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A GAUGE THEORY OF DAMAGE

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Abstract—The gauge theory of defects is formulated for solids with damage due to microcracking. The gauge damage potentials are introduced to model crack densities and currents through minimal replacement, while minimal coupling expands the elastic Lagrangian to include derivatives of the damage potentials. Stationarization of the action integral leads to field equations for the displacements and the damage potentials. The field equations are explicitly solved for the case of a 1-D solid (rod) under a quasi-static loading cycle and a permanent set is observed.

1. INTRODUCTION

The currently favorable approach to continuum damage mechanics [1–8] is to start with the Helmholtz free energy function for an elastic solid and expand it to include contributions from the damage variables. The evolution of the damage variables is determined through appropriately defined evolution laws, in a similar way to incremental plasticity. The gauge theory of damage, even though it uses a similar framework, is a fundamentally different way of modeling continuum damage. The Lagrangian (kinetic minus potential energy) in the gauge theory is modified to include spatial and time *derivatives* of the damage variables. It is a simple matter to check that, unless derivative dependence of the damage variables in the Lagrangian occurs, trivial algebraic relations derive as a result of stationarization. This leads to genuine Euler–Lagrange field equations for the damage variables, without the need of postulating evolution laws for them.

The subject of this work is to model the interactions between elastic fields and microcracking induced damage using the gauge theoretical approach [9] for broken global translational symmetries. A defect-free elastic solid is considered under isothermal loading with its Lagrangian being invariant under the global action of the group of Euclidean transformations and the group of material symmetries [10]. Compensating (gauge) potentials are introduced to render the action integral invariant under the local action of either group and their *derivatives* are used to form damage fields entering in the construction of the total Lagrangian. In particular, local invariance with respect to the group of rigid body translations gives rise to microcracking induced damage fields, due to the the loss of material continuity, while local invariance of the material symmetry group gives rise to defects associated with the lattice orientation, i.e. dislocations and disclinations. Here we will focus on local translations and the associated damage fields. In Section 2 the field equations for a general anisotropic effective medium with damage will be derived. In Section 3 the explicit solution of the field equations for a 1-D solid under a quasi-static loading cycle will be presented for various material constants combinations.

2. CONTINUUM GAUGE THEORY OF DAMAGE

Let us assume that in the absence of damage the material behaves as a hyperelastic solid with a Lagrangian given by

$$L_e = L_e(u^i,\ \partial_A u^i,\ \partial_t u^i) \tag{1}$$

where $u^i(X^A, t) = x^i(X^A, t) - X^B \delta^i_B$, $i = 1, 2, 3$ is the displacement field that defines the deformed configuration of the body. The Cartesian coordinates of the material points in the deformed configuration are $x^i(X^A, t)$, while X^A, $A = 1, 2, 3$ are the labels for the material points in the reference configuration. For simplicity the reference configuration will be taken to coincide with the initial configuration, i.e. $X^A = x^i(X^B, 0)\delta^A_i$. Both x^i and X^A are referred to

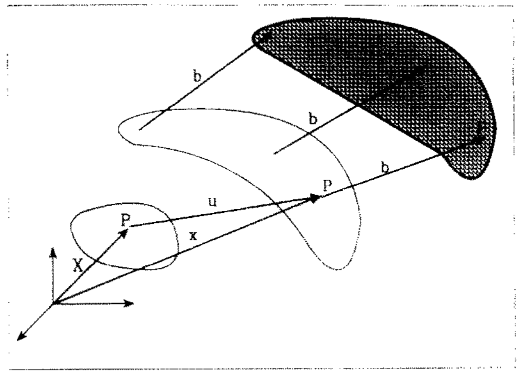

Fig. 1. Local translations superimposed on a deformed configuration.

the same coordinate frame, so that u^i is a vector field (Fig. 1). The usual additive decomposition of the Lagrangian into the kinetic minus the potential energy, and the requirement that rigid body rotations and translations (Euclidean transformations) leave the potential energy invariant, lead to the following form for the elastic Lagrangian:

$$L_e = \frac{1}{2}\,\partial_t u^i \delta_{ij}\,\partial_t u^j - \psi(e_{AB}) \tag{2}$$

