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MATERIAL AND SPATIAL GAUGE THEORIES OF SOLIDS—II. PROBLEMS WITH GIVEN MATERIAL DISLOCATION DENSITIES

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Abstract—Effects of material defects that arise from local action of the material translation symmetry group are analyzed. These problems are uncoupled by the assumptions that only the material translation group acts locally and that the material dislocation density 2-forms (material Cartan torsion) are both specified and 2-plane supported (elementary). All essential geometric quantities are evaluated. Since the material dislocation densities are specified, the only dynamical variables are the three displacement functions of the deformed body. Explicit forms of the dynamic equilibrium equations are obtained. These give new definitions of effective linear momentum and effective stress. Exact solutions of the field equations reproduce the stress field of straight screw dislocations, but the stress field for straight edge dislocations only obtain in the weak defect field limit. Both of these classes of problems arise through the local action of spatial material translations. The governing equations for problems with local action of the material time translation group are derived. They are shown to yield stress distributions that depend on the velocity components. Solutions of the field equations indicate that such models can represent relaxation phenomena.

1. INTRODUCTION AND NOTATION

A previous paper [1], henceforth referred to as I, laid the physical, geometric, and kinematic foundations of a guage theory for solids where both the spatial and the material symmetry groups were gauged on an equal footing. There we showed, contrary to previous thinking, that gauging the spatial (internal) symmetry group gave rise to virtual *microcracks* and *microrotations*, while gauging the material symmetry group gave rise to *dislocations* and *disclinations*. Since the material symmetry group has an active action on the base manifold (space of reference histories), the geometry and kinematics induced by local action of the material symmetry group is a good deal more complicated than that induced by the local action of the spatial symmetry group. There is an intrinsic unfamiliarity that lurks in the background when a gauge group acts on the base manifold. This can only be dispersed by careful analysis of specific problems where the answers are analytically simple and the predicted physical responses correspond to previously well analyzed situations.

Problems in the classical theory of dislocations are predicted on the validity of linear elasticity outside regions that support nonzero dislocation densities. Traditionally, one specifies the regions of nonzero dislocation density (i.e. regions occupied by dislocation cores) and the Burgers' vectors associated with the dislocation distributions. The problem is then a *semi-inverse* problem of computing the stress and displacement fields on the complements of the supports of the dislocation densities that result from the given distributions of dislocations. (Fully dynamic problems relating to how to compute the dislocation distributions that result from given loading histories are kinetic problems that are usually considered to be outside the domain of classical dislocation theory.) A recent publication [2] has shown that classical semi-inverse dislocation problems can be solved globally; that is, both inside and outside the dislocation cores provided the details of the dislocation density distributions are given. Since these problems form a substantive class that have conceptual significance and practical importance, they form the obvious candidates for analysis in the gauge theory of material symmetries.

The purposes of this paper are twofold. First, we provide a solid point of departure for understanding the full kinetic theory of material defects by obtaining exact global representations for elementary distributions of dislocation density by the semi-inverse method. These global representations will include those appropriate to straight screw and edge dislocation cores. We will also include semi-inverse representations for problems with elementary distributions of defects that arise from the local action of the material time translation group. Such defects have no classical analog. They are important, however, because breaking global time translation symmetry breaks global conservation of energy, and hence "time dislocation" effects provide means appropriate to modeling of dissipative mechanisms. Second, we compare the predictions of gauge theory for the material translation symmetry group with the predictions of classical dislocation theory. The agreement is sufficiently startling that it should convince most skeptics of the utility of the gauge-theoretic approach as a viable alternative. Be this as it may, the analysis of these classes of elementary problems in the more complicated context of gauge theory should provide a heightened understanding and appreciation of the geometry, the physics, and how they interrelate.

The notation used throughout this paper will be that introduced in [1]. Citation of equations from [1] will be by equation numbers prefixed with the Roman numeral I.

We restrict attention to problems in which only the 4-dimensional material translation group is allowed to act locally. This assumption has the simplifying consequence that the spatial distortion 1-forms B^i are exact 1-forms;

$$B^i = \mathrm{d}\chi^i(X^a), \qquad B^4 = \mathrm{d}X^4 = \mathrm{d}T. \tag{1.1}$$

Here $x^i = \chi^i(X^a)$, $t = X^4 = T$, define the mapping from the base space $\mathbb{R}^4$ of reference histories with coordinates $\{X^a \mid 1 \le a \le 4\}$ into the space of actual histories. Where necessary, we will use $\{X^A \mid 1 \le A \le 3\}$ for the spatial coordinates of $\mathbb{R}^4$, in which case $\{X^s\} = \{X^A, X^4\}$. Further, since we are disallowing local material rotations, the curvature 2-forms of the material $SO(3)$ sector vanish and the connection 1-forms for this sector vanish in the antiexact gauge. The fundamental coframe fields on $\mathbb{R}^4$ thus take the form

$$J^a = \mathcal{M}\langle \mathrm{d}X^a \rangle = J^a_b(X^c)\,\mathrm{d}X^b = \mathrm{d}X^a + \psi^a, \tag{1.2}$$

where

$$\psi^a = \psi^a_b(X^c)\,\mathrm{d}X^b \tag{1.3}$$

are the compensating 1-forms for the local action of the 4-dimensional translation group and $\mathcal{M}$ is the resulting minimal replacement operator. We therefore admit transformations of the coordinate covers of $\mathbb{R}^4$ of the form

$${}'X^A = R^A_B X^B + T^A(X^c), \qquad {}'X^4 = X^4 + T^4(X^c), \tag{1.4}$$

where R^A_B are the entries of a constant-valued orthogonal matrix, because these are the transformations that are induced by local action of the material translation group when the material rotation group is frozen in its global homogeneous phase.

