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A 1-DIMENSIONAL ELASTIC MODEL WITH TIME TRANSLATION GAUGE INVARIANCE

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Abstract—Elastic models with spatial translation gauge invariance are now recognized to provide useful alternative descriptions of the dynamics of materials with defects. The purpose of this paper is to investigate the implications and usefulness of elastic models with time translation gauge invariance. The analysis is confined to 1-dimensional models in order to simplify and illustrate the basic ideas. This gauge theory leads to explicit systems of constitutive relations for the effective stress and effective linear momentum that differ markedly from previous theories. The gauge degree of freedom provides an additional state variable that has some features in common with theories with internal variables. Explicit field equations are obtained for the gauge fields and the matter fields. They are second order, nonlinear partial differential equations. The sources for the gauge field equation evaluate in terms of the components of an effective momentum–energy complex. This shows that the time derivative of the gauge field is a potential function for the effective energy density and energy flux. Alternative forms of the field equations are also obtained. These are used to analyze a class of traveling wave solutions.

1. INTRODUCTION

As currently envisioned, an elastic body is naturally associated with two groups of symmetries. The first group is the *spatial symmetry group* of kinematically equivalent current configurations. This group is a Lie group that is the semidirect product of the 3-dimensional rotation group SO(3) with the 3-dimensional spatial translation group T(3). The second group is the *material symmetry group* that characterizes possible isotropy and homogeneity properties of the material. This group is contained in the semidirect product of the 3-dimensional rotation group O(3) with the 3-dimensional spatial translation group T(3) and the 1-dimensional time translation group T(1).

Work reported over the past decade (see [1] and the references cited therein) has shown that gauge theoretic constructs based on the local action of the spatial symmetry group can provide a sound theoretical basis for an analysis of the dynamics of defects in elastic materials. A series of three recent papers [2–4] has also shown that the local action of the maximal Lie subgroup of the material symmetry group can also provide models of defects in elastic materials, both by themselves and in conjunction with local action of the spatial symmetry group.

Gauge theories of spatial symmetry groups and material symmetry groups differ in several fundamental ways (see [2] for the details). Most of these can be traced to the fact that spatial symmetry groups act on the elastic state variables while leaving the independent spatial and time variables unchanged, while material symmetry groups act on the independent spatial and time variables. There is one further glaring difference, however. The material symmetry group can contain the 1-parameter Lie group of time translations while the spatial symmetry group cannot.

We are accustomed to associate global time translation invariance of a model with the existence of a conservation law for the energy of the body. Indeed, when the model derives from a variational principle, Noether's famous theorem leads rigorously to the existence of such a conservation law. On the other hand, local action of the time translation group is problematic, at best, and is not well understood. The purpose of this note is to attempt to clarify certain aspects of gauge theory that are associated with the local action of the time translation group. In order to eliminate all spurious effects, the analysis will be confined to the study of a (linear) model of a 1-dimensional elastic material whose admissible states stationarize a quadratic action integral that is invariant under the global action of the time translation group.

2. THE 1-DIMENSIONAL ELASTIC MEDIUM

A 1-dimensional, linear, elastic medium can be characterized by an action integral of the form

$$A[u]=\int_{t_0}^{t_1}\int_0^L \rho\left\{\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2-\frac{1}{2}c^2\left(\frac{\partial u}{\partial x}\right)^2\right\}\mathrm{d}x\wedge \mathrm{d}t, \tag{2.1}$$

where ρ is a constant with the dimensions of mass per unit length, c is a constant with the dimensions of velocity, and $u=u(x,t)$ is the state variable for the elastic material with dimension of a length. The Lagrangian function for this problem is

$$l=\frac{\rho}{2}\left\{\left(\frac{\partial u}{\partial t}\right)^2-c^2\left(\frac{\partial u}{\partial x}\right)^2\right\}, \tag{2.2}$$

and hence

$$p=\frac{\partial l}{\partial\left(\frac{\partial u}{\partial t}\right)}=\rho\frac{\partial u}{\partial t}=\text{linear momentum/unit length}, \tag{2.3}$$

$$\sigma=-\frac{\partial l}{\partial\left(\frac{\partial u}{\partial x}\right)}=\rho c^2\frac{\partial u}{\partial x}=\text{force potential (stress)}. \tag{2.4}$$

The Euler–Lagrange equation for the action integral (2.1) is therefore given by

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}; \tag{2.5}$$

namely, a 1-dimensional linear wave equation with wave speed c.

