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FINITE ELEMENT IMPLEMENTATION OF THE GAUGE THEORY OF DAMAGE

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Abstract—This work presents a finite element formulation of the guage theory of continuum damage as applied to solids with defects under mechanical loading. The Lagrangian density of the initially elastic body is expanded to include contributions from damage potentials that enter as compensating fields, necessary to restore local translational invariance of the action integral. The spatial finite element discretization is applied to the field variables, displacements and damage potentials, and the total Lagrangian is stantionarized with respect to variations of both fields. Additional constraints introduced by the antiexact gauge condition are imposed using a condensation of degrees of freedom method. The first part of the work includes the basic formulation, while the second part deals with the actual implementation and numerical examples pertinent to damage in composite laminates.

1. INTRODUCTION

The key aspect of damage at a microscope length scale is loss of material continuity due to the formation of material defects. Damage micro-mechanics investigates the detailed stress and strain fields around the vicinity of defects without definining any new field variables. With this approach, the distribution and shape of defects must be known *a priori*. Macro-mechanics of damage or continuum damage mechanics introduces internal state variables to fully describe the mechanical state of a material system in the presence of damage. While new field variables are introduced in the continuum approach, the same framework of continuum mechanics is maintained, with modifications needed at the constitutive description of material response and the evolution of internal variables.

The introduction of internal state variables to model creep damage in metals was initially made several decades ago by Kachanov [1]. Even though the focus earlier was on explaining creep behavior of metals, occurrence of damage in advanced composites, at levels of loading substantially lower than failure, initiated interest in modeling damage in composites [2–7]. The standard approach has been to define a set of internal state variables that may alter the constitutive behavior of the material, appropriate for modeling microcracking induced damage and then postulate their evolution laws. Arguments from thermodynamics and microphysical processes in conjunction with macroscopic experiments have been used to determine the functional form of the evolution laws and the additional material constants needed [8].

The current work utilizes the gauge theory of defects in solids [9] within the framework of continuum damage mechanics, to obtain the damage internal state variables as solutions to field equations [10]. The elastic Lagrangian in the gauge theory is modified to include spatial and time derivatives of the damage potentials resulting in Euler–Lagrange field equations for the damage internal state variables. These internal variables are compensating fields that restore the invariance of the Lagrangian under local translations, resulting from local loss of continuity at the microscale. This particular mechanism of introducing the damage potentials suggests a connection between the internal state variables and microcracking induced damage in solids.

The complexity of the resulting field equations makes the task of deriving closed form solutions of 2- and 3-D boundary value problems extremely difficult. In the first part of the present work (Sections 2–4) a finite element formulation of the gauge theory is presented and the field equations are numerically solved in finite domains with appropriate boundary conditions. Specific implementation to fibrous composites undergoing progressive damage with applied loading is discussed in Section 5. A preliminary version of this work for the quasi-static case has been given by Lagoudas and Huang [11].

2. GAUGE THEORY OF DAMAGE

Let the Cartesian coordinates of material points in the deformed configuration be $x_i(X_A, t)$, while X_A, $A = 1, 2, 3$, are the labels for the material points in the reference configuration. Assume that in the absence of damage, the material behaves as a hyperelastic solid with a Lagranian density given by

$$L_e = \frac{1}{2}\rho_0 \partial_t u_i \partial_t u_i - \psi(e_{AB}) \tag{1}$$

where $u_i(X_A, t) = x_i(X_A, t) - X_B\delta_{iB}$, $i = 1, 2, 3$, is the displacement field, ρ_0, is the mass density per unit undeformed volume, and the usual additive decomposition of the Lagrangian into the kinetic energy minus the Helmholtz free energy ψ is utilized. Note that ψ is the free energy per unit undeformed volume that depends on the Lagrange strain measure e_{AB} given by

$$e_{AB} = \frac{1}{2}(\partial_A u_i \delta_{iB} + \partial_B u_j \delta_{jA} + \partial_A u_k \partial_B u_k). \tag{2}$$

The Lagrangian density L is invariant under rigid body rotations and translations in the deformed configuration. If microcracking induced damage develops in the material, the body loses continuity and is not simply connected. This conceptually allows for local rearrangements of material particles without affecting the Lagrangian density. Local translations in the deformed configuration can be described by

$$'u_i(X_A, t) = u_i(X_A, t) + b_i(X_A, t) \tag{3}$$

where $b_i(X_A, t)$ is a local translation vector (space and time dependent). Since L is not invariant under local translations, to restore the invariance of L we replace the total displacement gradients $\partial_A u_i$ by the elastic distortion B_{iA} and the total velocity $\partial_t u_i$ by the material velocity B_i with the evaluation

$$B_{iA} = \partial_A u_i + \phi_{iA}, \qquad B_i = \partial_t u_i + \phi_i \tag{4}$$

where ϕ_{iA} and ϕ_i are the damage compensating potentials which restore the invariance of the Lagrangian under local rigid body translations in the deformed configuration. The damage potentials transform under local translations by the following transformation

$$'\phi_{iA} = \phi_{iA} - \partial_A b_i, \qquad '\phi_i = \phi_i - \partial_t b_i \tag{5}$$

leading to the invariance of the elastic distortion and material velocity, i.e.,

$$\begin{aligned} {}'B_{iA} &= \partial_A {}'u_i + {}'\phi_{iA} \\ &= \partial_A u_i + \partial_A b_i + \phi_{iA} - \partial_A b_i \\ &= B_{iA} \end{aligned} \tag{6}$$

and

$$\begin{aligned} {}'B_i &= \partial_t' u_i + {}'\phi_i \\ &= \partial_t u_i + \partial_t b_i + \phi_i - \partial_t b_i \\ &= B_i. \end{aligned} \tag{7}$$