We have already noted that the material curvature tensor vanishes throughout $\mathbb{R}^4$ because the material rotation group is assumed to be frozen in its global homogeneous phase. The only nontrivial differential concomitants of $\mathbb{R}^4$ are therefore the material Cartan torsion (material dislocation density and current) 2-forms Ξ^a (see I, Sect. 5). Since the connection 1-forms for the material rotation group vanish in the antiexact gauge, (I-5.9) show that

$$\Xi^a = \mathrm{d}J^a = \frac{1}{2}(\partial_b\psi^a_c - \partial_c\psi^a_b)\,\mathrm{d}X^b \wedge \mathrm{d}X^c. \tag{1.5}$$

We therefore have

$$\mathrm{d}\Xi^a = 0. \tag{1.6}$$

The entire set of Cartan equations of structure induced by the material symmetry group (the kinematic field equations for material defects) thus reduce to

$$\mathrm{d}J^a = \Xi^a, \qquad \mathrm{d}\Xi^a = 0. \tag{1.7}$$

2. ELEMENTARY MATERIAL DISLOCATION DENSITY DISTRIBUTIONS

A distribution of material dislocations is said to be *simple* if and only if its associated Cartan torsion 2-forms have the specification

$$\Xi^a = f(X^c)k^a\,\mathrm{d}X \wedge \mathrm{d}Y, \qquad \mathrm{d}k^a = 0 \tag{2.1}$$

for an appropriate orientation of the coordinate system for $\mathbb{R}^4$. Simple material dislocation density 2-forms may thus be envisioned as generated by distributions of "micro-dislocation lines" that are parallel to the Z-axis. Further, since the k's are constants, all of the material Burgers' vectors for this family of parallel micro-dislocation lines are parallel to the *four* dimensional vector $\{k^a\}$. The factor $f(X^c)$ thus plays the role of a generalized amplitude since we can obviously choose the vector $\{k^a\}$ to be a unit vector without loss of generality. Thus, if $\{k^a\}$ has only a Z-component, (2.1) should model a distribution of parallel, material, *screw micro-dislocations*. If $\{k^a\}$ has only an X-component, then (2.1) should model a distribution of parallel, material, *edge micro-dislocations*. Since we have also provided the capacity for local action of material time translations, we can also consider the case where $\{k^a\}$ has only a time component. In this case, (2.1) should model what we will term a distribution of parallel, material, *time micro-dislocations*. Since time micro-dislocations are outside the domain of classical dislocation theory, the study of elementary time micro-dislocations should provide a framework for understanding this new class of material defects.

In the present context, the kinematic equations demand that $\mathrm{d}\Xi^a = 0$. When the representations (2.1) are used, we thus obtain the conditions

$$0 = \mathrm{d}\Xi^a = k^a\,\mathrm{d}f(X^c) \wedge \mathrm{d}X \wedge \mathrm{d}Y. \tag{2.2}$$

Now, these conditions can be satisfied only when the function f does not depend on the arguments Z and T. Accordingly, distributions of elementary material dislocation density 2-forms have the representations

$$\Xi^a = k^a f(X, Y)\,\mathrm{d}X \wedge \mathrm{d}Y, \qquad \mathrm{d}k^a = 0. \tag{2.3}$$

The fundamental coframe fields are related to the dislocation density 2-forms by

$$\mathrm{d}J^a = \Xi^a = k^a f(X, Y)\,\mathrm{d}X \wedge \mathrm{d}Y. \tag{2.4}$$

Integrating these equations by use of the linear homotopy operator with center at the origin (see [2] and [3], Chap. 5) gives us

$$J^a = \mathrm{d}X^a + k^a F(X, Y)\{X\,\mathrm{d}Y - Y\,\mathrm{d}X\}, \tag{2.5}$$

where

$$F(X, Y) = \int_0^1 \lambda f(\lambda X, \lambda Y)\,\mathrm{d}\lambda. \tag{2.6}$$

If we introduce cylindrical polar coordinates in $\mathbb{R}^4$ by

$$X = R\cos\theta, \qquad Y = R\sin\theta, \qquad Z = Z, \tag{2.7}$$

the coframe fields take the possibly simpler form

$$J^a = \mathrm{d}X^a + k^a R^{-2} G(R, \theta)\{X\,\mathrm{d}Y - Y\,\mathrm{d}X\} = \mathrm{d}X^a + k^a G(R, \theta)\,\mathrm{d}\theta. \tag{2.8}$$

Here, the function $G(R, \theta)$ is given by

$$G(R, \theta) = \int_0^R s f(s\cos\theta, s\sin\theta)\,\mathrm{d}s. \tag{2.9}$$

Accordingly, if the support of $f(X, Y)$ is contained in the cylinder $R \le R_0$ (i.e. Ξ^a all vanish for $R > R_0$), we have

$$G(R, \theta) = G(R_0, \theta) \quad \text{for} \quad R > R_0. \tag{2.10}$$

In this event, (2.8) gives

$$J^a = \mathrm{d}X^a + k^a G(R_0, \theta)\,\mathrm{d}\theta \quad \text{for} \quad R > R_0. \tag{2.11}$$

