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THE INTERNAL MECHANICAL STATE OF SOLIDS WITH DEFECTS

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Abstract—In general, inelastic deformation is accompanied by internal mechanical stresses which also persist in the absence of external loads. A fundamental quantity for determining the state is the stress function tensor (or tensor potential), a field which can be calculated in terms of the incompatibility tensor, also a field. This tensor is not given a priori, but follows from the physics of the problem. Differential geometry is a most important and elegant tool to deal with the internal mechanical state, *both on the geometric and on the static side* (the concept of mutually dual strain state and stress state). If stress-free strain is the cause of the internal stresses, then Riemannian geometry is adequate. More general geometries describe defects, namely Riemann–Cartan geometry describes dislocations in the form of Cartan's torsion of the strain space, and nonmetric (affine) geometry the point defects vacancy, self-interstitial and shear fault in the form of nonmetricity (a tensor field) of the strain space. The specific response to dislocations is torque stress which arises as Cartan's torsion of the stress space, whereas that to point defects is moment stress without torque entering as the nonmetricity of the stress space. The fundamental duality between strain space and stress space gives the theory a particular symmetry. Physical realizations of such (material) spaces are the affine point structures, e.g. the Bravais lattices.

1. INTRODUCTION

The internal mechanical state is the stress–strain state which persists without the action of external forces or loads. In the relatively restricted field of the common linear elasticity theory it satisfies the field equations

$$-\varepsilon^{ikm}\varepsilon^{jln}\partial_k\partial_l\varepsilon_{mn} = \eta^{ij} \quad \text{compatibility} \tag{1}$$

$$\partial_i\sigma^{ij} = 0, \quad \sigma^{ij} = \sigma^{ji} \quad \text{equilibrium,} \tag{2}$$

and the constitutive law

$$\sigma^{ij} = c^{ijkl}\varepsilon_{kl}, \quad \varepsilon_{kl} = s_{klmn}\sigma^{mn}, \tag{3}$$

where

$$c^{ijkl}s_{klmn} = \tfrac{1}{2}(\delta^i_m\delta^j_n + \delta^i_n\delta^j_m). \tag{4}$$

We do not write down the boundary conditions for stress and strain, because their treatment is of a more technical nature and does not give us much physical insight.

In (1) the symmetric tensor $\boldsymbol{\eta}$ is the so-called incompatibility tensor. It measures the deviation from compatibility which is the special case for $\boldsymbol{\eta} = 0$. Incompatibility arises when nonfitting material elements are forced by elastic deformation to form a compact body. Only in the compatible case can the strain field be represented as the (symmetrized) gradient of the elastic vector field of displacements. This situation would arise if we considered the external load problem, in which, beside $\boldsymbol{\eta} = 0$, the volume density of external forces would occur, namely in eqn (2).

For the internal stress problem it is suggestive, although not the only possibility, to satisfy identically the equilibrium equations by the stress function ansatz (Beltrami, 1892; Kröner, 1954, 1955):

$$\sigma^{ij} = -\varepsilon^{ikm}\varepsilon^{jln}\partial_k\partial_l\chi_{mn}, \tag{5}$$

where for obvious reasons the symmetric tensor χ is called the *second-order stress function* or tensor potential. Note that Airy's stress function is a special case of χ.

If (5) is now substituted in (1) via part 2 of (3) we obtain the field equation for χ. For elastic isotropy

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{6}$$

$$\chi'_{ij} = \frac{1}{2\mu}\left(\chi_{ij} - \frac{\lambda}{2(2\lambda+\nu)}\chi^k_k\delta_{ij}\right). \tag{7}$$

If (5) with (3), (4) and (6) is substituted in (1) we obtain, after a simple calculation, the field equation for χ':

$$\nabla^4\chi'_{ij} = \eta_{ij}, \quad \partial_i\chi'_{ij} = 0. \tag{8}$$

The solution of this equation leads to the internal stress state via (7) and (5). Note that (8) has a particularly simple Green's function.