The components of the momentum–energy complex for this problem are given by

$$T^a_b=\frac{\partial l}{\partial\left(\frac{\partial u}{\partial x^a}\right)}\frac{\partial u}{\partial x^b}-\delta^a_b l. \tag{2.6}$$

Since the Lagrangian l does not depend explicitly on either x or t, an elementary calculation shows that any solution of the Euler–Lagrange equation (2.5) will necessarily satisfy the conservation laws

$$\frac{\partial}{\partial x^a}(T^a_b)=0,\qquad 1\le b\le 2. \tag{2.7}$$

Noting that the independence of l on x and t is equivalent to the statement that $l\,\mathrm{d}x\wedge\mathrm{d}t$ is invariant under the global Abelian Lie group of space translation $x\mapsto x+a$ and time translation $t\mapsto t+b$, Noether's first theorem states that the two conservation laws (2.7) are direct consequences of this invariance of $l\,\mathrm{d}x\wedge\mathrm{d}t$. Our primary interest in this paper is with the time translation group, so we will only exhibit the components of the momentum–energy complex that are relevant in this case:

$$T^x_t=-\rho c^2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}=-\sigma\frac{\partial u}{\partial t}=F_e=\text{energy flux}, \tag{2.8}$$

$$T^t_t=\frac{\rho}{2}\left\{\left(\frac{\partial u}{\partial t}\right)^2+c^2\left(\frac{\partial u}{\partial x}\right)\right\}=E_e=\text{energy density} \tag{2.9}$$

for the elastic body, respectively. The conservation law (2.7) for $b = t$ thus takes the form

$$\frac{\partial E_e}{\partial t} = -\frac{\partial F_e}{\partial x}; \tag{2.10}$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L E_e \,\mathrm{d}x = -\int_0^L \frac{\partial F_e}{\partial x}\,\mathrm{d}x = \left(\sigma\frac{\partial u}{\partial t}\right)(L, t) - \left(\sigma\frac{\partial u}{\partial t}\right)(0, t). \tag{2.11}$$

Thus, the total energy of the elastic body changes only because of the total work performed by the tractions on the boundaries of the body at $x = 0$ and $x = L$.

3. LOCAL ACTION OF THE TIME TRANSLATION GROUP

We now consider what happens when we replace the global action of the time translation group by a local action, T_τ, of that group. This means that points in the (x, t)-plane are subject to transformations of the form

$$T_\tau \mid {}'x = x, \qquad {}'t = t + \tau(x, t) \tag{3.1}$$

rather than just ${}'x = x$, ${}'t = t + b$ with $b = \text{constant}$. An elementary calculation shows that

$$\mathrm{d}'x \wedge \mathrm{d}'t = \left(1 + \frac{\partial \tau}{\partial t}\right)\mathrm{d}x \wedge \mathrm{d}t \tag{3.2}$$

and hence $l\,\mathrm{d}x \wedge \mathrm{d}t$ is not invariant under a general local time translation T_τ. The purpose of gauge theory is to restore the invariance of the action 2-form $l\,\mathrm{d}x \wedge \mathrm{d}t$. This is accomplished [2] by introducing a compensating 1-form field

$$\Psi = \psi_x(x, t)\,\mathrm{d}x + \psi_t(x, t)\,\mathrm{d}t \tag{3.3}$$

and the operation $\mathcal{M}$ of minimal replacement. The action of this operation on 1-forms is defined by

$$\mathcal{M}\langle \mathrm{d}x \rangle = \mathrm{d}x = J^x, \tag{3.4}$$

$$\mathcal{M}\langle \mathrm{d}t \rangle = \mathrm{d}t + \Psi = \psi_x\,\mathrm{d}x + (1 + \psi_t)\,\mathrm{d}t = J^t. \tag{3.5}$$

The reason why these constructs work is because we require the compensating 1-form field Ψ to transform according to the law

$${}'\Psi = \Psi - \mathrm{d}\tau \tag{3.6}$$

under the action of the gauge transformation T_τ. When (3.6) is written out, we have

$${}'\Psi = {}'\psi_x\,\mathrm{d}'x + {}'\psi_t\,\mathrm{d}'t = \left(\psi_x - \frac{\partial \tau}{\partial x}\right)\mathrm{d}x + \left(\psi_t - \frac{\partial \tau}{\partial t}\right)\mathrm{d}t. \tag{3.7}$$