The coframe fields are thus exact 1-forms for $R > R_0$, but are clearly not exact 1-forms for $R < R_0$ because

$$J^a = \mathrm{d}X^a + k^a G(R, \theta)\,\mathrm{d}\theta \quad \text{for} \quad R < R_0 \tag{2.12}$$

and (2.9) shows that $\partial G/\partial R = Rf(R\cos\theta, R\sin\theta) \neq 0$ for $f(X, Y) \neq 0$. Thus, *nontrivial distributions of elementary material dislocation densities of compact support give rise to* *fundamental coframe fields that are exact outside the support of the dislocation density distributions, but are not exact on the support of the dislocation density distributions.*

A result that we will need later is

$$J = \det(J^a_b) = \det(\delta^a_b + Fk^a(X\delta^2_b - Y\delta^1_b)) = 1 + (k^2X - k^1Y)F(X, Y)$$
$$= 1 + R^{-2}G(R, \theta)(k^2X - k^1Y). \tag{2.13}$$

3. FUNDAMENTAL FRAME FIELDS, TOTAL DISTORTION, AND EQUILIBRIUM EQUATIONS

The fundamental frame fields are vector fields

$$j_a = j^b_a(X^c)\, \partial_b \tag{3.1}$$

whose coefficient functions satisfy

$$J^b_a(X^e) j^a_c(X^e) = \delta^b_c. \tag{3.2}$$

The j's are thus obtained by inverting the matrix $((J^a_c))$. The results are a lengthy algebraic mess when the unit vector with components $\{k^a\}$ in (2.4) is in general position on the unit sphere. We will therefore take up specific cases in succeeding sections where the choice of $\{k^a\}$ allows us to obtain simple closed form expressions for the frame fields.

The reason why the fundamental frame fields are important is that minimal replacement of the configuration gradients, $\partial_a\chi^i$, gives the components of *total distortion* (see (I-4.21) and (I-4.24))

$$Y^i_a = \mathcal{M}\langle \partial_a\chi^i\rangle = j^b_a B^i_b, \tag{3.3}$$

and hence

$$\mathcal{M}\langle U(\partial_a\chi^i)\rangle = U(Y^i_a)J, \tag{3.4}$$

where $U(\partial_a\chi^i)$ is the classical strain energy function for the elastic phase of the material. There is, however, a significant simplification here. We have assumed that the spatial symmetry group is frozen in its global phase (i.e. it does not act locally). This means that

$$B^i = B^i_b(X^c)\, \mathrm{d}X^b = \partial_b\chi^i(X^c)\, \mathrm{d}X^b, \tag{3.5}$$

and hence (3.3) gives

$$Y^i_a = j^b_a\, \partial_b\chi^i = j_a\langle\chi^i\rangle. \tag{3.6}$$

Thus, the components of total distortion are uniquely determined by the j's and the mapping functions. If we introduce displacement functions $u^i(X^c)$ by

$$\chi^i = \delta^i_A X^A + u^i(X^c), \tag{3.7}$$

then (3.6) gives

$$Y^i_a = \delta^i_A j_a\langle X^A\rangle + j_a\langle u^i\rangle = j^i_a + j_a\langle u^i\rangle. \tag{3.8}$$

Any gauge theory starts with a Lagrangian function that is invariant under the global action of the gauge group. Since elementary distributions of material dislocations have specified frame and coframe fields and specified 2-forms of material dislocation density, the only quantities that are not determined are the displacement functions $u^i(X^c)$. Thus, the only terms that can be varied in the Lagrangian function for the problems under study are those that depend on the displacement functions. We may thus ignore all contributions in the total Lagrangian except those that arise through minimal replacement of the Lagrangian corresponding elastic phase of the material (i.e. we may ignore the free gauge field Lagrangian L_g). The Lagrangian for the elastic phase of the material has the form

$$L_0 = \frac{1}{2}\rho_0\, \partial_4\chi^i\delta_{ij}\, \partial_4\chi^j - U_0(\partial_A\chi^i). \tag{3.9}$$

If there are external loading devices that give rise to distributions of boundary tractions $\{T_i\}$, then the work performed by the assigned tractions is accounted for by adding a null Lagrangian of the form

$$\mathcal{N} = t_i^A(X^c)\, \partial_A \chi^i, \tag{3.10}$$

where $\{t_i^A(X^c)\}$ are functions that satisfy

$$\partial_A t_i^A = 0 \tag{3.11}$$

at all interior points of the body and the traction boundary conditions

$$t_i^A n_A = T_i. \tag{3.12}$$

Here, $\{n_A\}$ are the components of the outward oriented unit normal field of the spatial boundaries of the body. The total elastic Lagrangian is therefore of the form

$$L_e = \frac{1}{2} \rho_0\, \partial_4 \chi^i \delta_{ij}\, \partial_4 \chi^j - U_0(\partial_A \chi^i) + t_i^A\, \partial_A \chi^i. \tag{3.13}$$

The effective Lagrangian for bodies with specified distributions of elementary defects is obtained by minimal replacement: $L = \mathcal{M}\langle L_e \rangle$. This gives

$$L = \left\{ \frac{1}{2} \rho_0 Y_4^i \delta_{ij} Y_4^j - U(Y_A^i) + t_i^A Y_A^i \right\} J \tag{3.14}$$

because the multiplicative factor J comes from minimal replacement for the volume element of $\mathbb{R}^4$.

