Indian Institute of Technology Bhubaneswar

School of Infrastructure

Subject Name : Solid Mechanics	Subject Code: CE2L001

Tutorial No. 4

1. Different simple cases of transformations are illustrated in Fig. 1, where α and β are arbitrary scalar positive values. Consider the two-dimensional context and determine the deformation map, $\boldsymbol{\chi}(\boldsymbol{x})$, deformation gradient, $\boldsymbol{F}$, the strain tensor, $\boldsymbol{E} = (1/2)(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$, displacement vector, $\boldsymbol{u}$ and the linearized strain tensor, $\boldsymbol{E}_s$, for each case.

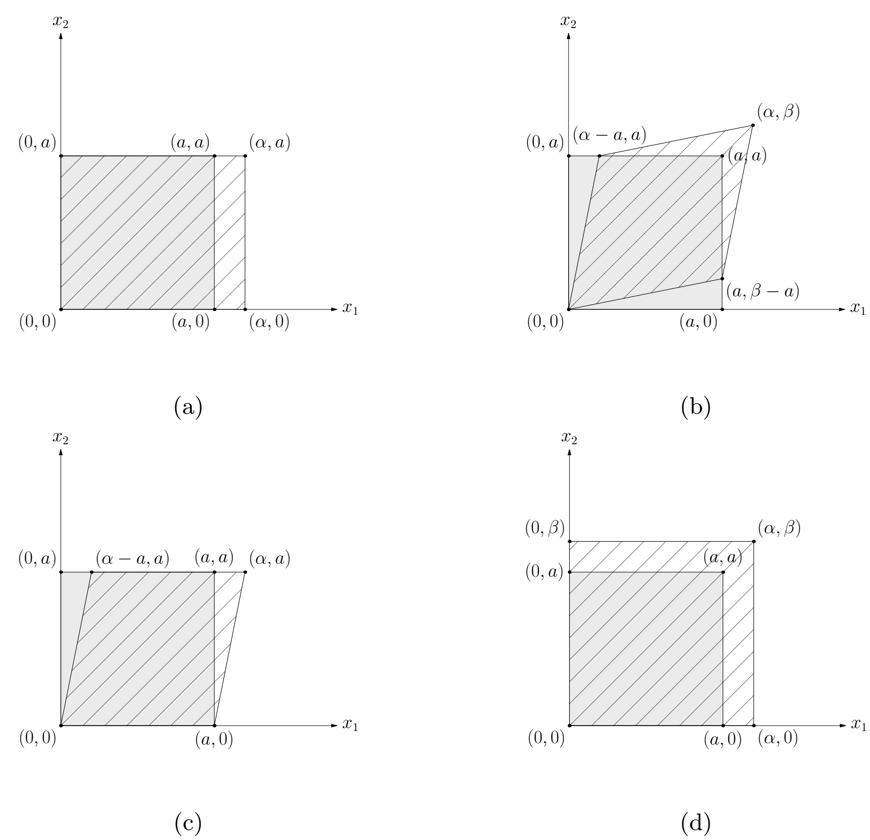

Figure 1: The undeformed and deformed configurations of a body under different cases of transformations with a, α, β denoting arbitrary scalar positive constants. The hatched area illustrates the deformed configurations.

Solution: Consider the displacement components to be a function of the coordinates x_1 and x_2. Then, a general expression for the displacement components can be written as

$$
\begin{aligned}
u_1(x_1, x_2) &= a + bx_1 + cx_2 + dx_1x_2 \\
u_2(x_1, x_2) &= p + qx_1 + rx_2 + sx_1x_2
\end{aligned}
$$

(a) From the deformed shape of the solid block in Fig.1 (a), the displacement along x_1 at each four corners of the solid block can be written as

$$
\begin{aligned}
u_1(0,0) &= a = 0 \\
u_1(a,0) &= a + ba = \alpha - a \implies b = (\alpha - a)/a \\
u_1(0,a) &= a + ca = 0 \implies c = 0 \\
u_1(a,a) &= a + ba + ca + da^2 = \alpha - a \\
&= (\alpha - a) + da^2 = (\alpha - a) \\
\implies d &= 0
\end{aligned} \tag{1}
$$