However, (3.1) gives $\mathrm{d}'x = \mathrm{d}x$, $\mathrm{d}'t = \mathrm{d}t + \partial\tau/\partial x\,\mathrm{d}x + \partial\tau/\partial t\,\mathrm{d}t$, and hence (3.7) is equivalent to

$$\left({}'\psi_x + {}'\psi_t\frac{\partial \tau}{\partial x}\right)\mathrm{d}x + {}'\psi_t\left(1 + \frac{\partial \tau}{\partial t}\right)\mathrm{d}t = \left(\psi_x - \frac{\partial \tau}{\partial x}\right)\mathrm{d}x + \left(\psi_t - \frac{\partial \tau}{\partial t}\right)\mathrm{d}t. \tag{3.8}$$

Equating the coefficients of $\mathrm{d}x$ and $\mathrm{d}t$ on both sides of this relation leads to the following transformation laws for the quantities $\{\psi_x, \psi_t\}$:

$${}'\psi_x = \psi_x - \frac{\partial \tau}{\partial x} - \frac{\partial \tau}{\partial x}\frac{\psi_t - \dfrac{\partial \tau}{\partial t}}{1 + \dfrac{\partial \tau}{\partial t}}, \qquad {}'\psi_t = \frac{\psi_t - \dfrac{\partial \tau}{\partial t}}{1 + \dfrac{\partial \tau}{\partial t}}. \tag{3.9}$$

There are two facts that should be noted about these transformation laws. First, if $\psi_x = \partial\xi(x, t)/\partial x$ and $\psi_t = \partial\xi(x, t)/\partial t$ (i.e. $\Psi = \mathrm{d}\xi$ is an exact 1-form), then the choice $\tau = \xi(x, t)$ gives $'\psi_x = '\psi_t = 0$. Thus, an exact compensating 1-form can be transformed away by an appropriately chosen local time translation T_τ. Second, if we choose $\tau(x, t)$ so that $\partial\tau(x, t)/\partial t = \psi_t(x, t)$, which is always possible, then (3.9) give $'\psi_x = \psi_x - \partial\tau/\partial x$ and $'\psi_t = 0$. Therefore, *it is always possible to transform to a Coulomb gauge for which $\psi_t = 0$ by an appropriate gauge transformation T_τ*. This result has particular relevance because (3.4), (3.5) and (3.9) show that J^x and J^t are invariant under gauge transformations:

$$
\begin{aligned}
'J^x &= \mathrm{d}'x = J^x,\\
'J^t &= \mathrm{d}'t + '\Psi = \mathrm{d}t + \mathrm{d}\tau + \Psi - \mathrm{d}\tau = \mathrm{d}t + \Psi = J^t.
\end{aligned} \tag{3.10}
$$

Accordingly, they can be evaluated in the Coulomb gauge without loss of generality. In particular, we have

$$\mathscr{M}\langle \mathrm{d}x \wedge \mathrm{d}t\rangle = J^x \wedge J^t = (1 + \psi_t)\, \mathrm{d}x \wedge \mathrm{d}t \tag{3.11}$$

is gauge invariant and that $\mathscr{M}\langle \mathrm{d}x \wedge \mathrm{d}t\rangle = \mathrm{d}x \wedge \mathrm{d}t$ in the Coulomb gauge where $\psi_t = 0$. Needless to say, use of the Coulomb gauge will significantly simplify many of the evaluations required in this discussion.

A gauge invariant basis $\{j_x, j_t\}$ dual to $\{J^x, J^t\}$ can be introduced by

$$
\begin{aligned}
j_x \rfloor J^x &= 1, \qquad j_x \rfloor J^t = 0,\\
j_t \rfloor J^x &= 0, \qquad j_t \rfloor J^t = 1,
\end{aligned}
$$

where $\rfloor$ denotes the inner product (see [5], Chap. 3). Explicit solution of these relations gives

$$j_x = \frac{\partial}{\partial x} - \frac{\psi_x}{1 + \psi_t}\frac{\partial}{\partial t}, \qquad j_t = \frac{1}{1 + \psi_t}\frac{\partial}{\partial t}. \tag{3.12}$$

These serve to define the minimal replacement operator's action on the derivatives of the state variable by

$$
\begin{aligned}
\mathscr{M}\left\langle\frac{\partial u}{\partial x}\right\rangle &= j_x\langle u\rangle = \frac{\partial u}{\partial x} - \frac{\psi_x}{1 + \psi_t}\frac{\partial u}{\partial t},\\
\mathscr{M}\left\langle\frac{\partial u}{\partial t}\right\rangle &= j_t\langle u\rangle = \frac{1}{1 + \psi_t}\frac{\partial u}{\partial t}.
\end{aligned} \tag{3.13}
$$