In the above ψ is the elastic strain energy function (Helmholtz free energy for isothermal processes) which depends on the Green–Lagrange strain measure

$$e_{AB} = \frac{1}{2}(\partial_A u^i \delta_{ij}\,\partial_B u^j + \delta_{Ai}\,\partial_B u^i + \delta_{Bi}\,\partial_A u^i) \tag{3}$$

Rigid body rotations imply that u^i transforms by

$$u^i = R^i_j x^j - R^i_j \delta^j_A X^A = R^i_j u^j \tag{4}$$

where R^i_j is an orthogonal matrix. Under such transformations the deformed configuration rotates simultaneously with the reference configuration or, equivalently, the coordinate frame rotates by $-R^i_j$. Similarly, under rigid body translations u^i transforms by

$$u^i = (x^i + b^i) - (\delta^i_A X^A + b^i) = u^i \tag{5}$$

where b^i is the translation of the reference and the deformed configurations (equivalently the coordinate frame translates by $-b^i$).

Let an elastic body occupy a region V with boundary S and let the body be subjected to surface tractions $T_i(X^A, t)$ on $S_\sigma \subset S$ and to initial and final momentum $P^0_i(X^A)$ at $t = t_0$ and $P^1_i(X^A)$ at $t = t_1$, respectively. The action integral for the body, in the absence of body forces, is defined by

$$A[u^i] = \int_{t_0}^{t_1}\int_V L_e\,\mathrm{d}v\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{S_\sigma} T_i u^i\,\mathrm{d}s\,\mathrm{d}t - \int_V P_i u^i\,\mathrm{d}v\bigg|_{t_0}^{t_1} \tag{6}$$

If tractions T_i are made equal to

$$T_i(X^A, t) = \partial_A Q_i^{AB} n_B + \partial_t Q_i^{4B} n_B \qquad \text{on } S_\sigma \tag{7}$$

and the initial and final momenta are given by

$$P^0_i(X^A) = \partial_A Q_i^{A4}(X^A, t_0) \qquad \text{in } V, \tag{8}$$

$$P^1_i(X^A) = \partial_A Q_i^{A4}(X^A, t_1) \qquad \text{in } V, \tag{9}$$

for some specified $Q_i^{AB}(X^C, t) = -Q_i^{BA}$, $Q_i^{A4}(X^C, t) = -Q_i^{4A}$, defined in V for all time, it can be easily seen that such a selection for T_i and P_i will give rise to a null Lagrangian [9], with the evaluation

$$N_e = \partial_B u^i\,\partial_A Q_i^{AB} + \partial_B u^i\,\partial_t Q_i^{4B} + \partial_t u^i\,\partial_A Q_i^{A4} \tag{10}$$

The action integral, therefore, given by

$$A[u^i] = \int_{t_0}^{t_1} \int_V (L_e + N_e)\, dv\, dt \tag{11}$$

completely characterizes an elastic material and will remain invariant under rigid body motions.

If microcracking induced damage develops in the material, the material locally loses continuity, even though at the macroscale it appears to be intact. In addition to the global translational invariance it is now possible to create states with the same elastic energy by local translations, namely translations depending on position X^A and time t. For example, one can locally translate the material points so that a crack is annihilated, while the creation of a similar one guarantees invariance of the elastic energy stored in the material. This additional freedom imbedded in the kinematics of deformable bodies derives from the microstructure and becomes observable in the macroscale as local translational invariance. Allowing the microstructure to occur, the first thing that needs to be modified is the representation of the elastic energy. The total deformation gradients $\partial_A u^i$ in L_e need to be replaced by the elastic distortion B^i_A, and the velocity $\partial_t u^i$ by the distortional velocity B^i, respectively defined by

$$B^i_A = \partial_A u^i + \varphi^i_A, \qquad B^i = \partial_t u^i + \varphi^i \tag{12}$$

where φ^i_A, φ^i are the damage compensating potentials which restore the invariance of the Lagrangian under local translations. Therefore, if $b^i(X^A, t)$ is the local translation vector, then

$$u^i(X^A, t) = u^i(X^A, t) + b^i(X^A, t) \tag{13}$$

$$\varphi^i_A = \varphi^i_A - \partial_A b^i, \qquad \varphi^i = \varphi^i - \partial_t b^i \tag{14}$$

