties and confusions is the inclusion of rotations and translations on an equal footing. Everyone seems to be reasonably "at home" with translations because local actions of translations give rise to *microcrack* and *microdislocation* density and current distributions. On the other hand, local actions of rotations give rise to *microrotation* and *microdisclination* density and current distributions that many investigators find problematic, at best. There is also an order of magnitude increase in the mathematical complexity of the theory when local rotations are included. This is because the gauge connections for the rotation subgroups act multiplicatively and the associated curvature 2-forms are intrinsically nonlinear. The situation is compounded for gauge theories of solids that are based on local action of both spatial and material symmetry groups. Clearly, a better understanding of gauge theories of solids can be obtained by first carefully analyzing what happens when only the translation groups are allowed to act locally (i.e. the rotation groups are "frozen" in their homogeneous phases). This is one of the purposes of this paper. The other is to obtain an understanding of how and in what ways gauge theories for material and spatial gauge groups differ.

A previous paper [1] laid the physical, geometric, and kinematic foundations for a gauge theory of solids where both spatial and material symmetry groups were gauged on an equal footing. Here, we concentrate on the other half of the gauge construction; namely, the problem of obtaining and analyzing the resulting field equations whose solutions predict possible material states. We make the explicit simplifying assumption that *only the translation subgroups are allowed to act locally*. Physically, this means that we restrict our attention to materials that require very large energies to activate spatial rotation defects (i.e. the "free field" Lagrangians for spatial rotation defects are multiplied by very large coupling constants). There are two ways in which material rotation defects can be eliminated. The first is where the material symmetry group contains only a discrete subgroup of the rotation group, such as in crystaline materials. Since discrete subgroups of the rotation group are not Lie groups, the neighborhood of the identity is a singleton and there are no smooth deformations (i.e. no gauge degrees of freedom). The second is where the material symmetry group contains a Lie subgroup of the orthogonal group but very large energies are required in order to activate material rotation defects. The equivalent kinematic restrictions are much simpler. They are contained in the requirements that the *admissible spatial symmetry transformations* G_s take the form

$$''\chi^i(X^c) = Q^i_j\chi^j(X^c) + b^i(X^c), \qquad ''t = t, \qquad \mathbf{Q}\mathbf{Q}^T = \mathbf{I}, \tag{1.1}$$

where $x^i = \chi^i(X^c)$, $t = X^4$ define the mapping from the space of reference histories $\mathbb{R}^4$ with coordinates $\{X^c\} = \{X^A, X^4\}$ into the space of current histories, while the *admissible material symmetry transformations* G_m are given by

$$''X^A = R^A_B X^B + T^A(X^c), \qquad ''X^4 = X^4 + T^4(X^c), \qquad R^A_C\delta_{AB}R^B_D = \delta_{CD}. \tag{1.2}$$

Since the matrices Q^i_j and R^A_B are constant-valued orthogonal matrices, the spatial and material rotation groups are frozen in their homogeneous (global) phases. This means that only spatial translations $T(3)$ can act locally on the space E_4 of current histories while both spatial and time translations $T(3) \oplus T(1)$ can act locally on the space $\mathbb{R}^4$ of reference histories. It is important that we include the action of the global, spatial and material rotations (i.e. the effects of the constant orthogonal matrices $\mathbf{Q}$ and $\mathbf{R}$) in order to correctly reflect the classical aspects of *objectivity* and *material frame indifference.*

In order to eliminate many repetitions, we will adopt the notation introduced in [1]. We also assume that the reader is familiar with gauge theoretic constructs in solid mechanics, as discussed in [1] and [3].

2. MINIMAL REPLACEMENT AND KINEMATIC RESULTS

We will use $\mathcal{M}$ to denote the minimal replacement operator of gauge theory. The reader is referred to [1] for demonstrations of the results summarized in this Section. Local action of the spatial translation group gives rise to fields of compensating 1-forms

$$\phi^i = \phi^i_a(X^c)\,\mathrm{d}X^a, \qquad 1 \le i \le 3. \tag{2.1}$$

These serve to evaluate the *elastic distortion* 1-forms $B^i = B^i_a(X^c)\,\mathrm{d}X^a$ through the relations

$$B^i = \mathcal{M}\langle \mathrm{d}\chi^i \rangle = \mathrm{d}\chi^i + \phi^i \tag{2.2}$$

because the rotation subgroup is frozen in its homogeneous phase (i.e. the connection 1-forms for the rotation group vanish throughout $\mathbb{R}^4$). It is precisely because the rotation subgroups are frozen in their homogeneous phases that the kinematic relations reported in [1] reduce to the particularly simple forms given here. The *spatial torsion* 2-forms are defined in terms of the elastic distortion 1-forms by

$$\Sigma^i = \mathrm{d}B^i = \frac{1}{2}(\partial_a\phi^i_b - \partial_b\phi^i_a)\,\mathrm{d}X^a \wedge \mathrm{d}X^b = \frac{1}{2}\Sigma^i_{ab}\,\mathrm{d}X^a \wedge \mathrm{d}X^b. \tag{2.3}$$

This minimal replacement action is the standard one for an internal symmetry group acting on a collection of matter fields $\chi^i(X^c)$.

The material translation symmetry group acts as an external symmetry group; that is, it acts on the base manifold $\mathbb{R}^4$ as well as on fields defined over the base manifold. The local action of the material translation group $T(3) \oplus T(1)$ gives rise to the compensating 1-forms

$$\psi^a = \psi^a_b(X^c)\,\mathrm{d}X^b, \qquad 1 \le a \le 4. \tag{2.4}$$

These serve to define the *fundamental coframe fields* by

$$J^a = J^a_b(X^c)\,\mathrm{d}X^b = \mathcal{M}\langle \mathrm{d}X^a \rangle = \mathrm{d}X^a + \psi^a, \qquad 1 \le a \le 4. \tag{2.5}$$

The fundamental coframe fields are assumed to satisfy the *regularity condition*

$$J = \det(J^a_b(X^c)) \ne 0 \tag{2.6}$$

at all points in the reference history $\mathbb{B}^4 \subset \mathbb{R}^4$ of the material body.

