Chapter 1

Strain analysis

1.1 Introduction

We are concerned here with *deformable bodies*, i.e. with continuum[1] bodies that can be strained: the relative positions of the *material* points are altered by some agents (forces, temperature etc.).

We will call *deformation* a change of position of the material points when this change is accompanied also by a mutual change of the relative positions. The description of the deformation (*strain analysis*) is based upon the introduction of some geometric quantities and algebraic operators, able to account for some properties of the deformation. All these points need to be specified mathematically.

1.2 Deformation gradient

We consider a solid continuum body which occupy the region Ω of the Euclidean space $\mathcal{E}$ (in short, we identify the body with Ω). Some agents strain Ω and deform it to the final configuration Ω_t. We use capital letters for denoting any quantity in Ω and small letters for Ω_t. The general situation is that sketched in Fig. 1.1.

Any point $P \in \Omega$ is transformed by the deformation into a *unique* point $p \in \Omega_t$:

$$p = f(P); \tag{1.1}$$

p is hence a function of point in Ω. Function f is said to be a *deformation* whenever it is a continuous and bijective function on Ω[2]. Bijectivity is essential to state a fundamental property of classical continuum mechanics: mass conservation.

[1]The notion of continuum body is primary here and it is left to the basic idea of a body whose fundamental property is that of occupying some space, i.e. a region $\Omega \subset \mathcal{E}$, the ordinary Euclidean space. We will denote by $\mathcal{V}$ the vector space associated with $\mathcal{E}$, called the *space of translations* $\mathbf{u}$, and by $Lin(\mathcal{V})$ the *linear space of second rank tensors over* $\mathcal{V}$, i.e. of all the linear transformations $\mathbf{L} : \mathcal{V} \to \mathcal{V}$.

[2]f is continuous in $P \in \Omega$ if, $\forall$ sequence $\{P_n \in \Omega, n \in \mathbb{N}\}$ that converges to P, the sequence $\{p_n = f(P_n), n \in \mathbb{N}\}$ converges to $f(P)$; f is continuous on Ω if it is continuous $\forall P \in \Omega$.

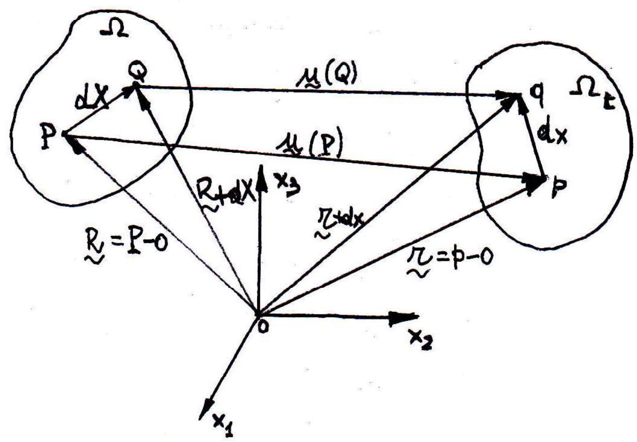

Figure 1.1: General sketch for the strain analysis

Ω is said to be the *reference configuration* and Ω_t the *actual configuration.* The vector

$$\mathbf{u}(P) = p - P = f(P) - P \tag{1.2}$$

is the *displacement vector*, a vector field on Ω; $\mathbf{R} = P - o$ and $\mathbf{r} = p - o$ are the position vectors respectively of P and p with respect to a fixed Cartesian frame.

The purpose of strain analysis is not only to study the displacement field $\mathbf{u}$, but, mainly, to analyse how matter deforms everywhere in Ω. For this, we try to study what happens in a material set close to any point $P \in \Omega$ and in particular how elementary geometric quantities defined on this set evolve during deformation.

To this purpose, let us introduce the concept of *fiber*: a fiber dX in the vicinity of $P \in \Omega$ is a vector composed by material points such that

$$dX = \alpha\mathbf{e}, \quad |\mathbf{e}| = 1, \;\; \alpha \to 0, \;\; \alpha \in \mathbb{R}^+. \tag{1.3}$$

A fiber

$$dX = Q - P \tag{1.4}$$

is hence a *small material vector* from $P \in \Omega$ to $Q \in \Omega$, with Q close to P. We are concerned with the following question: in which fiber dx is transformed by f the fiber dX? It is

$$dx = q - p = Q + \mathbf{u}(Q) - (P + \mathbf{u}(P)) = Q - P + \mathbf{u}(Q) - \mathbf{u}(P) = dX + \mathbf{u}(Q) - \mathbf{u}(P), \tag{1.5}$$

but

$$\mathbf{u}(Q) = \mathbf{u}(P) + \nabla\mathbf{u}(P)(Q - P) + o(Q - P)^2, \tag{1.6}$$

because Q is close to P. So, neglecting higher order terms, we get

$$\mathbf{u}(Q) = \mathbf{u}(P) + \nabla\mathbf{u}(P)dX \tag{1.7}$$

and finally

$$dx = [\mathbf{I} + \nabla\mathbf{u}(P)]dX. \tag{1.8}$$

$\nabla\mathbf{u}(P)$ is the *displacement gradient*; as a linear operator, $\nabla\mathbf{u}$ is a second-rank tensor[3]:

$$\nabla\mathbf{u} = u_{i,j}\ \mathbf{e}_i \otimes \mathbf{e}_j. \tag{1.9}$$

We pose

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}, \tag{1.10}$$

the *deformation gradient*. We thus obtain the formula

$$dx = \mathbf{F}dX \ \ \rightarrow \ \ F_{ij} = \delta_{ij} + u_{i,j} = \frac{dx_i}{dX_j}, \tag{1.11}$$

with δ_{ij} the Kronecker's symbol.

Generally speaking $\mathbf{F} \neq \mathbf{F}^\top$, so, though completely describing the deformation, $\mathbf{F}$ has not a *good algebraic structure*.

1.3 Geometric changes

We are interested in knowing how basic geometric quantities in the neighborhood of any point $P \in \Omega$ change during the deformation. This will allow to introduce other tensors that, though not able to completely describe the deformation, nonetheless have a better algebraic structure than $\mathbf{F}$.

1.3.1 Change in length

First, we investigate the changes of length of any fiber dX in P during the deformation: knowing $|dX|$, how long is $|dx|$? Putting

$$dX = |dX|\ \mathbf{e}, \quad |\mathbf{e}| = 1, \tag{1.12}$$

we have

$$|dx| = \sqrt{\mathbf{F}\ dX \cdot \mathbf{F}\ dX} = |dX|\sqrt{\mathbf{e}\cdot\mathbf{F}^\top\mathbf{F}\ \mathbf{e}}. \tag{1.13}$$

The *change in length* $\delta\ell(\mathbf{e})$ of a fiber in P parallel to $\mathbf{e}$ is defined as

$$\delta\ell(\mathbf{e}) := \frac{|dx| - |dX|}{|dX|} = \sqrt{\mathbf{e}\cdot\mathbf{F}^\top\mathbf{F}\ \mathbf{e}} - 1; \tag{1.14}$$

the *stretch* $\lambda(\mathbf{e})$ of the same fiber is

$$\lambda(\mathbf{e}) := \frac{|dx|}{|dX|} = 1 + \delta\ell(\mathbf{e}) = \sqrt{\mathbf{e}\cdot\mathbf{F}^\top\mathbf{F}\ \mathbf{e}}. \tag{1.15}$$

[3] The *dyad* $\mathbf{a}\otimes\mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the tensor such that $\forall\mathbf{v}\in\mathcal{V}, (\mathbf{a}\otimes\mathbf{b})\mathbf{v} = \mathbf{b}\cdot\mathbf{v}\ \mathbf{a}$. Given a orthonormal basis $e = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, any second-rank tensor $\mathbf{L}$ can be decomposed as a sum of nine dyads: $\mathbf{L} = L_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$, where the Cartesian components are given by $L_{ij} = \mathbf{e}_i\cdot\mathbf{L}\mathbf{e}_j$.

