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MATERIAL AND SPATIAL GAUGE THEORIES OF SOLIDS—I. GAUGE CONSTRUCTS, GEOMETRY, AND KINEMATICS

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Abstract—A gauge-theoretic approach for modeling continuous distributions of defects in solids is undertaken, based on local internal and external symmetries of an appropriate Lagrangian for those materials. The invariance of the Lagrangian under global action of internal (spatial) and external (material) symmetry groups is extended to invariance under local action of both groups of symmetries through an application of the minimal replacement construct of gauge theory. This is achieved by introduction of nontrivial spatial and material gauge connections. Both connections are used for the construction of two sets of Cartan equations of structure with two different types of torsion and curvature (spatial and material). The spatial torsion and curvature are shown to model "microcracks" and "microrotations", while the material torsion and curvature model dislocations, disclinations, and energy dissipation mechanisms.

1. SPATIAL SYMMETRIES AND FRAME INDIFFERENCE

We make the customary assumptions for modeling a hyperelastic body. In particular, we have a Lagrangian density L_e for the body that depends on the configuration of the body in its deformed state and on the configuration gradients. We will not make any explicit assumptions about the material symmetries of the body at this point. Thus, we can write the Lagrangian in the form

$$L_e := L_e(X^a, \chi^i, \partial_a \chi^i), \tag{1.1}$$

where $\{X^a \mid 1 \leq a \leq 4\}$ denote the coordinates of a point in the space $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ of reference histories (see [1]). We adopt the convention that $\{X^A \mid 1 \leq A \leq 3\}$ is a coordinate system on any time slice of $\mathbb{R}^4$, while X^4 is the time coordinate.

The space of configuration histories is denoted by $E_3 \times \mathbb{R}$ with the coordinate cover $\{x^i, t \mid 1 \leq i \leq 3\}$. The mapping χ, that is given by

$$x^i = \chi^i(X^a), \qquad t = X^4, \tag{1.2}$$

serves to define the history of the deformed body, and yields the configuration gradients $\partial_a \chi^i$ that appear on the right-hand side of (1.1). We may think of (1.2) as telling us how the deformed body is placed in physical (Euclidean) space E_3 at any given time $t = X^4 =$ constant.

Suppose that we transform the spatial coordinate cover of $E_3 \times \mathbb{R}$ by

$$'x^i = Q^i_j x^j + b^i, \quad 't = t, \tag{1.3}$$

where Q^i_j are the entries of a constant proper orthogonal matrix and b^i are the components of a constant vector. In other words, we operate on the vector in $E_3 \times \mathbb{R}$ with components $\{x^i, t\}$ from the left with an element of the matrix representation of the Lie group $SO(3) \rhd T(3)$ (see [1] for the details). The principle of *frame indifference* requires that L_e remain invariant under the transformation (1.3) because the deformed configuration has not changed; only the position of the observer has changed in a homogeneous and time independent fashion, and this should have no effect in Newtonian mechanics.

Suppose that we attach to each material point in the current configuration a coframe (basis for cotangent vectors) whose positioning depends on the material point $\{X^A\}$ and the time $\{X^4\}$. The deformed configuration history obviously remains unchanged under this process. Evaluations of the deformation gradients, $\partial_A \chi^i$ and $\partial_4 \chi^i$, are another matter altogether because we have

$$\chi^* \, \mathrm{d}x^i = \mathrm{d}\chi^i = \partial_a \chi^i \, \mathrm{d}X^a$$

which will necessarily be altered by a change of coframe. If the information content in the Lagrangian $L_e(X^a, \chi^i, \partial_a \chi^i)$ is to remain unchanged under changes of coframe, we have to make appropriate corrections. What is needed is a more sophisticated way of going from a material point to a neighboring material point; that is, we need a non-trivial *affine connection* for the cotangent space. The formal way of introducing the required affine connection is to replace the partial derivative by a covariant derivative that will compensate for all rotations and translations of the coframes. Symbolically, this is expressed by

$$\partial_a \chi^i \to D_a \chi^i = \partial_a \chi^i + \Gamma^i_{aj} \chi^j + \phi^i_a, \tag{1.4}$$

or, if we define $\{\hat{\chi}^r\} = \{\chi^1, \chi^2, \chi^3, 1\}^T$, then

$$\partial_a \hat{\chi}^r \to D_a \hat{\chi}^r = \partial_a \hat{\chi}^r + \hat{\Gamma}^r_{as} \hat{\chi}^s, \tag{1.5}$$

where $\hat{\Gamma}^r_{as}$ take values in the Lie algebra of $SO(3) \rhd T(3)$. These are the results that would be obtained if we viewed the choice of coframes to result from a virtual rigid body transformation of the base space $E_3 \times \mathbb{R}$ that varied from point to point (i.e., from a virtual, local action of $SO(3) \rhd T(3)$),

$$'x^i = Q^i_j(X^a) x^j + b^i(X^a). \tag{1.6}$$

The reader is referred to Edelen and Lagoudas [1, Sect. 4.2] for a detailed account of this construction. The Lie algebra valued connection $\hat{\Gamma}^r_{as}$ in (1.5) "subtracts" from $\partial_a \hat{\chi}^r$ the influence of the relative (local) rigid body motions of the coframes. The physical meaning of this *minimal replacement* operation is that the energy should not be influenced by the choice of coframes, and therefore $\partial_a \hat{\chi}^r$ cannot enter alone into the evaluation of the Lagrangian L_e. We must use $D_a \hat{\chi}^r$ instead, because $D_a \hat{\chi}^r$ contains the real information about local changes of the configuration under a deformation $\hat{\chi}^r$. In other words, if we use coframes that have been rotated and translated relative to each other to evaluate the deformation characterized by $\hat{\chi}^r$, we have to account for this by evaluating the local changes in the configuration according to (1.5). This will guarantee the invariance of L_e under local virtual coframe transformations.

Up to this point, we have not changed the deformed configuration, but only the way in which we evaluate the mapping functions $\hat{\chi}^r$ that represent the deformed configuration; namely with respect to relative translated and rotated coordinates that give rise to alternative choices of the local natural coframes. The requirement that L_e be frame indifferent has led to the minimal replacement construct; namely,

$$L_e(X^a, \chi^i, \partial_a \chi^i) \to L_e(X^a, \chi^i, D_a \chi^i). \tag{1.7}$$

This construct is the first step in applying a *generalized frame indifference principle* that will render L_e invariant under position and time dependent transformations. A trivial application of frame indifference now indicates that L_e should not depend explicitly on the arguments χ^i, and that it can depend on $D_a \chi^i$ only through the components of the minimally replaced right Cauchy–Green strain tensor and minimally replaced square of the velocity vector,

$$C_{AB} = D_A \chi^i \, \delta_{ij} D_B \chi^j, \qquad C_{44} = D_4 \chi^i \, \delta_{ij} D_4 \chi^j. \tag{1.9}$$

Consequently, to render L_e invariant under local coframe transformations, we first need to apply minimal replacement and then apply the usual frame indifference construction. Since we are dealing with infinitesimal transformations in the neighborhood of the identity of $SO(3) \rhd T(3)$, the order of applying the two operations is not important.

2. A GENERALIZED OBJECTIVITY PRINCIPLE

We now investigate active local rigid body motions of the deformed configuration for a fixed choice of coframes. Suppose that we have a smoothly deformed configuration and apply a rigid body motion to it; that is, for fixed coframes, the coordinates of the material points in the deformed configuration are given by $\{'\chi^i, 't\}$ with the evaluation

$$'x^i = '\chi^i = Q^i_j \chi^j + b^i, \qquad 't = t. \tag{2.1}$$

Notice that the mapping functions themselves have changed. The principle of *material objectivity* tells us that this transformation should not affect L_e because a rigid body motion of the deformed configuration cannot change the stored elastic energy of the material body. Without changing the global coframes, let us take Q^i_j and b^i to be particle dependent (i.e. to be functions of the X's). This operation will create new mapping functions $'\chi^i$ which are no longer objective because differentiation and local action of the group $SO(3) \triangleright T(3)$ do not commute. The physical reasoning here is that we should not try to evaluate the energy of the body by using $\partial_a{}'\chi^i$ because this is "contaminated" by the local rigid body motions of the material elements. Correction for the contaminating local rigid body motions is achieved by applying the minimal replacement construct, but with a different interpretation of $\hat{\Gamma}^r_{as}$. These connection coefficients are now introduced in order to compensate for the actual local rigid body motions of the material elements. As an example, we envision a material with "microcracks" that is examined with respect to a length scale that is large relative to the mean diameter of the microcracks. (The term "microcrack" should not be taken too seriously; it is used to signify that we are studying *spatial defects,* rather than materials with real cracks and all of the attendant accoutrements of fracture mechanics.) The deformed configuration of such a material will be characterized by the observable mapping functions $'\chi^i$, resulting from a smooth χ^i that has been transformed by local rotations and translations, but it is a mistake to use the configuration gradients $\partial_a{}'\chi^i$ to determine its stored energy. This fact is made clear by the following gedanken experiment.

Assume that a stress-free continuous body is loaded by boundary tractions in the elastic regime. Here and throughout the discussion of the gedanken experiment, loading and unloading processes are assumed to take place quasistatically so that we may neglect transient behavior properties. Let the material properties of the body be such that the body develops "microcracks" under the application of boundary tractions. This can be recognized by the fact that unloading does not take the body back to its original configuration. The unloaded state will thus contain residual stress fields. (Notice that cracks introduced as perfect cuts into the reference configuration are excluded here because they do not lead to residual stresses.) We assume that the stored energy is only due to elastic deformations, but clearly these cannot be captured by the configuration gradients because these would model a body that unloads completely. We therefore cut the deformed body in its unloaded state along the faces of the microcracks and extend these cuts until they meet. This cutting process disconnects the body into many small elements. From the continuum point of view, since there can be no distinct crack faces, the prescription is to disconnect the pieces until there are no remaining microcracks and then let each piece relax by removing all boundary tractions. This relaxation process will thus remove all of the residual stresses. Clearly, we can use the difference between the deformed state and the cut up relaxed companion states as a measure of the stored elastic energy. If $\mathrm{d}x^i$ denotes the distance between two neighboring material points in the deformed state and $\mathrm{d}X^A$ is the same relaxed material element in the reference state, then

$$\mathrm{d}x^i = B^i_A(X^B)\,\mathrm{d}X^A + B^i_4(X^A)\,\mathrm{d}T, \tag{2.2}$$

where B^i_A are the components of distortion of the material element that must be used for the evaluation of the stored elastic energy (the second set of terms on the right-hand side is absent in the static case). It is evident from the manner in which this relaxation process takes place and the way in which we introduce $'\chi^i$, that the disconnected pieces can fit together if they undergo local rigid body motions. In the case of infinitesimal rigid body motions we have

$$B^i_a - \Gamma^i_{aj}\chi^j - \phi^i_a = \partial_a \chi^i; \tag{2.3}$$

namely, if we superimpose local rigid body rotations and translations to the relaxed material pieces, we recover the reference state.