Use of replacement (4) in (2) and (1) results in

$$E_{AB} = \frac{1}{2}(\delta_{iA} B_{iB} + \delta_{iB} B_{iA} + B_{iA} B_{iB}) \tag{8}$$

and

$$\hat{L}_e = \frac{1}{2}\rho_0 B_i B_i - \psi(E_{AB}) \tag{9}$$

where E_{AB} and $\hat{L}_e$ are the replaced elastic strain tensor and elastic Lagrangian, respectively. The replaced elastic Lagrangian thus remains invariant under local translations that leave the elastic distortion and material velocity unchanged. The above minimal replacement construct has been applied to continua with defects by Kadic and Edelen [12], Edelen and Lagoudas [9] and Lagoudas and Edelen [13] and to continua with damage by Lagoudas [10].

Seeking a physical interpretation of the damage gauge potentials, one can think of them as describing the relative position of local frames translating with the material elements as they are being separated by damage. If the total displacement gradient is compensated to account for the relative separation of material points, the resulting elastic distortion will remain invariant under translation of the local frames caused by material separation, since their motion does not affect the elastic energy of the system that accounts only for stretching of the elastic bonds. When the local frames are frozen into position relative to the global frame, local invariance becomes the global translational invariance of L.

The kinetics associated with the damage potentials is introduced by adding to the elastic Lagrangian a part due to damage, which depends, in general, on the damage potentials, ϕ_{iA}, ϕ_i, and their derivatives [9]. Invariance under local translations results in dependence only on the damage fields D_{iAB}, D_{iA} given by

$$D_{iAB} = \partial_A \phi_{iB} - \partial_B \phi_{iA}, \qquad D_{iA} = \partial_A \phi_i - \partial_t \phi_{iA}. \tag{10}$$

In order to have an invariant Lagrangian under rigid body rotations in the deformed configuration, the final form of the total Lagrangian will be

$$L = \frac{1}{2}\rho_0 B_i B_i - \psi(E_{AB}) + L_{d1}(D_{iA} D_{iB}) - L_{d2}(D_{iAB} D_{iCD}). \tag{11}$$

Since D_{iAB} is a second order anti-symmetric tensor, a second order symmetric tensor $\hat{D}_{AB}$ can be formed by first introducing the dual vector d_{iA}

$$d_{iA} = \frac{1}{2}\epsilon_{ABC} D_{iBC} \tag{12}$$

where ϵ_{ABC} is the alternating tensor. If we define

$$\hat{D}_{AB} = d_{kA} d_{kB} \tag{13}$$

then $\hat{D}_{AB}$ is a second-order symmetric tensor and is invariant under rigid body rotations in the deformed configuration. The total Lagrangian finally becomes

$$L = \frac{1}{2} \rho_0 B_i B_i - \psi(E_{AB}) + L_{d1}(D_{iA} D_{iB}) - L_{d2}(\hat{D}_{AB}). \tag{14}$$

The Lagrangian function L must obey certain material symmetries imposed by the material constitution. The total quadratic Lagrangian of an anisotropic material has the form

$$L = \frac{1}{2} \rho_0 B_i B_i - \frac{1}{2} E_{AB} C_{ABCD} E_{CD} + \frac{1}{2} D_{iA} M_{AB} D_{iB} - S^1_{AB} \hat{D}_{AB} - \frac{1}{2} \hat{D}_{AB} S^2_{ABCD} \hat{D}_{CD} \tag{15}$$

where C_{ABCD} is the fourth order elastic stiffness tensor, M_{AB} is the second order intertia tensor associated with damage currents and S^1_{AB} and S^2_{ABCD} are the second and fourth order damage resistivity tensors, respectively. The elastic stiffness tensor and damage resistivity tensors have the following symmetries:

$$C_{ABCD} = C_{BACD} = C_{ABDC} = C_{CDAB}$$

$$S^1_{AB} = S^1_{BA} \tag{16}$$

$$S^2_{ABCD} = S^2_{BACD} = S^2_{ABDC} = S^2_{CDAB} \tag{17}$$

while the inertia tensor M_{AB} has diagonal symmetry. To fully characterize a general anisotropic material, one needs the mass density, ρ_0, 21 elastic constants, six independent constants for M_{AB}, six constants for S^1_{AB} and 21 constants for S^2_{ABCD}. Note that S^2_{ABCD} constants are associated with fourth order terms in the damage potentials, since $\hat{D}_{AB}$ are quadratic in the derivatives of the damage potentials, and thus can be neglected for linear materials. For quasi-static problems ρ_0 and M_{AB} are not needed. For transversely isotropic materials five independent constants for C_{ABCD} and two for S^1_{AB} are necessary, while for full isotropy two elastic constants and one constant for S^1_{AB} need to be supplied.