1.3.2 Change in angle

Be Θ the angle formed by two fibers $dX_i = |dX_i|\mathbf{e}_i,\ |\mathbf{e}_i| = 1,\ i = 1,2,$ in P; we are interested in knowing the angular change from Θ to θ, the angle formed by the deformed fibers dx_1 and dx_2.

We define the *change in angle* $\delta\theta(\mathbf{e}_1, \mathbf{e}_2)$ between the directions $\mathbf{e}_1$ and $\mathbf{e}_2$ the difference

$$\delta\theta(\mathbf{e}_1, \mathbf{e}_2) := \Theta - \theta; \tag{1.16}$$

remembering that

$$\cos\Theta = \frac{dX_1 \cdot dX_2}{|dX_1||dX_2|} = \mathbf{e}_1 \cdot \mathbf{e}_2,\ \cos\theta = \frac{dx_1 \cdot dx_2}{|dx_1||dx_2|} = \frac{\mathbf{F}\ dX_1 \cdot \mathbf{F}\ dX_2}{\lambda_1|dX_1|\lambda_2|dX_2|}, \tag{1.17}$$

we finally get

$$\delta\theta(\mathbf{e}_1, \mathbf{e}_2) = \arccos(\mathbf{e}_1 \cdot \mathbf{e}_2) - \arccos(\frac{\mathbf{e}_1 \cdot \mathbf{F}^\top\mathbf{F}\ \mathbf{e}_2}{\lambda_1\lambda_2}). \tag{1.18}$$

1.3.3 Change in volume

To study the volume changes around a point $P \in \Omega$, we consider the volume of the prism determined by three non coplanar fibers $dX_i = |dX_i|\mathbf{e}_i,\ |\mathbf{e}_i| = 1,\ i = 1,2,3,$ in P. The volume of the prism in Ω is

$$dV = dX_1 \cdot dX_2 \times dX_3, \tag{1.19}$$

while in Ω_t it is[4]

$$dv = dx_1 \cdot dx_2 \times dx_3 = \mathbf{F}dX_1 \cdot \mathbf{F}dX_2 \times \mathbf{F}dX_3 = \det\mathbf{F}\ dX_1 \cdot dX_2 \times dX_3, \tag{1.20}$$

i.e.

$$dv = \det\mathbf{F}\ dV. \tag{1.21}$$

We define *change in volume* in P the quantity

$$\delta v := \frac{dv - dV}{dV} = \det\mathbf{F} - 1. \tag{1.22}$$

To remark that because

$$\det\mathbf{F} = \frac{dv}{dV} \tag{1.23}$$

is a ratio of intrinsically positive quantities, it is necessarily

$$\det\mathbf{F} > 0. \tag{1.24}$$

We also remark that a deformation is locally *isochoric* $\iff \det\mathbf{F} = 1$.

[4] It can be proved that $\forall\mathbf{u},\mathbf{v},\mathbf{w} \in \mathcal{V}$ and $\forall\mathbf{L} \in Lin(\mathcal{V})$, $\mathbf{Lu} \cdot \mathbf{Lv} \times \mathbf{Lw} = \det\mathbf{L}\ (\mathbf{u} \cdot \mathbf{v} \times \mathbf{w})$. Because $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$ is the volume of the prism determined by $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, if $\det\mathbf{L} = 0$ then $\mathbf{L}$ annihilates the volume of the deformed prism, i.e. the original prism is changed into a flat figure.

1.3.4 Deformations

We can now precise mathematically the definition of *deformation*: *a function* $f(P) : \Omega \to \mathcal{E}$ *is a deformation if it is a continuous and bijective function of* P *on* Ω *and if* $\det \mathbf{F} > 0$ *everywhere in* Ω.

The inequality is strict: $\det \mathbf{F} = 0$ is not admissible because this should mean to transform a finite volume into a flat figure, with vanishing volume. Such a fact should not preserve bijectivity and conservation of the matter.

1.4 Pure deformations and rigid body motions

A deformation can be seen as a superposition of a *pure deformation* and of a *rigid body motion*, and these two parts can be split easily.

To this end, we use a classical decomposition of any second-rank tensor, splitting $\nabla\mathbf{u}$ in its *symmetric* and *skew* parts:

$$\nabla\mathbf{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}, \tag{1.25}$$

with

$$\boldsymbol{\varepsilon} = \frac{\nabla\mathbf{u} + \nabla^\top\mathbf{u}}{2}, \quad \boldsymbol{\omega} = \frac{\nabla\mathbf{u} - \nabla^\top\mathbf{u}}{2}, \tag{1.26}$$

being evidently

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^\top, \quad \boldsymbol{\omega} = -\boldsymbol{\omega}^\top. \tag{1.27}$$

Then,

$$\mathbf{F} = \mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\omega} \tag{1.28}$$

and

$$dx = (\mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\omega})\ dX = dX + \boldsymbol{\varepsilon}\ dX + \boldsymbol{\omega}\ dX. \tag{1.29}$$

So, any deformed fiber dx is equal to the originally undeformed one, dX, plus two modifying vectors; let us analyse them, starting with $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = -\boldsymbol{\omega}^\top \Rightarrow \exists\ \mathbf{v}_\omega \in \mathcal{V} :\ \boldsymbol{\omega}\ dX = \mathbf{v}_\omega \times dX, \tag{1.30}$$

$\mathbf{v}_\omega$ being the axial vector of $\boldsymbol{\omega}$. By the same definition of curl of a vector,

$$2\boldsymbol{\omega}\ dX = (\nabla\mathbf{u} - \nabla^\top\mathbf{u})\ dX = (\text{curl } \mathbf{u}) \times dX, \tag{1.31}$$

which gives also the relation

$$\mathbf{v}_\omega = \frac{1}{2}\text{curl } \mathbf{u}. \tag{1.32}$$

Let us now consider a particularly important case, that of *small displacements*; generally speaking, a rigid rotation is characterized by an amplitude, say φ, and by an axis of rotation, say $\mathbf{w}$, $|\mathbf{w}| = 1$. A general result, is that any rigid rotation can be represented by a tensor $\mathbf{R}$ which in terms of φ and $\mathbf{w}$ is given by

$$\mathbf{R} = \mathbf{I} + \sin\varphi\mathbf{W} + (1 - \cos\varphi)\mathbf{W}^2, \tag{1.33}$$

with $\mathbf{W} = -\mathbf{W}^\top$ the axial tensor of $\mathbf{w}$[5].

For small displacements, $\varphi \to 0$ so that

$$\mathbf{R} \simeq \mathbf{I} + \varphi \mathbf{W}; \tag{1.34}$$

so, comparing this result with eq. (1.29), we see that the term $\boldsymbol{\omega}\ dX$ represents a rigid motion in the assumption of small displacements. Hence, the term $\boldsymbol{\varepsilon}$ represents a pure deformation. For small displacement transformations, pure deformations are hence described by a symmetric tensor.

1.5 Small strain deformations

We now develop $\lambda, \delta\ell, \delta\theta$ and δv for the case of *small strain*: a deformation is said to be a *small strain deformation* if and only if

$$|u_{i,j}| \ll 1\ \forall i, j = 1, 2, 3. \tag{1.35}$$

We remark hence that the small displacement hypothesis concerns the smallness of vector $\mathbf{u}$, while the assumption of small strain that of $\nabla\mathbf{u}$. Nevertheless, whenever the above condition is satisfied, then all the components of $\boldsymbol{\omega}$ are small too, so also in this assumption $\varphi \to 0$, i.e. the small strain assumption is sufficient for interpreting the part $\boldsymbol{\omega}$ as a rigid motion.