Equation (8) contains an extra, or gauge, condition which has to do with the fact that χ is not uniquely defined by (5). In this respect, χ is analogous to the vector potential $\vec{A}$ of electrodynamics. Like $\vec{A}$, also χ is a fundamental quantity of the theory.

Naturally, the anisotropic stress function ansatz is much more involved. Its general form has been found only recently (Kröner, 1990a). Let

$$D_{ik}(\nabla) \equiv -c_{ijkl}\partial_j\partial_l \tag{9}$$

$$D^*_{ij}(\nabla) \equiv \tfrac{1}{2}\varepsilon_{ikm}\varepsilon_{jln}D_{kl}(\nabla)D_{mn}(\nabla) \tag{10}$$

$$f(\nabla) \equiv \tfrac{1}{6}\varepsilon_{ikm}\varepsilon_{jln}D_{ij}D_{kl}D_{mn} \tag{11}$$

be the fundamental differential operators of anisotropic linear elasticity theory. Look for a (symmetric) tensor potential ψ_{kl} such that

$$f(\nabla)\psi_{kl} = \eta_{kl}, \quad \partial_k\psi_{kl} = 0. \tag{12}$$

A routine calculation yields

$$\sigma_{ij} = Y_{ijkl}(\nabla)\psi_{kl}, \tag{13}$$

$$Y_{ijkl}(\nabla) \equiv \tfrac{1}{3}\varepsilon_{lps}\varepsilon_{kmr}c_{ijrs}c_{pqmn}D^*_{nq}(\nabla). \tag{14}$$

Since fourfold differentiations are needed to get the stress from the tensor potential ψ, this one might also be called the *fourth-order stress function* (tensor). Of course, the ansatz (13, 14) is also valid in the isotropic case, where $f(\nabla) \sim \nabla^6$.

Note that the scalar sixth-order differential operator $f(\nabla)$ is the same as that which arises in other anisotropic elastic problems, e.g. those with external loads. All knowledge accumulated in these cases (concerning f) can be utilized also in our present problem. In particular, much is known about the Green's function of the operator $f(\nabla)$.

Within the frame of continuum mechanics it is clear how the internal mechanical state can be calculated, once the distribution of stress–strain sources, i.e. the tensor field $\boldsymbol{\eta}$ is known. Usually, however, $\boldsymbol{\eta}$ is not given a priori but has to be found from the physics of the problem. To get some general view on this is the topic of the present article. On this occasion we can also learn something about the nonlinear theory. In fact, it has proven most useful to apply to our problem the mathematical language of differential geometry, which is basically nonlinear and well elaborated.

2. DIFFERENTIAL GEOMETRY IN CONTINUUM MECHANICS

The general concept of using differential geometry in continuum mechanics is that the material medium is treated as a continuous material space which is embedded into the three-dimensional physical, i.e. Euclidean, space and can be deformed there. This means that the geometries for two spaces must be distinguished. A fundamental question is then: if the geometry of the embedding space is Euclidean, what is the geometry of the material space, sometimes called the inner geometry?

The most important characteristic of the Euclidean space is that it has no curvature. If the material space had curvature, then it would not fit into the Euclidean space; hence the material space should be flat (have no curvature). Such spaces are also said to have teleparallelism. A flat space is not necessarily Euclidean, but can have additional structure. This has been studied above all under the condition of the so-called affinely connected, i.e. locally affine, spaces. The result is that a flat, locally affine space can have two types of structure which go beyond euclidean and even Riemannian geometry: the first one is *Cartan's torsion* and is representative of the elementary line defects (dislocations) in Bravais crystals, whereas the second one is the *nonmetricity* (see Schouten, 1954) and describes the elementary point defects (vacancy, self-interstitial, shear defect) in Bravais crystals. The distinct role of Bravais lattices with respect to affine differential geometry has to do with the fact that these lattices themselves have an affine structure.

