Indian Institute of Technology Bhubaneswar

School of Infrastructure

Session: Autumn 2025

Solid Mechanics (CE2L001) Class Test 1

<u>Notations :</u>
Zeroth-order tensors or scalars are represented by small letters. For eg. a
First-order tensors or vectors are represented by bold small letters. For eg. $\boldsymbol{a}$.
Second-order tensors are represented by bold capital letters. For eg. $\boldsymbol{A}$

1. (a) Explain the following statement through theoretical derivations and examples with illustrative figures. "Vector is independent of the coordinate system, but its components are not".

   (b) Consider two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ whose matrix of components relative to an orthonormal basis $\{\boldsymbol{e_i}\}$ are

$$[\boldsymbol{a}] = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} \text{ and } [\boldsymbol{b}] = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}.$$

   Compute (i) the magnitude of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ denoted by $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$, respectively.
   (ii) the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
   (iii) the area of the parallelogram bounded by $\boldsymbol{a}$ and $\boldsymbol{b}$.
   (iv) $\boldsymbol{b} \times \boldsymbol{a}$ and comment on the results.

Solution: (a) Vector is independent of the coordinate system, whereas its components (column matrix representation) changes from one coordinate system to another.

Let's consider a vector $\boldsymbol{u}$ of unit magnitude in a three-dimensional space being viewed from two different coordinate systems having orthonormal basis vectors $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}_i^*\}$, as shown in Fig. 1. Since, vector is independent of the coordinate system, $\boldsymbol{u} = u_i \boldsymbol{e}_i = u_i^* \boldsymbol{e}_i^*$.

The $\{\boldsymbol{e}_i^*\}$ basis vector is formed by rotating $\{\boldsymbol{e}_i\}$ by $\theta = 45°$ with respect to $\{\boldsymbol{e}_3\}$ axis, such that the vector $\boldsymbol{u}$ lies along $\boldsymbol{e}_1^*$.

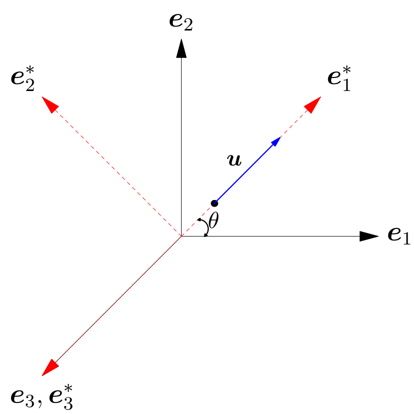


Figure 1: The representation of a vector $\boldsymbol{u}$, using two different coordinate systems

The components of the vector $\boldsymbol{u}$ with respect to each basis can be written as follows:

$$[\boldsymbol{u}]_{\{\boldsymbol{e}_i^*\}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [\boldsymbol{u}]_{\{\boldsymbol{e}_i\}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

The vector $\boldsymbol{u}$ can be represented as

$$\boldsymbol{u} = \boldsymbol{e}_1^* + 0\boldsymbol{e}_2^* + 0\boldsymbol{e}_3^* = \frac{1}{\sqrt{2}}\boldsymbol{e}_1 + \frac{1}{\sqrt{2}}\boldsymbol{e}_2 + 0\boldsymbol{e}_3.$$

The magnitude and direction of the vector, $\boldsymbol{u}$, remain unchanged in space.

(b) (i) The norm or magnitude of a vector can be determined as

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{1^2 + 4^2 + 6^2} = \sqrt{1 + 16 + 36} = \sqrt{53} \approx 7.28.$$

(ii) The angle (θ) between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be determined as

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right),$$

where $\boldsymbol{a} \cdot \boldsymbol{b}$ can be found as

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= 1 \times 2 + 4 \times 0 + 6 \times 4 = 2 + 0 + 24 = 26. \end{aligned} \tag{1}$$

One can determine $|\boldsymbol{b}|$ as

$$|\boldsymbol{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{2^2 + 0^2 + 4^2} = \sqrt{4 + 0 + 16} = \sqrt{20} \approx 4.47. \tag{2}$$

From Eq. (1), the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be determined as

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right) = \cos^{-1}\left(\frac{26}{\sqrt{53}\sqrt{20}}\right)$$
$$\implies \theta \approx 37.2^\circ.$$