Let us start developing $\lambda(\mathbf{e})$:

$$\begin{aligned} \lambda(\mathbf{e}) &= \sqrt{\mathbf{e}\cdot\mathbf{F}^\top\mathbf{F}\mathbf{e}} = \sqrt{\mathbf{e}\cdot(\mathbf{I}+\boldsymbol{\varepsilon}+\boldsymbol{\omega})^\top(\mathbf{I}+\boldsymbol{\varepsilon}+\boldsymbol{\omega})\mathbf{e}} \\ &= \sqrt{\mathbf{e}\cdot(\mathbf{I}+2\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}^2-\boldsymbol{\omega}\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}\boldsymbol{\omega}-\boldsymbol{\omega}^2)\mathbf{e}}. \end{aligned} \tag{1.36}$$

Terms like $\mathbf{e}\cdot\boldsymbol{\varepsilon}\boldsymbol{\omega}\mathbf{e}$ are of second order with respect to $2\mathbf{e}\cdot\boldsymbol{\varepsilon}\mathbf{e}$ to within the assumption of small strain:

$$2\mathbf{e}\cdot\boldsymbol{\varepsilon}\mathbf{e} = 2\varepsilon_{ij}e_ie_j = (u_{i,j}+u_{j,i})e_ie_j, \tag{1.37}$$

while

$$\mathbf{e}\cdot\boldsymbol{\varepsilon}\boldsymbol{\omega}\mathbf{e} = \varepsilon_{ik}\omega_{kj}e_ie_j = \frac{1}{4}(u_{i,k}+u_{k,i})(u_{k,j}-u_{j,k})e_ie_j. \tag{1.38}$$

As a consequence, for small strain deformations the terms $\boldsymbol{\varepsilon}\boldsymbol{\omega}, \boldsymbol{\omega}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^2$ and $\boldsymbol{\omega}^2$ can be discarded in front of $\boldsymbol{\varepsilon}$.

[5]To any $\mathbf{w} = (w_1, w_2, w_3) \in \mathcal{V}$ can be associated its *axial tensor* $\mathbf{W} = -\mathbf{W}^\top$ defined as

$$\mathbf{W} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix},$$

such that $\mathbf{w}\times\mathbf{v} = \mathbf{W}\mathbf{v}\ \forall\mathbf{v}\in\mathcal{V}$. It is easily checked that the only eigenvector of $\mathbf{W}$ is $\mathbf{w}$, relative to the unique real eigenvalue, 0. For this reason, $\mathbf{W}$ is called the *axial tensor* of $\mathbf{w}$ and reciprocally, $\mathbf{w}$ is said to be the *axial vector* or *axis* of $\mathbf{W}$.

So, still thanks to the smallness of $\nabla\mathbf{u}$, we get:

$$\begin{aligned}\lambda(\mathbf{e}) &\simeq \sqrt{\mathbf{e}\cdot(\mathbf{I}+2\boldsymbol{\varepsilon})\mathbf{e}} = \sqrt{1+2\mathbf{e}\cdot\boldsymbol{\varepsilon}\mathbf{e}}\\ &\simeq \sqrt{1+2\mathbf{e}\cdot\boldsymbol{\varepsilon}\mathbf{e}+(\mathbf{e}\cdot\boldsymbol{\varepsilon}\mathbf{e})^2} = \sqrt{(1+\mathbf{e}\cdot\boldsymbol{\varepsilon}\mathbf{e})^2},\end{aligned} \tag{1.39}$$

and finally

$$\lambda(\mathbf{e}) = 1+\mathbf{e}\cdot\boldsymbol{\varepsilon}\mathbf{e}. \tag{1.40}$$

It follows immediately that

$$\delta\ell(\mathbf{e}) = \lambda(\mathbf{e})-1 = \mathbf{e}\cdot\boldsymbol{\varepsilon}\mathbf{e}. \tag{1.41}$$

Let us now consider the change in angle in the assumption of small strain:

$$\mathbf{e}_1\cdot\mathbf{F}^\top\mathbf{F}\mathbf{e}_2 = \mathbf{e}_1\cdot(\mathbf{I}+2\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}^2-\boldsymbol{\omega}\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}\boldsymbol{\omega}+\boldsymbol{\omega}^2)\mathbf{e}_2 \simeq \mathbf{e}_1\cdot(\mathbf{I}+2\boldsymbol{\varepsilon})\mathbf{e}_2, \tag{1.42}$$

so

$$\begin{aligned}\delta\theta(\mathbf{e}_1,\mathbf{e}_2) &= \arccos(\mathbf{e}_1\cdot\mathbf{e}_2) - \arccos\frac{\mathbf{e}_1\cdot(\mathbf{I}+2\boldsymbol{\varepsilon})\mathbf{e}_2}{\lambda_1\lambda_2}\\ &= \Theta - \arccos\frac{\mathbf{e}_1\cdot\mathbf{e}_2+2\mathbf{e}_1\cdot\boldsymbol{\varepsilon}\mathbf{e}_2}{\lambda_1\lambda_2}.\end{aligned} \tag{1.43}$$

Finally, the change in volume:

$$\delta v = \det\mathbf{F}-1 = \det(\mathbf{I}+\boldsymbol{\varepsilon}+\boldsymbol{\omega})-1; \tag{1.44}$$

we use now the following general result of tensor algebra[6]:

$$\forall\mathbf{L}\in Lin(\mathcal{V}),\quad \det(\mathbf{I}+\mathbf{L}) = 1+\mathrm{tr}\mathbf{L}+\frac{\mathrm{tr}^2\mathbf{L}-\mathrm{tr}\mathbf{L}^2}{2}+\det\mathbf{L}. \tag{1.45}$$

Applying this result to the sum $\mathbf{I}+\boldsymbol{\varepsilon}+\boldsymbol{\omega}$, gives:

$$\delta v = \mathrm{tr}(\boldsymbol{\varepsilon}+\boldsymbol{\omega})+\frac{\mathrm{tr}^2(\boldsymbol{\varepsilon}+\boldsymbol{\omega})-\mathrm{tr}(\boldsymbol{\varepsilon}+\boldsymbol{\omega})^2}{2}+\det(\boldsymbol{\varepsilon}+\boldsymbol{\omega}), \tag{1.46}$$

and in the small strain assumption, one easily recognizes that the second and third term on the right hand side are negligible compared to the first one; hence

$$\delta v \simeq \mathrm{tr}(\boldsymbol{\varepsilon}+\boldsymbol{\omega}) = \mathrm{tr}\boldsymbol{\varepsilon}+\mathrm{tr}\boldsymbol{\omega}, \tag{1.47}$$

and because $\boldsymbol{\omega}=-\boldsymbol{\omega}^\top, \mathrm{tr}\boldsymbol{\omega}=0$, so finally

$$\delta v = \mathrm{tr}\boldsymbol{\varepsilon}. \tag{1.48}$$

We remark hence that the change in volume is a linear function of the ε_{ij} and that

$$\delta v = \mathrm{tr}\boldsymbol{\varepsilon} = \mathrm{tr}\frac{\nabla\mathbf{u}+\nabla^\top\mathbf{u}}{2} = \mathrm{tr}\nabla\mathbf{u} = \mathrm{div}\mathbf{u}, \tag{1.49}$$

[6]The proof of this result is rather long and tedious, but not difficult: it is sufficient to develop by components the terms on the left and right side of eq. (1.45) and remark, at the end, that they give the same global quantity

so a deformation is isochoric if and only if the displacement field is solenoidal.

In the end, we can notice that in the assumption of small strain, the rigid body part of the deformation, $\boldsymbol{\omega}$, does not take any part. $\boldsymbol{\varepsilon}$ is called the *infinitesimal strain tensor* or *tensor of small strains*; unlike $\mathbf{F}$, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^\top$ and, though it does not completely describe the deformation, it is sufficient to give us the relevant information about it in the assumption of small strain.