The displacement along x_2 and x_3 at each four corners of the solid block is zero. Thus the expression for the displacement components can be written as

$$
\begin{aligned}
u_1(x_1, x_2) &= \frac{(\alpha - a)}{a} x_1 \\
u_2(x_1, x_2) &= 0 \\
u_3(x_1, x_2) &= 0
\end{aligned} \tag{2}
$$

The linear elasticity tensor $\boldsymbol{F}$ can be written as

$$
\boldsymbol{E}_s = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)
$$

The components of the strain tensor can be obtained as

$$
\begin{aligned}
E_{11} &= \frac{\partial u_1}{\partial x_1} \\
E_{22} &= \frac{\partial u_2}{\partial x_2} \\
E_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)
\end{aligned} \tag{3}
$$

Thus from Eq.(2) and Eq.(3), the components of strain can be written as

$$
\begin{aligned}
E_{11} &= \frac{(\alpha - a)}{a} \\
E_{22} &= 0 \\
E_{12} &= 0
\end{aligned} \tag{4}
$$

Thus the strain tensor in component form can be written as

$$[\boldsymbol{E}] = \begin{bmatrix} \frac{(\alpha - a)}{a} & 0 \\ 0 & 0 \end{bmatrix} \tag{5}$$

(b) From the deformed shape of the solid block in Fig.1(b), the displacement along x_1 at each four corners of the solid block can be written as

$$\begin{aligned} u_1(0,0) &= a = 0 \\ u_1(a,0) &= a + ba = 0 \implies b = 0 \\ u_1(0,a) &= a + ca = \alpha - a \implies c = (\alpha - a)/a \\ u_1(a,a) &= a + ba + ca + da^2 = \alpha - a \\ &= (\alpha - a) + da^2 = (\alpha - a) \\ \implies d &= 0 \end{aligned} \tag{6}$$

The displacement along x_2 at each four corners of the solid block can be written as

$$\begin{aligned} u_2(0,0) &= p = 0 \\ u_2(a,0) &= p + qa = \beta - a \implies q = (\beta - a)/a \\ u_2(0,a) &= p + ra = 0 \implies r = 0 \\ u_2(a,a) &= p + qa + ra + sa^2 = \beta - a \\ &= (\beta - a) + sa^2 = (\beta - a) \\ \implies s &= 0 \end{aligned} \tag{7}$$

The displacement along x_3 at each four corners of the solid block is zero. Thus the expression for the displacement components can be written as

$$\begin{aligned} u_1(x_1, x_2) &= \frac{(\alpha - a)}{a} x_2 \\ u_2(x_1, x_2) &= \frac{(\beta - a)}{a} x_1 \\ u_3(x_1, x_2) &= 0 \end{aligned} \tag{8}$$

The linear elasticity tensor $\boldsymbol{F}$ can be written as

$$\boldsymbol{E}_s = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$$

The components of the strain tensor can be obtained as

$$\begin{aligned} E_{11} &= \frac{\partial u_1}{\partial x_1} \\ E_{22} &= \frac{\partial u_2}{\partial x_2} \end{aligned}$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \tag{9}$$

Thus from Eq.(8) and Eq.(9), the components of strain can be written as

$$\begin{aligned} E_{11} &= 0 \\ E_{22} &= 0 \\ E_{12} &= \frac{1}{2a}((\alpha - a) + (\beta - a)) \end{aligned} \tag{10}$$

Thus the strain tensor in component form can be written as

$$[\boldsymbol{E}] = \begin{bmatrix} 0 & \frac{1}{2a}((\alpha - a) + (\beta - a)) \\ \frac{1}{2a}((\alpha - a) + (\beta - a)) & 0 \end{bmatrix} \tag{11}$$