3. FINITE ELEMENT FORMULATION OF GAUGE THEORY OF DAMAGE

To facilitate the finite element formulation it is more convenient to introduce a matrix or vector representation of the different tensorial fields. The displacements u_i and the damage variables, ϕ_{iA} and ϕ_i, are first written as vectors $\{u\}$, $\{\phi\}$ and $\{\phi_i\}$ with

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \quad \{\phi\} = \begin{Bmatrix} \phi_{11} \\ \phi_{12} \\ \phi_{13} \\ \phi_{21} \\ \phi_{22} \\ \phi_{23} \\ \phi_{31} \\ \phi_{32} \\ \phi_{33} \end{Bmatrix}, \quad \{\phi_i\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \tag{18}$$

The displacement $\{u\}$ and damage potentials $\{\phi\}$ and $\{\phi_t\}$ within a finite element are interpolated from the element nodal degress of freedom, $\{U\}$, $\{\Phi\}$ and $\{\Phi_t\}$ [14] by

$$\{u\} = [N]\{U\}, \qquad \{\phi\} = [\hat{N}]\{\Phi\} \quad \text{and} \quad \{\phi_t\} = [N]\{\Phi_t\} \tag{19}$$

where $[N]$ is the shape function matrix for $\{U\}$ and $\{\Phi_t\}$, and $[\hat{N}]$ is the shape function matrix for $\{\Phi\}$. The shape functions depend on X_A only, while the nodal degrees of freedom are functions of time.

The strain E_{AB} from (8) is expanded as

$$E_{AB} = \frac{1}{2}(\partial_A u_i \delta_{iB} + \partial_B u_i \delta_{iA} + \partial_A u_i \partial_B u_i + \phi_{iA}\delta_{iB} + \phi_{iB}\delta_{iA} + \phi_{iA}\phi_{iB} + \partial_A u_i \phi_{iB} + \partial_B u_i \phi_{iA}). \tag{20}$$

Substitution of the expressions for $\{u\}$ and $\{\phi\}$ in equation (19) into equation (20) yields

$$\{\epsilon\} = [B]\{U\} + [Q][\hat{N}]\{\Phi\} + [B_{N1}]\{U\} + [B_{N2}]\{\Phi\} + [B_{N3}]\{\Phi\} \tag{21}$$

in which

$$\{\epsilon\} = [E_{11}, E_{22}, E_{33}, 2E_{23}, 2E_{31}, 2E_{12}]^T.$$

The explicit evaluations of the various matrices in equation (21) are given in the Appendix.

The two point damage tensor d_{iA} defined in equation (12) is expressed as

$$\{d_i\} = [D_i]\{\phi\}, \qquad i = 1, 2, 3 \tag{22}$$

while the symmetric damage tensor $\hat{D}_{AB}$ defined in equation (13) becomes

$$[\hat{D}] = \sum_{k=1}^{3} [D_k]\{\phi\}\{\phi\}^T[D_k]^T \tag{23}$$

and the matrices $[D_i]$ are given in the Appendix. Replacing $\{\phi\}$ by $\{\Phi\}$ from equation (19) yields

$$\begin{aligned}[\hat{D}] &= \sum_{k=1}^{3} [D_k][\hat{N}]\{\Phi\}\{\Phi\}^T[\hat{N}]^T[D_k]^T \\ &= \sum_{k=1}^{3} [D_k^*]\{\Phi\}\{\Phi\}^T[D_k^*]^T \end{aligned} \tag{24}$$

where

$$[D_k^*] = [D_k][\hat{N}]. \tag{25}$$

If we convert $[\hat{D}]$ into a 6×1 vector $\{\hat{d}\}$ then

$$\{\hat{d}\} = [\hat{D}^*]\{\Phi\} \tag{26}$$

with $[\hat{D}^*]$ given in the Appendix.

The material velocity B_i introduced in equation (4) can be expressed as

$$\{b\} = [N]\{\dot{U}\} + [N]\{\Phi_t\} \tag{27}$$

where $\{\dot{U}\}$ denotes the time derivative of $\{U\}$. Applying similar rules for D_{iA} defined in equation (10) gives

$$\{d_t\} = [D_t]\{\Phi_t\} - [\hat{N}]\{\dot{\Phi}\} \tag{28}$$

where $\{\dot{\Phi}\}$ is the time derivative of $\{\Phi\}$ and $[D_t]$ is given in the Appendix.

Substitution of the expressions for $\{\epsilon\}$, $\{b\}$, $\{\hat{d}\}$ and $\{d_{,t}\}$ into the total Lagrangian given by equation (15) yields

$$\begin{aligned} L &= \frac{1}{2}\rho\{b\}^T\{b\} - \frac{1}{2}\{\epsilon\}^T[C]\{\epsilon\} + \frac{1}{2}\{d_{,t}\}^T[M]\{d_{,t}\} - [S^1]\{\hat{d}\} - \frac{1}{2}\{\hat{d}\}^T[S^2]\{\hat{d}\} \\ &= \frac{1}{2}\rho(\{\dot{U}\}^T[N]^T[N]\{\dot{U}\} + 2\{\dot{U}\}^T[N]^T[N]\{\Phi_{,t}\} + \{\Phi_{,t}\}^T[N]^T[N]\{\Phi_{,t}\}) \\ &\quad - \frac{1}{2}[([B]+[B_{N1}])\{U\} + ([Q][\hat{N}] + [B_{N2}] + [B_{N3}])\{\Phi\}]^t[C] \\ &\quad \times [([B]+[B_{N1}])\{U\}([Q][\hat{N}] + [B_{N2}] + [B_{N3}])\{\Phi\}] \\ &\quad + \frac{1}{2}(\{\Phi_{,t}\}^T[D_{,t}]^T[M][D_{,t}]\{\Phi_{,t}\} - 2\{\dot{\Phi}\}^T[\hat{N}]^T[M][D_{,t}]\{\Phi_{,t}\} \\ &\quad + \{\dot{\Phi}\}^T[\hat{N}]^T[M][\hat{N}]\{\dot{\Phi}\}) - [S^1][\hat{D}^*]\{\Phi\} - \frac{1}{2}\{\Phi\}^T[\hat{D}^*]^T[S^2][\hat{D}^*]\{\Phi\}. \end{aligned} \tag{29}$$