and consequently

$$B^i_A = B^i_A, \qquad B^i = B^i \tag{15}$$

The above construct corresponds to the minimal replacement of the gauge theory of defects [9]. A similar idea, proposed by Kachanov [11], was the replacement of the surface area by an effective area in modeling creep and the following generalizations (i.e. [5]). The minimally replaced elastic Lagrangian has the form

$$\hat{L}_e = \frac{1}{2} B^i \delta_{ij} B^j - \psi(E_{AB}) \tag{16}$$

$$E_{AB} = \frac{1}{2}(B^i_A \delta_{ij} B^j_B + \delta_{Ai} B^i_B + \delta_{Bi} B^i_A) \tag{17}$$

and remains invariant under local translations which leave the elastic distortion unchanged.

The kinetics of the damage fields is introduced by adding to the elastic Lagrangian a part due to damage, which depends, in general, on the damage potentials, φ^i_A, φ^i, and their derivatives. Invariance under local translations results in dependence only on the damage fields D^i_{AB}, D^i_A given by

$$D^i_{AB} = \partial_A \varphi^i_b - \partial_B \varphi^i_A \qquad D^i_A = \partial_A \varphi^i - \partial_t \varphi^i_A \tag{18}$$

The additional requirement of invariance under rigid body rotations results in the final form of the total Lagrangian

$$L = \frac{1}{2} B^i \delta_{ij} B^j - \psi(E_{AB}) + L_d(D^i_A \delta_{ij} D^j_B,\, D^i_{AB} \delta_{ij} D^j_{CD}) \tag{19}$$

As an example, a quadratic Lagrangian has the explicit evaluation

$$L = \frac{1}{2} B^i \delta_{ij} B^j - \frac{1}{2} E_{AB} C^{ABCD} E_{CD} + \frac{1}{2} D^i_A \delta_{ij} M^{AB} D^j_B - \frac{1}{2} D^i_{AB} \delta_{ij} S^{ABCD} D^j_{CD} \tag{20}$$

where C^{ABCD} is the fourth order elastic stiffness tensor, M^{AB} is the second order inertia tensor associated with damage currents and S^{ABCD} is the fourth order damage resistivity tensor. For a general anisotropic material the elastic stiffness tensor has the usual symmetries, that is

$$C^{ABCD} = C^{BACD} = C^{ABDC} = C^{CDAB} \tag{21}$$

the inertia tensor M^{AB} has diagonal symmetry and the damage resistivity tensor has the symmetries

$$S^{ABCD} = -S^{BACD} = -S^{ABDC} = S^{CDAB} \tag{22}$$

since the damage fields are antisymmetric, namely $D^i_{AB} = -D^i_{BA}$. To fully characterize a general anisotropic material, therefore, one needs the mass density, ρ_0, 21 elastic constants, 6 independent components for M^{AB} and 6 independent constants for S^{ABCD}. For static problems ρ_0 and M^{AB} are not needed. For transverse isotropy five elastic constants and two independent components of S^{ABCD} are needed, while for an isotropic material the independent material constants reduce to the two Lamé constants and one independent component of S^{ABCD}.

The null Lagrangian N_e, after minimal replacement, takes the form

$$N = B^i_B\,\partial_A Q_i^{AB} + B^i_B\,\partial_t Q_i^{4B} + B^i\,\partial_A Q_i^{A4} \tag{23}$$

The action integral (11) finally becomes

$$A[u^i, \varphi^i_A, \varphi^i] = \int_{t_0}^{t_1}\int_V (L+N)\,\mathrm{d}v\,\mathrm{d}t \tag{24}$$