(c) From the deformed shape of the solid block in Fig.1(c), the displacement along x_1 at each four corners of the solid block can be written as

$$\begin{aligned} u_1(0,0) &= a = 0 \\ u_1(a,0) &= a + ba = 0 \implies b = 0 \\ u_1(0,a) &= a + ca = (\alpha - a) \implies c = (\alpha - a)/a \\ u_1(a,a) &= a + ba + ca + da^2 = \alpha - a \\ &= (\alpha - a) + da^2 = (\alpha - a) \\ \implies d = 0 & \end{aligned} \tag{12}$$

The displacement along x_2 and x_3 at each four corners of the solid block is zero. Thus the expression for the displacement components can be written as

$$\begin{aligned} u_1(x_1, x_2) &= \frac{(\alpha - a)}{a} x_2 \\ u_2(x_1, x_2) &= 0 \end{aligned}$$

The linear elasticity tensor $\boldsymbol{F}$ can be written as

$$\boldsymbol{E}_s = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$$

The components of the strain tensor can be obtained as

$$\begin{aligned} E_{11} &= \frac{\partial u_1}{\partial x_1} \\ E_{22} &= \frac{\partial u_2}{\partial x_2} \\ E_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \end{aligned} \tag{13}$$

Thus from Eq.(??) and Eq.(13), the components of strain can be written as

$$\begin{aligned} E_{11} &= 0 \\ E_{22} &= 0 \\ E_{12} &= \frac{(\alpha - a)}{a} \end{aligned} \tag{14}$$

Thus the strain tensor in component form can be written as

$$[\boldsymbol{E}] = \begin{bmatrix} 0 & \frac{(\alpha-a)}{a} \\ \frac{(\alpha-a)}{a} & 0 \end{bmatrix} \tag{15}$$

(d) From the deformed shape of the solid block in Fig.1(d), the displacement along x_1 at each four corners of the solid block can be written as

$$\begin{aligned} u_1(0,0) &= a = 0 \\ u_1(a,0) &= a + ba = \alpha - a \implies b = (\alpha - a)/a \\ u_1(0,a) &= a + ca = 0 \implies c = 0 \\ u_1(a,a) &= a + ba + ca + da^2 = \alpha - a \\ &= (\alpha - a) + da^2 = (\alpha - a) \\ \implies d &= 0 \end{aligned} \tag{16}$$

The displacement along x_2 at each four corners of the solid block can be written as

$$\begin{aligned} u_2(0,0) &= p = 0 \\ u_2(a,0) &= p + qa = 0 \implies b = 0 \\ u_2(0,a) &= p + ra = \beta - a \implies r = (\beta - a)/a \\ u_2(a,a) &= p + qa + ra + sa^2 = \beta - a \\ &= (\beta - a) + sa^2 = (\beta - a) \\ \implies s &= 0 \end{aligned} \tag{17}$$

The displacement along x_3 at each four corners of the solid block is zero. Thus the expression for the displacement components can be written as

$$\begin{aligned} u_1(x_1,x_2) &= \frac{(\alpha - a)}{a} x_1 \\ u_2(x_1,x_2) &= \frac{(\beta - a)}{a} x_2 \\ u_3(x_1,x_2) &= 0 \end{aligned} \tag{18}$$

The linear elasticity tensor $\boldsymbol{F}$ can be written as

$$\boldsymbol{E}_s = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$$

The components of the strain tensor can be obtained as

$$\begin{aligned} E_{11} &= \frac{\partial u_1}{\partial x_1} \\ E_{22} &= \frac{\partial u_2}{\partial x_2} \\ E_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \end{aligned} \tag{19}$$

Thus from Eq.(18) and Eq.(19), the components of strain can be written as

$$\begin{aligned} E_{11} &= (\alpha - a)/a \\ E_{22} &= (\beta - a)/a \\ E_{12} &= 0 \end{aligned} \tag{20}$$