Now, the components of stress in a gauge theory are defined by

$$\sigma_i^A = \mathcal{M}\left\langle \frac{\partial U_0}{\partial(\partial_A \chi^i)} \right\rangle = \frac{\partial U(Y_B^j)}{\partial Y_A^i}, \tag{3.15}$$

while the components of linear momentum have the evaluations

$$p_i = \mathcal{M}\left\langle \frac{\partial L_e}{\partial(\partial_4 \chi^i)} \right\rangle = \rho_0 Y_4^i. \tag{3.16}$$

On the other hand, we have

$$\begin{aligned} \frac{\partial L}{\partial(\partial_A \chi^i)} &= \frac{\partial L}{\partial Y_B^j} \frac{\partial Y_B^j}{\partial(\partial_A \chi^i)} + \frac{\partial L}{\partial Y_4^j} \frac{\partial Y_4^j}{\partial(\partial_A \chi^i)} \\ &= J(t_i^B - \sigma_i^B) j_B^A + p_i J j_4^A, \end{aligned} \tag{3.17}$$

and

$$\frac{\partial L}{\partial(\partial_4 \chi^i)} = J(t_i^B - \sigma_i^B) j_B^4 + p_i J j_4^4 \tag{3.18}$$

when (3.8) is used.

The *dynamic equilibrium equations* for the state variables $\{\chi^i\}$) for elementary distributions of material dislocations are the Euler–Lagrange equations

$$\partial_A \left\{ \frac{\partial L}{\partial(\partial_A \chi^i)} \right\} + \partial_4 \left\{ \frac{\partial L}{\partial(\partial_4 \chi^i)} \right\} = 0.$$

When the above evaluations are used, we obtain the rather complicated system of dynamic equilibrium equations

$$\partial_4 \{ J j_4^4 p_i + J(t_i^B - \sigma_i^B) j_B^4 \} = \partial_A \{ J(\sigma_i^B - t_i^B) j_B^A - J j_4^A p_i \}. \tag{3.19}$$

If the distributions of elementary material dislocation density vanish throughout the body (i.e. if $f(X, Y) = 0$), we have $J = 1$, $J_b^a = j_b^a = \delta_b^a$, and $Y_A^i = \partial_A \chi^i$. In this case, (3.19) reduce to the dynamic equilibrium equations for an elastic body because $\partial_A t_i^A = 0$. It is obvious from an inspection of (3.19) that the presence of nontrivial j's has a profound effect in addition to the fact that the stress is now a tensor-valued function of the components of toal distortion, $Y_a^i = j_a^b\, \partial_b \chi^i$. In particular, we may define components of *effective linear momentum* by

$$\mathcal{P}_i = J j_4^4 p_i + J j_B^4 (t_i^B - \sigma_i^B) \tag{3.20}$$

and components of *effective stress* by

$$\mathcal{S}_i^A = Jj_B^A(\sigma_i^B - t_i^B) - Jj_4^A p_i, \tag{3.21}$$

because the equations of dynamic equilibrium assume the standard form

$$\partial_4 \mathcal{P}_i = \partial_A \mathcal{S}_i^A. \tag{3.22}$$

There is therefore a stress contribution to the effective linear momentum, as indicated by (3.20), and a linear momentum contribution to the effective stress. These are direct consequences of the *unique* factorization $Y_a^i = j_a^b\, \partial_b \chi^i = j_a^b B_b^i$ of material gauge theory, and are anything but obvious from the more customary point of view. We know, however, that the factorization $Y_a^i = j_a^b\, \partial_b \chi^i$ is fundamental in the derivation of classical phenomenological plasticity theory [4]. What seems to have been missed in phenomenological plasticity are the collateral stress contributions to the effective linear momentum and the linear momentum contributions to the effective stress that are shown in (3.20) and (3.21). The reader should also note that these same effects will be present when we allow the defect fields to become dynamical variables rather than fields that are specified through the assumption of elementary defects. In effect, the assumptions of elementary defects allows us to ignore the fact that the defect variables are dynamical fields, so that we may concentrate on the effects of the presence of specified defects in solids. This, however, is exactly what is done in the classical theory of defects when solving problems with given distributions of edge and screw dislocations; the distributions are specified and we calculate the displacements and stresses needed to equilibrate the presence of the specified defects! The reader should note that the effective stress tensor $\mathcal{P}_i^A$ ceases to be symmetric even when $\sigma_i^A - t_i^A$ is symmetric.

Noting that classical dislocation theory is predicated on linear elasticity, some remarks would seem in place here concerning collateral linearization procedures for the geometric non-linearities inherent in a gauge theory for material translation groups. We start with the basic relations

$$J_b^a = \delta_b^a + \psi_b^a, \tag{3.23}$$

where the ψ's are the coefficients of the compensating 1-forms for local action of the translation group. Under the condition

$$\|\psi_b^a\| \ll 1, \tag{3.24}$$

we would have the approximate evaluation

$$J \approx 1 + \psi_c^c. \tag{3.25}$$

Evaluation of the inverse of the matrix $((J_b^a))$ to the same order gives

$$j_b^a \approx \delta_b^a - \psi_b^a. \tag{3.26}$$

The Y's then have the approximate evaluations

$$Y_a^i \approx \delta_a^i + \partial_a u^i - \psi_a^i - \psi_a^b\, \partial_b u^i. \tag{3.27}$$

If we further assume that the ψ's and the displacement gradients are of the same order, the last terms become negligible and we obtain

$$Y_a^i \approx \delta_a^i + \partial_a u^i - \psi_a^i. \tag{3.28}$$

We shall refer to this situation as the *weak defect field limit.* The relations (3.28) are exactly what we would obtain from minimal replacement for a *spatial* translation group if we were to set $\psi_a^i = -\phi_a^i$ (see [5], Chap. 3). We may thus conclude that all of the results reported for spatial gauge groups for solids become immediately available in the weak defect field limit. In particular, the solutions for edge and screw dislocations reported in [5] and [6] can be carried over as weak field approximate solutions. Now, the classic solution for a straight screw dislocation is well founded and totally reasonable outside the dislocation core. We will show in the next section that the classical solution for the screw dislocation can be obtained from the gauge theory for the material translation group *without approximation.* Things are not quite so nice for the classical edge dislocation solution, as is well known. This is also reflected here, for we shall show that the classical edge dislocation solution obtains only in the weak defect field limit.