(iii) The area of the parallelogram, A bounded by $\boldsymbol{a}$ and $\boldsymbol{b}$ can be determined

$$A = |\boldsymbol{a}\times\boldsymbol{b}|, \tag{3}$$

Let, $\boldsymbol{v} = \boldsymbol{a}\times\boldsymbol{b}$. The components of the vector $\boldsymbol{v} = v_i\boldsymbol{e}_i$ can be determined as $v_i = \epsilon_{ijk}a_jb_k$. Thus, the components along each basis vector can be determined as

$$\begin{aligned}
v_1 &= \epsilon_{123}a_2b_3 + \epsilon_{132}a_3b_2 = a_2b_3 - a_3b_2 = 4\times 4 - 6\times 0 = 16,\\
v_2 &= \epsilon_{231}a_3b_1 + \epsilon_{213}a_1b_3 = a_3b_1 - a_1b_3 = 6\times 2 + 1\times 4 = 8,\\
v_3 &= \epsilon_{312}a_1b_2 + \epsilon_{321}a_2b_1 = a_1b_2 - a_2b_1 = 1\times 0 - 4\times 2 = -8.
\end{aligned}$$

$$\implies \boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{v} = v_i\boldsymbol{e}_i = 16\boldsymbol{e}_1 + 8\boldsymbol{e}_2 - 8\boldsymbol{e}_3.$$

Thus the area, A can be found as

$$|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{16^2+8^2+(-8)^2} = \sqrt{256+64+64} \approx 19.6.$$

(iv) The vector $\boldsymbol{b}\times\boldsymbol{a}$ can be determined as

$$\boldsymbol{b}\times\boldsymbol{a} = -(\boldsymbol{a}\times\boldsymbol{b}) = -16\boldsymbol{e}_1 - 8\boldsymbol{e}_2 + 8\boldsymbol{e}_3.$$

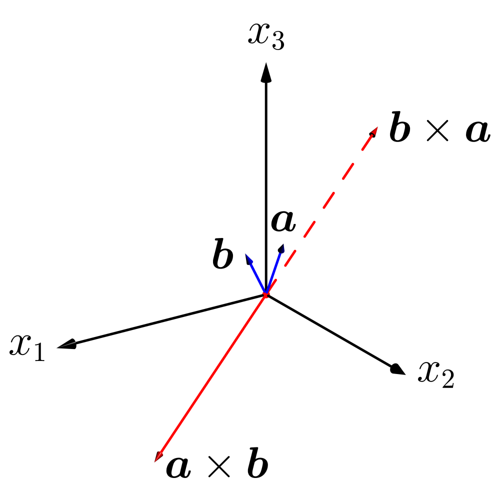


Figure 2: Schematic representation of the cross product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

2. (a) Define the Kronecker delta symbol (δ_{ij}) and the Permutation symbol (ϵ_{ijk}) that are frequently used in tensor calculus.

(b) Simplify the following expressions:
(i) $\delta_{ij}\,\delta_{jk}\,\delta_{kl}\,\delta_{lm}\,\delta_{mn}\,\delta_{nq}\,\delta_{qi}$.
(ii) $\epsilon_{1jk}\delta_{3j}v_k$.

Solution: (a) Kronecker delta symbol (δ_{ij}) allows a simple representation of the dot product (or scalar product) of orthonormal vectors in a right-handed basis system. Kronecker delta symbol (δ_{ij}) is defined as follows.

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{4}$$

The dot product between the basis vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ can be defined as $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$. The Kronecker delta, δ_{ij}, modifies (or contracts) the subscripts in the coefficients of an expression in which it appears as

$$\begin{aligned} a_i\delta_{ij} &= a_j \\ a_ib_j\delta_{ij} &= a_ib_i = a_jb_j \\ \delta_{ij}\delta_{ik} &= \delta_{jk} \end{aligned}$$

The permutation symbol, ϵ_{ijk} is coefficients of a rank-3 antisymmetric tensor and can be defined as

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } i,\ j,\ k \text{ are in cyclic order and not repeated} \\ 0, & \text{any two indices are the same} \\ -1, & \text{if } i,\ j,\ k \text{ are not in cyclic order and not repeated} \end{cases} \tag{5}$$