The stationarization of the action integral, associated with the total Lagrangian over the volume of a finite element, i.e.,

$$\delta A[u_i, \phi_{iA}, \phi_i] = \int_{t_0}^{t_1}\int_{V_e} \delta L \,\mathrm{d}t\,\mathrm{d}v = 0 \tag{30}$$

yields the following set of Euler–Lagrange equations in matrix form:

$$[M]\{\ddot{U}\} + [M]\{\dot{\Phi}_{,t}\} + [K_{11}]\{U\} + [K_{12}]\{\Phi\} = 0 \tag{31}$$

$$[L]\{\ddot{\Phi}\} - [G]\{\dot{\Phi}_{,t}\} + [K_{21}]\{U\} + [K_{22}]\{\Phi\} = 0 \tag{32}$$

$$[P]\{\Phi_{,t}\} + [M]\{\dot{U}\} - [G]^T\{\dot{\Phi}\} = 0 \tag{33}$$

where the different matrices appearing in the above matrix equations are given in the Appendix. The discretized equations are a system of algebraic equations for the space nodal values of the gauge potentials and the displacements, since the finite element discretization was carried out in space, while they involve second time derivatives of the fields.

A further simplification occurs if we explicitly solve from equation (33) for $\{\Phi_{,t}\}$, i.e.,

$$\{\Phi_{,t}\} = -[P]^{-1}[M]\{\dot{U}\} + [P]^{-1}[G]^T\{\dot{\Phi}\}. \tag{34}$$

Substitution of the above expression back into equations (31) and (32) yields

$$([M] - [M][P]^{-1}[M])\{\ddot{U}\} + [M][P]^{-1}[G]^T\{\ddot{\Phi}\} + [K_{11}]\{U\} + [K_{12}]\{\Phi\} = 0 \tag{35}$$

$$[G][P]^{-1}[M]\{\ddot{U}\} + ([L] - [G][P]^{-1}[G]^T)\{\ddot{\Phi}\} + [K_{21}]\{U\} + [K_{22}]\{\Phi\} = 0. \tag{36}$$

The above equations, written in a combined matrix notation, take the following form:

$$\begin{bmatrix} M - MP^{-1}M & MP^{-1}G^T \\ GP^{-1}M & L - GP^{-1}G^T \end{bmatrix}\begin{Bmatrix} \ddot{U} \\ \ddot{\Phi} \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{Bmatrix} U \\ \Phi \end{Bmatrix} = \{0\}. \tag{37}$$

In the presence of applied boundary tractions, T_i, and if the quasi-static loading assumption is made, the above matrix ordinary differential equations in time reduce to the following set of algebraic equations:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}\begin{Bmatrix} U \\ \Phi \end{Bmatrix} = \{F\} \tag{38}$$

where the added vector $\{F\}$ on the right-hand side of the equations denotes the element load vector and is given by

$$\{F\} = \begin{Bmatrix} \int_{s_\sigma} [N]^T\{T\}\,\mathrm{d}s \\ 0 \end{Bmatrix}. \tag{39}$$

4. ANTI-EXACT GAUGE CONDITION

In order to have a unique solution, the equations resulting from the finite element formulation have to be supplemented with the anti-exact gauge condition [9]. This extra constraint is necessary since the Euler–Lagrange equations satisfy differential identities. For the quasi-static case the anti-exact guage requirement reduces to

$$(X_A - X_{0A})\Phi_{iA} = 0 \tag{40}$$

where X_{0A} is chosen to be the center of the geometric configuration. The above constraints establish a unique decomposition of the total distortion into an elastic part and a part due to damage.

Procedures for imposing constraints in the finite element formulation include Lagrange multipliers, transformation equations and penalty functions [15]. The transformation of equations (condensation) method has been found to be the most convenient and efficient way of imposing the anti-exact guage constraints [16], and it will be briefly described here.

The constraint equations given by (40) are first written in matrix form, and they are then partitioned so that

$$[X_{\mathrm{r}} \quad X_{\mathrm{c}}]\begin{Bmatrix} D_{\mathrm{r}} \\ D_{\mathrm{c}} \end{Bmatrix} = \{0\} \tag{41}$$

where $\{D_{\mathrm{r}}\}$ and $\{D_{\mathrm{c}}\}$ are degrees of freedom to be retained (nine for each nodal point) and degrees of freedom to be condensed out (three for each nodal point), respectively. The solution for $\{D_{\mathrm{c}}\}$ yields