These minimal replacements for the derivatives are essential because the chain rule shows that $'u('x, 't) = u(x, t)$ and (3.1) thus imply

$$\frac{\partial u}{\partial x} = \frac{\partial' u}{\partial' x} + \frac{\partial' u}{\partial' t}\frac{\partial\tau}{\partial x}, \qquad \frac{\partial u}{\partial t} = \left(1 + \frac{\partial\tau}{\partial t}\right)\frac{\partial' u}{\partial' t}. \tag{3.14}$$

On the other hand, a direct calculation based on (3.9) and (3.13) shows that

$$
\begin{aligned}
\frac{\partial' u}{\partial' x} - \frac{'\psi_x}{1 + '\psi_t}\frac{\partial' u}{\partial' t} &= \frac{\partial u}{\partial x} - \frac{\psi_x}{1 + \psi_t}\frac{\partial u}{\partial t},\\
\frac{1}{1 + '\psi_t}\frac{\partial' u}{\partial' t} &= \frac{1}{1 + \psi_t}\frac{\partial u}{\partial t}.
\end{aligned} \tag{3.15}
$$

Accordingly, $j_x\langle u\rangle$ and $j_t\langle u\rangle$ are replacements for $\partial u/\partial x$ and $\partial u/\partial t$, respectively, because these new quantities are invariant under the action of local time translation transformation T_τ.

4. THE GAUGE INVARIANT ACTION INTEGRAL

The Lagrangian function l for our 1-dimensional elastic model is a function of $\partial u/\partial x$ and $\partial u/\partial t$, and hence it is not invariant under the action of a local time translation transformation

T_τ. Invariance under such transformations can be achieved by the now famous minimal replacement construct of gauge theory. The prescription is to replace the action 2-form $l\,\mathrm{d}x \wedge \mathrm{d}t$ by its minimal replacement $\mathcal{M}\langle l\,\mathrm{d}x \wedge \mathrm{d}t\rangle = \mathcal{M}\langle l\rangle \mathcal{M}\langle \mathrm{d}x \wedge \mathrm{d}t\rangle = \mathcal{L}_1\,\mathrm{d}x \wedge \mathrm{d}t$. Use of the relations established above gives us

$$\mathcal{L}_1 = \frac{\rho}{2}(1+\psi_t)\left\{\left(\frac{1}{1+\psi_t}\frac{\partial u}{\partial t}\right)^2 - c^2\left(\frac{\partial u}{\partial x} - \frac{\psi_x}{1+\psi_t}\frac{\partial u}{\partial t}\right)^2\right\}. \tag{4.1}$$

This modified Lagrangian function now depends on the compensating fields $\{\psi_x, \psi_t\}$ as well as on the original arguments $\partial u/\partial x$ and $\partial u/\partial t$, but there is no dependence on the derivatives of the compensating fields. This is tantamount to the fact that the Lagrangian (4.1) does not account for the energy required in setting up the compensating fields.

A proper accounting of the energy associated with establishing the compensating fields can be made by what is now referred to as minimal coupling. What is required is the construction of an action 2-form that depends on the derivatives of the compensating fields and which is invariant under the action of local time translation transformations T_τ. In view of the fact that T_τ induces the transformation $'\Psi = \Psi - \mathrm{d}\tau$, it is clear that $\mathrm{d}'\Psi = \mathrm{d}\Psi$; that is

$$\left(\frac{\partial' \psi_x}{\partial' t} - \frac{\partial' \psi_t}{\partial' x}\right)\mathrm{d}'x \wedge \mathrm{d}'t = \left(\frac{\partial \psi_x}{\partial t} - \frac{\partial \psi_t}{\partial x}\right)\mathrm{d}x \wedge \mathrm{d}t. \tag{4.2}$$

We thus see that

$$\mathcal{L}_\psi = \frac{\rho}{2}k(1+\psi_t)\left(\frac{\partial \psi_x}{\partial t} - \frac{\partial \psi_t}{\partial x}\right)^2, \tag{4.3}$$

where k is a coupling constant, and $\mathcal{L}_\psi$ is an T_τ-invariant Lagrangian function that can be used. The minimal coupling construct of standard gauge theory then suggests that the total Lagrangian $\mathcal{L}$ should be taken to be

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_\psi. \tag{4.4}$$

Since this Lagrangian is gauge invariant, we can evaluate it in the Coulomb gauge $\psi_t = 0$ without loss of generality. This gives us the total Lagrangian

$$\mathcal{L} = \frac{\rho}{2}\left\{\left(\frac{\partial u}{\partial t}\right)^2 - c^2\left(\frac{\partial u}{\partial x} - \psi_x\frac{\partial u}{\partial t}\right)^2 + k\left(\frac{\partial \psi_x}{\partial t}\right)^2\right\}, \tag{4.5}$$

for a 1-dimensional elastic model with time translation gauge invariance. The associated invariant action integral for a 1-dimensional elastic material with local time translations is therefore

$$A[u, \psi_x] = \int_0^T \int_0^L \mathcal{L}\,\mathrm{d}x \wedge \mathrm{d}t. \tag{4.6}$$