The above gedanken experiment can also be traced in the reverse order, giving us further insight. First, start with a perfect continuum in the reference state without residual stresses.

Cut it into small pieces which are then rigidly rotated and translated relative to each other. For the time being, this cutting process and the local rigid body motions do not involve any applied work since we disregard surface energy contributions and assume that the reference configuration is not acted on by external fields. This process is kinematically admissible as soon as we arrange the local translations and rotations so that there is no overlapping of material elements. The relaxed state of misfitting elements is thus formed. With successive elastic deformations, bring these elements to a perfect match, glue them together, let the body relax, and then apply the appropriate boundary tractions. This path is ref$\rightarrow$rel$\rightarrow$def (ref, rel, and def standing for reference, relaxed, and deformed configurations, respectively). Notice that the path ref$\rightarrow$rel is purely kinematical up to this point. There are therefore many different ways of creating internal stress states by merely choosing Γ^i_{aj} and ϕ^i_a appropriately. What is meant here is simply recasting (2.3) in terms of these transitions by the correspondence between

$$B = (\Gamma\chi + \phi) + \mathrm{d}\chi$$

and

$$\text{rel}\rightarrow\text{def} = \text{rel}\rightarrow\text{ref} + \text{ref}\rightarrow\text{def}.$$

Thus, in order to determine B^i_a completely, we can either "destroy" the material and measure B^i_a directly, or measure $\partial_a\chi^i$ and lift the relative rigid body rotations and translations (that are defined by the transition rel$\rightarrow$ref) from kinematical variables to dynamic variables. In other words, associate the local rigid body motions with the energetics and let a stationary energy principle choose the appropriate ones and assign them constitutive relations. With this remark, we end the gedanken experiment, having found a way to generalize the objectivity principle for materials with microstructure.

To summarize, implementation of the *generalized objectivity principle* requires a two step process. The first step is minimal replacement for the local action of the spatial symmetry group $SO(3)\rhd T(3)$. This step replaces the configuration gradients, $\partial_a\chi^i$, by the components of the elastic distortion, B^i_a, which transform covariantly under the local action of $SO(3)\rhd T(3)$. The second step follows exactly the same construct as the classical application of objectivity; namely, constructing an invariant Lagrangian L_e out of quantities that transform covariantly under the global action of $SO(3)\rhd T(3)$ with B^i_a appearing in L_e instead of $\partial_a\chi^i$. This results in L_e being independent of the arguments χ^i, while the dependence on B^i_a occurs only through the quantities

$$C_{ab} = B^i_a\,\delta_{ij}B^j_b = C_{ba}. \tag{2.4}$$

For static problems, these quantities reduce to minimal replacement applied to the components of the right Cauchy–Green strain tensor. It is clear from the previous argument that the order of applying minimal replacement and classical objectivity is immaterial, as it was for the generalized frame indifference case. We can therefore give a concise precis of the basic principle as follows:

$$\text{Generalized Objectivity} := \text{minimal replacement} + \text{classical objectivity}.$$

The basic difference between generalized frame indifference and generalized objectivity is that in the first we used rigidly moving coframes to describe a compatible deformation configuration, while in the second we used fixed coframes to describe an incompatible deformed configuration.

The fundamental group $G_s = SO(3)\rhd T(3)$ is the isometry group of Euclidean 3-dimensional space E_3 in which a material body finds itself at any instant of time (i.e. the space of histories is $E_3\times\mathbb{R}$). When geometric quantities are pulled back to the space of reference histories by the induced map χ^*, the group G_s acts as an *internal symmetry group* or gauge group of internal symmetries [1]. We thus have the induced local action of G_s on the state variables $\{\chi^i, B^i_a\}$, considered as fields on the space of reference histories $\mathbb{R}^4$, that is represented by

$$'\chi^i(X^b) = \chi^*\,'x^i = Q^i_j(X^b)\chi^j(X^b) + b^i(X^b), \qquad \chi^*\,'t = X^4, \tag{2.5}$$

$$'B^i_a(X^b) = Q^i_j(X^b)B^j_a(X^b). \tag{2.6}$$

The true action of G_s is spatial, however, because its primitive action occurs on time slices in the space of current histories $E_3 \times \mathbb{R}$, which is evidenced by the fact that G_s acts only on the spatial indices in (2.5) and (2.6). This is unequivocal in (2.5). The occurrence of the lower index "a" in (2.6) can confuse the issue unless we realize that the quantities $B^i_a(X^b)$ are components of a two-point tensor field [8],

$$B = B^i_a(X^b)\,\mathrm{d}X^a \otimes \frac{\partial}{\partial x^i}; \tag{2.7}$$

that is, a vector field on $E_3 \times \mathbb{R}$ with values in $\Lambda^1(\mathbb{R}^4)$ (the vector space of 1-forms on $\mathbb{R}^4$). Thus, since G_s acts on E_3, it acts on the vector index "i", as shown in (2.6), while leaving the index "a" for components of a 1-form over $\mathbb{R}^4$ unaffected. The point we wish to make here is that the group G_s is the isometry group of E_3 that the body inherits by finding itself in physical three dimensional Euclidean space at any given time. G_s is therefore entirely independent of the constituent properties of the material that inhabits E_3 at any given time. Regardless therefore of the microstructure and the material constitution of the body, the only possible defects that are allowed by minimal replacement for $SO(3) \rhd T(3)$ are 3 relative translations (*virtual microcracks*) and 3 relative rotations (*virtual microrotations*). In terms of real applications, these might correspond to actual microcracks in a ceramic material or fiber breaks and fiber rotations due to the failure of the fiber–matrix interface in a composite material.

These arguments provide ample reason for referring to G_s, in its inhomogeneous or locally acting phase, as the *spatial gauge group*, even though it acts as an internal symmetry group of the matter fields when pulled back to the space of reference histories. We therefore depart from the conceptualization and world view stated in [1] and [9]. In like manner, we will refer to defects that are modeled by local action of G_s as *spatial defects*. It is therefore the properties and structures of spatial defects that are treated in [1] and [9]. In the weak field (linear) approximation, the distinction between $\mathbb{R}^4$ and $E_3 \times \mathbb{R}$ becomes blurred, and this blurring led us to confuse spatial defects with dislocations and disclinations that are material defects, as we shall see presently.

3. MINIMAL COUPLING FOR THE SPATIAL GAUGE GROUP

The second important assumption in constructing a gauge theory is minimal coupling. Minimal replacement fixes the kinematics and introduces new field variables (gauge potentials) that are appropriate to the description of spatial defects, but it is minimal coupling that defines the kinetics of the interactions between the matter fields and the gauge potentials. Going back to the gedanken experiment with distributions of "microcracks", we realize that the affine connection and the mapping functions χ^i serve to determine the elastic distortion B^i_a and thereby correctly account for the elastic and kinetic energies stored in the system. The questions then outstanding are what controls the evolution of these fields and how can they be predicted from a continuum model without taking into consideration the underlying microstructure of the material. The simplest prescription is *minimal coupling*. This means that we form G_s-covariant quantities from derivatives of the gauge potentials and then use them to form another Lagrangian L_s that is added to the minimally replaced elastic Lagrangian L_e (see [1, 2, 9]). The new Lagrangian L_s then accounts for the various forms of energy depositions that are characteristic of the spatial defects. The essential meaning of the minimality of the coupling is that L_e and L_s are additive, although there are certain other collateral restrictions that must be observed.

In the case of Yang–Mills [2] gauge theory for a semisimple Lie group, the curvature formed by the connection associated with the gauge group is used to construct the minimally coupled Lagrangian. For the case of the non-semisimple gauge group $G_s = SO(3) \rhd T(3)$, it has been shown in [1, Chap. 5] that the appropriate covariant quantities are the components of G_s*-gauge curvature* θ^i_{abj} and the components of the G_s*-gauge torsion* Σ^i_{ab}. These are given by

$$\theta^i_{abj} = (\partial_a W^\alpha_b - \partial_b W^\alpha_a + C^\alpha_{\beta\gamma} W^\beta_a W^\gamma_b)\gamma^i_{\alpha j}, \tag{3.1}$$

$$\Sigma^i_{ab} = \theta^i_{abj}\chi^j + \partial_a \phi^i_b - \partial_b \phi^i_a + \gamma^i_{\alpha j}(W^\alpha_a \phi^j_b - W^\alpha_b \phi^j_a), \tag{3.2}$$

where $C^{\alpha}_{\beta\gamma}$ are the structure constants of $SO(3)$ and $\gamma^{i}_{\alpha j}$ are the generators of $so(3)$, the 3×3 matrix Lie algebra of $SO(3)$. Here, we have used the fact that $\Gamma^{i}_{aj} = W^{\alpha}_{a}\gamma^{i}_{\alpha j}$, where $W^{\alpha} = W^{\alpha}_{a}(X^{b})\,\mathrm{d}X^{a}$, $1 < \alpha < 3$, are the compensating 1-forms for the local action of $SO(3)$; that is, the connection for $SO(3)$ is a 1-form that takes its values in $so(3)$. The total Lagrangian thus becomes

$$L_{\text{total}} = L_{e}(X^{a}, B^{i}_{a}) + L_{s}(X^{a}, \theta^{i}_{abj}, \Sigma^{i}_{ab}). \tag{3.3}$$

In the case of "microcracks" considered in the gedanken experiment, L_s corresponds to the energy associated with the formation of "microcracks" per unit volume and per unit time. The larger L_s is for unit density of "microcracks", the more resistant the material becomes to the formation of "microcracks". A ceramic material, for example, is expected to be susceptible to the formation of virtual microcracks due to local translations, and the material parameters entering into the dependence of L_s on the G_s-gauge torsion will be small compared with those that enter into L_e. On the other hand, if the material is resistant to the formation of virtual microcracks, the material parameters entering into the dependence of L_s on the G_s-gauge torsion will be large compared to those that occur in L_e. Similar situations will obtain with respect to resistance to virtual microrotations and to the material parameters that enter into the dependence of L_s on the G_s-gauge curvature.