The associated *fundamental frame fields*

$$j_a = j^b_a(X^c)\partial_b = \mathcal{M}\langle \partial_a \rangle, \qquad 1 \le a \le 4, \tag{2.7}$$

are evaluated by $j_a \rfloor J^b = \delta_a^b$, where $\rfloor$ denotes the "pull down" or inner product (see [2], Chap. 3); that is

$$j_a^e(X^c)J_e^b(X^c) = \delta_a^b. \tag{2.8}$$

The fundamental frame fields $j_a = \mathcal{M}\langle \partial_a \rangle$ and coframe fields $J^a = \mathcal{M}\langle \mathrm{d}X^a \rangle$ define gauge-natural bases for vector and covector fields since they are the minimal replacement images of natural bases $\{\partial_a \mid 1 \le a \le 4\}$ and $\{\mathrm{d}X^a \mid 1 \le a \le 4\}$. The *material torsion* 2-forms have the evaluation

$$\Xi^a = \mathrm{d}J^a = \frac{1}{2}(\partial_b J_c^a - \partial_c J_b^a)\, \mathrm{d}X^b \wedge \mathrm{d}X^c = \frac{1}{2}\Xi_{bc}^a\, \mathrm{d}X^b \wedge \mathrm{d}X^c. \tag{2.9}$$

The total gauge group G is the direct product of the spatial gauge group and the material gauge group. Minimal replacement of the configuration gradients $\partial_a \chi^i$ thus gives the following evaluations for the components of *total distortion*

$$Y_a^i(X^c) = \mathcal{M}\langle \partial_a \chi^i \rangle = j_a^b(X^c)B_b^i(X^c). \tag{2.10}$$

This factorization of the total distortion into an elastic part, B_b^i, and a plastic part, j_a^b, has been commented upon in [1]. We make specific note, however, that this factorization is a derived consequence of the minimal replacement construct of gauge theory rather than a specific assumption, as is often the case in the current literature.

We will need explicit evaluations of the transformation properties of the various kinematic fields that are induced by the action of the total gauge group. The primitive forms of the gauge group action are given by (1.1) and (1.2). In order to simplify the transformation formula, we introduce the Jacobian matrix $\mathbf{A}$ and its inverse $\mathbf{a}$ by

$$A_c^b = \frac{\partial'' X^b}{\partial X^c}, \qquad a_c^b = \frac{\partial X^b}{\partial'' X^c}, \tag{2.11}$$

so that (1.2) gives

$$A_c^b = R_c^b + \frac{\partial T^b}{\partial X^c}.$$

Inclusions of such quantities is unavoidable because the material symmetry group acts on the base manifold $\mathbb{R}^4$. For example, the transformations of the fundamental coframe fields derived in [1] are $''J^a = R_c^a J^c$, where R_c^a are the components of the 4×4 matrix that is defined by

$$((R_c^a)) = \begin{bmatrix} R_B^A & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.12}$$

The matrix inverse of $\mathbf{R}$ will be denoted by $\mathbf{r} = ((r_b^a))$. Likewise the matrix inverse of $\mathbf{Q} = ((Q_B^A))$ will be denoted by $\mathbf{q} = ((q_B^A)) = \mathbf{Q}^T$. We therefore have

$$''J^a = {}''J_b^a\, \mathrm{d}''X^b = {}''J_b^a A_e^b\, \mathrm{d}X^e = R_c^a J^c = R_c^a J_e^c\, \mathrm{d}X^e;$$

that is, the J's have the explicit transformation laws

$$''J_r^a = R_c^a J_e^c a_r^e. \tag{2.13}$$

Accordingly, since $\det(R_c^a) = 1$, (2.6) and (2.13) give

$$''J = \det(''J_r^a) = J \det(a_r^e). \tag{2.14}$$

Since the 4-dimensional volume element

$$\pi = \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \wedge \mathrm{d}X^4 \tag{2.15}$$

has the transformation law

$$''\pi = \pi \det(A_b^a), \tag{2.16}$$

we have the *invariance* relation

$$''(J\pi) = {}''J''\pi = J\pi. \tag{2.17}$$

Similar arguments based on the transformation relations derived in [1] can be used to obtain the explicit transformation laws for the other field variables. We shall therefore simply state the results:

$$''j^a_b = r^e_b j^m_e A^a_m, \tag{2.18}$$

$$''B^i_r = Q^i_j B^j_a a^a_r, \tag{2.19}$$

$$''Y^i_a = Q^i_j Y^j_s r^s_a, \tag{2.20}$$

$$''\Sigma^i_{ab} = Q^i_j \Sigma^j_{ef} a^e_a a^f_b, \tag{2.21}$$

$$''\Xi^c_{ab} = R^c_m \Xi^m_{ef} a^e_a a^f_b. \tag{2.22}$$

3. MINIMAL COUPLING, THE TOTAL LAGRANGIAN, AND FIELD INTENSITIES

The second part of any gauge construct is *minimal coupling*. This construction customarily starts with a Lagrangian 4-form $L_0\pi$ for the matter fields that is invariant under the global action of the gauge group. For models of solids, the matter Lagrangian L_0 has the form

$$L_0 = T - U + N, \tag{3.1}$$

where

$$T = \frac{1}{2}\rho_0(X^A)\,\partial_4\chi^i\delta_{ij}\,\partial_4\chi^j \tag{3.2}$$

is the *kinetic energy density*,

$$U = U(\partial_a\chi^i) \tag{3.3}$$

is the *strain energy density*, and

$$N = t^A_i(X^c)\,\partial_A\chi^i - P_i(X^c)\,\partial_4\chi^i \tag{3.4}$$

is the *null Lagrangian* that accounts for the work done by the assigned boundary tractions and the assigned initial and final linear momenta. The functions $\{t^A_i(X^c), P_i(X^c)\}$ are explicit functions of position and time that satisfy

$$\partial_4 P_i = \partial_A t^A_i \tag{3.5}$$

at all interior points of the reference history $\mathbb{B}^4$ of the body. They satisfy the same traction boundary conditions

$$T_i = t^A_i n_A \tag{3.6}$$

as that for the actual problem under study, and the same initial and final linear momentum conditions. The reader is referred to ([3], Chap. 7) for a discussion of the use of null Lagrangians in this context. Now, it is clear that N is not invariant under the global action of the material translation group $T(3)\otimes T(1)$ because of the presence of the explicit functions $\{t^A_i(X^c), P_i(X^c)\}$ in N. Thus, although both T and U are invariant, L_0 is not invariant because of the presence of N on the right-hand side of (3.1). Use of (3.5) allows us to show that N can be written in the equivalent form

$$N = \partial_A(t^A_i\chi^i) - \partial_4(P_i\chi^i),$$

and hence the integral of N over $\mathbb{B}_4$ can be reduced to an integral over the boundary of $\mathbb{B}_4$ by use of the divergence theorem. Accordingly, changes in the action integral that come from the lack of invariance of N, under global action of the symmetry group for T and U, can be reduced to boundary integrals. All is thus not lost, for the invariance group for T and U (the Noetherian symmetry group for the Lagrangian $T - U$) goes over into a *Bessel-Hagen symmetry group* for the Lagrangian $L_0 = T - U + N$ (see [2], Section 7–4 with Bessel-Hagen symmetry = Noetherian symmetry of the second kind). We show in the appendix that the minimal replacement construct of classical gauge theory works for Bessel-Hagen symmetries as well as for Noetherian symmetries. If we assume that the t's transform like components of a

Piola–Kirchhoff stress tensor and the P's transform like components of linear momentum under global gauge transformations, then N becomes invariant and the global Bessel-Hagen symmetries of the Lagrangian L_0 become Noetherian symmetries (i.e. L_0 is invariant under global action of the symmetry group). The reader is referred to the Appendix for a further discussion of this matter.