Hamilton's principle, associated with the above action integral, states that of all displacements u^i and damage potentials φ^i_A and φ^i, the actual ones render the action integral stationary, i.e. the first variation δA of the action integral vanishes. The necessary conditions for vanishing of δA are the satisfaction of the Euler–Lagrange equations together with the appropriate boundary conditions. The Euler–Lagrange equations are

$$\partial_A \sigma_i^A = \partial_t p_i \tag{25}$$

$$\partial_A R_i^{AB} + \partial_t R_i^B = -\frac{1}{2}\sigma_i^B \tag{26}$$

$$\partial_A R_i^A = \frac{1}{2} p_i \tag{27}$$

while the boundary conditions become

$$\sigma_i^A n_A = (\partial_B Q_i^{BA} + \partial_t Q_i^{4A}) n_A \quad \text{on } S_\sigma \quad \text{for} \quad t_0 \le t \le t_1 \tag{28}$$

$$u^i = g^i(X^A, t) \quad \text{on } S_u \quad \text{for} \quad t_0 \le t \le t_1 \tag{29}$$

$$p_i = \partial_A Q_i^{A4}(X^A, t_0) \quad \text{in } V \quad \text{for} \quad t = t_0 \tag{30}$$

$$p_i = \partial_A Q_i^{A4}(X^A, t_1) \quad \text{in } V \quad \text{for} \quad t = t_1 \tag{31}$$

$$R_i^{AB} n_A = 0, \qquad R_i^A n_A = 0 \quad \text{on } S \quad \text{for} \quad t_0 \le t \le t_1 \tag{32}$$

$$R_i^A = 0 \quad \text{in } V \quad \text{at} \quad t = t_0,\ t = t_1 \tag{33}$$

An alternative to defining final momenta would be to assume that the initial displacements and damage potentials are prescribed by

$$u^i(X^A, 0) = 0, \qquad \varphi^i_A(X^A, 0) = 0, \qquad \varphi^i(X^A, 0) = 0 \tag{34}$$

that is, the reference configuration is stress-free and damage-free. The constitutive equations that define the first Piola–Kirchhoff stress and the damage forces derive from the derivatives of the Lagrangian and are given by

$$\sigma_i^A = -\frac{\partial L_e}{\partial B^i_A}, \qquad p_i = \frac{\partial L_e}{\partial B^i} \tag{35}$$

$$R_i^{AB} = -\frac{\partial L_d}{\partial D^i_{AB}}, \qquad R_i^A = \frac{\partial L_d}{\partial D^i_A} \tag{36}$$

The above field equations have to be supplemented with a gauge condition to make the selection of the damage potentials unique. It has been argued in our previous work that the antiexact gauge is the appropriate selection compatible with the observable quantities [9]. Without going into details we simply report that, if the origin of the coordinate system is chosen to be the center of the homotopy operator, the antiexact gauge condition is

$$H[\varphi^i_A, \varphi^i] = \int_0^1 [X^A \varphi^i_A(\eta X^B, \eta t) + t\varphi^i(\eta X^B, \eta t)]\,\mathrm{d}\eta = 0 \tag{37}$$

In the following problem the importance of the gauge selection will become apparent.

3. 1-D GAUGE THEORY OF DAMAGE

Assume that a 1-D solid exists which can develop damage along the spatial direction and varying with time. A real material that could be modeled as a 1-D solid is a composite rod, for example, a unidirectional fiber bundle with brittle fibers susceptible to damage. Statistical theories of strength have been developed for fiber bundles and it is widely known that a substantial number of fiber-breaks occurs for loads far below the strength limit.

The quadratic Lagrangian for the 1-D gauge theory of damage has the form

$$\begin{aligned} L + N = &\frac{1}{2}\rho_0(\partial_t u + \varphi_t)^2 - \frac{1}{2}\lambda(\partial_X u + \varphi_X)^2 + \frac{1}{2}s(\partial_X \varphi_t - \partial_t \varphi_X)^2 \\ &+ (\partial_X u + \varphi_X)\,\partial_t Q - (\partial_t u + \varphi_t)\,\partial_X Q \end{aligned} \tag{38}$$

where λ is the Lamé constant (in the absence of damage it coincides with the elastic axial Young's modulus) and s characterizes the resistivity of the material to damage. The independent variables are the spatial distance X, $-L \le X \le L$, and time t, $t_0 \le t \le t_1$, while the displacement $u(X, t)$ and the damage potentials $\varphi_X(X, t)$ and $\varphi_t(X, t)$ are the dependent variables in the variational formulation. The action integral takes the form

$$A[u, \varphi_X, \varphi_t] = \int_{-L}^{L} \int_{t_0}^{t_1} (L + N)\,\mathrm{d}X\,\mathrm{d}t \tag{39}$$