By definition, the subscripts of the permutation symbol can be permuted without changing its value; an interchange of any two subscripts will change the sign (hence, interchange of two subscripts twice keeps the value unchanged). The cross product of two base vectors $\boldsymbol{e}_j$ and $\boldsymbol{e}_k$ can be defined as

$$\boldsymbol{e}_j \times \boldsymbol{e}_k := \epsilon_{ijk}\boldsymbol{e}_i$$

Considering dot product with $\boldsymbol{e}_m$ on both sides of Eq. (2), one can get that

$$\begin{aligned} \boldsymbol{e}_m \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) &= \boldsymbol{e}_m \cdot (\epsilon_{ijk}\boldsymbol{e}_i) \\ &= \epsilon_{ijk}\,(\boldsymbol{e}_m \cdot \boldsymbol{e}_i) \\ &= \epsilon_{ijk}\,(\delta_{mi}) \\ &= \epsilon_{mjk} \\ \implies \boldsymbol{e}_m \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) &= \epsilon_{mjk}. \end{aligned}$$

Replacing index m, with i, we get

$$\epsilon_{ijk} = \boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) \tag{6}$$

(b) (i) To simplify the expression $\delta_{ij}\, \delta_{jk}\, \delta_{kl}\, \delta_{lm}\, \delta_{mn}\, \delta_{nq}\, \delta_{qi}$, one can use the contraction properties of the Kronecker delta symbol, δ_{ij}.

$$\begin{aligned}
\delta_{ij}\, \delta_{jk}\, \delta_{kl}\, \delta_{lm}\, \delta_{mn}\, \delta_{nq}\, \delta_{qi} &= \delta_{ij}\, \delta_{jk}\, \delta_{kl}\, \delta_{lm}\, \delta_{mn}\, \delta_{ni} \qquad (\text{since } \delta_{nq}\delta_{qi} = \delta_{ni}) \\
&= \delta_{ij}\, \delta_{jk}\, \delta_{kl}\, \delta_{lm}\, \delta_{mi} \qquad (\text{since } \delta_{mn}\delta_{ni} = \delta_{mi}) \\
&= \delta_{ij}\, \delta_{jk}\, \delta_{kl}\, \delta_{li} \qquad (\text{since } \delta_{lm}\delta_{mi} = \delta_{li}) \\
&= \delta_{ij}\, \delta_{jk}\, \delta_{ki} \qquad (\text{since } \delta_{kl}\delta_{li} = \delta_{ki}) \\
&= \delta_{ij}\, \delta_{ji} \qquad (\text{since } \delta_{jk}\delta_{ki} = \delta_{ji}) \\
&= \delta_{ii} \qquad (\text{since } \delta_{ij}\delta_{ji} = \delta_{ii}) \\
&= (\delta_{11} + \delta_{22} + \delta_{33}) \qquad (\text{since dummy index implies sum}) \\
&= 1 + 1 + 1 \quad (\text{using Eq. (4)}) \\
&= 3.
\end{aligned}$$

(ii) $\epsilon_{1jk}\delta_{3j}v_k$ can be expressed as follows:

$$\begin{aligned}
\epsilon_{1jk}\delta_{3j}v_k &= \epsilon_{13k}v_k \qquad (\text{using contraction property}, \epsilon_{1jk}\delta_{3j} = \epsilon_{13k}) \\
&= \epsilon_{131}v_1 + \epsilon_{132}v_2 + \epsilon_{133}v_3 \qquad (\text{since dummy index implies sum}) \\
&= 0 \times v_1 + (-1) \times v_2 + 0 \times v_3 \qquad (\text{using Eq. (5)}) \\
&= -v_2.
\end{aligned}$$

3. (a) Explain the concept of dyadic product using theoretical derivations and illustrative figures.

   (b) Prove the following tensor identities

   (i) $\boldsymbol{T}(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{T}\boldsymbol{a}) \otimes \boldsymbol{b}$.

   (ii) $(\boldsymbol{u} \otimes \boldsymbol{v})\, \boldsymbol{A} = \left(\boldsymbol{u} \otimes \boldsymbol{A}^T \boldsymbol{v}\right)$.