Local action of the gauge group destroys the invariance of the Lagrangian 4-form $(T - U)\pi$. Invariance is restored by minimal replacement. This means that the Lagrangian 4-form appropriate to states that undergo transformations induced by local action of the gauge group is given by

$$\mathscr{L}_1\pi = \mathcal{M}\langle L_0\pi\rangle. \tag{3.7}$$

Noting that $\mathcal{M}\langle\pi\rangle = J\pi$ and $\mathcal{M}\langle\partial_a\chi^i\rangle = Y^i_a$, we obtain

$$\mathscr{L}_1 = (\mathscr{T} - \mathscr{U} + \mathscr{N})J, \tag{3.8}$$

where

$$\mathscr{T} = \frac{1}{2}\rho_0 Y^i_4\delta_{ij}Y^j_4 \tag{3.9}$$

is the *gauge kinetic energy*,

$$\mathscr{U} = U(Y^i_A) \tag{3.10}$$

is the *gauge strain energy*, and

$$\mathscr{N} = t^A_i(X^c)Y^i_A - P_i(X^c)Y^i_4 \tag{3.11}$$

is the minimal replacement image of the null Lagrangian N. The reader should note that $\mathscr{N}$ is no longer a null Lagrangian, as an elementary calculation will easily show. The new part of the Lagrangian $\mathscr{L}_1$ that comes from $\mathscr{N}$ will thus make explicit contributions to the field equations. We show in the Appendix that $\mathscr{N}$ is invariant under local action of the gauge group, and thus (3.8) shows that $\mathscr{L}_1$ is invariant.

This is only part of the story, for the compensating fields $\{\psi^a_b, \phi^i_b\}$ occur in $\mathscr{L}_1$ only algebraically; that is, $\mathscr{L}_1$ does not contain derivatives of the compensating fields. The minimal coupling construct of gauge theory [3] tells us that we have to add to $\mathscr{L}_1$ a *free gauge field* Lagrangian $\mathscr{L}_g$ that accounts for the energy contributions of the compensating fields. This free gauge field Lagrangian must be invariant under the local action of the gauge group and contain all dependencies on the derivatives of the compensating fields. Since we have frozen the rotation subgroups in their homogeneous phases, the material and spatial gauge torsion 2-forms do not depend on the state variables explicitly. We may therefore apply the standard *minimal coupling* construct of gauge theory and write

$$\mathscr{L}_g = \mathscr{V}(X^a, \phi^i_a, \psi^a_b, \partial_c\phi^i_a, \partial_c\psi^a_b)J.$$

Here, we have included the factor J so that the resulting 4-form $\mathscr{L}_g\pi = \mathscr{V}J\pi$, "$(J\pi) = J\pi$, and the invariance of the Lagrangian 4-form result in the requirement that $\mathscr{V}$ *be a scalar-valued invariant function of its indicated arguments*. This invariance requirement significantly simplifies the possible arguments of $\mathscr{V}$, and we obtain

$$\mathscr{L}_g = \mathscr{V}(J^a_b, \Sigma^i_{ab}, \Xi^c_{ab})J. \tag{3.12}$$

The reader is warned that an allowed local action of spatial rotation groups would require us to use the alternative *minimal derivative coupling* construct, as spelled out in ([3], Section 5.3). Eliminating this added complexity is one of the advantages of considering materials for which the rotation subgroups are frozen in their homogeneous phases. The *total Lagrangian* $\mathscr{L}$ for material bodies considered in this paper thus takes the form

$$\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_g = (\mathscr{T} - \mathscr{U} + \mathscr{N} + \mathscr{V})J. \tag{3.13}$$

For later purposes, we also write (3.13) in the equivalent form

$$\mathscr{L} = \hat{\mathscr{L}}J, \tag{3.14}$$

and note that $\hat{\mathscr{L}}$, $\mathscr{T}$, $\mathscr{U}$, and $\mathscr{V}$ transform as scalar invariants under the local action of the gauge group.

4. FIELD INTENSITIES AND THEIR TRANSFORMATION LAWS

Stress and linear momentum in gauge theories of solids are defined as the images under minimal replacement of the corresponding quantities in elasticity theory (see [3], Chap. 5). We therefore have

$$\sigma_i^A = \mathcal{M}\left\langle \frac{\partial U}{\partial(\partial_A \chi^i)} \right\rangle = \frac{\partial \mathcal{U}}{\partial Y_A^i}, \tag{4.1}$$

and

$$p_i = \mathcal{M}\left\langle \frac{\partial T}{\partial(\partial_4 \chi^i)} \right\rangle = \frac{\partial \mathcal{T}}{\partial Y_4^i} = \rho_0 \delta_{ij} Y_4^j. \tag{4.2}$$

The *spatial translation defect intensity* (microcrack intensity) fields are defined by

$$\mathcal{R}_i^{ab} = \frac{\partial \mathcal{V}}{\partial \Sigma_{ab}^i}, \qquad \mathcal{R}_i^{ab} = -\mathcal{R}_i^{ba}. \tag{4.3}$$

These fields play the same role as those designated by R_i^{ab} in [3]. The change in notation has been made in order to incorporate the multiplicative factor J in the total Lagrangian $\mathcal{L}$ that comes from the action of the gauge group on the base manifold $\mathbb{R}^4$. Similarly, the *material translation defect intensity* (dislocation intensity) fields are defined by

$$\mathcal{Q}_c^{ab} = \frac{\partial \mathcal{V}}{\partial \Xi_{ab}^c}, \qquad \mathcal{Q}_c^{ab} = -\mathcal{Q}_c^{ba}. \tag{4.4}$$

The transformation laws for the various field intensities are direct consequences of the fact that $\mathcal{U}$, $\mathcal{T}$, $\mathcal{N}$, and $\mathcal{V}$ are scalar invariant functions of their arguments. Thus, for example, $''\mathcal{U}(''Y_A^i) = \mathcal{U}(Y_A^i)$, and $''\sigma_i^A = \partial''\mathcal{U}/\partial''Y_A^i$ show that

$$\sigma_i^A = \frac{\partial \mathcal{U}}{\partial Y_A^i} = \frac{\partial''\mathcal{U}}{\partial''Y_B^j}\frac{\partial''Y_B^j}{\partial Y_A^i} = {}''\sigma_j^B Q_i^j r_B^A$$

when (2.20) is used. We accordingly have

$$''\sigma_j^B = R_A^B \sigma_i^A q_j^i. \tag{4.5}$$

Similar arguments lead directly to the following transformation laws:

$$''p_j = p_i q_j^i, \tag{4.6}$$

$$''\mathcal{R}_i^{uv} = q_i^j \mathcal{R}_j^{ab} A_a^u A_b^v, \tag{4.7}$$

$$''\mathcal{Q}_w^{uv} = r_w^c \mathcal{Q}_c^{ab} A_a^u A_b^v. \tag{4.8}$$

We make explicit note that the $\mathcal{R}$-fields and the $\mathcal{Q}$-fields retain their antisymmetry in the upper pairs of indices.