and the Euler–Lagrange equations (25–27) reduce to

$$\lambda(\partial_X \partial_X u + \partial_X \varphi_X) = \rho_0(\partial_t \partial_t u + \partial_t \varphi_t) \tag{40}$$

$$s(\partial_t \partial_X \varphi_t - \partial_t \partial_t \varphi_X) = \lambda(\partial_X u + \varphi_X) - \partial_t Q \tag{41}$$

$$s(\partial_X \partial_X \varphi_t - \partial_X \partial_t \varphi_X) = \rho_0(\partial_t u + \varphi_t) - \partial_X Q \tag{42}$$

while the boundary conditions (28–33) become

$$\lambda(\partial_X u + \varphi_X) = \partial_t Q, \qquad X = -L, \qquad X = L, \qquad t_0 \le t \le t_1 \tag{43}$$

$$\partial_X \varphi_t - \partial_t \varphi_X = 0, \qquad X = -L, \qquad X = L, \qquad t_0 \le t \le t_1 \tag{44}$$

$$\rho_0(\partial_t u + \varphi_t) = \partial_X Q, \qquad t = t_0, \qquad t = t_1, \qquad -L \le X \le L \tag{45}$$

$$\partial_X \varphi_t - \partial_t \varphi_X = 0, \qquad t = t_0, \qquad t = t_1, \qquad -L \le X \le L \tag{46}$$

The field equations are not independent but they are connected through differential identities as a result of the local invariance. If we take the space derivative of (41) and subtract from the time derivative of (42), equation (40) is recovered. Therefore, it is essential to use a gauge condition to fix the arbitrariness in the selection of the damage potentials. To simplify the system of field equations, we introduce the new field variables $\hat{u}$, $\hat{\varphi}_X$, $\hat{\psi}_t$, such that

$$u = \hat{u} - \hat{\psi}_t, \qquad \varphi_X = \hat{\varphi}_X + \partial_X \hat{\psi}_t, \qquad \varphi_t = \partial_t \hat{\psi}_t \tag{47}$$

Substitution of (47) into (40–46) results in the total elimination of $\hat{\psi}_t$. Since equations (40–46) are satisfied for an arbitrary selection of $\hat{\psi}_t$, without loss of generality, $\hat{\psi}_t$ is chosen to be zero (Coulomb gauge). After introducing a new dependent variable $\xi(X, t) \equiv \partial_t \varphi_X$, the final set of differential equations with their boundary conditions becomes ($c = \sqrt{(\lambda/\rho_0)}$)

$$\frac{1}{c^2}\partial_t\,\partial_t u - \partial_X\,\partial_X u = \partial_X \varphi_X \tag{48}$$

$$\lambda(\partial_X u + \varphi_X) = \partial_t Q, \qquad X = -L, \qquad X = L, \qquad t_0 \le t \le t_1 \tag{49}$$

$$\rho_0\,\partial_t u = \partial_X Q, \qquad t = t_0, \qquad t = t_1, \qquad -L \le X \le L \tag{50}$$

$$\frac{1}{c^2}\partial_t\,\partial_t \xi - \partial_X\,\partial_X \xi + \frac{\lambda}{c^2 s}\xi = \frac{1}{s}\left(\frac{1}{c^2}\partial_t\,\partial_t Q - \partial_X\,\partial_X Q\right) \tag{51}$$

$$\xi = 0, \qquad X = -L, \qquad X = L, \qquad t_0 \le t \le t_1 \tag{52}$$

$$\xi = 0, \qquad t = t_0, \qquad t = t_1, \qquad -L \le X \le L \tag{53}$$

The above system of field equations has been decoupled into a single equation for ξ, i.e. (51) with boundary conditions (52), (53), and field equation (48) with boundary conditions (49), (50) for u, together with the definition of ξ.