Solution: (a) Tensor product or dyadic product of two vectors $\boldsymbol{a}$, and $\boldsymbol{b}$ are defined as

$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a},\ \forall\, \boldsymbol{c} \in V. \tag{7}$$

Note that $\boldsymbol{a} \otimes \boldsymbol{b}$, maps a third vector $\boldsymbol{c}$ to another vector $(\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$. Hence, if we could prove that the transformation is linear, $\boldsymbol{a} \otimes \boldsymbol{b}$ can be considered as a second-order tensor. For any arbitrary scalars

c, and $d \in \mathbb{R}$, and $\boldsymbol{x}$, and $\boldsymbol{y} \in V$ we have

$$\begin{aligned}(\boldsymbol{a} \otimes \boldsymbol{b})(c\boldsymbol{x} + d\boldsymbol{y}) &= [\boldsymbol{b} \cdot (c\boldsymbol{x} + d\boldsymbol{y})]\boldsymbol{a} \\ &= (c\boldsymbol{b} \cdot \boldsymbol{x} + d\boldsymbol{b} \cdot \boldsymbol{y})\boldsymbol{a} \\ &= c(\boldsymbol{b} \cdot \boldsymbol{x})\boldsymbol{a} + d(\boldsymbol{b} \cdot \boldsymbol{y})\boldsymbol{a}) \\ &= c[(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{x}] + d[(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{y}] \qquad (8)\end{aligned}$$

which proves that $\boldsymbol{a} \otimes \boldsymbol{b}$ is a linear transformation and hence $\boldsymbol{a} \otimes \boldsymbol{b}$ is a second-order tensor.

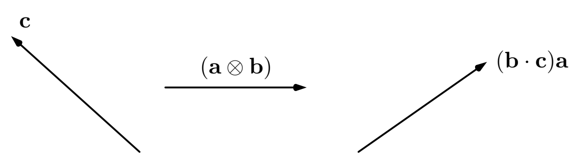


Figure 3: Illustration of $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$.

(b) Let $\boldsymbol{w} \in \mathcal{V}$ be any vector in the same vector space $\mathcal{V}$.

(i)

$$\begin{aligned}\boldsymbol{T}(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{w} &= \boldsymbol{T}(\boldsymbol{b} \cdot \boldsymbol{w})\boldsymbol{a} \quad \text{(using Eq. (7))} \\ &= (\boldsymbol{b} \cdot \boldsymbol{w})(\boldsymbol{T}\boldsymbol{a}) \quad \text{(since } (\boldsymbol{b} \cdot \boldsymbol{w}) \text{ is a scalar quantity)} \\ &= ((\boldsymbol{T}\boldsymbol{a}) \otimes \boldsymbol{b})\, \boldsymbol{w} \quad \text{(using Eq. (7)).}\end{aligned}$$

Since $\boldsymbol{w}$ is arbitrary, $\boldsymbol{T}(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{T}\boldsymbol{a}) \otimes \boldsymbol{b}$ (proved).

(ii)

$$\begin{aligned}(\boldsymbol{u} \otimes \boldsymbol{v})\, \boldsymbol{A}\boldsymbol{w} &= (\boldsymbol{v} \cdot \boldsymbol{A}\boldsymbol{w})\boldsymbol{u} \quad \text{(using Eq. (7))} \\ &= (\boldsymbol{A}^T\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u} \quad \text{(since } \boldsymbol{v} \cdot \boldsymbol{A}\boldsymbol{w} = \boldsymbol{A}^T\boldsymbol{v} \cdot \boldsymbol{w}) \\ &= \left(\boldsymbol{u} \otimes \boldsymbol{A}^T\boldsymbol{v}\right) \boldsymbol{w} \quad \text{(using Eq. (7)).}\end{aligned}$$

Since $\boldsymbol{w}$ is arbitrary, $(\boldsymbol{u} \otimes \boldsymbol{v})\, \boldsymbol{A} = \left(\boldsymbol{u} \otimes \boldsymbol{A}^T\boldsymbol{v}\right)$ (proved).