$$\{D_{\mathrm{c}}\} = [X_{\mathrm{rc}}]\{D_{\mathrm{r}}\} \tag{42}$$

where

$$[X_{\mathrm{rc}}] = -[X_{\mathrm{c}}]^{-1}[X_{\mathrm{r}}]. \tag{43}$$

If the original system of equations given by (38) is partitioned into

$$\begin{bmatrix} K_{\mathrm{rr}} & K_{\mathrm{rc}} \\ K_{\mathrm{cr}} & k_{\mathrm{cc}} \end{bmatrix}\begin{Bmatrix} D_{\mathrm{r}} \\ D_{\mathrm{c}} \end{Bmatrix} = \begin{Bmatrix} F_{\mathrm{r}} \\ F_{\mathrm{c}} \end{Bmatrix} \tag{44}$$

the final system of the condensed equations is given by

$$[K_{\mathrm{rr}} + X_{\mathrm{rc}}^T K_{\mathrm{cr}} + K_{\mathrm{rc}} X_{\mathrm{rc}} + X_{\mathrm{rc}}^T K_{\mathrm{cc}} X_{\mathrm{rc}}]\{D_{\mathrm{r}}\} = \{F_{\mathrm{r}} + X_{\mathrm{rc}}^T F_{\mathrm{c}}\}. \tag{45}$$

5. IMPLEMENTATION–APPLICATION TO COMPOSITES

Using ABAQUS UEL (User Element) capability, the finite formulation of the gauge theory described in the previous section can be easily incorporated into ABAQUS. Within the UEL subroutine, the element stiffness matrix, element load vector and the residual vector need to be updated. For a detailed discussion of the structure of the user supplied element, the actual implementation, patch test, etc. the reader is referred to Huang [16].

To be able to describe evolution of damage, the damage resistivity tensor $[S^1]$ and in the nonlinear case both $[S^1]$ and $[S^2]$ must be prescribed for the material system of interest. Having in mind applications to composites, where microcracking induced damage occurs in clearly defined ranges in the stress–strain diagram, a $[0/90]_s$ glass/epoxy system is selected. Assuming a state of plane stress, there is only one material parameter needed for the damage resistivity tensor S^1 in the case of transverse isotropy, to be denoted s_1. The value of the damage resistivity constant s_1 for the example presented here is obtained by fitting an experimental uniaxial stress–strain curve [17], and it has the value 5.0×10^6 N. The elastic material constants

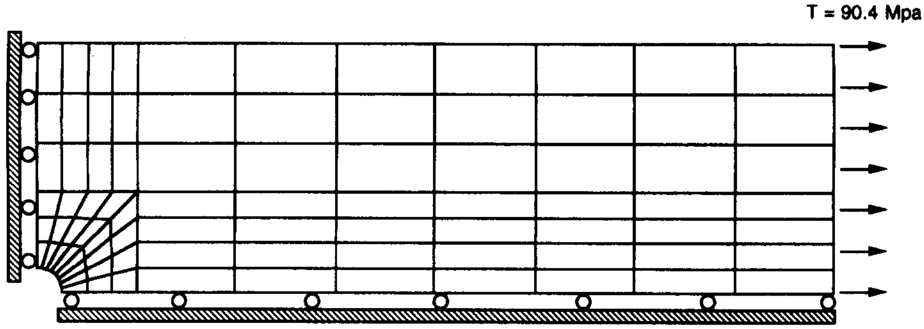

Fig. 1. Finite element mesh along with the boundary conditions and the applied traction.

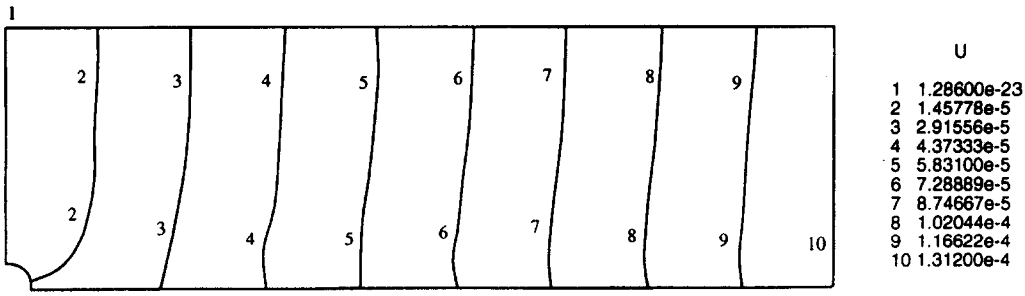

Fig. 2. Displacement u contour plot.

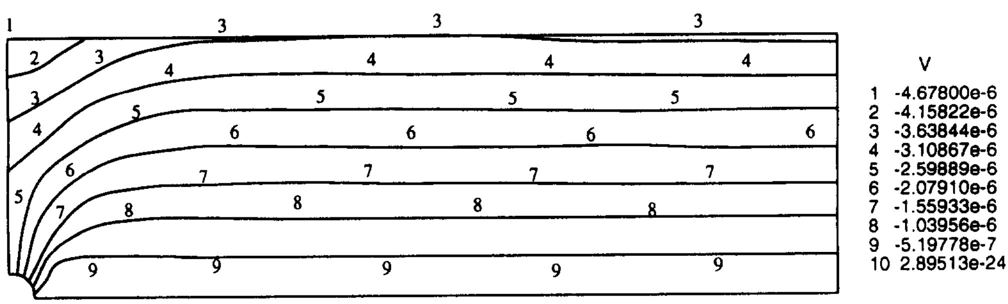

Fig. 3. Displacement v contour plot.

for the fiber are: $E_f = 72.3$ GPa, $\nu_f = 0.22$; the elastic constants for the matrix are: $E_m = 2.92$ GPa, $\nu_m = 0.35$.