The detailed construction of L_s will depend upon the specific material properties of the body. Accordingly, the recognized methods of modeling inhomogeneities and material symmetries will enter into the construction of L_s in exactly the same way that they enter into the construction of L_e. We therefore assume that L_e and L_s have the same material symmetry group G_m, and hence the total Lagrangian $L_e + L_s$ will also have G_m as its material symmetry group.

The view now emerge that the configuration history of a body that obtains from application of specific boundary tractions, initial data, and geometric boundary data will be implicitly controlled by the distributions of spatial defects. The determination of this configuration history becomes a classical stationarization problem in the gauge theory of spatial defects. This is because the total action integral of the body is to be rendered stationary with respect to choices of the matter fields $\chi^{i}(X^{b})$ and with respect to choices of the gauge potentials $W^{\alpha}_{a}(X^{b})$ and $\phi^{i}_{a}(X^{b})$. The explicit forms of the field equations that result from this stationarization process are given in [9], and in [1] where we have also included electromagnetic fields. We will not quote these field equations here because there are extensive changes that are demanded by the presence of another gauge group with very different gauge-theoretic properties.

4. MATERIAL SYMMETRIES AND THE MATERIAL GAUGE GROUP

We have explored the Yang–Mills gauge constructs for the spatial gauge group $G_s = SO(3) \rhd T(3)$ of local rigid body motions in the previous sections. This construct was seen to apply to all material bodies embedded in Euclidean space and leads to a consistent description of spatial defects. Because the choice of G_s is restricted by the properties of Euclidean space (the Euclidean group of isometries), only a limited number of defects can be described by a gauge theory based on G_s. On the other hand, most of the important defects in solids derive from disorderings of specific material symmetries, as exemplified by dislocations and disclinations in Bravais crystals. Our purposes in this section will be to deal with material symmetries, to construct the foundations for a gauge theory for a material gauge group, and to connect local symmetries of the material space with specific types of defects.

The first step in this process requires us to be more specific about what we mean by a material body, and by a reference history of a material body. A quick glance might suggest that we start with the notion of an abstract 3-dimensional differentiable manifold M_3, since such a starting point is sufficiently general as to satisfy everybody's tastes in the matter. Realization that we may want to allow the "material properties" of the body to change in the course of time precludes this possibility, and so we will start with the 4-dimensional differentiable manifold $\mathcal{B}$ that is the product of the real line $\mathbb{R}$ with an abstract 3-dimensional differentiable manifold M_3; that is, $\mathscr{B} = M_3 \times \mathbb{R}$. In order to connect this abstract setting with the notion of a physical body in the space of reference histories, we assume that $\mathscr{B}$ is immersed in $\mathbb{R}^4$ by an atlas $\mathscr{I}$ of local immersions so that the resulting image, $\mathbb{B}^4$, of $\mathscr{B}$ in $\mathbb{R}^4$ is a 4-dimensional cylinder with 1-dimensional generators that are parallel to the time axis. Thus, any two time slices of the body in $\mathbb{R}^4$ by hyperplanes $X^4 = k_1$ and $X^4 = k_2$ are isometric 3-dimensional regions $\mathbb{B}^3$; that is, $\mathbb{B}^4 = \mathbb{B}^3 \times \mathbb{R}$.

The abstract body $\mathscr{B}$, even when viewed from the vantage point of its image in $\mathbb{R}^4$ under $\mathscr{I}$, is a bare object that is yet unclothed with physical properties. Field theory allows us to clothe such bare objects by assigning a Lagrangian function whose 4-dimensional integral evaluates the physical action of the body. We will therefore assume that we have a Lagrangian that is defined on the image set, $\mathbb{B}^4$, of the body in $\mathbb{R}^4$. The specific properties of the body are then modeled by specifying this Lagrangian function. Modern field theory again comes to our aid by recognizing that specification of a Lagrangian is often achieved by specifying the symmetries of the Lagrangian (the material symmetry group). We therefore assume that our abstract body $\mathscr{B}$ carries a natural material symmetry group $\mathscr{G}$ which is carried over into the material symmetry group G_m on $\mathbb{B}^4$ by the atlas of immersions $\mathscr{I}$. We therefore view a material body as a triple $[\mathscr{B}, \mathscr{I}, \mathscr{G}]$ that gives rise to a cylindrical region $\mathbb{B}^4$ of $\mathbb{R}^4$, over which is defined a Lagrangian function, and a group G_m whose action on $\mathbb{B}^4$ induces the group of external symmetries of the Lagrangian function. The *material symmetry group* G_m is easy to find once we know the Lagrangian; simply compute the Noetherian symmetry group of the Lagrangian in the standard manner used in the calculus of variations (see [6], Chap. 7). The group G_m is mathematically distinct from the spatial symmetry group G_s, because the action of G_s results from transformations that do not change the independent variables $\{X^a\}$ while the action of G_m is uniquely determined by transformations of the independent variables $\{X^a\}$ themselves. In other words, G_m acts on the base manifold $\mathbb{B}^4$ of the body, while G_s acts on the range space of the matter fields χ^i that have $\mathbb{B}^4$ as their domain space.

As a motivation for applying gauge constructs to material symmetry groups, we return to the gedanken experiment described in Section 2. Let us pick up the argument at the point where we have cut the body up into small pieces and allowed each piece to relax so that there are no residual stresses. If there are only spatial defects present, we could fit all of the pieces together to form a reference configuration of the body by translating and rotating each piece in an appropriate fashion (i.e. by local action of the spatial symmetry group $G_s = SO(3) \triangleright T(3)$). Now, nothing guarantees that this will be the case. In fact, it is easy to imagine situations in which the pieces are of incommensurate sizes and shaped so that application of the best choice of the local action of G_s will result in misfits, gaps, and interstitials. In certain simple instances, local changes in the length scales would enable us to make the pieces fit together in a reference configuration, because the metric on the reference configuration is carried into a different replica on each piece. Such an incompatible relaxed state is customarily identified with a material with dislocations (see Kröner [3]). In this world view, a material with dislocations can be assigned a relaxed reference state that is compatible but materially inhomogeneous. The prescription here is to start with a homogeneous material, namely a material that is invariant under the global action of $T(3)$ on the reference configuration $\mathbb{B}^3$, and then insist on imposing invariance under local action of $T(3)$ by introducing appropriate changes in the natural length scales. What this means is that the atlas of immersions $\mathscr{I}$ of the abstrct body $\mathscr{B}$ will have to consist of more than just a singleton. Indeed, there will have to be a distinct element of $\mathscr{I}$ for each piece of the body that can not be fitted together by local action of G_s. Modern geometry is well aware of such situations; they are those for which the abstract body $\mathscr{B}$ has a nontrivial topological structure (see Rogula [10]). Indeed, compensation for the local action of $T(3)$ on $\mathbb{B}^3$ can be viewed as giving rise to the obstructions to trivialization of the topology of the preimage of $\mathbb{B}^3$; namely, obstructions to the existence of a global foliation of $\mathscr{B}$ by time-constant sections.

Thus far, the gedanken experiment has dealt with static situations. The history of a body is a dynamic process, however, and this has been provided for by constructing the reference configuration history $\mathbb{B}^4$ of the body and the fact that $\mathscr{B} = M_3 \times \mathbb{R}$ is a 4-dimensional differentiable manifold. It may therefore happen that the different pieces of the body in the gedanken experiment may fit together at one time, but not at another. It would therefore seem appropriate to include time translation $T(1)$ as well as the spatial translations $T(3)$ in our scheme.

We now proceed with the formal gauge construction for a material symmetry group G_m. In the interests of generality, the action of G_m on any time slice of $\mathbb{B}^4$ can be taken as a subgroup of the semidirect product of the special linear group with the 3-dimensional translation group, $SL(3, \mathbb{R}) \rhd T(3)$. Since we are dealing with Newtonian models, space and time remain separate constructs that are reflected in the fact that $\mathbb{B}^4 = \mathbb{B}^3 \times \mathbb{R}$. We therefore take as the general candidate

$$G_m = \{SL(3, \mathbb{R}) \rhd T(3)\} \times T(1); \tag{4.1}$$

that is, the material symmetry group is a subgroup of G_m. A useful decomposition of $SL(3, \mathbb{R})$ is the Iwasawa decomposition (see Barut and Raczka [4]) in which every element of $SL(3, \mathbb{R})$ has a unique decomposition as a product of elements of $SO(3)$, $D(3)$, and $W(3)$, where $D(3)$ is the (Abelian) dilatation group describing changes in the three length scales and where $W(3)$ is the (nilpotent) group describing simple shears. The Iwasawa decomposition thus breaks every unimodular matrix into the product of a proper orthogonal matrix, a diagonal matrix of determinant one, and an upper triangular matrix with a one at each diagonal element. We note that $SO(3)$ or a subgroup of $SO(3)$ is related to material symmetries of a solid, while $W(3)$ and $D(3)$ are related to various material symmetries of a fluid (see Wang [5]). Our primary interest in this series of papers is with what may loosely be referred to as solids. We therefore restrict our attention from now on to material bodies whose material symmetry groups are contained in

$$G_m = \{SO(3) \rhd T(3)\} \times T(1). \tag{4.2}$$

The action of the material symmetry group, G_m, on the reference history, $\mathbb{B}^4$, of the body is given by

$$'X^A = R^A_B X^B + T^A, \qquad 'X^4 = X^4 + T^4, \tag{4.3}$$

where R^A_B are the entries of a proper orthogonal matrix that belongs to the material symmetry group of the material. The starting Lagrangian function of the body is invariant under the global action of G_m. Minimal replacement for the maximal Lie subgroup of G_m will render the starting Lagrangian invariant under the local action of the maximal Lie subgroup of G_m, in which case (4.3) are replaced by

$$'X^A = R^A_B(X^b)X^A + T^A(X^b), \qquad 'X^4 = X^4 + T^4(X^b). \tag{4.4}$$

Restriction to the maximal Lie subgroup of the material symmetry group is clearly necessary, for only Lie groups have the group continuity properties required by gauge-theoretic constructs. From now on, we will use G_m to denote the maximal Lie subgroup of the material symmetry group, in the interests of simplicity. If this maximal Lie subgroup is the full material symmetry group, we may disregard this qualification. On the other hand, if the material symmetry group contains a finite subgroup, such as the discrete point group of a crystalline solid, then restriction to the maximal Lie subgroup of the material symmetry group will necessarily ignore all contributions from the discrete point group. The reader must therefore always bear in mind this restriction to the maximal Lie subgroup of the material symmetry group throughout this series of papers. With this understanding, we make explicit note that local action of the material symmetry group is the local action of a group that acts on the base manifold $\mathbb{R}^4$.