In the following of this text, we will assume always *small perturbations*, i.e. both the hypotheses of small displacements and small strain. Besides the possibility of completely describing the strain by tensor $\boldsymbol{\varepsilon}$, so discarding the part due to $\boldsymbol{\omega}$, this assumption let us consider as coincident the two configurations Ω and Ω_t, the reference and the actual one, because separated by a small displacement, in the sense that $|\mathbf{u}(P)| \ll d_\Omega \ \forall P \in \Omega$, where d_Ω is a characteristic dimension of Ω. Hence, $p = f(P) \simeq P \ \forall P \in \Omega$, so p can be approximated by P; this is the reason why in the following we will no more make the distinction between them and use always lower case letters for indicating points in Ω.

The possibility of approximating the actual configuration with the reference one has extremely important consequences in mechanics. In fact, in doing so, we tacitly postulate that the forces acting on Ω do not change their point of application and that the equilibrium equations are written in the reference configuration, which is known, and not in the actual, unknown one. This is false in principale, but in doing so, we eliminate one of the principal sources of nonlinearity: the dependence of the equilibrium equations from the unknown equilibrium configuration.

Of course, this can have some dramatical consequences, as it has. In particular, if on one side, along with the assumption of a linear behavior of the material, see Chapt. 3, this gives the nice property of *linearity* to the equilibrium problem of deformable bodies, on the other side it makes disappear some important phenomena of nonlinear equilibrium, like buckling and stability.

Nonetheless, several cases of practical interest are not affected by such phenomena and they satisfy with a high degree of approximation the small perturbations assumption; that is why we will use it in the following of this text for analyzing some important problems of the linear mechanics of deformable bodies.

1.6 Geometrical meaning of the ε_{ij}

We can now examine the geometrical meaning of the components ε_{ij} of $\boldsymbol{\varepsilon}$: let $\mathbf{e}_i$ and $\mathbf{e}_j$ be two vectors of a base for $\mathcal{V}$:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \ \ i,j = 1,2,3. \tag{1.50}$$

Then (no summation over i in the following equation):

$$\begin{aligned} \delta\ell(\mathbf{e}_i) &= \mathbf{e}_i \cdot \boldsymbol{\varepsilon}\mathbf{e}_i = \mathbf{e}_i \cdot \varepsilon_{hk}(\mathbf{e}_h \otimes \mathbf{e}_k)\mathbf{e}_i \\ &= \delta_{ik}\mathbf{e}_i \cdot \varepsilon_{hk}\mathbf{e}_h = \delta_{ih}\delta_{ik}\varepsilon_{hk} = \varepsilon_{ii}. \end{aligned} \tag{1.51}$$

So, the diagonal terms ε_{ii} represent the change in length of the fibers aligned with the axis $\mathbf{e}_i$; moreover (no summation over i and j in the following equation)

$$\delta\theta(\mathbf{e}_i,\mathbf{e}_j)=\frac{\pi}{2}-\arccos\frac{2\mathbf{e}_i\cdot\boldsymbol{\varepsilon}\mathbf{e}_j}{\lambda_i\lambda_j}=\arcsin\frac{2\varepsilon_{ij}}{(1+\varepsilon_{ii})(1+\varepsilon_{jj})}, \tag{1.52}$$

and because $|\varepsilon_{ij}|\ll 1\ \forall i,j$, then

$$\delta\theta(\mathbf{e}_i,\mathbf{e}_j)\simeq 2\varepsilon_{ij}: \tag{1.53}$$

the components of $\boldsymbol{\varepsilon}$ with distinct indices are half the shear deformation of the axes with corresponding indices.

1.7 Principal strains

An important consequence of the symmetry of $\boldsymbol{\varepsilon}$ is the existence of the *principal strains*, ensured by the spectral theorem[7]: there is a basis $v=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ composed of eigenvectors of $\mathbf{L}$, called *the principal directions of strain*, where

$$\boldsymbol{\varepsilon}=\varepsilon_i\mathbf{v}_i\otimes\mathbf{v}_i\ \rightarrow\ \boldsymbol{\varepsilon}=\left[\begin{array}{ccc}\varepsilon_1 & 0 & 0\\ 0 & \varepsilon_2 & 0\\ 0 & 0 & \varepsilon_3\end{array}\right]. \tag{1.54}$$

The terms on the diagonal are the *principal strains* and they coincide with the eigenvalues of $\boldsymbol{\varepsilon}$. Considering the results of the previous Section, it is then seen that in the basis of the principal directions the fibers aligned with the axes are simply stretched, not sheared: the principal directions of strain preserve their directions in the deformation and hence their mutual angles do not change.

We remark also that the change in volume is just the sum of the three eigenvectors of $\boldsymbol{\varepsilon}$:

$$\delta v=\varepsilon_1+\varepsilon_2+\varepsilon_3. \tag{1.55}$$

1.8 Spherical and deviatoric parts of $\boldsymbol{\varepsilon}$

An important decomposition of $\boldsymbol{\varepsilon}$, as of any other second-rank tensor, is into its *spherical*, $\boldsymbol{\varepsilon}_s$, and *deviatoric*, $\boldsymbol{\varepsilon}_d$, parts:

$$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_s+\boldsymbol{\varepsilon}_d, \tag{1.56}$$

with

$$\boldsymbol{\varepsilon}_s:=\frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon}\ \mathbf{I},\quad \boldsymbol{\varepsilon}_d:=\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_s. \tag{1.57}$$

[7]Spectral theorem: if a tensor $\mathbf{L}$ is symmetric, then it exists a basis of $\mathcal{V}$ composed by eigenvectors of $\mathbf{L}$ (for a demonstration, see the classical book of Halmos: *Finite-Dimensional Vector Spaces*, Springer 1987, p. 155). A consequence of this theorem is that $\mathbf{L}$ is diagonal in such a basis: in fact, be $v=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ a basis of eigenvectors of $\mathbf{L}$, $\mathbf{v}_i\cdot\mathbf{v}_j=\delta_{ij}\ \forall i,j$, and λ_k the eigenvalue corresponding to the eigenvector $\mathbf{v}_k$; then, $L_{ij}=\mathbf{v}_i\cdot\mathbf{L}\mathbf{v}_j=\lambda_j\mathbf{v}_i\cdot\mathbf{v}_j=\delta_{ij}\lambda_j\Rightarrow\mathbf{L}=\lambda_i\mathbf{v}_i\otimes\mathbf{v}_i$.

By the same definition, we obtain immediately that

$$\mathrm{tr}\boldsymbol{\varepsilon}_s = \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon}\ \mathrm{tr}\mathbf{I} = \mathrm{tr}\boldsymbol{\varepsilon}, \quad \mathrm{tr}\boldsymbol{\varepsilon}_d = 0, \tag{1.58}$$

i.e. all the change in volume are concentrated in the spherical part $\boldsymbol{\varepsilon}_s$, while $\boldsymbol{\varepsilon}_d$ describes an isochoric deformation giving hence only changes of shape that preserve the volume.

This decomposition is of some importance in different problems, namely for introducing one of the most used yielding criterion for isotropic elastic materials, see Sect. 4.12.

1.9 Compatibility equations

Once a displacement field $\mathbf{u}$ known, it is always possible, differentiating it, to get the strain field $\boldsymbol{\varepsilon}$: a displacement field always defines uniquely a strain field (the field $\mathbf{u}$ is here assumed to be at least of class C^1).

The converse is not true: given a field $\boldsymbol{\varepsilon}$, it is not always possible to find a displacement field $\mathbf{u}(p)$ to which it corresponds through

$$\boldsymbol{\varepsilon} = \frac{\nabla\mathbf{u} + \nabla^\top\mathbf{u}}{2}. \tag{1.59}$$

In fact, we have 3 unknown scalar fields $u_i(p)$ and 6 equations: the problem is over-determined. The question is hence: given the $\varepsilon_{ij}(p)$, which are the conditions that they must fulfill for being possible to find a *compatible* displacement field $\mathbf{u}(p)$, i.e. satisfying to the above equation?