5. EFFECTIVE FIELDS AND THE GAUGE FIELD EQUATIONS

Following current practices, we define the *effective linear momentum*, $\mathcal{P}$, by

$$\mathcal{P} = \frac{\partial \mathcal{L}}{\partial\left(\dfrac{\partial u}{\partial t}\right)} = \rho\left\{\frac{\partial u}{\partial t} + c^2\left(\frac{\partial u}{\partial x} - \psi_x\frac{\partial u}{\partial t}\right)\psi_x\right\}. \tag{5.1}$$

The reason for referring to this quantity as effective linear momentum is because the definition of the minimal replacement operator $\mathcal{M}$ and (2.3) give

$$\mathcal{M}\langle p\rangle = \mathcal{P}. \tag{5.2}$$

In like manner, we define the *effective force potential*, $\mathcal{S}$, by

$$\mathcal{S} = \mathcal{M}\langle \sigma \rangle = \rho c^2\left(\frac{\partial u}{\partial x} - \psi_x \frac{\partial u}{\partial t}\right). \tag{5.3}$$

The Euler–Lagrange field equation for the state variable $u(x, t)$ with Lagrangian function $\mathcal{L}$ then takes the simple form

$$\frac{\partial}{\partial t}\mathcal{P} = \frac{\partial}{\partial x}\mathcal{S}. \tag{5.4}$$

When this is written out with the help of (5.1) and (5.3) we obtain

$$\frac{\partial}{\partial t}\left\{\frac{\partial u}{\partial t} + c^2\left(\frac{\partial u}{\partial x} - \psi_x \frac{\partial u}{\partial t}\right)\psi_x\right\} = c^2 \frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial x} - \psi_x \frac{\partial u}{\partial t}\right\}. \tag{5.5}$$

Although this field equation is linear in $u(x, t)$, it is definitely nonlinear in the compensating field $\psi_x(x, t)$.

The defining relations for p, σ, $\mathcal{P}$ and $\mathcal{S}$ show that

$$\mathcal{P} = p + \mathcal{S}\psi_x, \qquad \mathcal{S} = \sigma - \rho c^2 \frac{\partial u}{\partial t}\psi_x. \tag{5.6}$$

It is therefore natural to define

$$\frac{\partial p}{\partial t} - \frac{\partial \sigma}{\partial x} = f_e = \text{equivalent elastic body force} \tag{5.7}$$

by direct analogy with the elastic case. The field equation (5.5) thus gives

$$\frac{f_e}{\rho c^2} = \frac{\partial}{\partial t}\left\{\left(\frac{\partial u}{\partial x} - \psi_x \frac{\partial u}{\partial t}\right)\psi_x\right\} + \frac{\partial}{\partial x}\left\{\psi_x \frac{\partial u}{\partial t}\right\}. \tag{5.8}$$

Yet another way of writing the field equation (5.5) is

$$\frac{\partial}{\partial t}\left\{\rho \frac{\partial u}{\partial t} + \mathcal{S}\psi_x\right\} = \frac{\partial \mathcal{S}}{\partial x}. \tag{5.9}$$

It is left to the reader to choose between these various forms of the field equation for $u(x, t)$ as his taste and predilections dictate.

It is a fairly simple matter to see that the Lagrangian (4.5) leads to the field equation

$$\rho k \frac{\partial^2 \psi_x}{\partial t^2} = \mathcal{S}\frac{\partial u}{\partial t}.$$

Written out in full, this gives

$$k \frac{\partial^2 \psi_x}{\partial t^2} = c^2\left\{\frac{\partial u}{\partial x} - \psi_x \frac{\partial u}{\partial t}\right\}\frac{\partial u}{\partial t}. \tag{5.10}$$

It is therefore linear in $\psi_x(x, t)$, but nonlinear in $u(x, t)$. Accordingly, the system of field equations (5.5) and (5.10) that have to be solved is a nonlinear system with all of its attendant complications. An examination of the field equations (5.5) and (5.10) reveals a very important elementary fact, however. *Any static (equilibrium) solution $u(x, t) = ax + b$ of the associated elasticity problem is an exact solution of (5.5) and (5.10) with $\psi_x(x, t) = 0$.* Thus, the effects of local time translations are only in evidence when there are departures from equilibrium configurations, which is reasonable. This shows, however, that the effects of local time translations can show up as soon as we attempt to move from one equilibrium configuration to another by a change in the external load environment!