5. FIELD EQUATIONS FOR THE MATTER FIELDS

Noting that the total Lagrangian $\mathcal{L}$ does not contain the arguments χ^i, the Euler–Lagrange equations for the matter fields reduce to

$$\partial_A\left\{\frac{\partial \mathcal{L}}{\partial(\partial_A \chi^i)}\right\} + \partial_4\left\{\frac{\partial \mathcal{L}}{\partial(\partial_4 \chi^i)}\right\} = 0. \tag{5.1}$$

Now, $\mathcal{L}$ depends on $\partial_a \chi^i$ only through the arguments Y_b^i, so that

$$\frac{\partial \mathcal{L}}{\partial(\partial_4 \chi^i)} = \frac{\partial \mathcal{L}}{\partial Y_B^j}\frac{\partial Y_B^j}{\partial(\partial_4 \chi^i)} + \frac{\partial \mathcal{L}}{\partial Y_4^j}\frac{\partial Y_4^j}{\partial(\partial_4 \chi^i)}.$$

Use of the evaluations and definitions of the field intensities given previously thus yield

$$\frac{\partial \mathcal{L}}{\partial(\partial_4 \chi^i)} = J j_4^4 (p_i - P_i) - J j_B^4 (\sigma_i^B - t_i^B). \tag{5.2}$$

An analogous argument gives the evaluations

$$\frac{\partial \mathscr{L}}{\partial(\partial_A \chi^i)} = -Jj_B^A(\sigma_i^B - t_i^B) + Jj_4^A(p_i - P_i). \tag{5.3}$$

A substitution of (5.2) and (5.3) into (5.1) thus yields the *matter field equations*

$$\partial_4\{Jj_4^4(p_i - P_i) - Jj_B^4(\sigma_i^B - t_i^B)\} = \partial_A\{Jj_B^A(\sigma_i^B - t_i^B) - Jj_4^A(p_i - P_i)\}. \tag{5.4}$$

These matter field equations are drastically different from those reported in [3] where only spatial gauge groups were considered. These differences are direct consequences of the action of the material gauge group on the base manifold $\mathbb{R}^4$. If the material gauge group is forced to act globally, we would have $J_b^a = \delta_b^a = j_b^a$ and $J = 1$. In this case (5.4) reduce exactly to the field equations for the matter fields given in [3] because $\partial_4 P_i = \partial_A t_i^A$. Thus, the matter field equations (5.4) are consistent with previously reported results.

We define the *effective linear momentum* $\mathscr{P}_i$ and the *effective stresses* $\mathscr{S}_i^A$ by

$$\mathscr{P}_i = Jj_4^4(p_i - P_i) - Jj_B^4(\sigma_i^B - t_i^B), \tag{5.5}$$

$$\mathscr{S}_i^A = Jj_B^A(\sigma_i^B - t_i^B) - Jj_4^A(p_i - P_i). \tag{5.6}$$

The matter field equations thus assume the simplified form

$$\partial_4 \mathscr{P}_i = \partial_A \mathscr{S}_i^A, \tag{5.7}$$

and (5.5) and (5.6) agree with the corresponding quantities defined in [3] in those situations in which the material translation group acts globally. Further, (5.2), (5.3), (5.5) and (5.6) show that

$$\mathscr{P}_i = \frac{\partial \mathscr{L}}{\partial(\partial_4 \chi^i)}, \qquad \mathscr{S}_i^A = -\frac{\partial \mathscr{L}}{\partial(\partial_A \chi^i)}. \tag{5.8}$$

The effective linear momenta and stresses thus have the standard variational meanings that are usually attached in Lagrangian field theory. In this sense, they may be thought of as canonical. We make particular note that (5.5) shows that there are stress contributions to the effective linear momenta and (5.6) show that there are linear momenta contributions to the effective stresses that are reminiscent of relativistic mechanics. The stress contributions to effective linear momenta disappear when $j_B^4 = 0$, and the linear momentum contributions to the effective stresses disappear when $j_4^A = 0$. Now, these conditions will be satisfied only when the material time translation subgroup acts globally and the compensating fields for the local action of the material space translations have no $\mathrm{d}X^4$-components. Under these circumstances, time slices of $\mathbb{R}^4$ are gauged by local action of the material space translation group, but this occurs in exactly the same fashion from one time slice to the next. This is just another way of saying that the material gauge group acts homogeneously with respect to time-dependence of the physical processes. Thus, the mixing of linear momenta and stresses in the definitions of the effective linear momenta and effective stresses have their origins in the mixing of spatial and time structures that arise from the gauging of the full 4-dimensional translation group of the base manifold $\mathbb{R}^4$.

The various quantities that occur in the matter field equations have been shown to have somewhat complicated laws of transformation under the action of the gauge group. The question thus arises as to whether the matter field equations are gauge covariant; that is, will they be satisfied in all gauges if they are satisfied in any one particular gauge? The easiest way to answer this question is to introduce the *effective stress-momentum* 3-forms by

$$\mathscr{X}_i = -\mathscr{S}_i^A \pi_A + \mathscr{P}_i \pi_4, \tag{5.9}$$

where the π_a are the canonical conjugate basis for 3-forms that are defined by

$$\pi_a = \partial_a \rfloor \pi. \tag{5.10}$$

The matter field equations can now be written in the equivalent form

$$\mathrm{d}\mathscr{X}_i = 0. \tag{5.11}$$

An elementary calculation shows that

$$''\pi_b = a_b^c \det(A_f^e)\pi_c \tag{5.12}$$

under the action of the gauge group. A straightforward calculation based on the above definitions and the previously established transformation laws shows that the $\mathscr{X}$'s have the simple transformation laws

$$''\mathscr{X}_i = q_i^j \mathscr{X}_j. \tag{5.13}$$

We therefore have

$$''(\mathrm{d}\mathscr{X}_i) = \mathrm{d}''\mathscr{X}_i = \mathrm{d}(q_i^j\mathscr{X}_j) = q_i^j\,\mathrm{d}\mathscr{X}_j, \tag{5.14}$$

because q_i^j are the components of a constant-valued orthogonal matrix. This shows that $\mathrm{d}\mathscr{X}_i = 0$ implies $\mathrm{d}''\mathscr{X}_i = 0$, and hence *the matter field equations are gauge covariant field equations.*