To this purpose, we apply the definition of strain components and operate some differentiations; for instance:

$$\varepsilon_{11} = u_{1,1}, \quad \varepsilon_{22} = u_{2,2}, \quad 2\varepsilon_{12} = u_{1,2} + u_{2,1}, \tag{1.60}$$

that differentiated twice as

$$\varepsilon_{11,22} = u_{1,122}, \quad \varepsilon_{22,11} = u_{2,211}, \quad 2\varepsilon_{12,12} = u_{1,212} + u_{2,112} \tag{1.61}$$

and summed up give the condition

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12}. \tag{1.62}$$

In a similar way, we get also

$$\begin{aligned} \varepsilon_{11,33} + \varepsilon_{33,11} &= 2\varepsilon_{13,13}, \\ \varepsilon_{22,33} + \varepsilon_{33,22} &= 2\varepsilon_{23,23}. \end{aligned} \tag{1.63}$$

Again,

$$\varepsilon_{11} = u_{1,1}, \quad 2\varepsilon_{12} = u_{1,2} + u_{2,1}, \quad 2\varepsilon_{13} = u_{1,3} + u_{3,1}, \quad 2\varepsilon_{23} = u_{2,3} + u_{3,2}, \tag{1.64}$$

differentiated twice as

$$\begin{aligned}\varepsilon_{11,23} = u_{1,123}, \quad 2\varepsilon_{12,13} = u_{1,213} + u_{2,113}, \\ 2\varepsilon_{13,12} = u_{1,312} + u_{3,112}, \quad 2\varepsilon_{23,11} = u_{2,311} + u_{3,211},\end{aligned} \tag{1.65}$$

and summed up give

$$\varepsilon_{12,13} + \varepsilon_{13,12} = \varepsilon_{23,11} + \varepsilon_{11,23}, \tag{1.66}$$

and similarly, permutating the indices,

$$\begin{aligned}\varepsilon_{12,23} + \varepsilon_{23,12} = \varepsilon_{13,22} + \varepsilon_{22,13}, \\ \varepsilon_{13,23} + \varepsilon_{23,13} = \varepsilon_{12,33} + \varepsilon_{33,12}.\end{aligned} \tag{1.67}$$

The 6 equations (1.62), (1.63), (1.66) and (1.67) are the *Saint Venant-Beltrami compatibility equations*; they must be satisfied by any strain field $\boldsymbol{\varepsilon}$ for it is a *real strain field*, in the sense of deriving by a displacement field through eq. (1.59).

The Saint Venant-Beltrami equations can be written in a compact form:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0; \tag{1.68}$$

these are 81 equations, but only the 6 Saint Venant-Beltrami equations are not identities, as it can be checked with some work but without difficulty.

1.10 Exercises

1. Study the following *simple* (i.e. such that $\nabla\mathbf{u} = \nabla^\top\mathbf{u}$) deformations:

   a) *extension* of amount α in the direction $\mathbf{e}$, $|\mathbf{e}| = 1$:

   $$\mathbf{u}(p) = \alpha\ (\mathbf{e}\otimes\mathbf{e})(p - p_0);$$

   b) *shear* of amount β with respect to the orthogonal directions $\mathbf{e}_1, \mathbf{e}_2$, $|\mathbf{e}_1| = |\mathbf{e}_2| = 1$:

   $$\mathbf{u}(p) = \beta(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1)(p - p_0);$$

   c) *dilatation* of amount γ:

   $$\mathbf{u}(p) = \gamma(p - p_0),$$

   with $\alpha, \beta, \gamma \in \mathbb{R}$, $|\alpha|, |\beta|, |\gamma| \ll 1$. For each case:

   i. write $\boldsymbol{\varepsilon}$;

   ii. determine δv;

   iii. determine the change of volume of a cube with the sides parallel to the axes;

   iv. determine $\delta\ell$ and λ for the sides of such a cube;

   v. determine $\delta\theta$ for each couple of sides of the same cube;

   vi. calculate the principal strains;

vii. calculate the principal directions of strain.

2. Show that it is always possible to decompose $\boldsymbol{\varepsilon}$ into a dilatation and an isochoric combination of 3 extensions plus 3 shears (such a decomposition has important applications in the theory of strength of isotropic elastic materials).

3. For the displacement field

$$\mathbf{u}(p) = \alpha(X_2 + X_3)\mathbf{e}_1 + \alpha(X_1 + X_3)\mathbf{e}_2 + \beta(X_1 + X_2)\mathbf{e}_3, \ \alpha, \beta \in \mathbb{R},$$

   i. determine the conditions on α, β for this field describe an infinitesimal strain;

   ii. find $\boldsymbol{\varepsilon}$;

   iii. find the change in length and angle of the 3 vectors of the base;

   iv. decompose the deformation into a dilatation plus 3 extensions and 3 shears.

4. The deformation described in cylindrical coordinates by

$$r = R, \ \theta = \Theta + \alpha Z, \ z = Z, \ \alpha \in \mathbb{R},$$

   is called a *torsion*;

   i. justify why it is called so, studying the displacement field of a circular cylinder of axis Z;

   ii. calculate $\mathbf{F}$ and $\nabla\mathbf{u}$;

   iii. show that the transformation is isochoric;

   iv. determine the condition to be satisfied by α for the deformation to be infinitesimal;

   v. find $\boldsymbol{\varepsilon}$;

   vi. calculate the displacement field in the case of small strain;

   vii. calculate the change in length and angle of the vectors of the cylindrical base;

   viii. calculate the displacement field $\mathbf{u}$ in Cartesian coordinates and deduce from it $\nabla\mathbf{u}$ and $\boldsymbol{\varepsilon}$.

5. For the deformation described in spherical coordinates by

$$\begin{aligned} r &= R(1 - \alpha|\phi^2 - \pi\phi|), \\ \varphi &= \phi, \qquad\qquad\qquad \alpha \in \mathbb{R}, \\ \theta &= \Theta, \end{aligned}$$

   i. represent graphically a sphere after deformation, for both the cases of $\alpha < 0$ and $\alpha > 0$;

   ii. find the displacement field $\mathbf{u}$;

   iii. calculate $\nabla\mathbf{u}$ and $\mathbf{F}$;

iv. determine the conditions on α for the transformation be really a deformation;

v. determine the conditions on α for the transformation be an infinitesimal deformation;

vi. determine $\boldsymbol{\varepsilon}$;

vii. calculate the change in length and angle for the vectors of the spherical base;

viii. calculate the change in length and angle for a point on the polar axis and on the equatorial plane;

ix. calculate the global change of volume for a sphere of radius ρ for both the cases of finite and infinitesimal strain.

6. Show that, just for any other second-rank symmetric tensor, among the eigenvalues of $\boldsymbol{\varepsilon}$ there are the highest value, the lowest one and a value which is a stationary point, with respect to the direction, of the change in length of a fiber.

7. Show that

   i.
$$|\boldsymbol{\varepsilon}|^2 + |\boldsymbol{\omega}|^2 = |\nabla \mathbf{u}|^2;$$

   ii.
$$|\boldsymbol{\varepsilon}|^2 - |\boldsymbol{\omega}|^2 = \nabla \mathbf{u} \cdot \nabla^\top \mathbf{u}.$$

8. Be $\mathbf{u}$ of class at least C^2 and assume that $\mathbf{u} = \mathbf{o}$ on $\partial\Omega$. Then, show the *Korn's inequality*:
$$\int_\Omega |\nabla \mathbf{u}|^2 d\omega \leq 2 \int_\Omega |\boldsymbol{\varepsilon}|^2 d\omega.$$

9. A *plane strain* is a situation where
$$\mathbf{u} = u_i(x_1, x_2)\mathbf{e}_i,\ \ i = 1, 2;$$

   i. write $\boldsymbol{\varepsilon}$ for such a case;

   ii. show that the six equations of Saint Venant-Beltrami reduce to only one and write it.