4. THE STRAIGHT SCREW DISLOCATION

The simplest problem to consider is that corresponding to the straight screw dislocation. We therefore consider a body of infinite spatial extent that is traction free at spatial infinity. This means that we may take $t_i^B = 0$ because $T_i = 0$. In order to have a distribution of elementary screw microdislocations, we take

$$k^a = \delta_3^a. \tag{4.1}$$

Confinement of the screw microdislocations to a core region $R \leq R_0$ can be obtained by the requirement

$$f(X, Y) = 0 \quad \text{for} \quad R > R_0. \tag{4.2}$$

We shall further assume that the distribution of screw microdislocation density has cylindrical symmetry so that

$$f(X, Y) = \mathscr{F}(R), \qquad R^2 = X^2 + Y^2, \tag{4.3}$$

and $\mathscr{F}(R)$ vanishes for $R > R_0$. These assumptions give us

$$F(X, Y) = R^{-2}\int_0^R s\mathscr{F}(s)\,\mathrm{d}s \quad \text{for} \quad R \leq R_0 \tag{4.4}$$

and

$$F(X, Y) = R^{-2}\int_0^{R_0} s\mathscr{F}(s)\,\mathrm{d}s = \frac{b}{2\pi}R^{-2} \quad \text{for} \quad R > R_0, \tag{4.5}$$

where the constant b is determined by

$$b = 2\pi\int_0^{R_0} s\mathscr{F}(s)\,\mathrm{d}s. \tag{4.6}$$

The fundamental coframe fields for this problem can be obtained by substituting (4.1) and (4.4) into (2.5). We therefore have

$$J^1 = \mathrm{d}X, \qquad J^2 = \mathrm{d}Y, \qquad J^4 = \mathrm{d}T, \qquad J^3 = \mathrm{d}Z + F(X\,\mathrm{d}Y - Y\,\mathrm{d}X), \tag{4.7}$$

and hence

$$J = \det(J_b^a) = 1. \tag{4.8}$$

Now, (4.5) and (4.7) show that

$$J^3 = \mathrm{d}Z + \frac{b}{2\pi}\mathrm{d}\theta \quad \text{for} \quad R > R_0.$$

We therefore have

$$b = \int_\gamma J^3$$

where γ is any closed contour in the (X, Y)-plane that contains the circle $R = R_0$. The constant b is therefore the total Burgers' vector for the screw dislocation core distribution. Accordingly, (4.5) and (4.7) show that the J's only depend on the first moment b if the screw microdislocation distribution in the region $R > R_0$. On the other hand, (4.4) shows that J^3 and hence the J's depend explicitly on the screw microdislocation distribution $\mathscr{F}(R)$ in the core region $R \leq R_0$.

The matrix $((J_b^a))$ is sufficiently simple that it can be inverted in simple closed form. This gives the fundamental frame fields

$$j_1 = \partial_X + FY\,\partial_Z, \qquad j_2 = \partial_Y - FX\,\partial_Z, \qquad j_3 = \partial_Z, \qquad j_4 = \partial_T. \tag{4.9}$$

Accordingly, $Y^i_a = j_a\langle\chi^i\rangle$ give the following explicit evaluations:

$$Y^i_1 = (\partial_X + FY\,\partial_Z)\langle X^i\rangle, \qquad Y^i_2 = (\partial_Y - FX\,\partial_Z)\langle\chi^i\rangle, \tag{4.10}$$

$$Y^i_3 = \partial_Z\langle\chi^i\rangle, \qquad Y^i_4 = \partial_T\langle\chi^i\rangle. \tag{4.11}$$

If we introduce the elastic displacement functions $\{u^i\}$ by (3.7), we obtain

$$Y^i_1 = \delta^i_1 + \partial_X u^i + FY\delta^i_3 + FY\,\partial_Z u^i, \tag{4.12}$$

$$Y^i_2 = \delta^i_2 + \partial_Y u^i - FX\delta^i_3 - FX\,\partial_Z u^i, \tag{4.13}$$

$$Y^i_3 = \delta^i_3 + \partial_Z u^i, \qquad Y^i_4 = \partial_T u^i. \tag{4.14}$$

Now that we have the j's, we can write out the dynamic equilibrium equations. Since F is independent of time and there are no tractions at infinity in the (X, Y)-plane, the problem is a static one. We may therefore set $\partial_T u^i = 0$, in which case we have the following static equilibrium equations:

$$0 = \partial_X\sigma^1_1 + \partial_Y\sigma^2_1 + \partial_Z\{F(Y\sigma^1_1 - X\sigma^2_1) + \sigma^3_1\}, \tag{4.15}$$

$$0 = \partial_X\sigma^1_2 + \partial_Y\sigma^2_2 + \partial_Z\{F(Y\sigma^1_2 - X\sigma^2_2) + \sigma^3_2\}, \tag{4.16}$$

$$0 = \partial_X\sigma^1_3 + \partial_Y\sigma^2_3 + \partial_Z\{F(Y\sigma^1_3 - X\sigma^2_3) + \sigma^3_3\}. \tag{4.17}$$

The problem has thus boiled down to specifying the dependence of the stresses on the Y's.