The natural boundary conditions associated with the χ-fields can be obtained from the terms in the variation of the action integral that reduce to boundary integrals, as spelled out in [3]. For the problem at hand, we obtain

$$(\mathscr{X}_i \delta\chi^i)|_{\partial B_4} = 0. \tag{5.15}$$

A characteristic feature of gauge theories of solids is that the field variables χ^i can not be assigned Dirichlet data because the χ's are not gauge covariant quantities. Therefore, we can not assume that the variations of the χ's satisfy homogeneous Dirichlet data on $\partial\mathbb{B}_4$. Accordingly, (5.15) can be satisfied only when we impose the homogeneous Neumann conditions

$$(\mathscr{X}_i)|_{\partial\mathbb{B}_4} = 0. \tag{5.16}$$

Moting that $\mathbb{B}_4$ is a 4-dimensional region that is a cylinder with generators parallel to the time axis, (5.16) yield the "standard" initial conditions

$$(\mathscr{P}_i)|_{T=T_0} = 0, \qquad (\mathscr{P}_i)|_{T=T_1} = 0, \tag{5.17}$$

and the homogeneous traction boundary conditions

$$(\mathscr{S}_i^A n_A)|_{\partial\mathbb{B}_3} = 0. \tag{5.18}$$

6. FIELD EQUATIONS FOR THE SPATIAL DEFECT FIELDS

The Euler–Lagrange equations for the spatial compensating fields ϕ_a^i are

$$\partial_a\left\{\frac{\partial\mathscr{L}}{\partial(\partial_a\phi_b^i)}\right\} = \frac{\partial\mathscr{L}}{\partial\phi_b^i}. \tag{6.1}$$

Now, $\mathscr{L}$ depends on ϕ_b^i only through B_b^i that occur in the arguments Y_c^i. We therefore have

$$\frac{\partial\mathscr{L}}{\partial\phi_b^i} = \frac{\partial\mathscr{L}}{\partial Y_c^j}\frac{\partial Y_c^j}{\partial\phi_b^i} = \frac{\partial\mathscr{L}}{\partial Y_E^j}\frac{\partial Y_E^j}{\partial\phi_b^i} + \frac{\partial\mathscr{L}}{\partial Y_4^j}\frac{\partial Y_4^j}{\partial\phi_b^i}.$$

The constitutive relations (4.1), (4.2), the evaluations of the Y's given by (2.10), and the relations (2.5) and (2.8) thus yield

$$\frac{\partial\mathscr{L}}{\partial\phi_b^i} = -Jj_E^b(\sigma_i^E - t_i^E) + Jj_4^b(p_i - P_i) = -\delta_B^b\mathscr{S}_i^B + \delta_4^b\mathscr{P}_i, \tag{6.2}$$

where the last equality follows from (5.5) and (5.6). On the other hand, the only place where derivatives of the ϕ's occur is through the arguments Σ_{ab}^i of $\mathscr{V}$. An elementary calculation and the antisymmetry of $\mathscr{R}_i^{ab}$ in its upper indices shows that

$$\frac{\partial\mathscr{L}}{\partial(\partial_a\phi_b^i)} = 2J\mathscr{R}_i^{ab}. \tag{6.3}$$

The field equations for the spatial compensating fields ϕ^i_b, *the equations of balance of spatial defects,* are therefore given by

$$2\,\partial_a\{J\mathcal{R}_i^{ab}\} = -\delta^b_B\mathcal{S}_i^B + \delta^b_4\mathcal{P}_i. \tag{6.4}$$

The first thing to be noted is that the effective stresses $\mathcal{S}_i^B$ and the effective momenta $\mathcal{P}_i$, rather than the stress σ_i^B and momenta p_i, act as sources in the balance of spatial defects. This substantiates the fact that effective stresses and momenta are the canonical quantities in gauge theory.

The field equations (6.4) necessarily differ from those reported in [3] because of the presence of the local mateial gauge group. However, when $J^a_b = \delta^a_b$ (i.e. when material defects are absent), $J = 1$, $j^a_b = \delta^a_b$, and (6.4) reduce exactly to the field equations reported in [3]. The field equations (6.4) are also *gauge covariant,* as is easily seen by studying the properties of the spatial intensity 2-forms

$$\mathcal{R}_i = \frac{1}{2}J\mathcal{R}_i^{ab}\pi_{ab}, \tag{6.5}$$

where $\pi_{ab} = -\pi_{ba}$ are the conjugate basis for 2-forms that are defined by

$$\pi_{ab} = \partial_a \rfloor \pi_b = \partial_a \rfloor \partial_b \rfloor \pi. \tag{6.6}$$

In fact, we can write the field equations (6.4) in the equivalent gauge covariant form

$$2\,\mathrm{d}\mathcal{R}_i = \mathcal{X}_i, \tag{6.7}$$

which shows the similarity with the general form of the field equations for spatial translational defects reported in [3]. Thus, satisfaction of the equations of balance of spatial defects in any one gauge implies that they are satisfied in any other choice of gauge.

The left-hand sides of (6.4) involve the quantities $\mathcal{R}_i^{ab}$ that are antisymmetric in the index pair (a, b). Accordingly, we have the identities

$$\partial_b\partial_a\{J\mathcal{R}_i^{ab}\} = 0. \tag{6.8}$$

The right-hand sides of the field equations (6.4) must therefore satisfy the integrability conditions

$$0 = -\partial_B\mathcal{S}_i^B + \partial_4\mathcal{P}_i. \tag{6.9}$$

These, however, are just the matter field equations (5.7). We have therefore established the following basic result. *The system of field equations consisting of the matter field equations and the equations of balance of spatial defects entails no additional integrability conditions.*

The required boundary conditions for the ϕ-fields are most easily expressed by introducing the 1-forms $\delta\phi^i = \delta\phi^i_a\,\mathrm{d}X^a$ of variations. We then have the requirements

$$(\mathcal{R}_i \wedge \delta\phi^i)|_{\partial B_4} = 0. \tag{6.10}$$

Since the ϕ's are internal variables, they are uncontrollable by external agencies, and hence the values of the $\delta\phi$'s can not be controlled on the boundaries. This has the effect that (6.10) must be required to hold for all evaluations of the $\delta\phi$'s on the boundary; that is, the $\mathcal{R}$-fields have to satisfy homogeneous Neumann conditions on the boundaries.

7. FIELD EQUATIONS FOR THE MATERIAL DEFECT FIELDS

Calculation of the field equations for the compensating fields ψ^a_b for local action of the material translation group is a more complicated matter. First, let us recall that $\mathcal{L} = \hat{\mathcal{L}}J$, with $\hat{\mathcal{L}} = \mathcal{T} - \mathcal{U} + \mathcal{N} + \mathcal{V}$, while derivatives of the ψ-fields occur only in $\mathcal{V}$. On the other hand, the ψ-fields occur in J and in $\mathcal{T}$, $\mathcal{U}$, and $\mathcal{N}$ because $Y^i_a = j^b_a B^i_b$, and also in the arguments of $\mathcal{V}$. Since the Euler–Lagrange equations for the ψ-fields are

$$\partial_c\left\{\frac{\partial\mathcal{L}}{\partial(\partial_c\psi^a_b)}\right\} = \frac{\partial\mathcal{L}}{\partial\psi^a_b}, \tag{7.1}$$

computation of the right-hand sides are significantly more complicated. Evaluations of the left-hand sides follow exactly the argument given in the last section. We therefore have

$$\frac{\partial \mathscr{L}}{\partial(\partial_c \psi^a_b)} = 2J\mathscr{Q}^{cb}_a. \tag{7.2}$$