4. (a) In two dimensions, any orthogonal tensor can be expressed as

$$\boldsymbol{R} = \cos\theta\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \sin\theta\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \sin\theta\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \cos\theta\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2$$

(i) Show that $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$ (where $\boldsymbol{I}$ is the second order identity tensor), i.e. prove that the inverse of an orthogonal tensor is its transpose.
(ii) Show that $|\boldsymbol{v}| = |\boldsymbol{R}\boldsymbol{v}|$ for all $\boldsymbol{v}$; i.e. an orthogonal tensor does not change the length of a vector.

(b) Consider a two-dimensional orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ in which a two-dimensional tensor $\boldsymbol{T}$ has the representation

$$\boldsymbol{T} = T_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad \text{where } \ i,j = 1,2.$$

The component matrix of $\boldsymbol{T}$ has values

$$[\boldsymbol{T}]_{\{\boldsymbol{e}_i\}} = \begin{bmatrix} 20 & 10 \\ 10 & 10 \end{bmatrix}.$$

Consider a second basis $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*\}$ which is related to $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ by

$$\boldsymbol{e}_1^* = \frac{1}{2}\boldsymbol{e}_1 + \frac{\sqrt{3}}{2}\boldsymbol{e}_2, \qquad \boldsymbol{e}_2^* = -\frac{\sqrt{3}}{2}\boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{e}_2.$$

Find the values of $[\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}}$ in the $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*\}$ basis.

Solution:

(a) If $\boldsymbol{R} = R_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, the $\boldsymbol{R}^T = R_{ji}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. For given $\boldsymbol{R}$, one can find $\boldsymbol{R}^T$ as

$$\boldsymbol{R}^T = \cos\theta\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sin\theta\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \sin\theta\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \cos\theta\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_2$$

(i)

$$\begin{aligned}
\boldsymbol{R}^T\boldsymbol{R} = {} & (\cos\theta\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sin\theta\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \sin\theta\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \cos\theta\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) \\
& (\cos\theta\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \sin\theta\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \sin\theta\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \cos\theta\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) \\
& = \cos^2\theta\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sin^2\theta\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \\
& + \cos^2\theta\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \sin^2\theta\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \qquad (\text{using } (\boldsymbol{a}\otimes\boldsymbol{b})(\boldsymbol{c}\otimes\boldsymbol{d}) = (\boldsymbol{b}\cdot\boldsymbol{c})(\boldsymbol{a}\otimes\boldsymbol{d}) \text{ and } \boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij}) \\
& = (\cos^2\theta + \sin^2\theta)\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + (\cos^2\theta + \sin^2\theta)\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \\
& = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \qquad \big(\text{using } \ \cos^2\theta + \sin^2\theta = 1\big) \\
& = \boldsymbol{I} \qquad (\text{since } \boldsymbol{I} = \boldsymbol{e}_i \otimes \boldsymbol{e}_i)
\end{aligned}$$

Hence, $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$ (proved).

(ii) Let $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2$ whose maginitude is $|\boldsymbol{v}| = \sqrt{v_1^2 + v_2^2}$. Then:

$$\begin{aligned}
\boldsymbol{R}\boldsymbol{v} &= (\cos\theta\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \sin\theta\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \sin\theta\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \cos\theta\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_2)\,(v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2) \\
&= (v_1\cos\theta - v_2\sin\theta)\boldsymbol{e}_1 + (v_1\sin\theta + v_2\cos\theta)\boldsymbol{e}_2 \qquad (\text{using } (\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a} \text{ and } \boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij})
\end{aligned}$$

Now, compute the magnitude of $\boldsymbol{Rv}$ as

$$\begin{aligned}
|\boldsymbol{Rv}| &= \sqrt{(v_1\cos\theta - v_2\sin\theta)^2 + (v_1\sin\theta + v_2\cos\theta)^2} \\
&= \sqrt{v_1^2\cos^2\theta - 2v_1v_2\cos\theta\sin\theta + v_2^2\sin^2\theta + v_1^2\sin^2\theta + 2v_1v_2\cos\theta\sin\theta + v_2^2\cos^2\theta} \\
&= \sqrt{v_1^2(\cos^2\theta + \sin^2\theta) + v_2^2(\sin^2\theta + \cos^2\theta)} \\
&= \sqrt{v_1^2 + v_2^2} \\
&= |\boldsymbol{v}|
\end{aligned}$$

Hence, $|\boldsymbol{v}| = |\boldsymbol{Rv}|$ (proved).