The minimal replacement construct in the case of a gauge group that acts on the base manifold $\mathbb{B}^4 \subset \mathbb{R}^4$, in contrast to minimal replacement for an internal symmetry group such as G_s, will affect both the 4-dimensional volume element $\mu = \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \wedge \mathrm{d}X^4$ and the partial derivative operation ∂_a (see Edelen [6]). With G_m given by (4.2), we have

$$\mathrm{d}X^A \to DX^A = J^A_a\, \mathrm{d}X^a = (\delta^A_a + \Upsilon^A_{aB}X^B + \psi^A_a)\, \mathrm{d}X^a, \tag{4.5}$$

$$\mathrm{d}X^4 \to DX^4 = J^4_a\, \mathrm{d}X^a = (\delta^4_a + \psi^4_a)\, \mathrm{d}X^a, \tag{4.6}$$

$$\mu \to J\mu = \det(J^a_b)\mu, \tag{4.7}$$

$$\partial_a \chi^i \to y^i_a = j^b_a \, \partial_b \chi^i, \tag{4.8}$$

where j^b_a are the components of the matrix inverse of the matrix $\mathbf{J} = ((J^a_b))$ and

$$Y^A_{aB} = M^\alpha_a \gamma^A_{\alpha B} \tag{4.9}$$

are the components of the connection matrix for $SO(3)$ that takes values in the matrix Lie algebra of $SO(3)$ with basis $\{\gamma^A_{1B}, {}^A_{2B}, {}^A_{3B}\}$. Here,

$$M^\alpha = M^\alpha_a(X^b) \, \mathrm{d}X^a \tag{4.10}$$

are the compensating 1-forms for the local action of $SO(3)$ and

$$\psi^A = \psi^A_a(X^b) \, \mathrm{d}X^a, \qquad \psi^4 = \psi^4_a(X^b) \, \mathrm{d}X^a \tag{4.11}$$

are the compensating 1-forms for the local action of the group $T(3)$ of translations on B^3, and the group $T(1)$ of time translations, respectively. These representations are obtained by using the faithful matrix representation of G_m as a matrix subgroup of $GL(5, \mathbb{R})$. Here, in the interests of simplicity, we have assumed that the material symmetry group contains $SO(3)$. If the material symmetry group only contains a Lie subgroup of $SO(3)$, such as would be the case for orthotropic materials, a similar analysis can also be made by using the faithful matrix representation of the material symmetry group as a matrix subgroup of $GL(5, \mathbb{R})$. In this event, there will be fewer group parameters and hence there will be fewer compensating 1-forms for the local action of the material symmetry group. We leave the details of such constructions to subsequent communications.

The important fact to be noted at this juncture is that minimal replacement for the material symmetry group has replaced the natural coframe basis $\{\mathrm{d}X^1, \mathrm{d}X^2, \mathrm{d}X^3, \mathrm{d}X^4\}$ for $\mathbb{R}^4$ by what may be termed the *material coframe basis* $\{J^1, J^2, J^3, J^4\}$. This material coframe basis carries all of the information about defects inherent in the material. For example, circuit integrals of the coframe basis 1-forms around boundaries of 2-dimensional regions can be used to define intrinsic material Burgers' vectors. The reader should note that the material coframe basis is independent of mappings χ from the space of reference histories into the space of current histories, and hence the information carried by the material coframe basis is universal with respect to spatial realizations of the material body. As such, they serve as the material analog of Kröner's internal observers.

This is not all of the story because we also have the gauge structure that is introduced by local action of the spatial gauge group G_s. What we actually have is a total gauge group

$$G = G_m \times G_s \tag{4.12}$$

which is the direct product of the material gauge group and the spatial gauge group. Care has to be exercised here, because we have

$$\mathrm{d}\chi^i = \partial_a \chi^i \, \mathrm{d}X^a, \qquad \mathrm{d}t = \delta^4_a \, \mathrm{d}X^a \tag{4.13}$$

for the global action of G. In order to simplify matters, let $\mathcal{M}$ denote the operation of minimal replacement (see [7]) for the total gauge group G. We then have

$$\mathcal{M}\langle \mathrm{d}\chi^i \rangle = B^i_a \, \mathrm{d}X^a = B^i, \qquad \mathcal{M}\langle \mathrm{d}t \rangle = \delta^4_a \, \mathrm{d}X^a = B^4, \tag{4.14}$$

$$\mathcal{M}\langle \mathrm{d}X^a \rangle = J^a_b \, \mathrm{d}X^b = J^a, \tag{4.15}$$

while (4.13) gives

$$\mathcal{M}\langle \mathrm{d}\chi^i \rangle = \mathcal{M}\langle \partial_a \chi^i \rangle \mathcal{M}\langle \mathrm{d}X^a \rangle, \qquad \mathcal{M}\langle \mathrm{d}t \rangle = \mathcal{M}\langle \delta^4_a \rangle \mathcal{M}\langle \mathrm{d}X^a \rangle. \tag{4.16}$$

Thus, if we set

$$\mathcal{M}\langle \partial_a \chi^i \rangle = Y^i_a, \qquad \mathcal{M}\langle \delta^4_a \rangle = \mathcal{M}\langle \partial_a t \rangle = Y^4_a, \tag{4.17}$$

(4.14) through (4.16) give the evaluations

$$B^i_a = Y^i_b J^b_a, \qquad B^4_a = Y^4_b J^b_a. \tag{4.18}$$

In terms of the various 1-forms introduced above, (4.18) take the simpler form

$$B^i = Y^i_b J^b, \qquad B^4 = Y^4_b J^b. \tag{4.19}$$

This means that the basis $\{B^i, B^4 \mid 1 \le i \le 3\}$ for $\Lambda^1(\mathbb{B}^4)$ with values in $T(E_3 \times \mathbb{R})$, that is obtained from minimal replacement for G_s, and the basis $\{J^b \mid 1 \le b \le 4\}$ for $\Lambda^1(\mathbb{B}^4)$ with values in $T(\mathbb{B}^4)$, that is obtained from minimal replacement for G_m, are related by (4.19). The quantities $\{Y^i_b, Y^4_b\}$ are thus seen to be genuine two-point tensor fields, as are the original quantities $\{\partial_a\chi^i, \partial_a t\}$ which they replace through minimal replacement. When we put all of this together, minimal replacement for the total gauge group $G = G_m \times G_s$ gives

$$\mu \to J\mu = \det(J^a_b)\mu, \tag{4.20}$$

$$\partial_a\chi^i \to Y^i_a = j^c_a B^i_c = j^c_a(\partial_c\chi^i + \Gamma^i_{cj}\chi^j + \phi^i_c), \tag{4.21}$$

$$\partial_a t \to Y^4_a = j^c_a B^4_c = j^c_a\, \delta^4_c = j^4_a, \tag{4.22}$$

while the Lagrangian 4-form $L\mu$ undergoes the replacement

$$L_e(X^a, \partial_a\chi^i)\mu \to L_e(X^a, Y^i_a)J\mu. \tag{4.23}$$

The replacement relations (4.21) and (4.23) show that the configuration gradients $\partial_a\chi^i$ are replaced by what we will term the components of *total distortion*; namely,

$$Y^i_a = j^c_a B^i_c. \tag{4.24}$$

This shows that the components of total distortion have an explicit factorization in terms of the components of *elastic distortion*, B^i_c, and what may be termed the components of *plastic distortion*, j^c_a. Indeed, the components of elastic distortion arise through minimal replacement of the configuration gradients for local action of the spatial gauge group G_s when the material gauge group G_m is frozen in its homogeneous (global) phase. The components of elastic distortion thus characterize the spatial defects that are independent of the material properties of the body. They are therefore universal with respect to all materials. On the other hand, the components of plastic distortion, j^c_a, arise through minimal replacement for the local action of the material gauge group G_m when the spatial gauge group G_s is frozen in its homogeneous (global) phase and χ is the identity map. *The components of plastic distortion thus characterize the material, or topological defects of the body that are inherited from the microstructures out of which the body is constructed.* They are thus universal with respect to how the body is placed in physical space and time by the mapping χ. The fact of particular interest here, is that minimal replacement for the total gauge group $G = G_m \times G_s$ leads to precisely the factorization (4.24). Factorization of this general form has often been posited in phenomenalogical plasticity on an *ad hoc* basis [12], but without the realization that the plastic distortion is characteristic of material topological defects. The added understanding and justification for the factorization (4.24) that is engendered by minimal replacement for the total gauge group $G_m \times G_s$ adds both substance and weight to the gauge-theoretic arguments.