For a calculation of the right-hand sides of (7.1) we start by noting that $\mathscr{L} = J\hat{\mathscr{L}}$ gives

$$\frac{\partial \mathscr{L}}{\partial \psi^a_b} = \frac{\partial J}{\partial \psi^a_b}\hat{\mathscr{L}} + J\frac{\partial \hat{\mathscr{L}}}{\partial \psi^a_b} = Jj^b_a\hat{\mathscr{L}} + J\frac{\partial \hat{\mathscr{L}}}{\partial \psi^a_b}. \tag{7.3}$$

On the other hand $\hat{\mathscr{L}} = \mathscr{T} - \mathscr{U} + \mathscr{N} + \mathscr{V}$ gives

$$\frac{\partial \hat{\mathscr{L}}}{\partial \psi^a_b} = \frac{\partial \mathscr{V}}{\partial \psi^a_b} + \frac{\partial}{\partial Y^j_e}(\mathscr{T} - \mathscr{U} + \mathscr{N})\frac{\partial Y^j_e}{\partial \psi^a_b} \tag{7.4}$$

because $\mathscr{T}$, $\mathscr{U}$, and $\mathscr{N}$ depend on ψ^a_b only through the Y's. When we use the constitutive relations listed previously, we obtain

$$\frac{\partial \hat{\mathscr{L}}}{\partial \psi^a_b} = \frac{\partial \mathscr{V}}{\partial \psi^a_b} + (p_j - P_j)\frac{\partial Y^j_4}{\partial \psi^a_b} - (\sigma^A_j - t^A_j)\frac{\partial Y^j_A}{\partial \psi^a_b}. \tag{7.5}$$

However, $Y^j_a = j^b_a B^j_b$, and the j's are the inverses of the J's, while $J^a_b = \delta^a_b + \psi^a_b$. A lengthy, but straightforward calculation then yields

$$\frac{\partial Y^j_e}{\partial \psi^a_b} = -Y^j_a j^b_e. \tag{7.6}$$

When all of these various relations are combined and we introduce $\hat{\mathscr{L}}_1 = \mathscr{T} - \mathscr{U} + \mathscr{N}$, we obtain

$$\frac{\partial \mathscr{L}}{\partial \psi^a_b} = J\frac{\partial \mathscr{V}}{\partial \psi^a_b} + J\mathscr{V}j^b_a - J\mathscr{T}^e_a j^b_e, \tag{7.7}$$

where

$$\mathscr{T}^e_a = \mathscr{M}\langle T^e_a\rangle = \mathscr{M}\left\langle \frac{\partial \mathscr{L}_0}{\partial(\partial_e \chi^j)}\partial_a \chi^j - \delta^e_a \mathscr{L}_0\right\rangle = \frac{\partial \hat{\mathscr{L}}_1}{\partial Y^j_e}Y^j_a - \delta^e_a \hat{\mathscr{L}}_1$$

is the minimal replacement of the momentum-energy complex for the matter fields; that is $\mathscr{T}^e_a$ is the *gauge-theoretic momentum-energy complex*. It has the explicit evaluation

$$\mathscr{T}^e_a = (p_i - P_i)\delta^e_4 Y^i_a - (\sigma^A_i - t^A_i)\delta^e_A Y^i_a - \delta^e_a(\mathscr{T} - \mathscr{U} + \mathscr{N}). \tag{7.8}$$

We now have all of the pieces in order to evaluate the field equations for the compensating fields ψ^a_b. The equations of *balance of material defects* are

$$2\,\partial_c\{J\mathscr{Q}^{cb}_a\} = J\left\{\frac{\partial \mathscr{V}}{\partial \psi^a_b} + \mathscr{V}j^b_a\right\} - J\mathscr{T}^e_a j^b_e. \tag{7.9}$$

These are the new field equations that come from local action of the material translation group.

There is an obvious and inescapable difference between (7.9) and the balance of spatial defects. The equations of balance of spatial defects have source terms that are linear in the effective stress and momenta, while the balance of material defects have source terms that depend on the gauge-theoretic momentum-energy complex; that is, the sources are intrinsically nonlinear in the matter fields. Thus, the balance of spatial defects may be approximated by linear equations under certain circumstances (see [3], Chap. 9), while the balance of material defects have no elementary linearizations. *Evolution of material defects are inherently nonlinear processes.*

Another significant difference is related to the integrability conditions demanded by the equations of balance of material defects. Noting that $\mathscr{Q}^{cb}_a = -\mathscr{Q}^{bc}_a$, the left-hand sides of (7.9) satisfy the identity

$$\partial_b\partial_c\{J\mathscr{Q}^{cb}_a\} = 0.$$

Accordingly, the terms on the right-hand sides of (7.9) must satisfy the relations

$$\partial_b\left\{J\left(\frac{\partial\mathcal{V}}{\partial\psi_b^a}+\mathcal{V}j_a^b\right)\right\}=\partial_b\{J\mathcal{T}_a^e j_e^b\} \tag{7.10}$$

if the field equations (7.9) are to be solvable.

The classical momentum-energy complex for the matter fields is such that any solution of the classical Euler–Lagrange equations will necessarily satisfy the equations of classical balance of field momentum-energy (see [2], Theorem 7–3.2)

$$\partial_e T_a^e=-(\partial_a t_i^A)\partial_A\chi^i+(\partial_a P_i)\partial_4\chi^i. \tag{7.11}$$

The right-hand sides of these equations are not zero because the original matter Lagrangian L_0 depends explicitly on the independent variables through the functions $P_i(X^c)$ and $t_i^A(X^c)$ that appear in the null Lagrangian $\mathcal{N}$. On the other hand

$$\mathcal{M}\langle\partial_e T_a^e\rangle=j_e^b\,\partial_b\mathcal{T}_a^e=j_e\langle\mathcal{T}_a^e\rangle, \tag{7.12}$$

and hence $j_e\langle\mathcal{T}_a^e\rangle$ are the gauge-theoretic generalization of the "divergence quantities" $\partial_e T_a^e$ of the classical theory. Now, a rearrangement of the terms in (7.10) give the explicit evaluations

$$j_e\langle\mathcal{T}_a^e\rangle=J^{-1}\,\partial_b\left\{J\left(\frac{\partial\mathcal{V}}{\partial\psi_b^a}+\mathcal{V}j_a^b\right)\right\}-J^{-1}\mathcal{T}_a^e\,\partial_b\{Jj_e^b\}. \tag{7.13}$$

These equations may thus be interpreted as equations of *gauge balance of field momentum-energy*. In particular, the classical equations of balance of field momentum-energy have gone over into equations of gauge balance of gauge-theoretic momentum-energy with the specific source terms that appear on the right-hand sides of (7.13). When $\mathcal{V}$ is constant and $J_b^a=\delta_b^a$, (7.13) reduce to (7.11), but we are then back in the domain of validity of elasticity theory.