(b) From the definition $\boldsymbol{e}_i^* = \boldsymbol{Q}\boldsymbol{e}_i$, the components Q_{ij} of orthogonal transformation tensor $\boldsymbol{Q} = Q_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ can be defined as

$$Q_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{Q}\boldsymbol{e}_j = \boldsymbol{e}_i \cdot \boldsymbol{e}_j^*. \tag{9}$$

Since, tensor is independent of coordinates system, $\boxed{\boldsymbol{T} = T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = T_{ij}^*\boldsymbol{e}_i^* \otimes \boldsymbol{e}_j^*}$. Thus the components T_{ij}^* can be expressed as

$$\begin{aligned}
T_{ij}^* &= \boldsymbol{e}_i^* \cdot \big(\boldsymbol{T}\boldsymbol{e}_j^*\big) \\
&= \boldsymbol{e}_i^* \cdot \big(T_{kl}(\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\boldsymbol{e}_j^*\big) \\
&= \boldsymbol{e}_i^* \cdot \big(T_{kl}(\boldsymbol{e}_j^* \cdot \boldsymbol{e}_l)\boldsymbol{e}_k\big) \qquad \text{(since } (\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\boldsymbol{e}_j^* = (\boldsymbol{e}_j^* \cdot \boldsymbol{e}_l)\boldsymbol{e}_k\text{)} \\
&= T_{kl}\,\big((\boldsymbol{e}_k \cdot \boldsymbol{e}_i^*)(\boldsymbol{e}_l \cdot \boldsymbol{e}_j^*)\big) \\
\implies T_{ij}^* &= Q_{ki}T_{kl}Q_{lj} \qquad \text{(using Eq. (9))}
\end{aligned} \tag{10}$$

Thus, one can find the components of the tensor $\boldsymbol{T}$ in $\{\boldsymbol{e}_i^*\}$ basis, i.e., $[\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}} := \big(T_{ij}^*\big)_{i,j\in(1,2,3)}$, in terms of the components of the tensor $\boldsymbol{T}$ in $\{\boldsymbol{e}_i\}$ basis, i.e., $[\boldsymbol{T}]_{\{\boldsymbol{e}_i\}} := (T_{ij})_{i,j\in(1,2,3)}$, as

$$[\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}} = [\boldsymbol{Q}]^T[\boldsymbol{T}]_{\{\boldsymbol{e}_i\}}[\boldsymbol{Q}]. \tag{11}$$

The components of $\boldsymbol{Q}$ in matrix form can be computed as

$$\begin{aligned}
[\boldsymbol{Q}] &= \begin{bmatrix} \boldsymbol{e}_1 \cdot \boldsymbol{e}_1^* & \boldsymbol{e}_1 \cdot \boldsymbol{e}_2^* \\ \boldsymbol{e}_2 \cdot \boldsymbol{e}_1^* & \boldsymbol{e}_2 \cdot \boldsymbol{e}_2^* \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{e}_1 \cdot \left(\frac{1}{2}\boldsymbol{e}_1 + \frac{\sqrt{3}}{2}\boldsymbol{e}_2\right) & \boldsymbol{e}_1 \cdot \left(-\frac{\sqrt{3}}{2}\boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{e}_2\right) \\ \boldsymbol{e}_2 \cdot \left(\frac{1}{2}\boldsymbol{e}_1 + \frac{\sqrt{3}}{2}\boldsymbol{e}_2\right) & \boldsymbol{e}_2 \cdot \left(-\frac{\sqrt{3}}{2}\boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{e}_2\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad \text{(using } \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}\text{)}
\end{aligned}$$

Thus, one can find the components of the tensor $\boldsymbol{T}$ in $\{\boldsymbol{e}_i^*\}$ basis, $[\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}} := \left(T_{ij}^*\right)_{i,j\in(1,2)}$ as

$$\begin{aligned}[\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}} &= [\boldsymbol{Q}]^T[\boldsymbol{T}]_{\{\boldsymbol{e}_i\}}[\boldsymbol{Q}] \\ &= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 20 & 10 \\ 10 & 10 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{25+10\sqrt{3}}{2} & \frac{-10-5\sqrt{3}}{2} \\ \frac{-10-5\sqrt{3}}{2} & \frac{35-10\sqrt{3}}{2} \end{bmatrix}.\end{aligned}$$

Hence, $\boxed{[\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}} = \begin{bmatrix} \frac{25+10\sqrt{3}}{2} & \frac{-10-5\sqrt{3}}{2} \\ \frac{-10-5\sqrt{3}}{2} & \frac{35-10\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 21.16 & -9.33 \\ -9.33 & 8.84 \end{bmatrix}}$.