Classical dislocation theory is predicated on linear elasticity. Therefore, without further discussion or arguments, we set

$$\sigma_{\dot{A}B} = \frac{\lambda}{2} e^F_F \delta_{AB} + \mu e_{AB} \tag{4.18}$$

with

$$e_{AB} = Y_{\dot{A}B} + Y_{\dot{B}A} - 2\delta_{AB}; \tag{4.19}$$

that is, we take

$$U = \frac{\lambda}{8}(e^F_F)^2 + \frac{\mu}{4} e_{EF} e^{EF}.$$

The relations (4.19) define the linear engineering strain tensor of linear elasticity, to within a factor of two, while (4.18) give the standard linear stress–strain law. The strain tensor e_{AB} is known to be only an approximation of strain measures

$$E_{AB} = Y^i_A \delta_{ij} Y^j_B - \delta_{AB}.$$

On the other hand, the stress–strain relations (4.18) can be taken to be exact for a class of engineering materials.

An elementary calculation based on (4.15) through (4.19) shows that the equations of equilibrium are satisfied with

$$u^1 = u^2 = u^3 = 0. \tag{4.20}$$

The solution is not trivial, however, for the matrix of stresses has the evaluation

$$((\sigma_{\dot{A}B})) = \mu F \begin{bmatrix} 0 & 0 & Y \\ 0 & 0 & -X \\ Y & -X & 0 \end{bmatrix}. \tag{4.21}$$

For $R > R_0$, $F = (b/2\pi)R^{-2}$, and (4.21) agrees exactly in the region $R > R_0$ with the stress distribution for a screw dislocation outside of the straight dislocation core parallel to the Z-axis. In contrast to the classical solution, however, (4.21) is also valid inside the dislocation core (i.e. in the region $R \leq R_0$) and is everywhere finite if the microdislocation density $\mathcal{F}(R)$ is a bounded function. This follows from the fact that (4.5) gives $F(0) = \mathcal{F}(0)/2$ when $\mathcal{F}$ is bounded. This solution also agrees exactly with that obtained in [2], to which the reader is referred for a discussion of the meanings and interpretations of the differences between such solutions and the classical ones based on the Volterra displacement boundary value problem.

We note, in particular, that the solution for the straight screw dislocation has been obtained without explicit linearization of the geometric relations between coframe and frame fields generated by the local action of the translation group.

5. THE STRAIGHT EDGE DISLOCATION

Distributions of straight edge microdislocations are modeled by taking

$$k^a = \delta_1^a \tag{5.1}$$

for a body of infinite spatial extent that is traction free at infinity. Under these circumstances, we may take $t_i^A = 0$. We assume that the microdislocation density $f(X, Y)$ vanishes for $R > R_0$, so that the dislocations are confined to the core region $R \leq R_0$. The results obtained in Section 2 then show that the fundamental coframe fields are given by

$$J^1 = \mathrm{d}X + F(X, Y)\{X\,\mathrm{d}Y - Y\,\mathrm{d}X\}, \qquad J^2 = \mathrm{d}Y, \qquad J^3 = \mathrm{d}Z, \qquad J^4 = \mathrm{d}T. \tag{5.2}$$

We therefore have

$$J = \det(J_b^a) = 1 - FY. \tag{5.3}$$

For this problem, we have $J^1 = \mathrm{d}X + G(R_0, \theta)\,\mathrm{d}\theta$ for $R > R_0$. If we take $F(X, Y)$ to be a function of R only, then

$$J^1 = \mathrm{d}X + \frac{b}{2\pi}\,\mathrm{d}\theta$$

with b given by (4.6). In this case, b is evaluated by

$$b = \int_\gamma J^1,$$

for any closed contour in the (X, Y)-plane that contains the circle $R = R_0$. The constant b is thus the total Burgers' vector for the edge dislocation core region. This is the customary situation in the classical description of an edge dislocation, so we will assume that $F(X, Y)$ is a function of R only. The reader should note, however, that $F(X, Y)$ can be allowed to depend on both R and θ, and this may actually be necessary in order to obtain a correct model of an edge dislocation in material gauge theory. Indeed, the primitive notion of an edge dislocation has a naturally associated skewing of the (X, Y)-plane that is suggestive of an intrinsic dependence of the core dislocation density distribution on the angle θ. For such general $F(X, Y)$,

$$b = \int_\gamma J^1 = \int_0^{2\pi} G(R_0, \theta)\,\mathrm{d}\theta$$

would still define the total Burgers' vector for the edge dislocation core, but J^1 now has the more complicated evaluation $J^1 = \mathrm{d}X + G(R_0, \theta)\,\mathrm{d}\theta$; that is, $F = R^{-2}G(R_0, \theta)$ for $R > R_0$ which shows the explicit dependence on θ.