The above material parameters have been used to predict damage in a composite laminated plate with a central hole. The width of the plate is 20 mm and the central hole is 2 mm in diameter. This is the same setup as in [18] who experimentally determined the strain around the central hole as the applied tension loading was raised from 0 to 90.4 MPa. Figure 1 shows the mesh arrangement along with the boundary conditions and applied traction T for the central hole specimen (due to symmetry only one-quarter of the specimen is meshed).

Figures 2 and 3 show the u and v displacement contour plots at the end of the monotonic loading path which was uniaxial tension up to maximum value T. The gauge potentials ϕ_{11} and ϕ_{22} generated by the applied loading are plotted in Figs 4 and 5. It is seen from Fig. 4 that most of the damage due to opening of microcracks in the loading direction occurred near the upper half of the quarter hole, which is consistent with the resulting stress concentration there. This mode of damage is expected to take place mostly in the 90° plies in the form of matrix cracking. The gauge potential ϕ_{22} shown in Fig. 5 corresponds to opening of microcracks in the transverse to loading direction (matrix cracking in the 0° plies) and it is much smaller than the

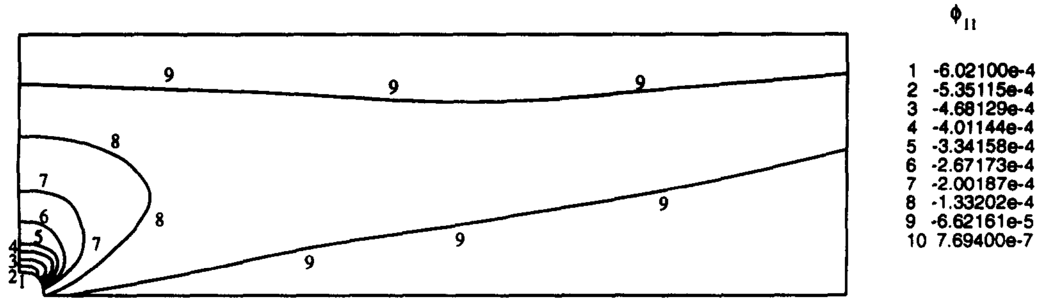

Fig. 4. Gauge potential ϕ_{11} contour plot.

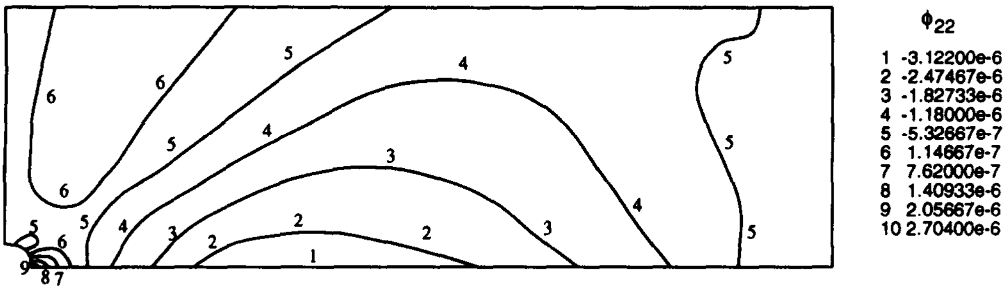

Fig. 5. Gauge potential ϕ_{22} contour plot.

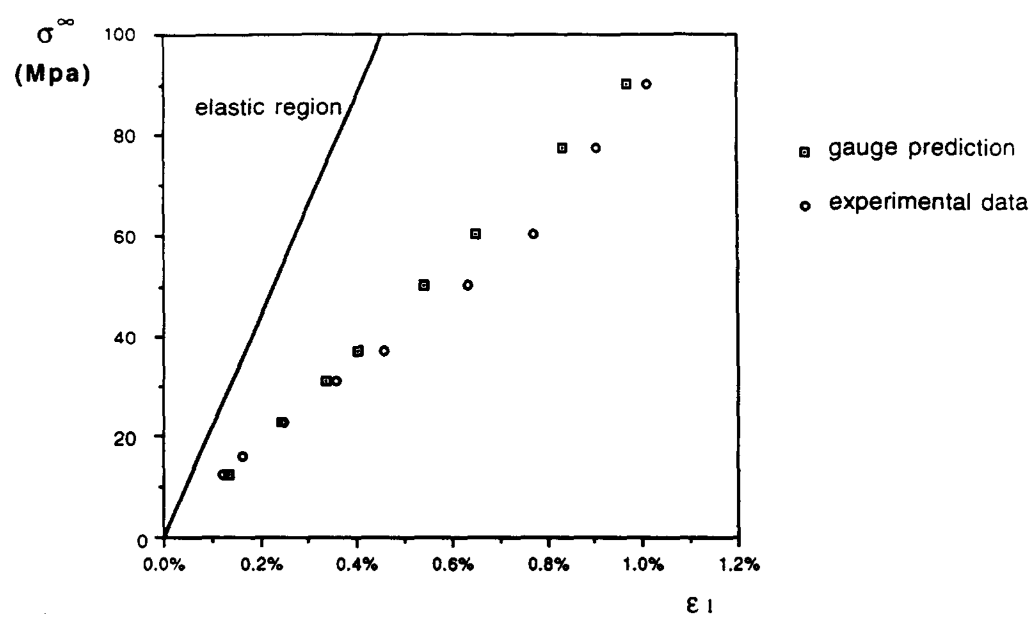

Fig. 6. Total axial strain at the top of quarter hole vs remote applied stress σ^{∞}.

damage in the 90° plies. To compare the model predictions with experimental results obtained by [18], the strain on the top of the quarter hole is plotted versus the remote applied stress in Fig. 6. The agreement between the gauge theory prediction and the experimental results is good. The resulting values of the remaining two gauge potentials ϕ_{12} and ϕ_{21} are small relative to ϕ_{11} for the applied loading and contour plots of these gauge potentials are not shown.