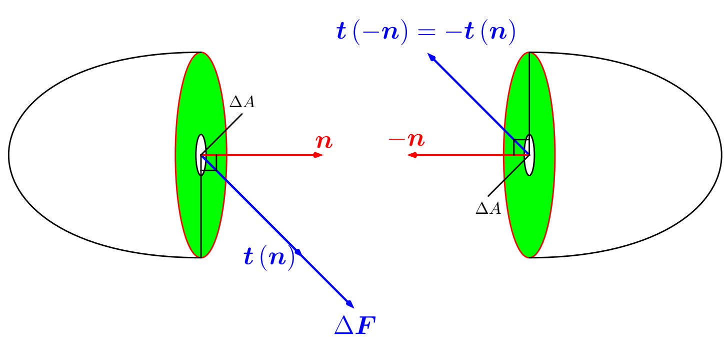
5. (a) Explain the concept of traction vector. Explain the relation between the traction vector and the stress tensor for an ideal solid and an ideal fluid using theoretical derivations and illustrative figures.

(b) Explain the following questions using illustrative figures and practical examples.

To represent the stress state at a point of a solid body, why a second-order tensor called "stress tensor" is introduced? What was the difficulty to represent the stress state at a point of a solid body using a vector?

Solution: (a) Traction vector $\boldsymbol{t}(\boldsymbol{n})$ is defined as the force acting on an area with normal $\boldsymbol{n}$. At a point of a body the traction vector is defined as

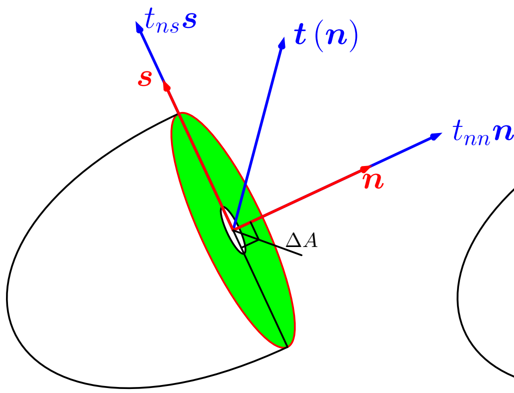
$$\boldsymbol{t}(\boldsymbol{n}) = \lim_{\Delta A\to 0} \frac{\Delta \boldsymbol{F}}{\Delta A} \tag{12}$$



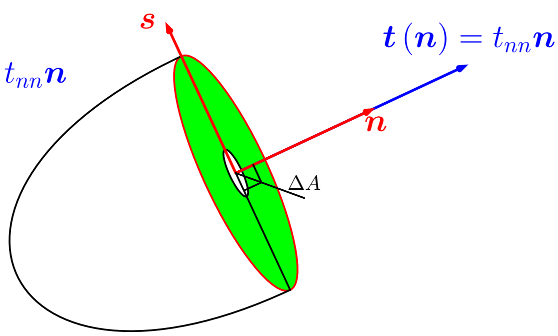
A property of the traction vector is that it must follow Newton's third law for action and reaction. Therefore, in the same point of a body the stress vector on the area with normal $\boldsymbol{n}$ and normal $-\boldsymbol{n}$ must be opposite. This means that

$$\boldsymbol{t}(-\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{n}). \tag{13}$$

The traction vector on an arbitrary plane with unit normal $\boldsymbol{n}$ can be expressed as $\boxed{\boldsymbol{t}(\boldsymbol{n}) = \boldsymbol{\sigma}\boldsymbol{n}}$. For an ideal fluid shear resistance is negligible ($t_{ns} \approx 0$) and $\boxed{\boldsymbol{t}(\boldsymbol{n}) = t_{nn}\boldsymbol{n}}$, where $t_{nn} \in \mathbb{R}$ is a scalar quantity (see the figure below). For an ideal solid, there is a significant shear resistance and $\boxed{\boldsymbol{t}(\boldsymbol{n}) = t_{nn}\boldsymbol{n} + t_{ns}\boldsymbol{s}}$ (see the figure below). The traction vector has both shear and normal components.