To model quasi-static loading we select

$$Q = -\frac{\lambda a}{\nu\varepsilon}\cos(\nu\varepsilon t) \tag{54}$$

where a is the amplitude of loading, ε is a small parameter which controls the rate of loading, and $\nu = 2\pi c/L$. According to the above definition of Q, the boundary conditions (49) and (52) become (Fig. 2)

$$\partial_t Q = a\lambda\sin(\nu\varepsilon t), \qquad X = -L, L \tag{55}$$

$$\xi = 0, \qquad X = -L, L \tag{56}$$

while the initial and final conditions (50) and (53) are dropped and εt is considered to be a loading parameter. We also introduce the normalizations

$$\tau = \varepsilon\frac{\nu}{\pi}t = \varepsilon\frac{2c}{L}t, \qquad x = \frac{X}{L} \tag{57}$$

and expand the fields in powers of the small parameter ε by

$$\varphi_X = \varphi_X^0 + \varepsilon\varphi_X^1 + O(\varepsilon^2) \tag{58}$$

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2) \tag{59}$$

$$\xi = \varepsilon\xi_1 + O(\varepsilon^2) \tag{60}$$

The first order terms of (51) are

$$-\frac{c^2 s}{L^2}\partial_x\,\partial_x\xi_1 + \lambda\xi_1 = a\lambda\nu\cos(\pi\tau) \tag{61}$$

where $\partial_x = L\partial_X$, with solution

$$\xi_1 = a\nu\cos(\pi\tau)\left(1 - \frac{\cosh\gamma x}{\cosh\gamma}\right) \tag{62}$$

that satisfies the boundary condition (56). The ratio of material constants γ is defined by

$$\frac{\gamma}{L} = \sqrt{\left(\frac{\lambda}{c^2 s}\right)} \tag{63}$$

The leading order terms of equation (48) are

$$-\frac{1}{L}\partial_x\,\partial_x u_0 = \partial_x\varphi_X^0 \tag{64}$$

and, after integration and satisfaction of the boundary condition (55), equation (64) results into

$$\partial_x u_0 = L[a\sin(\pi\tau) - \varphi_X^0] \tag{65}$$

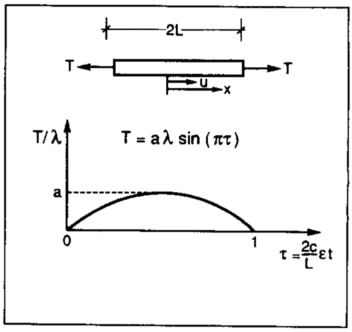

Fig. 2. Quasi-static loading of a 1-D continuum.

The damage potential φ_X^0 is found from (62) integrating $\xi_1 = (\nu/\pi)\,\partial_\tau\varphi_X^0$, with the evaluation

$$\varphi_X^0 = a\sin(\pi\tau)\left(1 - \frac{\cosh\gamma x}{\cosh\gamma}\right) + h(x) \tag{66}$$

The antiexact gauge requires that

$$H\varphi_X^0 = x\int_0^1 \varphi_X^0(\eta x, \eta\tau)\,\mathrm{d}\eta = 0 \tag{67}$$

which is used to determine the unknown function $h(x)$.

Substitution of φ_X^0 into (65) and integration with respect to x, with the removal of the integration constant by fixing the displacements at $x = 0$, finally yields the total displacement

$$\begin{aligned} u_0(x, \tau; \gamma) = aLx\Bigg\{&\sin(\pi\tau)\frac{\sinh(\gamma x)}{\gamma x\cosh\gamma} - \frac{1}{\pi\tau}[\cos(\pi\tau) - 1] \\ &- \frac{1}{\cosh\gamma[(\gamma x)^2 + (\pi\tau)^2]}[\gamma x\sinh(\gamma x)\sin(\pi\tau) - \pi\tau\cosh(\gamma x)\cos(\pi\tau) + \pi\tau]\Bigg\} \end{aligned} \tag{68}$$

For various values of γ the above result is plotted in Figs 3–5 for traction amplitude $a = 0.01$. The value of $\gamma/L = 0.1$ in Fig. 3 represents the case where the damage resistivity of the material is large and the response is elastic, the material points returning to their original position at the end of unloading. It can be shown analytically from (68) that in the limit as $\gamma \to 0$ $(s \to \infty)$, $u_0/L \to ax\sin(\pi\tau)$, which is the elastic response. Figure 4 considers the case where the damage resistivity is comparable to the elastic response, i.e. $\gamma/L = 1$, and the