3. RIEMANNIAN GEOMETRY AND STRESS FREE STRAIN

3.1. *The strain space*

It has been known for a long time that the compatibility equations of elasticity theory can be formulated as the vanishing of the curvature tensor K^k_{nml} formed with a Riemannian connection g^k_{ml}, a Christoffel symbol. This connection is a special case of the connection Γ^k_{ml} of the general affinely connected space. We have

$$g^h_{ml} \equiv g^{hk} g_{mlk} \equiv \tfrac{1}{2} g^{hk} (\partial_m g_{kl} - \partial_k g_{lm} + \partial_l g_{mk}), \tag{15}$$

with g_{kl} as the metric tensor. The curvature tensors are

$$K^k_{nml} \equiv 2(\partial_n g^k_{ml} + g^k_{np} g^p_{ml})_{[nm]} \tag{16}$$

for the Riemannian geometry and

$$R^k_{nml} \equiv 2(\partial_n \Gamma^k_{ml} + \Gamma^k_{np} \Gamma^p_{ml})_{[nm]} \tag{17}$$

for the more general affine space. The subscript $[n, m]$ denotes antisymmetrization in n, m. Note that Riemannian geometry is characterized by $\Gamma^k_{ml} = g^k_{ml}$. If we use, in well-known notation, $\varepsilon_{kl} = (g_{kl} - \delta_{kl})/2$ for the strain tensor in Eulerian description, then $K^k_{nml} = 0$ is indeed the compatibility equation for the strain. It can easily be written down in full nonlinearity.

The general solution of $K^k_{nml} = 0$, expressed in g^k_{ml}, is (check!)

$$g^k_{ml} \equiv A^k_{k'} \partial_m A^{k'}_l, \quad A^k_{k'} A^{k'}_l = \delta^k_l, A^{l'}_{k} A^k_{k'} = \delta^{l'}_{k'}, \quad k, k' = 1, 2, 3. \tag{18}$$

The (3×3)-matrix $\mathbf{A}$ is subject to the condition that by definition g^k_{ml} is symmetric in m, l. It is convenient and admissible to understand $\mathbf{A}$ as the matrix of a point transformation (a deformation) from an initial to a current state. If in the latter we use a coordinate cover (k') which is dragged along from the cover (k) of the initial system, then

$$\mathrm{d}x^{k'} = A^{k'}_k \,\mathrm{d}x^k, \quad \mathrm{d}x^k = A^k_{k'} \,\mathrm{d}x^{k'}. \tag{19}$$

Because of the symmetry of g^k_{ml}

$$\partial_m A_l^{k'} - \partial_l A_m^{k'} = 0. \tag{20}$$

This means compatibility. In fact, (20) implies

$$A_l^{k'} = \delta_l^{k'} + \partial_l u^{k'} \tag{21}$$

where $\mathbf{u}$ is the displacement field and $A_l^{k'}$ the deformation gradient. The equation $\mathbf{g} = \mathbf{A} \cdot \mathbf{A}^T$ is easily proved. Here $\mathbf{A}^T$ is the transposed matrix.

So far nothing has been said in this section about the physical nature of the displacement field. If the displacements are elastic, i.e. their gradient is related to the stress, then the problem is one of external sources which we do not consider.

Suppose that now the strain $\boldsymbol{\varepsilon}$ is a superposition of elastic strain and some spontaneous stress-free strain, say $\boldsymbol{\varepsilon}^*$. Then the elastic and stress-free strain taken separately do not satisfy the compatibility equations; only the total strain, which is a sum of the single strains in the linearized theory, does this. In this case, the incompatibility tensor needed in the formalism of eqns (1)–(4) can be calculated as

$$\eta^{ij} = \varepsilon^{ikm}\varepsilon^{jln}\partial_k\partial_l\varepsilon^*_{mn}. \tag{22}$$

It is not particularly difficult to obtain the corresponding result for the nonlinear case, but we do not need this for our further reasoning.

The causes for stress-free strain may be multifarious—temperature, magnetization, electric polarization and others. However, we have a different situation if defects are the sources of internal stress. This problem is treated in the next section.