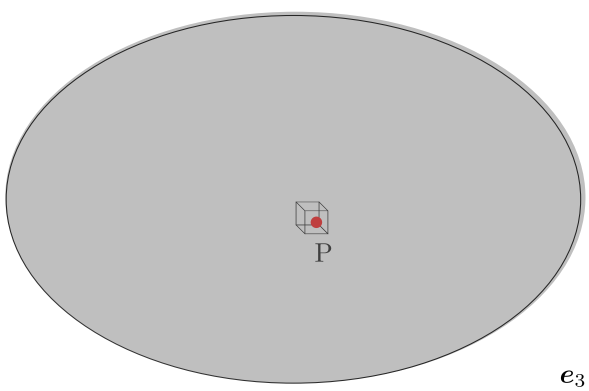


(a) Ideal solid

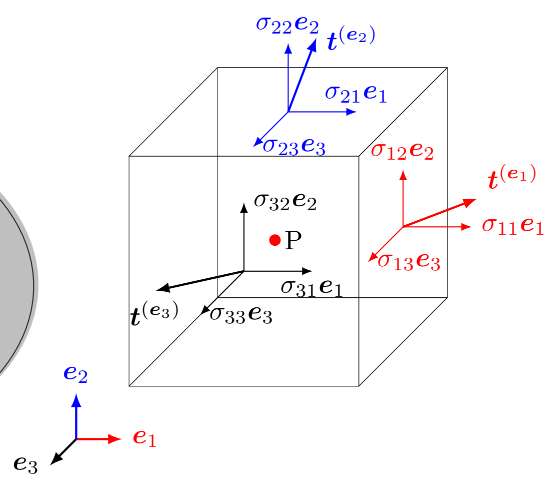


(b) Ideal fluid

(b) Representing the stress state at a point in a solid body requires describing the forces acting on various planes passing through that point. A vector alone is insufficient for this purpose because the stress state at a point involves both the magnitude and direction of forces on different planes, which vary depending on the orientation of the plane.



(a) To define stress state at a point P of a solid body, an infinitesimal volume with three independent planes is considered.



(b) Stress state at point P of a solid body is defined by the stress tensor $\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$.

Difficulty with Vectors: A vector can describe the force acting on a single plane, but it cannot capture the complex state of stress that involves multiple planes with different orientations. The stress state at a point is characterized by:

Normal stresses: Stress components $(\sigma_{11}, \sigma_{22}, \sigma_{33})$ causing forces perpendicular to the plane.

Shear stresses: Stress components $(\sigma_{12} = \sigma_{21}, \sigma_{13} = \sigma_{31}, \sigma_{23} = \sigma_{32})$ causing forces parallel to the plane.

These stresses depend on the orientation of the plane, and a vector cannot encapsulate this directional dependence for all possible planes.

Solution: Stress Tensor: A second-order tensor, known as the stress tensor $\boldsymbol{\sigma}$, is introduced to represent the stress state. The stress tensor describes the traction vector on any plane passing through a point, given by $\boldsymbol{t}^{(\boldsymbol{n})} := \boldsymbol{t}(\boldsymbol{n}) = \boldsymbol{\sigma}\boldsymbol{n}$, where $\boldsymbol{t}$ is the traction vector (force per unit area) on a plane with normal vector $\boldsymbol{n}$. The traction vector can be found on three mutually perpendicular planes by applying the stress tensor $\boldsymbol{\sigma}$ to the surface normal vectors as given below.

$$\boldsymbol{\sigma}\boldsymbol{e}_1 = \boldsymbol{t}^{(\boldsymbol{e}_1)}, \quad \boldsymbol{\sigma}\boldsymbol{e}_2 = \boldsymbol{t}^{(\boldsymbol{e}_2)}, \quad \boldsymbol{\sigma}\boldsymbol{e}_3 = \boldsymbol{t}^{(\boldsymbol{e}_3)}.$$