5. TRANSFORMATIONS INDUCED BY THE GAUGE GROUP AND CARTAN EQUATIONS

The occurrence of the quantities Y^i_a as the replacements for $\partial_a\chi^i$ in the matter Lagrangian, as indicated by (4.23), may seem problematic. A clear view of why such quantities should occur can be obtained from the following arguments. Let $\{j_a \mid 1 \le a \le 4\}$ be the basis for $T(\mathbb{B}^4)$ that is dual to the G_m-gauge covariant coframe basis $\{J^a \mid 1 \le a \le 4\}$ for $\Lambda^1(\mathbb{B}^4)$. We therefore have

$$j_a \rfloor J^b = \delta^b_a, \qquad j_a = j^c_a\, \partial_c, \tag{5.1}$$

where $\rfloor$ denotes inner multiplication (see [6], Chap. 3). Acting on the basic relations

$$B^i = Y^i_a J^a \tag{5.2}$$

with $j_c \rfloor$ shows that the Y's are evaluated by

$$Y^i_b = j_b \rfloor B^i. \tag{5.3}$$

The quantities $\{Y^i_b\}$ are thus *scalars* as far as coordinate transformations of $\mathbb{B}^4$ are concerned. This has very important ramifications in gauge theory for $G_m \times G_s$, as we now proceed to show.

The gauge group G_m induces the transformations

$$'X^A = R^A_B(X^b)X^B + T^A(X^b), \qquad 'X^4 = X^4 + T^4(X^b) \tag{5.4}$$

of the coordinate cover of $\mathbb{B}^4$, so that the new coordinate cover can be a very complicated one. Accordingly, the natural basis $(\mathrm{d}X^a \mid 1 \le a \le 4\}$ for $\Lambda^1(\mathbb{B}^4)$ thus transforms according to the complicated transformation relations

$$\mathrm{d}'X^a = \frac{\partial' X^a}{\partial X^b}\,\mathrm{d}X^b. \tag{5.5}$$

On the other hand, the gauge covariant basis $\{J^a\}$ and the dual basis $\{j_a\}$ transform in the gauge covariant fashions

$$'J^a = R^a_b J^b, \qquad 'j_a = r^b_a j_b, \tag{5.6}$$

where r^b_a are the entries of the 4×4 matrix that is the inverse of the 4×4 matrix

$$((R^a_b)) = \begin{bmatrix} R^A_B & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.7}$$

If we introduce the 4×4 conection matrix of 1-forms for the $SO(3)$ sector of G_m by

$$((\Upsilon^a_b)) = \begin{bmatrix} \Upsilon^A_B & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.8}$$

then the gauge covariant basis $\{J^a\}$ gives rise to the following *material Cartan equations of structure* (see [1] and [6], Chap. 5):

$$\mathrm{d}J^a + \Upsilon^a_b \wedge J^b = \Xi^a, \qquad \mathrm{d}\Xi^a + \Upsilon^a_b \wedge \Xi^b = \Delta^a_b \wedge J^b, \tag{5.9}$$

$$\mathrm{d}\Upsilon^a_b + \Upsilon^a_c \wedge \Upsilon^c_b = \Delta^a_b, \qquad \mathrm{d}\Delta^a_b + \Upsilon^a_c \wedge \Delta^c_b = \Delta^a_c \wedge \Upsilon^c_b. \tag{5.10}$$

Here, Ξ^a are the 2-forms of *G_m-gauge torsion* and Δ^a_b are the 2-forms of *G_m-gauge curvature,* with the linear G_m-gauge transformation laws

$$'\Xi^a = R^a_b \Xi^b, \qquad '\Delta^a_b = R^a_c\,\Delta^c_e r^e_b, \tag{5.11}$$

while the material compensating 1-forms have the affine G_m-gauge transformation laws

$$'\Upsilon^a_b = (R^a_c \Upsilon^c_e - \mathrm{d}R^a_e) r^e_b, \qquad '\psi^a = R^a_b \psi^b - {'\Upsilon^a_b} T^b - \mathrm{d}T^a. \tag{5.12}$$

Thus, all of the quantities $\{J^a, j_a, \Xi^a, \Delta^a_b\}$ transform linearly under the action of the G_m-gauge group, while $\{\Upsilon^a_b, \psi^a\}$ transform linearly only when $\mathrm{d}R^a_e = 0$ and $T^a = 0$.

These deceptively simple transformation laws conceal an unwanted complication because they involve the Jacobian matrices $\partial' X^a/\partial X^b$ when written out in terms of components. For example, we have

$$'J^a = {'J^a_b}\,\mathrm{d}'X^b = {'J^a_b}\frac{\partial' X^b}{\partial X^c}\,\mathrm{d}X^c = R^a_b J^b = R^a_b J^b_c\,\mathrm{d}X^c;$$

that is,

$$'J^a_b \frac{\partial' X^b}{\partial X^c} = R^a_b J^b_c.$$

Exactly the same thing happens with the 1-forms B^i that obtain from minimal replacement for the spatial gauge group G_s. The resulting transformation law is (see [1])

$$'B^i = Q^i_j B^j \tag{5.13}$$

when

$$'\chi^i = Q^i_j \chi^j + b^i. \tag{5.14}$$

However, because the total gauge group is $G_m \times G_s$, the coordinate transformations (5.4) that are induced by G_m, together with those from G_s, give

$$'B^i = 'B^i_a \, \mathrm{d}'X^a = 'B^i_a \frac{\partial' X^a}{\partial X^c} \mathrm{d}X^c = Q^i_j B^j = Q^i_j B^j_c \, \mathrm{d}X^c;$$

that is,

$$'B^i_a \frac{\partial' X^a}{\partial X^c} = Q^i_j B^j_c. \tag{5.15}$$

The components of the elastic distortion, B^i_a, are thus aware of the transformations induced on the base manifold $\mathbb{B}^4$ by the action of G_m. Similar remarks also apply to the quantities that arise through the *spatial Cartan equations of structure* that are induced by the local action of G_s (see [1]):

$$\mathrm{d}B^i + \Gamma^i_j \wedge B^j = \Sigma^i, \qquad \mathrm{d}\Sigma^i + \Gamma^i_j \wedge \Sigma^j = \Theta^i_j \wedge B^j, \tag{5.16}$$

$$\mathrm{d}\Gamma^i_j + \Gamma^i_k \wedge \Gamma^k_j = \Theta^i_j, \qquad \mathrm{d}\Theta^i_j + \Gamma^i_k \wedge \Theta^k_j = \Theta^i_k \wedge \Gamma^k_j. \tag{5.17}$$

Here, Σ^i are the 2-forms of the *G_s-gauge torsion* and the quantities Θ^i_j are the 2-forms of *G_s-gauge curvature* with the transformation laws

$$'\Sigma^i = Q^i_j \Sigma^j, \qquad '\Theta^i_j = Q^i_k \Theta^k_m q^m_j, \qquad Q^i_j q^j_k = \delta^i_k. \tag{5.18}$$

With this information at hand, the basic relations $B^i = Y^i_a J^a$, that is, $Y^i_b = j_b \rfloor B^i$, give the combined gauge and induced coordinate transformation laws

$$'Y^i_a = Q^i_j Y^j_b r^b_a. \tag{5.19}$$

The pesky Jacobian matrices $\partial' X^a / \partial X^b$ have thus disappeared because the Y's transform as scalars under changes of coordinate covers of $\mathbb{B}^4$. The position and time dependent transformation matrices $\{Q^i_j\}$ and $\{r^b_a\}$ are still present in (5.19), *but they occur in exactly the same fashion as would obtain if the gauge group acted only globally* (i.e., when $\{Q^i_j\}$ and $\{r^b_a\}$ are constant matrices). In fact, this is precisely why minimal replacement is used, for this construct has the property that is preserves the validity of linear transformation laws when global action of a group is replaced by local action of that group. From the converse side, if one insists on the continued validity of (5.19) when the group action is allowed to be local, one is led directly to the minimal replacement construct provided internal geometric consistency is demanded. It thus seems abundantly clear that the quantities $\{Y^i_a\}$ are indeed the correct quantities to replace the customary configuration gradients $\{\partial_a \chi^i\}$ when the total group $G = G_m \times G_s$ is allowed to act locally.

Now that we know the transformation laws for the Y's namely (5.19), it is an easy matter to see that the gauge covariant derivatives of the Y's have the evaluations

$$D_b Y^i_a = \partial_b Y^i_a + \Gamma^i_{bj} Y^j_a - Y^i_c \Upsilon^c_{ba}. \tag{5.20}$$

We note, in particular, that the gauge covariant derivatives of the Y's involve both the components of linear connection Γ^i_{bj} that come from the local action of the $SO(3)$ sector of G_s and the components of linear connection Υ^c_{ba} that come from the local action of the $SO(3)$ sector of G_m. This shows that the Y's carry the linear connection structure of the total gauge group $G = G_m \times G_s$.