Here we recall a result well-known from Einstein's relativity theory, namely that every Riemann tensor can, without loss of information, be replaced by the Einstein tensor, denoted by E^{ij}:

$$E^{ij} = \tfrac{1}{4}\varepsilon^{inm}\varepsilon^{jlk}K_{nmlk}. \tag{23}$$

This formula may be used in three dimensions only.

Due to its definition the Riemann tensor satisfies two identities, in recent literature called the first and second Bianchi identities. Written for the Einstein tensor they are

$$E^{ij} = E^{ji}, \quad \nabla_i E^{ij} = 0, \tag{24}$$

with ∇_i the symbol of covariant (with respect to g^k_{ml}) differentiation. Every tensor satisfying (24) can be considered as the Einstein tensor of some Riemannian space defined with the help of $\mathbf{E}$.

3.2. *The stress space*

Now compare (24) with (2). The form of these equations is rather similar, the difference lying in the two kinds of differentiation.

Note that this difference disappears in the linear approximation, where $\nabla_i \to \partial_i$. The similarity has become the basis of Schaefer's (1953) analogy between the statics (in the absence of body forces) and the linearized static theory of general relativity. Recall that in Eulerian (Cartesian) coordinates (2) holds in the nonlinear theory, too.

In Schaefer's analogy, which was later extended by Minagawa (1962) and by Kondo (1962), the stress tensor $\boldsymbol{\sigma}$ is the Einstein tensor of a space which is now called stress space or, even better, though less convenient, stress function space. The point is that like the strain, so also does the stress function play the role of a metric. This follows from the comparison of (1) and (5) where $\boldsymbol{\varepsilon}$ is analog to $\boldsymbol{\chi}$ and $\boldsymbol{\eta}$ to $\boldsymbol{\sigma}$.

The analogy would not be very useful, were it restricted to the linearized theory. I have shown recently (Kröner, 1987), that by a slight modification the analogy can be made true also for nonlinear media in the situation of Riemannian geometry. It was proved that a stress tensor Σ^{ij}, called the Riemann stress, can be defined such that the equilibrium equations assume the form

$$\nabla_i^* \Sigma^{ij} = 0, \quad \Sigma^{ij} = \Sigma^{ji}. \tag{25}$$

Here ∇^* is the symbol of covariant differentiation, but now not in the strain space, where we use ∇_i, but in the stress space defined by $\mathbf{\Sigma}$. In fact, (25) can be understood as the Bianchi identities of the stress space. Therefore it is possible to introduce a Christoffel symbol, say χ^k_{ml}, whose metric is closely related to the former stress function. The fulfillment of (25) implies that of (2) and vice versa. This means that Σ^{ij} and σ^{ij} can be mutually converted (Kröner, 1987).

It is common to define stress tensors by forces acting on area elements. The question then arises whether area elements in the reference or current state are meant. For the definition of Riemann stress we need an area element of the riemannian stress space. Thus the Riemann stress tensor represents the contact force acting on the area element of the Riemannian space which itself is determined by the stress. Of course, the geometry is that of the current state.

At first sight the concept of Riemann stress appears strange, in particular as to practicability. It has, however, the agreeable quality, from a theoretical standpoint, that differential geometry can be used for static problems just as it is used in the strain space. All static and geometric equations are then exact. Note that the stress space has not the same units as the strain space. For instance, the metric χ is not dimensionless. This apparent shortcoming can easily be reconciled.

Consider the differential form

$$\mathrm{d}\vec{x} = \mathrm{d}\vec{x} \cdot \mathbf{A} \tag{26}$$

of (19), now written in a new form. In (26) $\mathrm{d}\vec{x}'$ is the relative "placement" of two neighboring points of the material medium, whose relative position in some initial state is $\mathrm{d}\vec{x}$.