6. CONCLUSIONS

Phenomenological models of continuum damage mechanics require determination of material constants related to damage variables, whose evolution laws must be postulated. For the gauge theory of damage, for a plane stress state and transverse isotropy, one material parameter s_1 needs to be prescribed from a uniaxial stress–strain curve. Using this single parameter, the gauge theory is capable of determining the damage state at every point of a material system subject to general in-plane loading conditions, provided that the resulting field equations are solved in the domain of interest.

The solution of the field equations in terms of displacements has been rendered unique by imposing the anti-exact gauge condition. Even though a change in the choice of gauge will result in a different displacement field, the stresses will remain invariant under any choice of gauge due to the way the minimal replacement construct is applied. As a suggestion for future work, a formulation of the gauge theory in terms of the elastic distortion as the only set of field variables might be preferential, since such an approach would not require the explicit enforcement of the anti-exact gauge condition.

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APPENDIX

The different matrices defined in the finite element implementation of the gauge theory of damage are given below:

$$[B]=\begin{bmatrix} N_{1,1} & 0 & 0 & \cdots \\ 0 & N_{1,2} & 0 & \cdots \\ 0 & 0 & N_{1,3} & \cdots \\ 0 & N_{1,3} & N_{1,2} & \cdots \\ N_{1,3} & 0 & N_{1,1} & \cdots \\ N_{1,2} & N_{1,1} & 0 & \cdots \end{bmatrix}$$

$$[Q]=\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&1&0\\0&0&1&0&0&0&1&0&0\\0&1&0&1&0&0&0&0&0\end{bmatrix}$$

$$[B_{N1}]=\begin{bmatrix}l_{11}N_{1,1}&l_{21}N_{1,1}&l_{31}N_{1,1}&\cdots\\l_{12}N_{1,2}&l_{22}N_{1,2}&l_{32}N_{1,2}&\cdots\\l_{13}N_{1,3}&l_{23}N_{1,3}&l_{33}N_{1,3}&\cdots\\2l_{12}N_{1,3}&2l_{22}N_{1,3}&2l_{32}N_{1,3}&\cdots\\2l_{11}N_{1,3}&2l_{21}N_{1,3}&2l_{31}N_{1,3}&\cdots\\2l_{11}N_{1,2}&2l_{21}N_{1,2}&2l_{31}N_{1,2}&\cdots\end{bmatrix}$$

$$\textit{with}\quad l_{iA}=\frac{1}{2}\sum_{k=1}^{n}N_{k,A}U_i^k,$$

$$[B_{N2}]=\begin{bmatrix}l_{11}\hat{N}_1&0&0&l_{21}\hat{N}_1&0&0&l_{31}\hat{N}_1&0&0&\cdots\\0&l_{12}\hat{N}_1&0&0&l_{22}\hat{N}_1&0&0&l_{32}\hat{N}_1&0&\cdots\\0&0&l_{13}\hat{N}_1&0&0&l_{23}\hat{N}_1&0&0&l_{33}\hat{N}_1&\cdots\\0&0&2l_{12}\hat{N}_1&0&0&2l_{22}\hat{N}_1&0&0&2l_{32}\hat{N}_1&\cdots\\0&0&2l_{11}\hat{N}_1&0&0&2l_{21}\hat{N}_1&0&0&2l_{31}\hat{N}_1&\cdots\\0&2l_{11}\hat{N}_1&0&0&2l_{21}\hat{N}_1&0&0&2l_{31}\hat{N}_1&0&\cdots\end{bmatrix}$$

$$\textit{with}\quad l_i=\frac{1}{2}\sum_{k=1}^{n}\hat{N}_k\Phi_{iA}^k,$$

$$[B_{N3}]=\begin{bmatrix}l_{11}\hat{N}_1&0&0&l_{21}\hat{N}_1&0&0&l_{31}\hat{N}_1&0&0&\cdots\\0&l_{12}\hat{N}_1&0&0&l_{22}\hat{N}_1&0&0&l_{32}\hat{N}_1&0&\cdots\\0&0&l_{13}\hat{N}_1&0&0&l_{23}\hat{N}_1&0&0&l_{33}\hat{N}_1&\cdots\\0&l_{13}\hat{N}_1&l_{12}\hat{N}_1&0&l_{23}\hat{N}_1&l_{22}\hat{N}_1&0&l_{33}\hat{N}_1&l_{32}\hat{N}_1&\cdots\\l_{13}\hat{N}_1&0&l_{11}\hat{N}_1&l_{23}\hat{N}_1&0&l_{21}\hat{N}_1&l_{33}\hat{N}_1&0&l_{31}\hat{N}_1&\cdots\\l_{12}\hat{N}_1&l_{11}\hat{N}_1&0&l_{22}\hat{N}_1&l_{21}\hat{N}_1&0&l_{32}\hat{N}_1&l_{31}\hat{N}_1&0&\cdots\end{bmatrix}$$