The field equations (5.5) and (5.10) are the Euler–Lagrange equations for the action functional (4.6) with Lagrangian $\mathscr{L}$. As such, there is a momentum–energy complex with components

$$\mathscr{T}^a_b = \frac{\partial \mathscr{L}}{\partial\left(\frac{\partial u}{\partial x^a}\right)} \frac{\partial u}{\partial x^b} + \frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \psi_x}{\partial x^a}\right)} \frac{\partial \psi_x}{\partial x^b} - \delta^a_b \mathscr{L} \tag{5.11}$$

associated with this problem. Noting that $\mathscr{L}$ is independent of x and t, any solution of the field equations (5.5) and (5.10) will result in satisfaction of the conservation laws

$$\frac{\partial}{\partial x^a} \mathscr{T}^a_b = 0, \qquad 1 \le b \le 2. \tag{5.12}$$

In particular, we have

$$\mathscr{T}^x_t = -\mathscr{S} \frac{\partial u}{\partial t}, \tag{5.13}$$

$$\mathscr{T}^t_t = \frac{\rho}{2}\left\{\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x} - \psi_x \frac{\partial u}{\partial t}\right)^2 + k\left(\frac{\partial \psi_x}{\partial t}\right)^2\right\} + \mathscr{S}\psi_x \frac{\partial u}{\partial t}, \tag{5.14}$$

which satisfy

$$\frac{\partial}{\partial t} \mathscr{T}^t_t + \frac{\partial}{\partial x} \mathscr{T}^x_t = 0. \tag{5.15}$$

We may accordingly identify $\mathscr{T}^x_t$ with the *effective energy flux* and $\mathscr{T}^t_t$ with the *effective energy density*, in which case (5.15) becomes an equation of conservation of effective energy for a 1-dimensional elastic model with time translation gauge invariance.

A comparison of (5.13) with the field equation (5.9) shows that

$$\frac{\partial}{\partial t}\left\{\rho k \frac{\partial \psi_x}{\partial t}\right\} = -\mathscr{T}^x_t. \tag{5.16}$$

Accordingly, (5.15) shows that we also have

$$\frac{\partial}{\partial x}\left\{\rho k \frac{\partial \psi_x}{\partial t}\right\} = \mathscr{T}^t_t \tag{5.17}$$

whenever the field equations (5.5) and (5.10) are satisfied. These relations show that $\rho k\, \partial \psi_x / \partial t$ *may be interpreted as a potential for the effective energy quantities* $\mathscr{T}^x_t$ *and* $\mathscr{T}^t_t$. The compensating field ψ_x thus has a direct interpretation in terms of the energetics of the model.

An alternative way of writing $\mathscr{T}^x_t$ and $\mathscr{T}^t_t$ is to use (2.8) and (2.9) to eliminate a number of terms in (5.13) and (5.14). This gives

$$\mathscr{T}^x_t = T^x_t + \rho c^2 \psi_x \left(\frac{\partial u}{\partial t}\right)^2, \tag{5.18}$$

$$\mathscr{T}^t_t = T^t_t + \frac{\rho}{2}\left\{k\left(\frac{\partial \psi_x}{\partial t}\right)^2 - c^2 \psi_x^2 \left(\frac{\partial u}{\partial t}\right)^2\right\}. \tag{5.19}$$

There are two reasons for obtaining these relations. First, (5.18) and (5.19) show that the interpretations of $\mathscr{T}^x_t$ and $\mathscr{T}^t_t$ as components of an effective momentum-energy complex are reasonable. Second, the known interpretations of T^x_t and T^t_t for a 1-dimensional elastic model show that

$$\frac{\partial}{\partial t} T^t_t + \frac{\partial}{\partial x} T^x_t = r_e \tag{5.20}$$

can be interpreted as the rate of energy production for the elastic model. The relations (5.18)–(5.20) thus give

$$r_e = -\frac{\rho}{2}\frac{\partial}{\partial t}\left\{k\left(\frac{\partial\psi_x}{\partial t}\right)^2 - c^2\psi_x^2\left(\frac{\partial u}{\partial t}\right)^2\right\} - \rho c^2\frac{\partial}{\partial x}\left\{\psi_x\left(\frac{\partial u}{\partial t}\right)^2\right\} \tag{5.21}$$

in the presence of a compensating field ψ_x for local time translations. In particular, we see that $r_e = 0$ whenever $\psi_x(x, t) = 0$.