This complicated formalism can be simplified if we define the gauge covariant exterior derivative D by

$$DB^i = \mathrm{d}B^i + \Gamma^i_j \wedge B^j, \qquad DJ^a = \mathrm{d}J^a + \Upsilon^a_b \wedge J^b, \qquad DY^i_a = (D_b Y^i_a)\, \mathrm{d}X^b. \tag{5.21}$$

We then have

$$DB^i = \Sigma^i, \qquad DJ^a = \Xi^a, \tag{5.22}$$

from the material Cartan equations of structure, and hence gauge covariant exterior differentiation of the fundamental relations $B^i = Y^i_a J^a$ gives us the relations

$$\Sigma^i = DY^i_a \wedge J^a + Y^i_a \Xi^a \tag{5.23}$$

between the G_s-gauge torsion Σ^i and the G_m-gauge torsion Ξ^a. These relations show that the Σ^i is not simply the resolution of Ξ^a with respect to the G_s-covariant basis $\{J^a\}$ because of the additional terms on the right-hand side of (5.23) that involve DY^i_a. Similar relations can be derived from the remaining equations of structure:

$$D\Sigma^i = \Theta^i_j \wedge B^j, \qquad D\Xi^a = \Delta^a_b \wedge J^b, \tag{5.24}$$

$$D\Theta^i_j = 0, \qquad D\Delta^a_b = 0, \tag{5.25}$$

the latter usually being referred to as the *generalized Bianchi identities.*

6. KINEMATIC RELATIONS

It has been shown in [1] and [9] that the spatial Cartan equations of structure induced by G_s are the basic kinematic relations for a body with spatial defects. Indeed, the solution of the identification problem given in Section 4.2 of [1] showed that B^i are the (spatial) distortion 1-forms, Σ^i are the (spatial) dislocation density and current 2-forms, etc. The conceptual process of replacing G_s by the total gauge group $G = G_m \times G_s$ carries with it a corresponding enlargement of the underlying kinematic relations. This enlarged kinematic structure is contained by the Cartan equations of structure for both G_s and G_m; namely, by (5.9), (5.10) and (5.16), (5.17). These equations will thus enter throughout this series of papers. Arguments identical to those given in [1] then show that J^a may be identified with the material distortion 1-forms, Ξ^a may be identified with the material dislocation density and current 2-forms, etc. The one essential difference here is that we now have nontrivial J^4 and Ξ^4, so that there are material time-distortion and material time-dislocation density and current 2-forms. These will prove to be of particular significance in modeling dissipative aspects of moving material defects. Solutions of the spatial and material Cartan equations of structure in the antiexact gauge (see [1, Sect. 4.5]) lead to canonical forms for the spatial and material defect structures that will prove to be essential in order to determine the observables of the theory.

There are additional kinematic relations that we will need in succeeding papers in this series. We define the quantities h_{ab} and h^{ab} by

$$h_{ab} = J^A_a\, \delta_{AB} J^B_b, \qquad h^{ab} = j^a_A\, \delta^{AB} j^b_B. \tag{6.1}$$

Each of these constitute the components of a symmetric 4×4 matrix of rank three by Sylvester's theorem of inertia for quadratic forms. They will play the roles of components of covariant and contravariant "metric" tensors on $\mathbb{B}^4$, albeit singular ones because they are only of rank 3. In fact, an elementary calculation and use of the relations $J^a_b j^b_c = \delta^a_c$ yield

$$h_{ab} h^{bc} = \delta^c_a - J^4_a j^c_4, \tag{6.2}$$

as might be expected.

The linear connection 1-forms Υ^a_b are *anholonomic* linear connection 1-forms [11, 13] because they act on the enumeration index for the gauge covariant basis elements $\{J^a\}$ for $\Lambda^1(\mathbb{B}^4)$; that is

$$DJ^a = \mathrm{d}J^a + \Upsilon^a_b \wedge J^b.$$

The corresponding *holonomic* linear connection 1-forms $\hat{\Upsilon}^a_b$ for $\mathbb{B}^4$, that are induced by the local action of G_m, are given by (see [11, Sect. 6])

$$\hat{\Upsilon}^a_b = J^e_b \Upsilon^c_e j^a_c - j^a_c\, \mathrm{d}J^c_b = \hat{\Upsilon}^a_{cb}\, \mathrm{d}X^c. \tag{6.3}$$

The 4-dimensional subset $\mathbb{B}^4$ of the space of reference histories $\mathbb{R}^4$ is thus endowed with a nontrivial linear connection as a consequence of the local action of the material gauge group

G_m! Remembering that each $J^a = J^a_b \, dX^b$ is a covector and each $j_a = j^b_a \, \partial_b$ is a vector, their covariant derivatives have the evaluations

$$\nabla_c J^a_b = \partial_c J^a_b - \hat{\Upsilon}^e_{cb} J^a_e, \qquad \nabla_c j^a_b = \partial_c j^a_b + \hat{\Upsilon}^a_{ce} j^e_b. \tag{6.4}$$

We can therefore define total covariant derivatives $\overset{T}{\nabla}_c$ by accounting for the effects of the local action of G_m through the connection coefficients Υ^a_{bc}. When this is done and (6.3) are used, we have

$$\overset{T}{\nabla}_c J^a_b = \nabla_c J^a_b + \Upsilon^a_{ce} J^e_b = 0, \tag{6.5}$$

$$\overset{T}{\nabla}_c j^a_b = \nabla_c j^a_b - \Upsilon^e_{cb} j^a_e = 0. \tag{6.6}$$

Accordingly, (6.1), (6.5), and (6.6) show that

$$\overset{T}{\nabla}_c h_{ab} = 0, \qquad \overset{T}{\nabla}_c h^{ab} = 0. \tag{6.7}$$

On the other hand $\overset{T}{\nabla}_c$ and ∇_c agree when applied to ordinary tensor fields on $\mathbb{B}^4$, and hence we have

$$\nabla_c h_{ab} = 0, \qquad \nabla_c h^{ab} = 0. \tag{6.8}$$

The space $\mathbb{B}^4$ is therefore a 4-dimensional Riemann-Cartan space [13] with metric tensor $\{h_{ab}\}$ of rank 3. Thus, even though $\mathbb{B}^4$ has a singular metric tensor, it has a well defined linear connection that is uniquely determined by the minimal replacement construct for the material gauge group. This fact, more than any other, should dispel the misconception that a linear connection for a Riemann–Cartan space that is metrically compatible (i.e., satisfies (6.8)) necessarily involves the linear connection determined by Christoffel symbols. Such a situation is clearly impossible here due to the singular nature of the metric tensor $\{h_{ab}\}$. This conclusion should not be confused with the fact that the space $\mathbb{B}^4$ has nontrivial torsion, as we shall now demonstrate.

The standard definition of the components of the *classic torsion tensor* $\{S^a_{bc}\}$, of a holonomic linear connection with components $\hat{\Upsilon}^a_{bc}$, is

$$S^a_{bc} = \hat{\Upsilon}^a_{[bc]}. \tag{6.9}$$

When (6.3) is used, we obtain the relations

$$S^a_{bc} = j^a_e \Xi^e_{bc}, \qquad \Xi^a_{bc} = J^a_e S^e_{bc}. \tag{6.10}$$

These are the relations between the classical torsion of a linear connection and the G_m-gauge torsion $\Xi^e = \frac{1}{2}\Xi^e_{ab} \, dX^a \wedge dX^b$ that bring the two seemingly different torsion structures into harmony. In particular, this establishes the fact that $\mathbb{B}^4$ *is a (singular metric) Riemann–Cartan space with nontrivial torsion whenever the G_m-gauge torsion 2-forms Ξ^a have at least one nontrivial component*! If we define classic torsion 2-forms by

$$S^a = S^a_{bc} \, dX^b \wedge dX^c, \tag{6.11}$$

(6.10) can be written in the equivalent fashion

$$S^a = j^a_e \Xi^e, \qquad \Xi^a = J^a_e S^e. \tag{6.12}$$

Since the material Cartan equations of structure serve to identify Ξ^e with the 2-forms of material dislocation density and current, (6.12) recover the relations between the holonomic torsion tensor of $\mathbb{B}^4$ and the material dislocation densities and currents reported in the literature [3, 13]. A substitution of (6.12) into (5.23) gives us the relations

$$\Sigma^i = DY^i_a \wedge J^a + Y^i_a J^a_e S^e. \tag{6.13}$$

Hopefully, these explicit relations between the various torsion structures will dispel the confusion on this subject that appears to be rampant.

Next, we note that $J = \det(J^a_b)$ transforms as a scalar density under transformations of $\mathbb{B}^4$. We accordingly have

$$\overset{T}{\nabla}_c J = \nabla_c J = \partial_c J - \hat{Y}^e_{ce} J = 0. \tag{6.14}$$

This result will feature prominently in succeeding papers because the minimally replaced matter Lagrangian is of the form $L_e(X^a, Y^i_a)J$, and hence the Euler–Lagrange equations for this Lagrangian will involve $\partial_c J$.

We have seen that $L_e(X^a, Y^i_a)$ can depend on the Y's only through the components of the *total right Cauchy–Green strain matrix*

$$C_{ab} = Y^i_a\, \delta_{ij} Y^j_b, \tag{6.15}$$

as a consequence of invariance of the original matter Lagrangian under spatial rotations (i.e., $C_{ab} = \mathcal{M}\langle \partial_a \chi^i\, \delta_{ij}\, \partial_b \chi^j \rangle$). We make particular note of the fact that each component of the total right Cauchy-Green strain matrix transforms as a scalar under all transformations of the total gauge group $G_m \times G_s$. When the evaluations (4.21) are substituted into (6.15), we have

$$C_{ab} = j^c_a B^i_c\, \delta_{ij} B^j_d j^d_b = j^c_a \overset{s}{C}_{cd} j^d_b, \tag{6.16}$$

where

$$\overset{s}{C}_{ab} = B^i_a\, \delta_{ij} B^j_b \tag{6.17}$$

is the *spatial right Cauchy–Green strain tensor*. The reader should note that the components of the spatial right Cauchy–Green strain tensor transform as the components of a second order covariant tensor field under transformations of $\mathbb{B}^4$. Using the fact that $\{j_a\}$ is the basis dual to $\{J^a\}$, an elementary calculation based on (6.16) gives us

$$\overset{s}{C}_{ab} = J^c_a C_{cd} J^d_b,$$

which are the parallel of the fundamental relations $B^i = Y^i_a J^a$. The relations between total and spatial right Cauchy–Green strain components are therefore conjugacy relations rather than direct factorization, because they must reflect the fact that the total right Cauchy–Green strain is represented by components of a matrix of scalars on $\mathbb{B}^4$, while the spatial right Cauchy–Green strain is represented by the components of a second order covariant tensor field on $\mathbb{B}^4$. The *total engineering strain scalar matrix*, in the presence of defects, has the evaluation

$$E_{ab} = C_{ab} - j^c_a h_{cd} J^d_b, \tag{6.18}$$

because

$$j^c_a h_{cd} j^c_b = j^c_a J^A_c\, \delta_{AB} J^B_d j^d_b = \delta^A_a\, \delta_{AB}\, \delta^B_b.$$

When (6.1) and (6.16) are used, (6.18) gives

$$E_{ab} = j^c_a(\overset{s}{C}_{cd} - h_{cd}) j^d_b = j^c_a \overset{s}{E}_{cd} j^d_b, \tag{6.19}$$

where

$$\overset{s}{E}_{cd} = \overset{s}{C}_{cd} - h_{cd} \tag{6.20}$$

are the components of the *spatial engineering strain tensor*. They transform as the components of a second order covariant tensor field under transformations of $\mathbb{B}^4$, as indeed they must. The reader should take particular note that the spatial engineering strain tensor involves the material metric tensor $\{h_{cd}\}$ of $\mathbb{B}^4$, and hence this strain tensor is a measure of change in lengths that are measured relative to the length scales induced on $\mathbb{B}^4$ by the local action of the material gauge group. If the material gauge group is restricted to its homogeneous phase (it acts only globally), then (6.20) reduces to the standard form for the spatial engineering strain tensor (see [1]). When material (i.e., reference topological) defects are present, they give rise to the new metric structure with metric tensor $\{h_{ab}\}$, and this new metric structure is what has to be used in computing changes in length. The strain-wise kinematics induced by minimal replacement for the total gauge group $G_m \times G_s$ is thus both consistent and meaningful.