$$\textit{with}\quad L_{iA}=\sum_{k=1}^{n}N_{k,A}U_i^k,$$

$$B_{N31}=\begin{bmatrix}l_{11}N_{1,1}&l_{21}N_{1,1}&l_{31}N_{1,1}&\cdots\\l_{12}N_{1,2}&l_{22}N_{1,2}&l_{32}N_{1,2}&\cdots\\l_{13}N_{1,3}&l_{23}N_{1,3}&l_{33}N_{1,2}&\cdots\\l_{13}N_{1,2}+l_{12}N_{1,3}&l_{23}N_{1,2}+l_{22}N_{1,3}&l_{33}N_{1,2}+l_{32}N_{1,3}&\cdots\\l_{13}N_{1,1}+l_{11}N_{1,3}&l_{23}N_{1,1}+l_{22}N_{1,3}&l_{33}N_{1,1}+l_{32}N_{1,3}&\cdots\\l_{13}N_{1,1}+l_{12}N_{1,2}&l_{23}N_{1,1}+l_{22}N_{1,2}&l_{33}N_{1,1}+l_{32}N_{1,2}&\cdots\end{bmatrix}$$

$$\textit{with}\quad l_{iA}=\frac{1}{2}\sum_{k=1}^{n}\hat{N}_k\Phi_{iA}^k.$$

$$[D_1]=\left[\begin{array}{ccc|c|c}\frac{-\partial}{\partial X_2}&\frac{\partial}{\partial X_1}&0&&\\0&\frac{-\partial}{\partial X_3}&\frac{\partial}{\partial X_2}&0&0\\\frac{\partial}{\partial X_3}&0&-\frac{\partial}{\partial X_1}&&\end{array}\right],\qquad [D_2]=\left[\begin{array}{c|ccc|c}&\frac{-\partial}{\partial X_2}&\frac{\partial}{\partial X_1}&0&\\0&0&\frac{-\partial}{\partial X_3}&\frac{\partial}{\partial X_2}&0\\&\frac{\partial}{\partial X_3}&0&-\frac{\partial}{\partial X_1}&\end{array}\right]$$

$$[D_3]=\left[\begin{array}{c|c|ccc}&&\frac{-\partial}{\partial X_2}&\frac{\partial}{\partial X_1}&0\\0&0&0&\frac{-\partial}{\partial X_3}&\frac{\partial}{\partial X_2}\\&&\frac{\partial}{\partial X_3}&0&-\frac{\partial}{\partial X_1}\end{array}\right]$$

$$[\hat{D}^*] = \begin{bmatrix} D^{*k}_{1J}\Phi_{,J}D^{*k}_{11} & D^{*k}_{1J}\Phi_{,J}D^{*k}_{12} & \cdots & D^{*k}_{1J}\Phi_{,J}D^{*k}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 2D^{*k}_{1J}\Phi_{,J}D^{*k}_{21} & 2D^{*k}_{1J}\Phi_{,J}D^{*k}_{22} & \cdots & 2D^{*k}_{1J}\Phi_{,J}D^{*}_{k}2n \end{bmatrix}$$

$$[D_t] = \begin{bmatrix} N_{1,1} & 0 & 0 & \cdots \\ N_{1,2} & 0 & 0 & \cdots \\ N_{1,3} & 0 & 0 & \cdots \\ 0 & N_{1,1} & 0 & \cdots \\ 0 & N_{1,2} & 0 & \cdots \\ 0 & N_{1,3} & 0 & \cdots \\ 0 & 0 & N_{1,1} & \cdots \\ 0 & 0 & N_{1,2} & \cdots \\ 0 & 0 & N_{1,3} & \cdots \end{bmatrix}$$

$$[M] = \int_{V_e} \rho_0 [N][N]^T \, \mathrm{d}v$$

$$[L] = \int_{V_e} [\hat{N}]^T [M][\hat{N}] \, \mathrm{d}v$$

$$[G] = \int_{V_e} [\hat{N}]^T [M][D_t] \, \mathrm{d}v$$

$$[P] = \int_{V_e} [D_t]^T [M][D_t] \, \mathrm{d}v$$

$$[K_{11}] = \int_{V_e} ([B] + 2[B_{N1}] + [B_{N31}])^T [C]([B] + [B_{N1}]) \, \mathrm{d}v$$

$$[K_{12}] = \int_{V_e} ([B] + 2[B_{N1}] + [B_{N31}])^{\mathrm{T}} [C]([Q][\hat{N}] + (B_{N2})] + [B_{N3}]) \, \mathrm{d}v$$

$$[K_{21}] = \int_{V_e} ([Q][\hat{N}] + 2[B_{N2}] + [B_{N3}])^T [C]([B] + [B_{N1}]) \, \mathrm{d}v$$

$$[K_{22}] = \int_{V_e} \{([Q][\hat{N}] + 2[B_{N2}] + [B_{N3}])^T [C]([Q][\hat{N}] + [B_{N2}] + [B_{N3}]) + 2[D^*_k]^T [S^1][D^*_k] + 2[\hat{D}^*]^T [S^2][\hat{D}^*]\} \, \mathrm{d}v.$$

In the above V_e denotes the element volume.