6. AN ALTERNATIVE FORMULATION

If we solve the constitutive relations (5.1) and (5.3) for $\partial u/\partial x$ and $\partial u/\partial t$, we obtain

$$\rho\frac{\partial u}{\partial x} = \psi_x\mathcal{P} + (c^{-2} - \psi_x^2)\mathcal{S}, \tag{6.1}$$

$$\rho\frac{\partial u}{\partial t} = \mathcal{P} - \psi_x\mathcal{S}. \tag{6.2}$$

These relations can be used to express the field equations (5.5) and (5.10) in the equivalent form

$$\frac{\partial\mathcal{P}}{\partial t} = \frac{\partial\mathcal{S}}{\partial x}, \tag{6.3}$$

$$\rho k\frac{\partial^2\psi_x}{\partial t^2} = \mathcal{S}(\mathcal{P} - \psi_x\mathcal{S}). \tag{6.4}$$

However, the equations (6.1) and (6.2) can be eliminated provided we adjoin their integrability (compatibility) condition

$$\frac{\partial}{\partial x}\{\mathcal{P} - \psi_x\mathcal{S}\} = \frac{\partial}{\partial t}\{\psi_x\mathcal{P} + (c^{-2} - \psi_x^2)\mathcal{S}\}. \tag{6.5}$$

In thus follows that the original system of field equations (5.5) and (5.10) is equivalent to the new system (6.3)–(6.5). However, (6.3) always admits the potential solution

$$\mathcal{P} = \frac{\partial\xi}{\partial x}, \qquad \mathcal{S} = \frac{\partial\xi}{\partial t} \tag{6.6}$$

in terms of a smooth potential function $\xi(x, t)$. When (6.6) are substituted into (6.4) and (6.5), we obtain the resulting complete system of equivalent field equations

$$\rho k\frac{\partial^2\psi_x}{\partial t^2} = \frac{\partial\xi}{\partial t}\left(\frac{\partial\xi}{\partial x} - \psi_x\frac{\partial\xi}{\partial t}\right), \tag{6.7}$$

$$\frac{\partial}{\partial x}\left\{\frac{\partial\xi}{\partial x} - \psi_x\frac{\partial\xi}{\partial t}\right\} = \frac{\partial}{\partial t}\left\{\psi_x\frac{\partial\xi}{\partial x} + (c^{-2} - \psi_x^2)\frac{\partial\xi}{\partial t}\right\}. \tag{6.8}$$

The remaining field quantities $u(x, t)$, $\mathcal{P}(x, t)$, and $\mathcal{S}(x, t)$ can then be calculated from (6.1), (6.2) and (6.6) once we know $\xi(x, t)$ and $\psi_x(x, t)$. In fact, (6.1), (6.2) and (6.6) yield

$$\rho\frac{\partial u}{\partial x} = \psi_x\frac{\partial\xi}{\partial x} + (c^{-2} - \psi_x^2)\frac{\partial\xi}{\partial t}, \tag{6.9}$$

$$\rho\frac{\partial u}{\partial t} = \frac{\partial\xi}{\partial x} - \psi_x\frac{\partial\xi}{\partial t}. \tag{6.10}$$

7. TRAVELING WAVE SOLUTIONS

One of the reasons for rewriting the field equations in the form given by (6.7) and (6.8) is that this form of the field equations allows us to investigate the existence of traveling wave solutions with relative ease. Let us set

$$\xi = \alpha(\eta), \qquad \psi_x = \beta(\eta), \tag{7.1}$$

where

$$\eta = x + at, \tag{7.2}$$

and introduce the notation $\mathrm{d}f(\eta)/\mathrm{d}\eta = f'$. A direct substitution of (7.1) into the field equations (6.7) and (6.8) yields

$$\rho k a \beta'' = (\alpha')^2(1 - a\beta), \tag{7.3}$$

$$a\frac{\mathrm{d}}{\mathrm{d}\eta}\{\beta\alpha' + (c^{-2} - \beta^2)a\alpha'\} = \frac{\mathrm{d}}{\mathrm{d}\eta}\{\alpha' - a\beta\alpha'\}. \tag{7.4}$$

The second of these integrates to give

$$\alpha' = \frac{-k_1}{(a\beta - 1)^2 - \left(\frac{a}{c}\right)^2}, \tag{7.5}$$

where k_1 is an integration constant. When (7.5) is substituted into (7.3) and we make the substitution

$$\gamma(\eta) = a\beta(\eta) - 1, \tag{7.6}$$

we obtain

$$\rho k \gamma'' = \frac{-k_1^2 \gamma}{\left\{\gamma^2 - \left(\frac{a}{c}\right)^2\right\}^2}. \tag{7.7}$$