We note, in particular, that the classical law of conservation of energy of a classic elastic body without externally applied tractions has gone over into the following equation of gauge balance of energy (simply take $a=4$),

$$j_e\langle\mathcal{T}_4^e\rangle=J^{-1}\,\partial_b\left\{J\left(\frac{\partial\mathcal{V}}{\psi_b^4}+\mathcal{V}j_4^b\right)\right\}-J^{-1}\mathcal{T}_4^e\,\partial_b\{Jj_e^b\}, \tag{7.14}$$

and that

$$\mathcal{T}_4^4=\mathcal{T}+\mathcal{U}-\mathcal{N}+\frac{\partial\mathcal{N}}{\partial Y_4^i}Y_4^i=\mathcal{M}\left\langle T+U-N+\frac{\partial N}{\partial(\partial_4\chi^i)}\partial_4\chi^i\right\rangle \tag{7.15}$$

is the minimal replacement equivalent of the matter field energy density. Thus, according to (7.14), there is the source of total field energy density that comes from the free gauge field Lagrangian $\mathcal{V}$, and the "convective" energy source that comes from the fact that Jj_e^b is not a constant tensor density (i.e. that all geometric quantities are referred to the fundamental frame and coframe fields that vary from point to point in the body's history). Material dislocations thus provide ample opportunities to model intrinsically nonconservative systems, as has already been indicated in [4]. Whether or not such systems will be strictly dissipative is another matter altogether, and will be dealt with in a later paper in this series.

Gauge covariance of the field equations (7.9) can be established by introducing the 2-forms of *material defect intensity*

$$\mathcal{Q}_a=\frac{1}{2}J\mathcal{Q}_a^{cb}\pi_{cb}; \tag{7.16}$$

that is, the field equations (7.9) have the equivalent gauge covariant presentation

$$2\,\mathrm{d}\mathcal{Q}_a=\left\{J\left(\frac{\partial\mathcal{V}}{\partial\psi_b^a}+\mathcal{V}j_a^b\right)-J\mathcal{T}_a^e j_e^b\right\}\pi_b. \tag{7.17}$$

Thus, satisfaction of the equations of balance of material defects in any one gauge implies satisfaction of these field equations in all choices of gauge.

Boundary conditions for the ψ-fields are obtained in exactly the same way as that used for the ϕ-fields. They take the form

$$(\mathcal{Q}_a \wedge \delta\psi^a)|_{\partial\mathbb{B}_4} = 0 \tag{7.18}$$

for all 1-form variations $\delta\psi^a = \delta\psi^a_b \, dX^b$ of the ψ-fields.

8. ADMISSIBLE FORMS FOR THE FREE GAUGE FIELD LAGRANGIAN

The free gauge field Lagrangian $\mathcal{V}J$ is constructed from a function $\mathcal{V} = \mathcal{V}(J^a_b, \Sigma^i_{ab}, \Xi^c_{ab})$ that is required to be *invariant* in value under the action of the gauge group. This means that $\mathcal{V}$ has to satisfy

$$\mathcal{V}({}''J^d_e, \Sigma^j_{ef}, {}''\Xi^d_{ef}) = \mathcal{V}(J^a_b, \Sigma^i_{ab}, \Xi^c_{ab}), \tag{8.1}$$

when the "new" fields are expressed in terms of the "old" fields by the transformation laws given in Section 2. A glance at (2.21) and (2.22) shows that the transformation laws for the Σ's and Ξ's are quadratic in the inverse Jacobian matrices a^b_c of the coordinate transformations of $\mathbb{R}^4$ induced by local action of the material translation group. Now, the j's are well defined functions of the J's, while (2.18) shows that the transformation laws for the j's are linear in the Jacobian matrices A^b_c. We can therefore eliminate the dependence on Jacobian matrices and their inverses by introducing the new quantitites

$$\hat{\Sigma}^i_{ab} = \Sigma^i_{cd} j^c_a j^d_b, \qquad \hat{\Xi}^c_{ab} = \Xi^c_{de} j^d_a j^e_b. \tag{8.2}$$

These quantities exhibit the antisymmetries

$$\hat{\Sigma}^i_{ab} = -\hat{\Sigma}^i_{ba}, \qquad \hat{\Xi}^c_{ab} = -\Xi^c_{ba}, \tag{8.3}$$

and obey the transformation laws

$$''\hat{\Sigma}^i_{ab} = Q^i_j \hat{\Sigma}^j_{de} r^d_a r^e_b, \qquad ''\hat{\Xi}^c_{ab} = R^c_f \hat{\Xi}^f_{de} r^d_a r^e_b. \tag{8.4}$$

Since the Q's, R's, and r's are constant-valued orthogonal matrices, we see from (8.4) that the $\hat{\Sigma}$'s and $\hat{\Xi}$'s transform as scalars under coordinate transformations of $\mathbb{R}^4$ that are generated by local action of the material translation group. They are thus acceptable arguments for a scalar-valued invariant function $\mathcal{V}$.

In the interests of simplicity, and for practical necessity, we restrict our attention to the construction of invariants of degree two or less in the $\hat{\Sigma}$'s and the $\hat{\Xi}$'s. Because **Q** and **R** are distinct orthogonal matrices, there can be no mixing between the $\hat{\Sigma}$'s and the $\hat{\Xi}$'s. The first of (8.4) shows that there are no linear invariants in the $\hat{\Sigma}$'s. When we recall the special nature of the matrix **R** given by (2.12), we see that the space and time indices naturally *split*. The quadratic invariants of the spatial quantities have the following form:

$$\mathcal{V}_s = k_1 \hat{\Sigma}^i_{AB} \delta_{ij} \hat{\Sigma}^j_{CD} \delta^{AC} \delta^{BD} + k_2 \hat{\Sigma}^i_{A4} \delta_{ij} \hat{\Sigma}^j_{B4} \delta^{AB}. \tag{8.5}$$

An examination of the second of (8.4) shows that there is the linear invariant $\hat{\Xi}^A_{A4}$. Inclusion of such a linear invariant will simply shift the "zero point" of the fields, so we will ignore it. Now, the transformation laws for the $\hat{\Xi}$'s involve only the orthogonal matrix **R**, and its inverse **r**, and the space and time parts of the $\hat{\Xi}$'s also split because of the special form of the **R** matrix given by (2.12). We therefore have the following quadratic invariants of the material quantities:

$$\begin{aligned}\mathcal{V}_m = {} & K_1 \hat{\Xi}^E_{AB} \delta_{EF} \hat{\Xi}^F_{CD} \delta^{AC} \delta^{BD} + K_2 \hat{\Xi}^E_{EA} \delta^{AB} \hat{\Xi}^F_{FB} + K_3 \hat{\Xi}^E_{A4} \delta_{EF} \hat{\Xi}^F_{B4} \delta^{AB} \\ & + K_4 \hat{\Xi}^4_{AB} \hat{\Xi}^4_{CD} \delta^{AC} \delta^{BD} + K_5 \hat{\Xi}^4_{A4} \delta^{AB} \hat{\Xi}^4_{B4}.\end{aligned} \tag{8.6}$$