10. Consider the change in length of a fiber $\mathbf{x} = \alpha\mathbf{e},\ |\mathbf{e}| = 1$,
$$\delta\ell = \frac{1}{\alpha^2} \mathbf{x} \cdot \boldsymbol{\varepsilon}\mathbf{x},$$
and the quadratic form
$$\mathbf{x} \cdot \boldsymbol{\varepsilon}\mathbf{x} = \pm k^2,\ \ k \in \mathbb{R}.$$
This defines a quadric, the *strain quadric of Cauchy*. Then,
$$\delta\ell = \pm\frac{k^2}{\alpha^2},$$

so the change in length of any fiber is inversely proportional to its square norm, i.e. to the square of the distance of the origin of the fiber from the quadric itself. Find the strain quadric for the cases of extension, shear and dilatation studied in exercise 1, and for a plane strain as defined in the previous exercise.

Chapter 2

Stress analysis

2.1 Forces

We are now concerned with *forces* as possible, though not unique, *agents of deformation*. About forces, we admit they are intuitively understood (we do not care here of their real, ultimate physical nature, of no importance for our context; it is sufficient for us to know that forces produce displacements and hence deformations) and that they are *represented by vectors*. There are different types of forces and it is important to understand that the *interior parts of a body* Ω *exchange forces between them*.

The general situation that we examine is that of a body Ω of which we consider a *material part* $\beta \subset \Omega$, with frontier $\partial\beta$ and outward unit normal $\mathbf{n}$, see Fig. 2.1. A material part is a subset of Ω composed by a set of material points, i.e., during deformation, the points remain exactly the same and their quantity is preserved.

Generally speaking, some forces act upon β and they can be of two types:

i. *volume* or *body* forces: these forces are directly applied to the material points in β for the simple reason that they exist. They are *remote* forces, result of the presence of one or more force fields: gravitational, electrostatic, magnetic etc. As such, these forces normally depend upon the position and they admit a *density*:

   - a *volume* density $\mathbf{b}=\mathbf{b}(p)$, or
   - a *mass* density $\mathbf{r}=\mathbf{r}(p) \rightarrow \mathbf{b}=\rho\ \mathbf{r}$,

   with ρ the volume mass (density of the matter). These forces are extensive quantities, so the total remote force acting upon β is

$$\mathbf{F}_\beta = \int_\beta \mathbf{b}\ dv = \int_\beta \rho\ \mathbf{r}\ dv; \tag{2.1}$$

ii. *surface* forces: these are the forces that Ω exchange with the environment, by contact through its boundary $\partial\Omega$, like pressure or thrusts exerted by some devices or other bodies, or the forces that β exchange with the rest of Ω still by contact through its frontier $\partial\beta$, called also *interior* forces; these last are the direct consequence of the same idea of continuum.

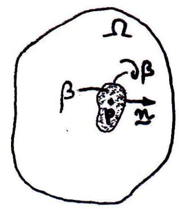

Figure 2.1: Material part.

The surface forces too admit a density, in this case of course a *surface density*, $\mathbf{t}$:

$$\mathbf{f}_\beta = \int_{\partial\beta} \mathbf{t} \, ds. \tag{2.2}$$

The density of surface forces $\mathbf{t}$ is called *traction* or *stress vector*. About $\mathbf{t}$, we admit the *Cauchy's postulate*: $\mathbf{t}$ is a function of the *actual* position and of the outward normal to $\partial\beta$:

$$\mathbf{t} = \mathbf{t}(p, \mathbf{n}). \tag{2.3}$$

The above statements deserve some remarks:

- there exist also attractive body forces that interior parts of a same body mutually exchange; such forces are neglected in the classical theory, but can be of course of an extreme importance in other fields, like astronomy and geophysics;
- the volume forces and the surface forces acting upon the boundary $\partial\Omega$ are *external* forces; they are considered to be *known*;
- the interior forces are *unknown* and to determine these last once the external forces known is the *major problem of continuum mechanics*;
- the Cauchy's postulate is a strong assumption: two different surfaces $\partial\beta_1$ and $\partial\beta_2$ sharing in p the same normal $\mathbf{n}$, share also the same traction $\mathbf{t}$; in particular, $\mathbf{t}$ does not depend upon the curvature of the surfaces in p;
- considering that through any point $p \in \partial\beta$ the matter exchanges only interior forces and not also interior couples is an implicit assumption that defines a class of materials, the so-called *classical continuum bodies à la Cauchy*; several classical materials can be well represented by this model, e.g. metallic alloys, wood, concrete etc, but not other ones, called *polar bodies*, like some polymers, for which the introduction of surface couples exchanged by interior parts of the body is necessary for a satisfactory description of its behavior; in this text, we will refer only to classical Cauchy bodies.

2.2 The Cauchy's theorem

The Cauchy's postulate does not specify in which way $\mathbf{t}$ is a function of $\mathbf{n}$. This is done by the

Theorem (Cauchy's theorem on stress)**.** *Traction* $\mathbf{t}$ *is a linear function of* $\mathbf{n}$*, i.e. it exists a second-rank tensor* $\boldsymbol{\sigma}$*, the Cauchy's stress tensor, such that*

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}. \tag{2.4}$$

Proof. Let us see the classical proof based on the use of the so-called *tetrahedron of Cauchy*. We consider at a point $p \in \Omega$ a tetrahedron like in Fig. 2.2, where p is the axes origin and the fourth face, whose normal is $\mathbf{n}$, is inclined with respect to the three faces passing by the axes. Be δ the distance of p from the inclined face. For δ sufficiently

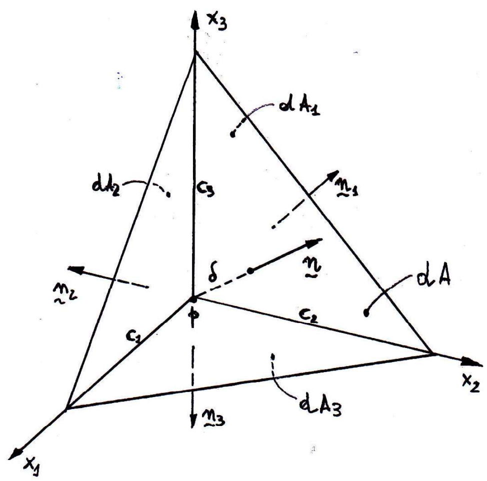

Figure 2.2: The tetrahedron of Cauchy.

small, all the tetrahedron is in Ω; be dA the area of the inclined face, with outward unit normal $\mathbf{n}$, while dA_i is the area of the face orthogonal to axis x_i, of outward unit normal $\mathbf{n}_i = -\mathbf{e}_i$. Be $\mathbf{t} = (t_1, t_2, t_3)$ the traction on the inclined surface and $\mathbf{b}$ the body force.

About the area of the surfaces of the tetrahedron, we know that[1]

$$dA_i = dA\ \mathbf{n} \cdot \mathbf{e}_i \ \ \forall i = 1, 2, 3, \tag{2.5}$$

[1]The result in eq. (2.5) is known as *theorem of the cosine for the surfaces.* To prove it, we name c_i the length of the side of the tetrahedron along the axis x_i; then

$$dA_i = \frac{1}{2} c_j c_k, \ \ i, j, k = 1, 2, 3, \ \ i \neq j \neq k,$$

and

$$dA = \frac{1}{2} |(-c_1, c_2, 0) \times (-c_1, 0, c_3)| = \frac{1}{2}\sqrt{c_1^2 c_2^2 + c_2^2 c_3^2 + c_3^2 c_1^2}.$$

The normal $\mathbf{n}$ to dA is given by

$$\mathbf{n} = \frac{(-c_1, c_2, 0) \times (-c_1, 0, c_3)}{|(-c_1, c_2, 0) \times (-c_1, 0, c_3)|} = \frac{1}{\sqrt{c_1^2 c_2^2 + c_2^2 c_3^2 + c_3^2 c_1^2}} (c_2 c_3, c_1 c_3, c_1 c_2)$$

so we get

$$\mathbf{n} \cdot \mathbf{e}_i = \frac{dA_i}{dA},$$

i.e. eq. (2.5).