In the stress space we assign a relative force

$$\mathrm{d}\vec{f} = \mathrm{d}\vec{x}' \cdot \varphi \tag{27}$$

to the relative placement $\mathrm{d}\vec{x}'$ and in this way define the (3×3)-matrix φ. Since we develop a differential geometry, it is convenient to give $\mathrm{d}\vec{f}$ the dimension of a length—then eqns (26) and (27) are equivalent, except for the different meaning of the letters. We may give $\mathrm{d}\vec{f}$ and φ the new dimensions by the introduction of "new" $\mathrm{d}\vec{f}$ and φ which result from the "old" $\mathrm{d}\vec{f}$ and φ by multiplication with a constant of dimension $[\text{force/length}]^{-1}$. We then form the metric fundamental form with the new quantities as

$$\mathrm{d}\vec{f} \cdot \mathrm{d}\vec{f} = \mathrm{d}\vec{x} \cdot \varphi \cdot \varphi^{\mathrm{T}} \cdot \mathrm{d}x^{\mathrm{T}} \equiv \mathrm{d}\vec{x} \cdot \chi \cdot \mathrm{d}\vec{x}^{\mathrm{T}}, \tag{28}$$

where all quantities have the dimensions of common differential geometry. As we had $\mathbf{g} = \mathbf{A} \cdot \mathbf{A}^{\mathrm{T}}$, we now have $\boldsymbol{\chi} = \boldsymbol{\varphi} \cdot \boldsymbol{\varphi}^{\mathrm{T}}$. In this section we have studied the relation between riemannian geometry and elasticity, in particular with the theory of the internal state. Riemannian geometry is relevant to situations where, due to some physical conditions, a stress-free strain develops. This is not the case of defects. To treat these, we have to go beyond Riemannian geometry. This will be done in the next section.

4. AFFINE DIFFERENTIAL GEOMETRY AND DEFECTS

4.1. *The strain space*

Bravais crystals are *affine* structures and therefore distinct with respect to *affine* differential geometry. To discuss some features of this geometry we introduce the abbreviation (Schouten, 1954)

$$\varphi_{\{mlk\}} \equiv \varphi_{mlk} - \varphi_{lkm} + \varphi_{kml}, \tag{29}$$

valid for any quantity with three lower indices.

We define also Cartan's torsion by

$$S_{mlk} \equiv \Gamma_{[ml]k} \tag{30}$$

with $\Gamma^h_{ml} \equiv g^{hk}\Gamma_{mlk}$ as the connection and

$$Q_{mlk} \equiv -\nabla_m g_{lk} \tag{31}$$

as the nonmetricity of the (material) space. Here ∇_m is the symbol of covariant differentiation, now with respect to the connection Γ_{mlk} rather than to the Christoffel symbol used in the Riemannian geometry. Finally we define

$$G_{mlk} \equiv \tfrac{1}{2}\partial_m g_{lk}. \tag{32}$$

Utilizing the permutation prescription of (29) we can write down and prove easily the identity (Schouten, 1954)

$$\Gamma_{mlk} = G_{\{mkl\}} - S_{\{mkl\}} + \tfrac{1}{2}Q_{\{mkl\}} \tag{33}$$

or

$$\mathbf{\Gamma} = \{\mathbf{G}\} - \{\mathbf{S}\} + \tfrac{1}{2}\{\mathbf{Q}\}, \tag{34}$$

which is valid for **G**, **S** and **Q** as defined by (30)–(32). It is easy to prove that **S** and **Q**, hence also $\{\mathbf{S}\}$ and $\{\mathbf{Q}\}$ are tensors. S_{mkl} is antisymmetric in m, k, and Q_{mkl} is symmetric in k, l. Hence $\{\mathbf{S}\}$ and $\{\mathbf{Q}\}$ describe different things. $\{\mathbf{G}\}$, finally, is a Riemann connection, i.e. a Christoffel symbol, therefore not a tensor.