The total quadratic scalar invariant $\mathcal{V}$ therefore takes the form

$$\mathcal{V} = \mathcal{V}_s + \mathcal{V}_m. \tag{8.7}$$

We have noted in [1] that the quantities

$$h^{ab} = j^a_A \delta^{AB} j^b_B \tag{8.8}$$

play the role of components of a contravariant metric tensor, albeit one of rank 3. This fact can be used to write some of the invariants in (8.5) and (8.6) in a form directly analogous to quantities that occur in the Lagrangian for electromagnetic fields. In fact, we have

$$\hat{\Sigma}^i_{AB}\delta_{ij}\hat{\Sigma}^j_{CD}\delta^{AC}\delta^{BD} = \Sigma^i_{ab}\delta_{ij}\Sigma^j_{cd}h^{ac}h^{bd}, \tag{8.9}$$

$$\hat{\Xi}^E_{AB}\delta_{EF}\hat{\Xi}^F_{CD}\delta^{AC}\delta^{BD} = \Xi^E_{ab}\delta_{EF}\Xi^F_{cd}h^{ac}h^{bd}, \tag{8.10}$$

$$\hat{\Xi}^4_{AB}\hat{\Xi}^4_{CD}\delta^{AC}\delta^{BD} = \Xi^4_{ab}\Xi^4_{cd}h^{ac}h^{bd}. \tag{8.11}$$

The remaining invariants in the lists (8.5) and (8.6) come about precisely because h^{ab} is a 4×4 matrix of rank 3 such that $h^{ab}J^4_b = 0$. The generalized "time direction" defined by the 1-form $J^4 = J^4_a\,\mathrm{d}X^a$ can thus support additional invariants of the Σ's and Ξ's. This is the source of the extra invariants that appear in $\mathcal{V}$, but which are absent in classical electrodynamics where the metric tensor is of rank equal to 4.

Now that we have an explicit representation for the free field Lagrangian $\mathcal{V}J$, we can obtain explicit evaluations of the field intensity quantities $\mathcal{R}^{ab}_i$, $\mathcal{Q}^{ab}_c$, and the derivatives $\partial\mathcal{V}/\partial\psi^a_b$. These latter arise because $\mathcal{V}$ depends on the fields j^a_b, which are themselves functions of $J^a_b = \delta^a_b + \psi^a_b$. We leave these explicit evaluations to succeeding papers for the simple reason that they are lengthy and involved when all of the coupling constants k_1, k_2, K_1, K_2, K_3, K_4, and K_5 are nonzero. Fortunately, useful classes of problems can be analyzed with only a few of the coupling constants nonzero.

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APPENDIX

Bessel–Hagen Symmetries and Minimal Replacement

The null Lagrangian at the classical level used in this paper has the evaluation

$$N = t^A_i(X^c)\,\partial_A\chi^i - P_i(X^c)\,\partial_4\chi^i = \partial_A(t^A_i\chi^i) - \partial_4(P_i\chi^i) \tag{A1}$$

because the functions $\{t^A_i(X^c), P_i(X^c)\}$ are required to satisfy the relations

$$\partial_A t^A_i = \partial_4 P_i \tag{A2}$$

throughout the interior of $\mathbb{B}_4$. The action integral associated with N reduces to a boundary integral, by the divergence theorem, and hence changes in the action integral under global action of the gauge group reduce to boundary integrals. The global gauge group is thus a Noetherian symmetry group of the second kind (see [2], Sect. 7–4 and 7–5), or a Bessel-Hagen symmetry group [5].

It is clear from (A1) that arbitrary assignments of the functions $\{t^A_i(X^c), P_i(X^c)\}$ will result in N not being invariant under global transformations of the gauge group (i.e. the gauge group contains the group of translations on $\mathbb{R}^4$). Now, suppose that t^A_i are assumed to transform like components of Piola–Kirchhoff stress and P_i are assumed to transform like components of linear momentum under global action of the symmetry group; that is, we have

$$'t^B_j('X^e) = R^B_A t^A_i(X^c)q^i_j, \qquad 'P_j('X^e) = P_i(X^c)q^i_j, \tag{A3}$$

where we have used the single prime to remind us that the underlying transformations are global. In this event, N will be invariant under global action of the gauge group and the Bessel-Hagen symmetry group becomes an ordinary Noetherian symmetry group. In practice, the t's and P's will be known for a given coordinate cover of $\mathbb{R}^4$. Since the material gauge group is an active group that moves points around in $\mathbb{R}^4$, while the t's and P's have physical meaning only for a given location of $\mathbb{B}_4$ in $\mathbb{R}^4$, we may interpret (A3) as telling us how to change the evaluations of the t's and P's when move $\mathbb{B}_4$ around in $\mathbb{R}^4$ by the homogeneous action of the gauge group. When interpreted in this way, the t's and the P's will carry the boundary tractions, the initial momenta, and the final momenta along with the $\mathbb{B}_4$ as it is moved around in $\mathbb{R}^4$ by the global action of the gauge group. Imposition of the transformation relations (A3) are thus both physically meaningful and appropriate. The reader should note, however, that satisfaction of (A3) tells us nothing about how $\{t_i^A(X^c), P_i(X^c)\}$ transform under local action of the gauge group.

If we apply minimal replacement to N, we obtain

$$\mathcal{N} = (t_i^A(X^c)Y_A^i - P_i(X^c)Y_4^i)J. \tag{A4}$$

Let $\{''t_j^B(''X^e), ''P_j(''X^e)\}$ be the image of $\{t_i^A(X^c), P_i(X^c)\}$ under local action of the gauge group. We then have

$$''(\mathcal{N}\pi) = (''t_j^B(''X^e)''Y_B^j - ''P_j(''X^e)''Y_4^j)''J''\pi \tag{A5}$$

under local action of the gauge group. However, the Y's have the transformation laws (2.20), and hence (A5) will yield the invariance relation

$$''(\mathcal{N}\pi) = \mathcal{N}\pi \tag{A6}$$

provided we require the t's and the P's have the transformation laws

$$''t_j^B(''X^e) = R_A^B t_i^A(X^c)q_j^i, \qquad ''P_j(''X^e) = P_i(X^c)q_j^i. \tag{A7}$$

These, however, are just the gauge extension of the transformation laws (A3). Thus, even if we had not imposed (A3), and assumed only that the t's and p's were given for only realization of $\mathbb{B}_4$ in $\mathbb{R}^4$, the gauging process allows us to require satisfaction of the transformation laws (A7) and thereby obtain invariance of the minimally replaced Lagrangian 4-form $\mathcal{L}_1\pi$.