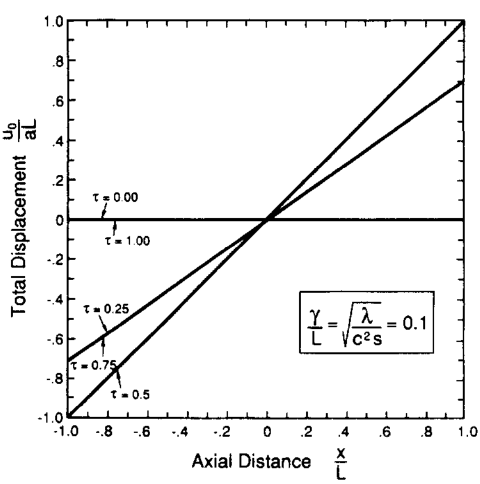

Fig. 3. Displacement response of a 1-D continuum with large damage resistance under quasi-static loading.

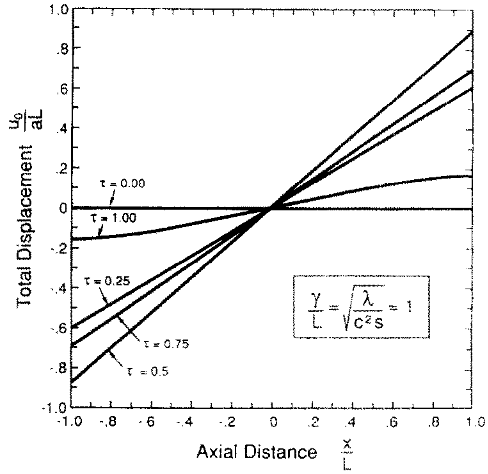

Fig. 4. Displacement response of a 1-D continuum with moderate damage resistance under quasi-static loading.

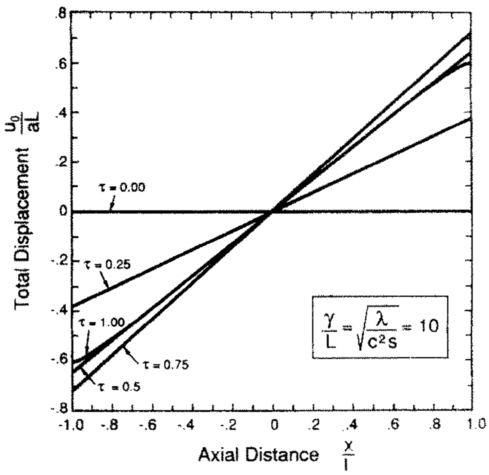

Fig. 5. Displacement response of a 1-D continuum with small damage resistance under quasi-static loading.

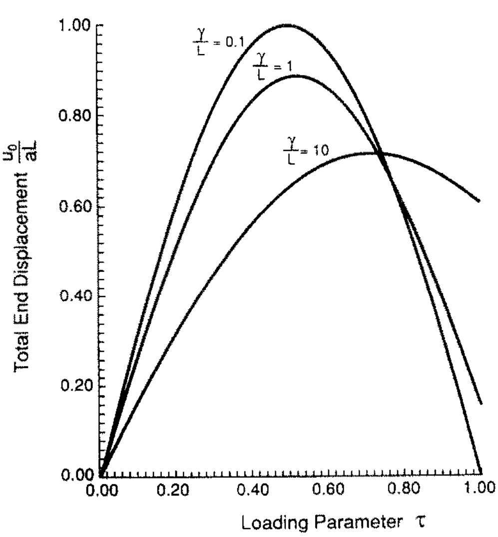

Fig. 6. Total end displacement during the loading cycle ($\gamma/L = \sqrt{\lambda/c^2 s}$).

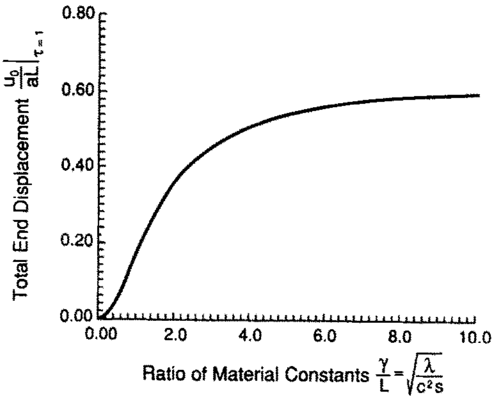

Fig. 7. Permanent set at the end of unloading.

appearance of a permanent set is evident at the end of unloading. A material compliant to damage is shown in Fig. 5, with $\gamma/L = 10$, and the irrecoverable part of the displacement at the end of unloading is dominant.