We now come to the classical kinematic quantities; namely, velocity and mass flux in the current configuration. When the total gauge group acts globally (i.e., in the classical situation), the spatial velocity vector field, v, is defined by (see [1, Sect. 1.4])

$$v = \chi_*(\partial_4) = \frac{\partial \chi^i}{\partial T}\frac{\partial}{\partial x^i} + \frac{\partial}{\partial t}. \tag{6.21}$$

Accordingly, local action of the total gauge group induces the *total spatial velocity vector field*

$$\begin{aligned} V &= \mathcal{M}\langle v\rangle = \mathcal{M}\langle \chi_*(\partial_4)\rangle = j_4^a \mathcal{M}\langle \chi_*(\partial_a)\rangle \\ &= j_4^a\left\{B_a^i \frac{\partial}{\partial x^i} + B_a^4 \frac{\partial}{\partial t}\right\} = Y_4^i \frac{\partial}{\partial x^i} + Y_4^4 \frac{\partial}{\partial t}. \end{aligned} \tag{6.22}$$

Again, we have the emergence of the Y's that we have come to expect from local action of the total group. If the material symmetry group acts only globally, we have $j_4^a = \delta_4^a$, and hence the total spatial velocity vector field reduces to

$$V = B_4^i \frac{\partial}{\partial x^i} + \frac{\partial}{\partial t},$$

because $B_a^4 = \delta_a^4$. We therefore obtain agreement with the analysis presented in [1]. However, as soon as there is local action of the material symmetry group, the spatial velocity vector assumes the more complicated form given by (6.22).

There is an alternative way of writing (6.22) that will prove to be useful. Using the next to the last equality in (6.22), we have

$$V = j_4^A B_A^i \frac{\partial}{\partial x^i} + j_4^4\left(B_4^i \frac{\partial}{\partial x^i} + B_4^4 \frac{\partial}{\partial t}\right). \tag{6.23}$$

Using the results established in [1] and in the previous discussions, we may define the *elastic spatial velocity vector field* by

$$V_e = B_4^i \frac{\partial}{\partial x^i} + B_4^4 \frac{\partial}{\partial t} = B_4^i \frac{\partial}{\partial x^i} + \frac{\partial}{\partial t} \tag{6.24}$$

because $B_4^4 = 1$. An inspection of (6.23) suggests that we define the *plastic spatial velocity vector field* by

$$V_p = j_4^A B_A^i \frac{\partial}{\partial x^i}, \tag{6.25}$$

in which case (6.23) gives

$$V = V_p + j_4^4 V_e. \tag{6.26}$$

This decomposition of the total spatial velocity vector field into elastic and plastic parts is mathematically unique and physically reasonable. Particular note should be taken of the fact that the plastic spatial velocity has no time component (i.e., it is purely spatial in nature), and hence it need not vanish in steady processes.

The classical spatial form of the mass flux 3-form (i.e., when the gauge group acts globally) is given by

$$\mathcal{O} = (\chi_*\partial_4) \rfloor \rho\pi(x), \tag{6.27}$$

where $\pi(x) = dx^1 \wedge dx^2 \wedge dx^3 \wedge dt$ and ρ is the mass density. Local action of the total gauge group thus gives the modified spatial form of the mass flux 3-form

$$\mathcal{P} = \mathcal{M}\langle\mathcal{O}\rangle = \rho V \rfloor \pi(x) = \rho\left(Y_4^i \frac{\partial}{\partial x^i} + Y_4^4 \frac{\partial}{\partial t}\right) \rfloor \pi(x). \tag{6.28}$$

The law of conservation of mass thus assumes the form $d\mathcal{P} = 0$ under local action of the total gauge group; that is

$$0=\frac{\partial}{\partial x^a}(\rho Y_4^a)=\frac{\partial}{\partial x^i}(\rho j_4^b B_b^i)+\frac{\partial}{\partial t}(\rho j_4^b B_b^4), \tag{6.29}$$

after eliminating the nonzero 4-form $\pi(x)$. This can also be written in the equivalent form

$$0=\frac{\partial}{\partial x^a}(\rho V^a) \tag{6.30}$$

with $V = V^a\, \partial/\partial x^a$. Thus, if we use (6.26) we have

$$0=\frac{\partial}{\partial x^a}(\rho V_p^a)+\frac{\partial}{\partial x^a}(\rho j_4^4 V_e^a), \tag{6.31}$$

which shows the contributions from the plastic as well as from the elastic spatial velocity fields. Such modifications of the continuity equation have been suggested in the past, but usually from a heuristic basis. Here, they are a direct consequence of the minimal replacement construct associated with the local action of the total gauge group $G = G_m \times G_s$. The material form of the continuity equation is substantially more complicated, and will be deferred until later.

A marked departure from previous thinking has been incorporated in this theory by allowing local action of the one parameter group of time translations. In view of the classical conjucacy between energy and time, local action of the time translation group may be anticipated to give rise to means for modeling dissipation mechanisms in engineering materials. Final proof of this must be delayed to a succeeding paper, since it will obviously involve the kinetics that are predicted by the field equations. There are kinematic arguments that can be given at this point, however, that are both suggestive and indicative. These kinematic arguments will also engender a feel for the new possibilities that open up by gauging the time translation group.

In the interests of simplicity, we confine the discussion to the case where only the time translation group is allowed to act locally; that is, $G = T(1)$. Under these circumstances we have

$$B_a^i=\partial_a\chi^i=\delta_a^i+\partial_a u^i, \tag{6.32}$$

where we have introduced the elastic displacement functions $\{u^i\}$ by $\chi^i = \delta_A^i X^A + u^i$, and the gauge induced coframe basis

$$J^1=\mathrm{d}X^1,\qquad J^2=\mathrm{d}X^2,\qquad J^3=\mathrm{d}X^3, \tag{6.33}$$

$$J^4=\mathrm{d}X^4+\psi_a^4\,\mathrm{d}X^a. \tag{6.34}$$

These latter relations follow from the fact that there is only one compensating 1-form $\psi^4 = \psi_a^4\, \mathrm{d}X^a$ for the 1-parameter gauge group $T(1)$. Collecting the terms in (6.34) together in the form

$$J^4=\psi_A^4\,\mathrm{d}X^A+(1+\psi_4^4)\,\mathrm{d}X^4,$$

it is clear that the problem can be simplified if we can manage to have $\psi_4^4 = 0$. This can always be achieved by a gauge transformation, as is evident from the gauge transformation relations (5.12), which reduce to

$$'\psi^4=\psi^4-(\partial_a T^4)\,\mathrm{d}X^a; \tag{6.35}$$

simply take $\partial_4 T^4 = \psi_4^4$. This choice of gauge will be referred to as the *Coulomb gauge* in analogy with electrodynamics. Thus, for the Coulomb gauge, we have

$$J^4=\zeta_A\,\mathrm{d}X^A+\mathrm{d}T. \tag{6.36}$$

We therefore have $J = \det(J_b^a) = 1$, and the gauge induced basis for frame fields, $j_b = j_b^a\, \partial_a$, $j_b \rfloor J^a = \delta_b^a$, is given by

$$j_1=\partial_1-\zeta_1\,\partial_4,\qquad j_2=\partial_2-\zeta_2\,\partial_4,\qquad j_3=\partial_3-\zeta_3\,\partial_4,\qquad j_4=\partial_4. \tag{6.37}$$

This particularly simple representation for the j's comes about because the matrix $((J^a_b))$ has the shape of a partition matrix that is easily inverted in closed form.

Now that we know the j's and the B's, the Y's can be calculated explicitly:

$$Y^i_a = j^c_a B^i_c = j^c_a \, \partial_c \chi^i = j_a \langle \chi^i \rangle. \tag{6.38}$$

However, Y^i_a are the minimal replacement of $\partial_a \chi^i$, and j_b are the operators that are obtained from minimal replacement for the natural frame basis ∂_a. We thus see that (6.38) are exactly what we should expect from the gauge-theoretic viewpoint. In order to see the full content of (6.38), it is useful to write them out in terms of the elastic displacement functions (i.e., by using $\chi^i = \delta^i_A X^A + u^i$) and the ζ's:

$$Y^i_A = \delta^i_A + \partial_A u^i - \zeta_A \, \partial_4 u^i, \qquad Y^i_4 = \partial_4 u^i. \tag{6.39}$$

Use of (6.15) to calculate the total right Cauchy–Green strain tensor gives us

$$C_{AB} = \{(\partial_A - \zeta_A \, \partial_4 \langle \chi^i \rangle\} \, \delta_{ij} \{(\partial_B - \zeta_B \, \partial_4) \langle \chi^j \rangle\}. \tag{6.40}$$

Since $h_{ab} = \delta^A_a \, \delta_{AB} \, \delta^B_b$ in this case, the total engineering strain is given by

$$E_{AB} = C_{AB} - \delta_{AB}, \tag{6.41}$$

the *linear engineering strain* approximation gives

$$e_{AB} = (\partial_A u^i - \zeta_A \, \partial_4 u^i) \, \delta_{iB} + \delta_{Ai}(\partial_B u^i - \zeta_B \, \partial_4 u^i); \tag{6.42}$$

that is, the linear engineering strain becomes velocity dependent! The kinematic argument stops here. If, however, we were to start with a linear stress-strain law, then minimal replacement for local action of the group of time translations will give an evaluation of the stress in terms of the classical strain ε_{AB} and the velocity because $\mathcal{M}\langle \sigma(\varepsilon_{AB}) \rangle = \sigma(e_{AB})$. The relations (6.42) involve the three functions ζ_A, however, which are unknown functions of position and time at this point, so they could all be zero. Determination of these functions will be achieved only after we have included the gauge field Lagrangian L_g (minimal coupling) and rendered the total action stationary with respect to the six field variables $\{u^i(X^b), \ \zeta_A(X^b)\}$. Conclusive proof that gauging the time translation group will have to be postponed to the next paper in this series.