This equation can be integrated once to obtain

$$\rho k (\gamma')^2 = k_2 + \frac{k_1^2}{\gamma^2 - \left(\frac{a}{c}\right)^2} \tag{7.8}$$

where k_2 is an integration constant. This equation can be integrated once more to obtain

$$\sqrt{\rho k}\int \sqrt{\frac{\gamma^2 - \left(\frac{a}{c}\right)^2}{k_1^2 + k_2\left\{\gamma^2 - \left(\frac{a}{c}\right)^2\right\}}}\,\mathrm{d}\gamma = k_3 \pm \eta. \tag{7.9}$$

where k_3 is an integration constant. For instance, if $a/c = k_1 = k_2 = 1$, then we obtain

$$\sqrt{\rho k}\left\{\sqrt{\gamma^2 - 1} - \arccos\left(\frac{1}{\gamma}\right)\right\} = k_3 \pm \eta.$$

Once (7.9) is solved for $\gamma(\eta)$, (7.5) can be integrated to obtain

$$\alpha = k_4 - k_1 \int \frac{\mathrm{d}\eta}{\gamma(\eta)^2 - \left(\frac{a}{c}\right)^2} \tag{7.10}$$

where k_4 is another constant of integration. Since there are four constants of integration, k_1, k_2, k_3, and k_4, we have established the following result. *The field equations (6.7) and (6.8)* *admit a 4-fold infinity of traveling wave solutions of the form*

$$\xi = \alpha(x + at; k_1, k_2, k_3, k_4), \qquad \psi_x = \beta(x + at; k_1, k_2, k_3) \tag{7.11}$$

for any value of the wave speed "a". There is thus a drastic difference between a 1-dimensional elastic model and a 1-dimensional elastic model with time translation gauge invariance; simply note that the only solutions of the 1-dimensional elastic model with any wave speed $a \neq c$ are of the form $u(x, t) = K_1 + K_2(x + at)$.

8. AN UNEXPECTED RESULT

Elementary dimension analysis shows that ψ_x has the dimensions of time divided by length. This suggests that we look for solutions of (6.7) and (6.8) for which

$$\psi_x = \left(\frac{\partial \xi}{\partial x}\right) \Big/ \left(\frac{\partial \xi}{\partial t}\right). \tag{8.1}$$

When (8.1) is substituted into the field equations (6.7) and (6.8), they can be integrated directly to yield

$$\xi = a(x)t + b(x), \qquad \psi_x = \frac{t\dfrac{\mathrm{d}a(x)}{\mathrm{d}x} + \dfrac{\mathrm{d}b(x)}{\mathrm{d}x}}{a(x)}, \tag{8.2}$$

where $a(x)$ and $b(x)$ are arbitrary smooth functions. We therefore have

$$\mathscr{P} = t\frac{\mathrm{d}a(x)}{\mathrm{d}x} + \frac{\mathrm{d}b(x)}{\mathrm{d}x}, \qquad \mathscr{S} = a(x) \tag{8.3}$$

from (6.6), While (6.9) and (6.10) yield

$$\rho\frac{\partial u}{\partial t} = 0, \qquad \rho c^2\frac{\partial u}{\partial x} = a(x). \tag{8.4}$$

Accordingly, (8.4) yield

$$\rho c^2 u(x, t) = k_1 + \int a(x)\, \mathrm{d}x, \tag{8.5}$$

where k_1 is an integration constant. This shows that *a 1-dimensional elastic body with time translation gauge invariance can admit any time independent, smooth displacement field (time independent strain field) we wish*; simply choose the function $a(x)$ appropriately. The reason for this is that $\partial u/\partial t = 0$ does not imply $\mathscr{P} = 0$, and hence $\partial\mathscr{P}/\partial t = \partial\mathscr{S}/\partial x$ will equilibrate $\mathscr{S} = a(x)$, as (8.3) clearly indicate.

One way around this embarrassment of solutions is to impose the natural, homogeneous, Neumann data

$$\frac{\partial \psi_x}{\partial t}(x, 0) = 0 \tag{8.6}$$

for the compensating field ψ_x. This has the effect of demanding that $a(x) = \text{constant}$, by (8.2), in which case we obtain the standard 1-dimensional, linear, elasticity result that $\partial u/\partial t(x, t) = 0$ implies $u(x, t) = k_1 x + k_2$ by (8.5). This should stand as fair warning that a detailed analysis of admissible boundary conditions for the compensating fields is a necessary part of any future theory that incorporates time translation gauge invariance.

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