In the last section we have discussed the Christoffel symbol in connection with stress-free strain. For simplicity we shall now assume that such stress-free strain is absent. We shall, however, also now study a situation of teleparallelism, i.e. $R_{nmlk} = 0$. The general solution of this equation has the same form as that of $K_{nmlk} = 0$, namely

$$\Gamma^k_{ml} = A^k_{k'}\,\partial_m A^{k'}_l, \tag{35}$$

where the matrix **A** is now arbitrary, because $\mathbf{\Gamma}$ is no longer symmetric. The form (35) of the connection determines also the form of the torsion as

$$S^k_{ml} = \tfrac{1}{2}A^k_{k'}(\partial_m A^{k'}_l - \partial_l A^{k'}_m). \tag{36}$$

Vanishing of S yields the former result $A^{k'}_l = \delta^{k'}_l + \partial_l u^{k'}$, which means that nonvanishing torsion leads to an incompatible situation.

All this has been discussed in the literature and is now well-known. Let us then come to the less-known case of the nonmetricity $\{\mathbf{Q}\}$. Our starting point is Schouten's third identity for the curvature tensor,

$$R_{nm(lk)} = \nabla_{[n}Q_{m]lb} + S^p_{nm}Q_{plk}. \tag{37}$$

Note that all identities of the curvature tensor originate from definition (17). In our case of teleparallelism $R_{nm(lk)} = 0$ and after a short calculation (37) becomes

$$(\partial_n Q_{mlk} - \Gamma^p_{nl} Q_{mpk} - \Gamma^p_{nk} Q_{mlp})_{[nm]} = 0. \tag{38}$$

$\mathbf{\Gamma}$ is now given by (35). Every function $\mathbf{Q}$ satisfying (38) has the form $-\nabla_m g_{lk}$ and is therefore admissible in (34).

The linearized general solution of (38) is

$$Q_{mlk} = \partial_m h_{lk} \tag{39}$$

with an arbitrary symmetric tensor field h_{lk}. Now recall that the tensor $\mathbf{g}$ occurring in $\{\mathbf{G}\}$ is given by $\mathbf{g} = \mathbf{A}\cdot\mathbf{A}^{\mathrm{T}}$ for the connection $-G_{\{mkl\}} + S_{\{mkl\}}$. For such a *metric connection* (Schouten, 1954) the length measurement is directly bound to the law of parallel displacement with the help of the connection. For instance, when a lattice vector is parallel displaced (using $\mathbf{\Gamma}$) along itself, say 1000 times, then its start and goal are separated by 1000 atomic spacings. The result of this counting is measured by g_{kl}. Because the result of the measurement by parallel displacement and by counting lattice steps is the same, we say that the space is metric with respect to the connection $\mathbf{\Gamma}$.

This is no longer so in the nonmetric situation. Now $\mathbf{g} \neq \mathbf{A}\cdot\mathbf{A}^{\mathrm{T}}$ because of the entering of $\mathbf{Q}$ (or $\mathbf{h}$) in (33). We now have, in linear approximation,

$$\mathbf{g} = \mathbf{A}\cdot\mathbf{A}^{\mathrm{T}} + \mathbf{h}, \tag{40}$$

so that $\mathbf{h}$ is a measure of how much the metric $\mathbf{g}$ must be changed due to a nonmetric contribution to the connection. Physically, the nonmetricity enters in the form of the elementary points defects. For instance, vacancies which are not regarded in the step counting shorten the distance between two points. It follows that $\mathbf{h}$, thus $\mathbf{Q}$, is a measure of the density of the point defects. All this has been worked out recently (Kröner, 1990) by a more formal nonlinear consideration which, however, has not yet been matched to the present theory.

4.2. *The stress space*

Since here we deal with the more general affine geometry, the Bianchi identities also take a more general form. They read (Schouten, 1954):

$$(R^k_{nml} - 2\nabla_n S^k_{ml} + 4S^p_{nm} S^k_{lp})_{[nml]} = 0 \tag{41}$$

and

$$(\nabla_p R^k_{nml} - 2S^q_{pn} R^k_{mql})_{[pnm]} = 0. \tag{42}$$

The general affine curvature (or non-Riemannian curvature) tensor $\mathbf{R}$ cannot be replaced by an Einstein tensor—only that part of $\mathbf{R}$ which is antisymmetric in both n, m and l, k.