Thus the strain tensor in component form can be written as

$$[\boldsymbol{E}] = \begin{bmatrix} (\alpha - a)/a & 0 \\ 0 & (\beta - a)/a \end{bmatrix} \tag{21}$$

2. Consider a homogeneous deformation corresponds to a strain field where the strain is the same at all points in a material body. Consider a prismatic, uniform thickness bar of initial length l_o undergoing a homogeneous deformation as shown in Fig. 2. Determine (a) deformation gradient, $\boldsymbol{F}$, (b) finite strain, $\boldsymbol{E}$ and (c) linearized strain, $\boldsymbol{E}_s$ (d) Demonstrate how the finite strain component reduces to linearized strain component through a numerical example.

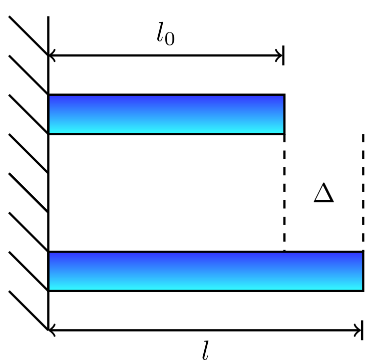

Figure 2: Undeformed and deformed element in the homogenous strain field in the bar.

Solution: (a) From Fig. 2, one can write that

$$\boldsymbol{F}\,(l_0\,\boldsymbol{e}_1) = l\,\boldsymbol{e}_1 \implies \boldsymbol{F}\,\boldsymbol{e}_1 = \left(\frac{l}{l_o}\right)\boldsymbol{e}_1, \qquad \boldsymbol{F}\,\boldsymbol{e}_2 = \boldsymbol{e}_2, \qquad \boldsymbol{F}\,\boldsymbol{e}_3 = \boldsymbol{e}_3. \tag{22}$$

One can find the deformation gradient, $\boldsymbol{F}$, using any one of the following methods.

Method I: For $\boldsymbol{F} = F_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, with $\{\boldsymbol{e}_i\}$ denoting the orthonormal basis, one can find the components as

$$F_{ij} = \boldsymbol{e}_i \cdot (\boldsymbol{F}\boldsymbol{e}_j) \tag{23}$$

By using Eq. (22) and Eq. (23),

$$F_{11} = \boldsymbol{e}_1 \cdot (\boldsymbol{F}\,\boldsymbol{e}_1) = l/l_0, \qquad F_{12} = \boldsymbol{e}_1 \cdot (\boldsymbol{F}\,\boldsymbol{e}_2) = 0, \qquad F_{13} = \boldsymbol{e}_1 \cdot (\boldsymbol{F}\,\boldsymbol{e}_3) = 0.$$

$$F_{21} = \boldsymbol{e}_2 \cdot (\boldsymbol{F}\,\boldsymbol{e}_1) = 0, \qquad F_{22} = \boldsymbol{e}_2 \cdot (\boldsymbol{F}\,\boldsymbol{e}_2) = 1, \qquad F_{23} = \boldsymbol{e}_2 \cdot (\boldsymbol{F}\,\boldsymbol{e}_3) = 0,$$

$$F_{31} = \boldsymbol{e}_3 \cdot (\boldsymbol{F}\,\boldsymbol{e}_1) = 0, \qquad F_{32} = \boldsymbol{e}_3 \cdot (\boldsymbol{F}\,\boldsymbol{e}_2) = 0, \qquad F_{33} = \boldsymbol{e}_3 \cdot (\boldsymbol{F}\,\boldsymbol{e}_3) = 1.$$

Hence,

$$\boldsymbol{F} = \left(\frac{l}{l_0}\right)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

Method II: From Eq. (22), one can write that

$$\boldsymbol{F}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 = \left(\frac{l}{l_0}\right)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \qquad \boldsymbol{F}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 = \boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \qquad \boldsymbol{F}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 = \boldsymbol{e}_3 \otimes \boldsymbol{e}_3.$$