Since the matrix $((J_b^a))$ is easily inverted, we obtain the following simple evaluations for the fundamental frame fields:

$$j_1 = \frac{1}{1 - YF}\partial_X, \qquad j_2 = \partial_Y - \frac{XF}{1 - YF}\partial_X, \qquad j_3 = \partial_Z, \qquad j_4 = \partial_T. \tag{5.4}$$

Now that we have the j's, a direct calculation gives the evaluations

$$Y_1^i = \frac{1}{1 - YF}(\delta_1^i + \partial_X u^i), \qquad Y_3^i = \delta_3^i + \partial_Z u^i, \qquad Y_4^i = \partial_T u^i, \tag{5.5}$$

$$Y_2^i = \delta_2^i + \partial_Y u^i - \frac{XF}{1 - YF}(\delta_1^i + \partial_X u^i). \tag{5.6}$$

This problem is obviously a static one, so we may set $\partial_T u^i = 0$. Substituting the above results into the dynamic equations of equilibrium gives us the following system of static equilibrium equations:

$$0 = \partial_X\{\sigma_1^1 - XF\sigma_1^2\} + \partial_Y\{(1 - YF)\sigma_1^2\} + \partial_Z\{(1 - YF)\sigma_1^3\}, \tag{5.7}$$

$$0 = \partial_X\{\sigma_2^1 - XF\sigma_2^2\} + \partial_Y\{(1 - YF)\sigma_2^2\} + \partial_Z\{(1 - YF)\sigma_2^3\} \tag{5.8}$$

$$0 = \partial_X\{\sigma_3^1 - XF\sigma_3^2\} + \partial_Y\{(1 - YF)\sigma_3^2\} + \partial_Z\{(1 - YF)\sigma_3^3\}. \tag{5.9}$$

It is now an easy matter to see from these equations that the effective stress tensor $\mathcal{S}_B^A$ is *not* a symmetric tensor. The intrinsic geometric nonlinearity of the problem is thus directly manifested at this elementary level. In fact, it is easily seen that the antisymmetry persists unless XF and YF are considered negligible compared to unity (small defect amplitude limit). Now, XF and YF will be negligible compared to unity when $\|\psi_b^a\| \ll 1$. We saw in Section 3, however, that in these circumstances the problem reduces to the corresponding problem for spatial defects in the weak field limit. In this limit, the results given in [6] show that the solution to this problem corresponds with that of an edge dislocation in classical dislocation theory. Perhaps, the reason why the classical solution for the edge dislocation is not altogether satisfactory is because it does not obtain from the field equations of the material translation group unless we make very strong additional linearization assumptions.

The constitutive relations to be used in the equilibrium equations will again be taken to be those of linear elasticity; namely (4.18) and (4.19). In this event, (5.7)–(5.9) are a system of 3 equations for the determination of the three displacement functions $\{u^i\}$. Since the function F depends only on X and Y, an inspection of the resulting field equations shows that we may take all of the displacement functions to be independent of Z. The field equations thus reduce to

$$0 = \partial_X\{\sigma_{XX} - XF\sigma_{XY}\} + \partial_Y\{(1 - YF)\sigma_{XY}\}, \tag{5.10}$$

$$0 = \partial_X\{\sigma_{XY} - XF\sigma_{YY}\} + \partial_Y\{(1 - YF)\sigma_{YY}\}. \tag{5.11}$$

Even in this reduced form, these equations together with the linear constitutive relations, give linear field equations for the unknowns u^i with coefficients that are complicated functions of X and Y. This makes them very hard to solve, even for the simplest choices of the function $F(X, Y)$. We will therefore leave them to the reader for an unfilled month or so, and settle for the fact that the version of these equations in the weak defect limit has a solution that agrees with that the classical solution for an edge dislocation.

6. ELEMENTARY DEFECTS DUE TO LOCAL ACTION OF THE TIME TRANSLATION GROUP

Elementary defects that arise through local action of the time translation group are characterized by

$$k^a = \delta_4^a. \tag{6.1}$$

Use of the results given in Sections 2 and 3 shows that the fundamental coframe fields are given by

$$J^1 = \mathrm{d}X, \qquad J^2 = \mathrm{d}Y, \qquad J^3 = \mathrm{d}Z, \qquad J^4 = \mathrm{d}T - YF\,\mathrm{d}X + XF\,\mathrm{d}Y. \tag{6.2}$$

We therefore have

$$J = \det(J_b^a) = 1. \tag{6.3}$$

The problems to be considered in this section are intrinsically different from those of previous sections, both mathematically and physically. We will therefore not assume that the distribution of material time dislocation microdensity is confined to a core region of the (X, Y)-plane. The function $F(X, Y)$ is therefore left unrestricted except for obvious smoothness conditions.

The matrix $((J_b^a))$ again inverts with relative ease to give the fundamental frame fields

$$j_1 = \partial_X + YF\,\partial_Y, \qquad j_2 = \partial_Y - XF\,\partial_T, \qquad j_3 = \partial_Z, \qquad j_4 = \partial_T. \tag{6.4}$$

It is then a simple matter to compute the components of total distortion:

$$Y_1^i = (\partial_X + YF\,\partial_T)\langle\chi^i\rangle, \qquad Y_2^i = (\partial_Y - XF\,\partial_T)\langle\chi^i\rangle,$$
$$Y_3^i = \partial_Z\langle\chi^i\rangle, \qquad Y_4^i = \partial_T\langle\chi^i\rangle. \tag{6.5}$$

Introducing displacement functions by $\chi^i = \delta_A^i X^A + u^i(X^b)$, we therefore have

$$Y_1^i = \delta_1^i + \partial_X u^i + YF\,\partial_T u^i, \qquad Y_3^i = \delta_3^i + \partial_Z u^i,$$
$$Y_2^i = \delta_2^i + \partial_Y u^i - XF\,\partial_T u^i, \qquad Y_4^i = \partial_T u^i. \tag{6.6}$$

Since the Y's replace the customary configuration gradients, (6.6) shows that *the resulting strains are explicitly velocity dependent.* This is in strong contrast with classical theory.