If, however, $\mathbf{Q} = 0$, then covariant derivation and raising and lowering of indices commute and $\mathbf{R}$ becomes antisymmetric also in l, k. Lowering k and multiplying by $\varepsilon^{ilk}\varepsilon^{jnm}$ in (41) and (42) leads to

$$\mathrm{div}_\Gamma\, \tilde{\mathbf{S}} + 2\vec{E} = 0 \quad \text{1st Bianchi} \tag{43}$$

$$\mathrm{div}_\Gamma\, \mathbf{E} = 0 \quad \text{2nd Bianchi,} \tag{44}$$

where

$$(\mathrm{div}_\Gamma)_i \equiv \nabla_i + 2S^p_{ip} \tag{45}$$

is the divergence operation in a space with torsion and

$$\vec{E}_k = \tfrac{1}{2}\varepsilon_{ijk}E^{ij}, \quad \tilde{S}^{nk} = -\mathrm{e}^{nml}S^k_{ml}. \tag{46}$$

As shown by Stojanović (1963) and by Kröner (1963) eqns (43) and (44) have exactly the form of the equilibrium equations of a material medium with force and torque stresses if $\mathbf{E}$ is now interpreted as the (in general nonsymmetric) Riemann–Cartan force stress and $\tilde{\mathbf{S}}$ as the (Riemann–Cartan) torque stress. The general solution of these equations is easily written down in terms of stress functions and torsion of the stress space, and obviously, the latter represents the torque stress. It should be clear by now that the concept of stress space works also in the presence of dislocations. The torque stresses are the specific response to dislocations. If $\mathbf{S} = 0$, but $\mathbf{Q} \neq 0$, then the Bianchi identities reduce to

$$R^k_{[nml]} = 0, \quad \nabla_{[p}R^k_{nm]l} = 0 \tag{47}$$

where, however, covariant derivation and raising and lowering of indices no longer commute. Therefore k in part 2 of (47) cannot simply be taken down. If within the concept of stress space (47) is considered as the equilibrium equations, then the general solution contains a nonmetricity tensor which represents the specific response to the presence of geometric nonmetricity, i.e. to point defects. This response has the quality of moment stresses without torque.

5. CONCLUSION

In this work we have tried to show that differential geometry is a useful tool to deal with certain types of solids with defects. We have restricted ourselves to solids with a microscopically affine constitution. Such solids are realized best by the Bravais crystals. We have chosen this type of solid because it is relatively simple, and therefore also the differential geometry used is simple. This is the well-explored geometry of locally affine spaces. Both the geometry and the statics were discussed in terms of this geometry, making use of the fundamental duality between strain space and stress space. This duality has its origin in the general equations of mechanics where position and momentum are recognized as dual quantities. In this description the theory of the internal mechanical state achieves a very high symmetry. It is amusing to think that this theory could have some relevance for a theory of the universe which is a physical system in which we live as observers who perceive internal states only.

The theory shown here admits almost any extension. Solids with more complex (than affine) structure, and also liquid crystals can be described by a more involved geometry. In all such theories defects play a fundamental role, and in fact they determine to a large degree the material's properties.

A fundamental quantity in our theory is the connection $\mathbf{\Gamma}$, which defines what is parallel in the considered space. The wish, then, is natural to have a visual impression of this parallelism which somehow should be inherent in the material's structure. For the Bravais crystal the relation between the crystalline order and the law of parallelism is immediate. In crystallography and physics all vectors within one of the three sets of primitive lattice vectors have always been considered as parallel. The required teleparallelism implies that if two vectors at distant points are parallel, then the parallelism has an absolute character. This view was taken by Kondo (1952) and by Bilby *et al.* (1955).