The variation of the total end displacement $u_0(x = 1, \tau; \gamma)$ during the loading cycle is shown in Fig. 6. Since $u_0(x = 1, \tau; \gamma) = -u_0(x = -1, \tau; \gamma)$, the end displacement is half of the total permanent elongation of the rod. It is clearly shown in the figure that a permanent set develops as γ increases. To find the irrecoverable part of the displacement, we calculate the displacement at the limit as $\tau \to 1$ $(t \to \infty)$. It can be easily found from (68) that

$$u_0(x,\ \tau = 1;\ \gamma) = aLx\left[\frac{2}{\pi} - \frac{\pi}{(\gamma x)^2 + \pi^2}\frac{1 + \cosh(\gamma x)}{\cosh \gamma}\right] \tag{69}$$

The above asymptotic result is shown in Fig. 7 for the end displacement, $x = 1$, as a function of the ratio γ/L. It is observed that, as $\gamma \to 0$, $u_0(x = 1,\ \tau = 1;\ \gamma = 0) = 0$, while as $\gamma \to \infty$, $u_0(x = 1,\ \tau = 1;\ \gamma \to \infty) \to 2aL/\pi$. The permanent set, therefore, can be fully characterized by the single curve of Fig. 7, by scaling the displacement by the total length L and the amplitude of the applied loading a.

Differentiating u_0/L with respect to x we can obtain an expression for the total strain $\varepsilon^t(x,\ \tau;\ \gamma)$. The irrecoverable inelastic strain is obtained from (65), as

$$\varepsilon^{in} = \varepsilon^t - \varepsilon^e = \partial_x u_0/L - a\sin(\pi\tau) = -\varphi^0_X \tag{70}$$

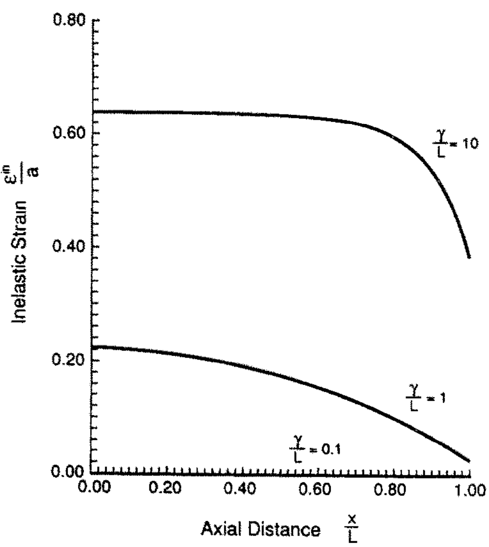

Fig. 8. Inelastic strain at the end of unloading ($\gamma/L = \sqrt{\lambda/c^2 s}$).

For the 1-D solid considered, the elastic strain is fully recovered at the end of the loading cycle ($\tau = 1$), as is evident from above. The inelastic strain at the end of loading is therefore equal to the total strain in the rod and has the evaluation

$$\varepsilon^{in}(x, \tau = 1; \gamma) = a\left\{\frac{2}{\pi} + \frac{2\gamma^2 x^2 \pi}{[(\gamma x)^2 + \pi^2]^2} \frac{1 + \cosh(\gamma x)}{\cosh \gamma} - \frac{\pi}{(\gamma x)^2 + \pi^2} \frac{1 + \cosh(\gamma x) + \gamma x \sinh(\gamma x)}{\cosh \gamma}\right\} \tag{71}$$

The above result is plotted in Fig. 8 for various values of γ. It can be easily shown that as $\gamma \to \infty$, the inelastic strain approaches the value $2a/\pi$.

4. CONCLUSIONS

An internal state variable theory of damage based on the gauge principles of minimal replacement and minimal coupling has been developed. The field equations have been analytically solved for a 1-D solid under quasi-static loading. At the end of the loading cycle a permanent set has been observed, the extent of which depends on the ratio of material parameters.

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