The equations of dynamic equilibrium can be obtained by substituting these results into (3.19). Noting that (6.4) gives $j_B^A = \delta_B^A$, $j_4^A = 0$, $j_4^4 = j_3^4 = 0$, and $j_1^4 = YF$, $j_2^4 = -XF$, we obtain

$$\partial_T\{\rho_0\,\partial_T u_i + YF(t_i^1 - \sigma_i^1) - XF(t_i^2 - \sigma_i^2)\} = \partial_A(\sigma_i^A - t_i^A) = \partial_A\sigma_i^A, \tag{6.7}$$

because $\partial_A t_i^A = 0$. Here we have used the standard constitutive relations

$$p_i = \rho_0\,\partial_T u_i \tag{6.8}$$

for the components of linear momentum. The effective stresses for these problems are the standard ones, namely

$$\mathscr{S}_i^A = \sigma_i^A - t_i^A. \tag{6.9}$$

On the other hand, the components of effective linear momentum,

$$\mathscr{P}_i = \rho_0\,\partial_T u_i + YF(t_i^1 - \sigma_i^1) - XF(t_i^2 - \sigma_i^2), \tag{6.10}$$

differ markedly from (6.8).

The explicit forms of the dynamic equilibrium equations are obtained by substituting the constitutive relations (4.18) and (4.19) into (6.7). The resulting equations, although linear in the unknown displacement functions $\{u^i\}$, are both lengthy and complicated with variable coefficients all over the place. We shall therefore look at a particularly simple problem in this category.

Consider a static, plane strain problem in linear elasticity with displacement functions $\{w_1(X, Y), w_2(X, Y), w_3(X, Y)\}$ and associated stresses $\{t_i^A\}$. Since the function F is independent of Z, we can take

$$u_1 = w_1, \qquad u_2 = w_2, \qquad u_3 = w_3 + \zeta(X, Y, T), \tag{6.11}$$

in which case $\partial_T u_3 = \partial_T\zeta = \dot\zeta$ because $\dot w_i = 0$. The only nonzero components of $\sigma_{AB} - t_{AB}$ are then

$$\sigma_{13} - t_{13} = \sigma_{31} - t_{31} = \mu\{\partial_X\zeta + YF\dot\zeta\},$$
$$\sigma_{23} - t_{23} = \sigma_{32} - t_{32} = \mu\{\partial_Y\zeta - XF\dot\zeta\}. \tag{6.12}$$

Accordingly, since the body is axially symmetric, the new problem and the static plane strain problem will satisfy the same traction boundary conditions on the cylindrical spatial boundary. Things will change on the boundary planes $Z = \pm l$ both because $u_3 = w_3 + \zeta$ and because of the new stresses (6.12).

If we set $v^2 = \rho_0/\mu$ and note that $X = R\cos\theta$, $Y = R\sin\theta$ imply

$$(Y\,\partial_X - X\,\partial_Y)H = -\frac{\partial H}{\partial\theta},$$

then the dynamic equilibrium equations reduce to the single linear equation

$$(v^2 - R^2F^2)\ddot\zeta + \frac{\partial F}{\partial\theta}\dot\zeta + 2F\frac{\partial\dot\zeta}{\partial\theta} = \nabla^2\zeta. \tag{6.13}$$

This equation can be solved, in principle, for reasonable assignments of the function $F(X, Y)$ and appropriate initial and boundary data. If we look for time-wise separable solutions of the form

$$\zeta(X, Y, T) = \alpha(T)\beta(X, Y), \tag{6.14}$$

we obtain

$$(v^2 - R^2F^2)\beta\ddot{\alpha} + \left(\frac{\partial F}{\partial \theta}\beta + 2F\frac{\partial \beta}{\partial \theta}\right)\dot{\alpha} = (\nabla^2\beta)\alpha. \tag{6.15}$$

Accordingly, such solutions will exist only if there exist constants K_1 and K_2 such that

$$\nabla^2\beta = -K_1(v^2 - R^2F^2), \tag{6.16}$$

$$\beta\frac{\partial F}{\partial \theta} + 2\frac{\partial \beta}{\partial \theta}F = K_2(v^2 - R^2F^2). \tag{6.17}$$

Since $F(X, Y)$ has yet to be determined, we may view (6.16) and (6.17) as a coupled nonlinear system for the determination of $F(X, Y)$ and $\beta(X, Y)$. Thus, F and β must conspire in order to have time-wise separable solutions.

When these conditions are satisfied, we obtain

$$\ddot{\alpha} + K_2\dot{\alpha} + K_1\alpha = 0 \tag{6.18}$$

for the determination of $\alpha(T)$. Various choices of the constants K_1 and K_2 will thus lead to

$$u_3 = w_3 + \alpha(T)\beta(X, Y) \tag{6.19}$$

relaxing, oscillating, or growing relative to the corresponding plain strain problem. *This behavior will lead to relaxing, oscillating, or growing tractions on the planes $Z = \pm l$ while preserving the tractions of the plain strain problem on the cylindrical faces of the body*! This acquits our goal of showing that local action of the time translation group provides a basis for modeling relaxation phenomena in engineering materials.

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