while the volume of the tetrahedron is

$$dV = \frac{1}{3}\delta\ dA. \tag{2.6}$$

We write now the balance of the forces acting upon the tetrahedron, using the *Euler's axiom*: when a body Ω is in equilibrium, then all of its material parts β are in equilibrium. Then, imaging the tetrahedron as a separated part of Ω, it will be in equilibrium under the action of the body forces and of the surface (contact) forces that it exchanges with the rest of Ω through its four surfaces. This gives the balance equation:

$$\mathbf{t}\ dA + \mathbf{t}_i dA_i + \mathbf{b}\ dV = \mathbf{o}, \tag{2.7}$$

and, for the above formulae for the areas and volume we get, after dividing by dA,

$$\mathbf{t} + \mathbf{t}_i\ \mathbf{n}\cdot\mathbf{e}_i + \frac{1}{3}\mathbf{b}\ \delta = \mathbf{o}. \tag{2.8}$$

Hence, when $\delta \to 0$, the point p tends to the surface dA whose normal is $\mathbf{n}$ and the body forces vanish; because $\mathbf{n}_i = -\mathbf{e}_i$, we obtain

$$\mathbf{t} = -\mathbf{t}_i\ \mathbf{n}\cdot\mathbf{e}_i = -(\mathbf{t}_i \otimes \mathbf{e}_i)\mathbf{n} = (\mathbf{t}_i \otimes \mathbf{n}_i)\mathbf{n}. \tag{2.9}$$

We put

$$\boldsymbol{\sigma} = \mathbf{t}_i \otimes \mathbf{n}_i, \tag{2.10}$$

the *Cauchy's stress tensor in p*, and finally

$$\mathbf{t} = \boldsymbol{\sigma}\ \mathbf{n}. \tag{2.11}$$

$\square$

From eq. (2.9) we have also

$$\sigma_{ij} = \mathbf{e}_i\cdot(\mathbf{t}_k \otimes \mathbf{n}_k)\mathbf{e}_j = \mathbf{t}_k\cdot\mathbf{e}_i\ \mathbf{n}_k\cdot\mathbf{e}_j = (\mathbf{t}_k)_i(\mathbf{n}_k)_j. \tag{2.12}$$

Of course, if we take $\mathbf{n} = \mathbf{e}_i$, then $\mathbf{t} = \mathbf{t}_i$, as it must be. Just as for any other second rank tensor, given a base $e = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\boldsymbol{\sigma} = \sigma_{ij}\ \mathbf{e}_i \otimes \mathbf{e}_j, \tag{2.13}$$

with

$$\sigma_{ij} = \mathbf{e}_i\cdot\boldsymbol{\sigma}\ \mathbf{e}_j. \tag{2.14}$$

It is important to remark that $\boldsymbol{\sigma}$ is a function of the place and time, not of $\mathbf{n}$:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(p, t). \tag{2.15}$$

As already done, the dependence upon time, always existing, is left tacitly understood in the equations.

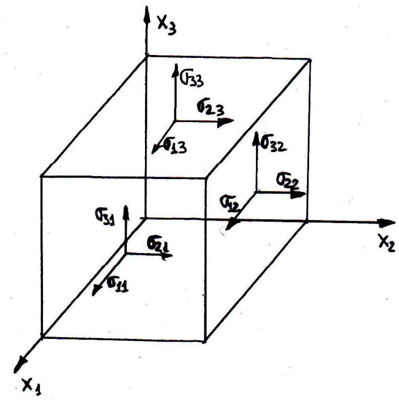

Figure 2.3: The components σ_{ij}.

2.3 Stress components

Let us apply the Cauchy's theorem to surface elements whose normal is parallel to one of the axes, $\mathbf{n} = \mathbf{e}_k$:

$$\mathbf{t}^{(k)} = \sigma_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_k = \sigma_{ij}\ \delta_{jk}\ \mathbf{e}_i = \sigma_{ik}\ \mathbf{e}_i, \tag{2.16}$$

so

$$\mathbf{t}^{(k)} = \sigma_{ik}\ \mathbf{e}_i = (\sigma_{1k}, \sigma_{2k}, \sigma_{3k}); \tag{2.17}$$

this result shows that the k-th column of the matrix representing $\boldsymbol{\sigma}$ in the base $e = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is composed by the Cartesian components of the traction acting upon the surface whose normal is $\mathbf{e}_k$. Graphically, the situation is depicted in Fig. 2.3. We remark the position of the indexes: the first one gives the direction of the component of the traction acting upon a surface whose normal is the axis indicated by the second index (e.g. σ_{13} is the component along x_1 of the traction acting upon a surface whose normal is $\mathbf{e}_3$).

To remark that the above nomenclature comes directly from the mere application of the equations; some authors chose to swap the indexes: in σ_{ij}, i is the direction of the normal to the surface upon which the traction acts, while j is the direction of the component σ_{ij} of the traction. This is not so important, because $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$, as we will see below.

Looking at Fig. 2.3, it is clear why:

- the components with equal indexes σ_{ii} are called *normal stresses*: they give the component of the traction upon a surface that is normal, i.e. perpendicular, to the same surface; because in eq. (2.4) $\mathbf{n}$ is the *outward* unit normal, a normal stress σ_{ii} is positive if it is a tension, negative if a compression; normal stresses form the diagonal of the matrix representing $\boldsymbol{\sigma}$;
- the components with different indexes σ_{ij}, $i \neq j$ are called *shear stresses*: they give a component of the traction upon a surface orthogonal to an axis that is tangential to the same surface; they are the out-of-diagonal components of the matrix representing $\boldsymbol{\sigma}$.

More generally, for each element of surface of unit normal $\mathbf{n}$, the traction $\mathbf{t} = \boldsymbol{\sigma}\ \mathbf{n}$ can be decomposed into two mutually orthogonal vectors, see Fig. 2.4:

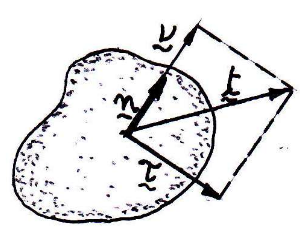

Figure 2.4: Normal, $\boldsymbol{\nu}$, and tangential, $\boldsymbol{\tau}$, stresses.

- the *normal stress* $\boldsymbol{\nu}$:

$$\boldsymbol{\nu} = (\mathbf{t}\cdot\mathbf{n})\mathbf{n} = (\mathbf{n}\otimes\mathbf{n})\mathbf{t} = (\mathbf{n}\otimes\mathbf{n})\boldsymbol{\sigma}\ \mathbf{n}; \tag{2.18}$$

- the *tangential stress* $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \mathbf{t} - \boldsymbol{\nu} = (\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\mathbf{t} = (\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\boldsymbol{\sigma}\ \mathbf{n}. \tag{2.19}$$

2.4 Balance equations

We can now write the balance equations for any part β of Ω. The Euler's axioms stipulate that $\forall\beta\subset\Omega$, the force resultant and the moment resultant are null. Let us start examining first the force resultant:

$$\int_\beta \mathbf{b}\ dv + \int_{\partial\beta}\mathbf{t}\ ds = \int_\beta \rho\ \ddot{p}\ dv\ \ \forall\beta\subset\Omega. \tag{2.20}$$

Applying the Cauchy's theorem we get

$$\int_\beta \mathbf{b} - \rho\ \ddot{p}\ dv + \int_{\partial\beta}\boldsymbol{\sigma}\ \mathbf{n}\ ds = \mathbf{o}\ \ \forall\beta\subset\Omega, \tag{2.21}$$

and for the tensor form of the Gauss theorem

$$\int_\beta (\mathbf{b} - \rho\ \ddot{p} + \mathrm{div}\boldsymbol{\sigma})\ dv = \mathbf{o}\ \ \forall\beta\subset\Omega. \tag{2.22}$$

The only possibility for this integral to be null $\forall\beta\subset\Omega$ is the integrand to be identically null:

$$\mathbf{b} + \mathrm{div}\boldsymbol{\sigma} = \rho\ \ddot{p}\quad \forall p\in\Omega. \tag{2.23}$$