The main characteristics of crystals, namely the existence of three primitive crystallographic directions and the countability of lattice steps entered our theory through the definition of dislocation and point defect. We have not discussed these definitions in this work. They become meaningless if we restrict ourselves to Riemannian geometry, i.e. we eliminate the defects. Hereby we also eliminate the crystallinity, which no longer enters the theory. Now remember that the stresses occurring in the Riemannian case originated from stress-free strain which, perhaps, should be classified as external rather than internal. When doing so, then the line and point defects are the only sources of internal stress, but they do not lead to uniquely defined strain (or metric). In fact, setting $\mathbf{S} = \mathbf{Q} = 0$ and also $\mathbf{R} = 0$

(teleparallelism), we have $\mathbf{\Gamma} = \{\mathbf{G}\}$ and the pertaining curvature vanishes. Under these conditions the $\boldsymbol{\varepsilon}$ occurring in $\mathbf{G}$ may be any (symmetrized) gradient field, as for instance it would arise if the crystal is deformed from outside. As internal observers we cannot perceive such an outside deformation. For us, therefore, the strain can be subjected to some side condition like divergence freedom. This is exactly and for the same reason the situation we had with the stress function tensor $\boldsymbol{\chi}$ which we have declared analogous to $\boldsymbol{\varepsilon}$. The last result, namely the analogy also in the side conditions of strain and stress state, seems to support our interpretation of defect theory in terms of differential geometry. This interpretation states that from the standpoint of the internal observer both the statics and the geometry (or kinematics) of affine structures (Bravais crystals) are described adequately by the differential geometry of affinely connected spaces.

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REFERENCES

Beltrami, E. (1892). Osservazioni sulla nota precedente (Morera). *Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fiz. Mat. Natur.* V. Ser. **1**(1), 141–142.

Bilby, B. A., Bullough, R. and Smith, E. (1955). Continuous distributions of dislocations: a new application of the methods of non-Riemannian geometry. *Proc. Roy. Soc.* (London) **A231,** 263–273.

Kondo, K. (1952). On the geometrical and physical foundations of the theory of yielding. *Proc. 2nd Japan Nat. Congr. Appl. Mech.*, pp. 41–47.

Kondo, K. (1962). Micro-non-holonomic structure of the stress function space, with reference to temperature, elastic and plastic matter constants. In *RAAG Memoirs*, Vol. 3, p. 148. Gakujutsu Bunken Fukyu-kai, Tokyo.

Kröner, E. (1954). Die Spannungsfunktionen der dreidimensionalen isotropen Elastizitätstheorie. *Z. Physik* **139,** 175–188.

Kröner, E. (1955). *Berichtigung Z. Physik* **143,** 374.

Kröner, E. (1963). Zum statischen Grundgesetz der Versetzungstheorie. *Ann. Phys.* (Leipzig) (7) **11,** 13–21.

Kröner, E. (1987). Stress space and strain space in continuum mechanics. *Phys. Stat. Sol.* (b) **144,** 39–44.

Kröner, E. (1990a). Three dimensional stress functions in anisotropic elasticity. *Rend. di Mat.* Ser. VII, **10,** 773–784.

Kröner, E. (1990b). The differential geometry of elementary point and line defects in Bravais crystals. *Int. J. Theor. Phys.* **29,** 1219–1237.

Minagawa, S. (1962). Reimannian three-dimensional stress-function space. In *RAAG Memoirs*, Vol. 3, pp. 135–147.

Minagawa, S. (1968). On some generalized features of stress-function space with reference to the problem of stresses in a curved membrane. In *RAAG Memoirs*, Vol. 4, pp. 159–170. Gakujutsu Bunken Fukyu-kai, Tokyo.

Schaefer, H. (1953). Die Spannungsfunktionen des dreidimensionalen Kontinuums und des elastischen Körpers. *Z. Angew. Math. Mech.* **33,** 356–362.

Schouten, H. (1954). *Ricci-Calculus*. Springer, Heidelberg.

Stojanović, R. (1963). Equilibrium conditions for internal stresses in non-Euclidean continua and stress spaces. *Int. J. Engng Sci.* **19,** 323–327.