7. MINIMAL DERIVATIVE COUPLING FOR THE GAUGE GROUP $G_m \times G_s$

The minimal coupling construct for the total gauge group $G = G_m \times G_s$ leads to the addition of a G-invariant Lagrangian L_g to the minimally replaced matter Lagrangian. The total Lagrangian is thus given by

$$L = L_e(X^a, Y^i_a)J + L_g(X^a, J^a_b, \theta^i_{abj}, \Sigma^i_{ab}, \Delta^c_{abd}, \Xi^c_{ab})J. \tag{7.1}$$

The associated G-invariant action integral thus has the evaluation

$$\mathcal{A} = \int_{B^4} (L_e + L_g) J \mu. \tag{7.2}$$

Here, we have used the fact that B^i_a are excluded as possible arguments of L_g by the requirements of minimal derivative coupling to the matter fields (see [1], Sect. 5.3]). Explicit evaluations of the G_s-curvature components θ^i_{abj} and G_s-torsion components Σ^i_{ab} have been stated previously. When we use the representation (4.9) for the linear connection of the $SO(3)$ sector of G_m, *the G_m-curvature components* Δ^c_{abd} and the *G_m-torsion components* Ξ^c_{ab} have the evaluations

$$\Delta^c_{abd} = (\partial_a M^\alpha_b - \partial_b M^\alpha_a + C^\alpha_{\beta\rho} M^\beta_a M^\rho_b) \gamma^c_{\alpha d}, \tag{7.3}$$

$$\Xi^c_{ab} = \Delta^c_{abd} X^d + \partial_a \psi^c_b - \partial_b \psi^c_a + (M^\alpha_a \psi^e_b - M^\alpha_b \psi^e_a) \gamma^c_{\alpha e}. \tag{7.4}$$

Thus, all quantities that appear in the total Lagrangian are functions of the following list of field variables and their derivatives:

$$\sum = \{\chi^i, \phi^i_a, W^\alpha_a, \psi^a_b, M^\alpha_b\}. \tag{7.5}$$

These field variables and their derivatives occur in the total Lagrangian only in the specific combinations shown in (7.1), because only these combinations have linear transformation laws under the local action of the total gauge group G.

The fact that L_g must be G-invariant places strong restrictions on the possible ways in which the various arguments of L_g can occur. In particular, the $\{a, b\}$ indices on θ^i_{abj}, Σ^i_{ab}, Δ^c_{abd}, and Ξ^c_{ab} are tensor indices under transformations of $\mathbb{B}^4$. These must be converted into scalar indices by contractions with j^a_f in order to be admissible as arguments of a function in the ordinary sense. Therefore, L_g can depend on the various curvature and torsion quantities only through the arguments

$$\tilde{\theta}^i_{fgj} = j^a_f j^b_g \theta^i_{abj}, \qquad \tilde{\Sigma}^i_{fg} = j^a_f j^b_g \Sigma^i_{ab}, \tag{7.6}$$

$$\tilde{\Delta}^c_{fgd} = j^a_f j^b_g \Delta^c_{abd}, \qquad \tilde{\Xi}^c_{fg} = j^a_f j^b_g \Xi^c_{ab}. \tag{7.7}$$

We can then proceed to form the list of scalars

$$\prod = \{\tilde{\Sigma}^i_{ab}\,\delta_{ij}\tilde{\Sigma}^j_{cd},\ \tilde{\theta}^i_{abj}\tilde{\theta}^j_{cdi},\ \tilde{\Xi}^a_{cd},\ \tilde{\Delta}^c_{abd}, \ldots\}. \tag{7.8}$$

All of the scalars in this list are invariant under the action of G_s and under the coordinate transformations induced on $\mathbb{B}^4$ by the action of G_m. However, all of the indices $\{a, b, c, d\}$ in the list (7.8) undergo anholonomic transformations induced by the local action of G_m. For example, we have

$$'\tilde{\Xi}^a_{bc} = R^a_d r^e_b r^f_c \tilde{\Xi}^d_{ef}.$$

Accordingly, G-invariant scalars have to be constructed from the list (7.8) by forming products so that the various occurrences of R^a_b and r^a_b will cancel out because $R^a_b r^b_c = \delta^a_c$. Remembering the particular form that R^a_b has [see (5.7)] and the privileged nature of the time variable X^4, formation of G_m-invariants out of the list (7.8) places strong restrictions on the possible arguments of the Lagrangian function. For example, if all of the special orthogonal group is included in G_m (i.e., the material is isotropic), then there are no invariants that are linear in $\tilde{\Sigma}^i_{ab}$ and the only invariants that are quadratic in $\tilde{\Sigma}^i_{ab}$ are

$$\tilde{\Sigma}^i_{AB}\,\delta_{ij}\tilde{\Sigma}^j_{CD}\,\delta^{AC}\delta^{BD}, \qquad \tilde{\Sigma}^i_{4A}\,\delta_{ij}\tilde{\Sigma}^j_{4B}\,\delta^{AB}. \tag{7.9}$$

This is because the $\tilde{\Sigma}$'s have the transformation laws

$$'\tilde{\Sigma}^i_{ab} = Q^i_j r^c_a r^d_b \tilde{\Sigma}^j_{cd}$$

so that they transform as components of a vector under spatial rotations (i.e., under action by the $SO(3)$ sector of G_s). The invariants (7.9) agree with those given in [1] when account is taken of the presence of the additional group G_m. They may also be written in the equivalent fashion

$$\Sigma^i_{ab}\,\delta_{ij}\Sigma^j_{cd}h^{ac}h^{bd}, \qquad \Sigma^i_{ra}\,\delta_{ij}\Sigma^j_{sb}h^{ab}j^r_4 j^s_4. \tag{7.10}$$

Similar arguments prevail with respect to the remaining arguments in the lists (7.5) and (7.8). The list of invariants is much richer for these arguments because all of the indices for the $\tilde{\Xi}$'s and the $\tilde{\Delta}$'s live in the same index space. Thus, for example, we have the linear invariants

$$\tilde{\Xi}^A_{4A}, \qquad \tilde{\Delta}^A_{ABC}\,\delta^{BC}, \tag{7.11}$$

while the list of quadratic invariants is extensive. We will not go further into the subject at this point. Succeeding papers in this series will consider materials with explicit material symmetry groups, at which time we will state appropriate choices for the G-invariant free gauge field Lagrangian L_g. The discussion given here is to alert the reader to the underlying computational problems that are involved in constructing G-invariant Lagrangians.

8. DISCUSSION

The kinematic foundations of a gauge theory of solids with the gauge group $G_m \times G_s$ have been established in this paper. The essential new aspect is that the theory allows for the local action of the material symmetry group. The effects of this enlargement are pervasive. They allow us to accommodate nontrivial topologies for the material such as those perceived in the classic literature [3, 10]. Indeed, the reference history $\mathbb{B}^4$ of the body becomes a Riemann-Cartan space with both curvature and torsion. The intrinsic structure of this space has characteristic peculiarities that are anything but customary. For example, the metric tensor $\{h_{ab}\}$ for the 4-dimensional space $\mathbb{B}^4$ only has rank 3, and yet there is a well defined linear connection that satisfies the Ricci lemma, $\nabla_a h_{bc} = 0$ for any and all choices of the compensating fields for local action of the material symmetry group.

The new degrees of freedom engendered by local action of the material symmetry group increase the permitted dynamic interplay in a major way. This comes from the many new invariants that can appear in the evaluation of the free gauge field Lagrangian L_g. New effects also appear through the inclusion of the new compensating fields in the matter Lagrangian L_e because of the new arguments Y_a^i rather than B_a^i that replace $\partial_a \chi^i$. These new arguments provide the explicit direct factorization $Y_a^i = j_a^b B_b^i$ into a "plastic" distortion $\{j_a^b\}$ and an elastic distortion $\{B_a^i\}$ that should prove to be fundamental in achieving a new level of understanding of phenomenalogical plasticity. In fact, even at this point there is a new perspective, for the plastic distortion $\{j_a^b\}$ is different from the identity precisely because of material defects that reflect the nontrivial topology of the material.

There is one very important point that needs to be made at this juncture. The analysis of the effects of local action of the material symmetry group G_m have been made under the assumption that G_m is a Lie group that is a subgroup of $\{SO(3) \triangleright T(3)\} \times T(1)$. Now, the various crystal classes have material symmetry groups that contain only discrete subgroups of $SO(3)$. Since a discrete subgroup of $SO(3)$ is not a Lie group, it can not participate in the gauging process (i.e. it does not contain elements in an arbitrarily small neighborhood of the identity element which can be allowed to become position and time dependent). Thus, all that will survive for the material gauge group for such materials will be $T(3) \times T(1)$. What this says is that materials whose material symmetry groups contain discrete subgroups of $SO(3)$ can not admit material disclination effects. This observation is in sharp contrast with the corresponding situation with spatial defects, because the spatial gauge group is necessarily the full Lie group $SO(3) \triangleright T(3)$.

The next paper in this series will derive the governing field equations and study their consequences. These will include integrability conditions, conservation laws, and admissible initial and boundary conditions. A succeeding paper will examine asymptotic representations, linearizations, and solutions of a class of definitive problems.

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