These are the *Cauchy-Poisson equations of motion for classical continuum bodies*. They generalize to each point of a deformable body the second principle of dynamics of Newton. In case of equilibrium, $\ddot{p} = \mathbf{o}$ and we obtain the *equilibrium equations*

$$\mathbf{b} + \mathrm{div}\boldsymbol{\sigma} = \mathbf{o}\quad \forall p\in\Omega. \tag{2.24}$$

In terms of components, the above equations read like

$$b_i + \sigma_{ij,j} = \rho\, \ddot{p}_i, \quad i, j = 1, 2, 3. \tag{2.25}$$

Let us now turn the attention on the moment resultant on β:

$$\int_\beta (p - o) \times \mathbf{b}\ dv + \int_{\partial\beta} (p - o) \times \mathbf{t}\ ds = \int_\beta \rho (p - o) \times \ddot{p}\ dv\ \ \forall \beta \subset \Omega. \tag{2.26}$$

Still using the Cauchy's theorem we get

$$\int_\beta (p - o) \times (\mathbf{b} - \rho\, \ddot{p})\ dv + \int_{\partial\beta} (p - o) \times (\boldsymbol{\sigma}\mathbf{n})\ ds = \mathbf{o}\ \ \forall \beta \subset \Omega \tag{2.27}$$

and introducing, first, the axial tensor $\mathbf{W}$ of $(p - o)$

$$\int_\beta \mathbf{W}(\mathbf{b} - \rho\, \ddot{p})\ dv + \int_{\partial\beta} \mathbf{W}(\boldsymbol{\sigma}\mathbf{n})\ ds = \mathbf{o}\ \ \forall \beta \subset \Omega, \tag{2.28}$$

then the motion equation and the Gauss theorem, we obtain

$$\int_\beta \mathrm{div}(\mathbf{W}\boldsymbol{\sigma}) - \mathbf{W}\mathrm{div}\boldsymbol{\sigma}\ dv = \mathbf{o}\ \ \forall \beta \subset \Omega, \tag{2.29}$$

that, for being true $\forall \beta \subset \Omega$, gives the condition

$$\mathrm{div}(\mathbf{W}\boldsymbol{\sigma}) = \mathbf{W}\mathrm{div}\boldsymbol{\sigma}\ \ \forall p \in \Omega. \tag{2.30}$$

We now develop:

$$\begin{aligned} \mathrm{div}(\mathbf{W}\boldsymbol{\sigma}) &= (\mathbf{W}\boldsymbol{\sigma})_{ij,j}\ \mathbf{e}_i = (W_{ik}\ \sigma_{kj})_{,j}\ \mathbf{e}_i \\ &= W_{ik,j}\ \sigma_{kj}\ \mathbf{e}_i + W_{ik}\ \sigma_{kj,j}\ \mathbf{e}_i = W_{ik,j}\ \sigma_{kj}\ \mathbf{e}_i + \mathbf{W}\ \mathrm{div}\boldsymbol{\sigma}, \end{aligned} \tag{2.31}$$

and injecting this result into eq. (2.30) gives

$$W_{ik,j}\sigma_{kj} = 0\ \forall i = 1, 2, 3. \tag{2.32}$$

For a generic point $p = (p_1, p_2, p_3) \in \Omega$,

$$\mathbf{W} = \left[\begin{array}{ccc} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{array}\right], \tag{2.33}$$

so that $W_{12,3} = -1, W_{13,2} = 1$ etc. Injecting these results into eq. (2.32) for $i = 1, 2, 3$ gives

$$\begin{array}{l} i = 1\ \rightarrow\ \sigma_{23} = \sigma_{32}, \\ i = 2\ \rightarrow\ \sigma_{13} = \sigma_{31}, \qquad \Rightarrow \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^\top. \\ i = 3\ \rightarrow\ \sigma_{12} = \sigma_{21}, \end{array} \tag{2.34}$$

So, for classical continuum bodies, the balance of the couples corresponds to the symmetry of $\boldsymbol{\sigma}$.

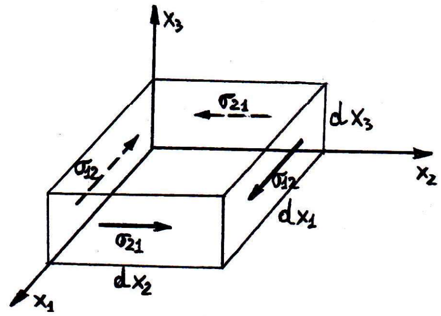

Figure 2.5: Reciprocity of the shear stresses.

There are at least two other ways to prove the *reciprocity of the shear stresses*, i.e. the symmetry of $\boldsymbol{\sigma}$, both of them more *mechanical* then the previous one. In the first one, we consider a parallelepiped with the faces parallel to the axes, like in Fig. 2.5. If, e.g., we focus on the balance of the torque around axis x_3, body forces and tractions on the horizontal faces give higher order contributions and can be discarded, so we have

$$(\sigma_{21}+\frac{\partial\sigma_{21}}{\partial x_1}dx_1)dx_1dx_2dx_3=(\sigma_{12}+\frac{\partial\sigma_{12}}{\partial x_2}dx_2)dx_1dx_2dx_3, \tag{2.35}$$

and neglecting higher order terms we get $\sigma_{12}=\sigma_{21}$; in a similar way we obtain also $\sigma_{13}=\sigma_{31}$ and $\sigma_{23}=\sigma_{32}$.

The other method to prove the symmetry of $\boldsymbol{\sigma}$ is based upon the use of the classical *Principle of Virtual Displacements*[2]: for each possible infinitesimal rigid displacement field $\mathbf{w}$, the balance equations are satisfied if and only if

$$\int_{\partial\beta}\mathbf{t}\cdot\mathbf{w}\;ds+\int_{\beta}(\mathbf{b}-\rho\,\ddot{p})\cdot\mathbf{w}\;dv=0. \tag{2.36}$$

In fact, using the Cauchy's and Gauss's theorems we have

$$\begin{aligned}\int_{\partial\beta}\mathbf{t}\cdot\mathbf{w}\;ds&=\int_{\partial\beta}\boldsymbol{\sigma}\,\mathbf{n}\cdot\mathbf{w}\;ds=\int_{\partial\beta}\boldsymbol{\sigma}^\top\mathbf{w}\cdot\mathbf{n}\;ds\\&=\int_{\beta}\operatorname{div}(\boldsymbol{\sigma}^\top\mathbf{w})dv=\int_{\beta}(\mathbf{w}\cdot\operatorname{div}\boldsymbol{\sigma}+\boldsymbol{\sigma}\cdot\nabla\mathbf{w})dv.\end{aligned} \tag{2.37}$$

Using the equation of mouvement (2.23) for expressing div$\boldsymbol{\sigma}$, we have

$$\int_{\partial\beta}\mathbf{t}\cdot\mathbf{w}\;ds+\int_{\beta}(\mathbf{b}-\rho\,\ddot{p})\cdot\mathbf{w}\;dv=\int_{\beta}\boldsymbol{\sigma}\cdot\nabla\mathbf{w}\;dv\quad\forall\beta\subset\Omega. \tag{2.38}$$

The left-hand member is null for a body at equilibrium, for the Principle of Virtual Displacements; so, because the above equation must be satisfied $\forall\beta\subset\Omega$, we obtain the condition

$$\boldsymbol{\sigma}\cdot\nabla\mathbf{w}=0\quad\forall p\in\Omega, \tag{2.39}$$

[2]The Principle of Virtual Displacements as used here is just the principle as known, usually, for rigid bodies mechanics; the key point for the principle in this form is the virtual displacement field to be infinitesimal and rigid; in such a circumstance, as used here, the principle is exactly the same used in classical rigid mechanics. A more general form of the Principle of Virtual Displacements exists for deformable bodies, it is presented in Section 2.7.