Adding all the above three components,

$$\begin{aligned}
\boldsymbol{F}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3) &= \left(\frac{l}{l_0}\right)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \\
\implies \boldsymbol{F}\boldsymbol{I} &= \left(\frac{l}{l_0}\right)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \quad (\text{since } \boldsymbol{I} = \boldsymbol{e}_i \otimes \boldsymbol{e}_i) \\
\implies \boldsymbol{F} &= \left(\frac{l}{l_0}\right)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \quad (\text{since } \boldsymbol{F}\boldsymbol{I} = \boldsymbol{F})
\end{aligned}$$

Method III: A general expression for the displacement components in a one-dimensional (1D) bar can be written as,

$$x_1'(x_1) = a + bx_1, \qquad x_2'(x_J) = x_2, \qquad x_3'(x_1) = x_3.$$

Applying boundary conditions,

$$x_1'(0) = 0 \implies a = 0.$$
$$x_1'(l_0) = l \implies b\,l_0 = l \implies b = \frac{l}{l_0}.$$

Hence,

$$x_1' = \left(\frac{l}{l_0}\right) x_1, \quad x_2'(x_1) = x_2, \quad x_3'(x_1) = x_3.$$

The deformation gradient, $\boldsymbol{F}$, can be written in its component form as

$$[\boldsymbol{F}] = \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} & \frac{\partial x_1'}{\partial x_3} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} & \frac{\partial x_2'}{\partial x_3} \\ \frac{\partial x_3'}{\partial x_1} & \frac{\partial x_3'}{\partial x_2} & \frac{\partial x_3'}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \left(\frac{l}{l_0}\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1. \end{bmatrix}$$

Hence, $\boldsymbol{F} = \left(\frac{l}{l_0}\right) \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + 1\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + 1\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$.

(b) Finite Cauchy-Green strain, $\boldsymbol{E} = (1/2)\left(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I}\right)$ can be computed using its component form as follows.

$$[\boldsymbol{F}]^T[\boldsymbol{F}] = \begin{bmatrix} \left(\frac{l}{l_0}\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{l}{l_0}\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \left(\frac{l}{l_0}\right)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\boldsymbol{E}] = \frac{1}{2}\left(\begin{bmatrix} \left(\frac{l}{l_0}\right)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} \left(\frac{l^2 - l_0^2}{2l_0^2}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $\boldsymbol{E} = \left(\frac{l^2-l_0^2}{2l_0^2}\right) \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 = \epsilon_{11}^{\text{Cauchy}}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1$, where $\epsilon_{11}^{\text{Cauchy}} = \frac{l^2-l_0^2}{2l_0^2}$. (c) Linearized strain, $\boldsymbol{E}_s = \frac{1}{2}\left(\boldsymbol{F}^T + \boldsymbol{F} - 2\boldsymbol{I}\right)$ can be computed using its component form as

$$[\boldsymbol{E}_s] = \frac{1}{2}\left([\boldsymbol{F}^T] + [\boldsymbol{F}] - 2[\boldsymbol{I}]\right).$$

For the present problem, $[\boldsymbol{F}^T] = [\boldsymbol{F}]$.

$$[\boldsymbol{E}_s] = \frac{1}{2}\left(\begin{bmatrix} 2\left(\frac{l}{l_0}\right) & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) = \begin{bmatrix} \left(\frac{l-l_0}{l_0}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $\boldsymbol{E}_s = \left(\frac{l-l_0}{l_0}\right) \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 = \epsilon_{11}^{\text{eng}}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1$, where engineering strain component, $\epsilon_{11}^{\text{eng}} = \left(\frac{l-l_o}{l_o}\right)$. One can see how the Cauchy-Green finite strain component reduces to a linearized strain component as

$$\begin{aligned} \epsilon_{11}^{\text{Cauchy}} &= \frac{l^2 - l_0^2}{2l_0^2} = \frac{(l-l_0)(l+l_0)}{2l_0^2} \\ &= \frac{(l-l_0)(2l_0)}{2l_0^2} \quad (\text{ if } \Delta \text{ is very less, then } l + l_0 \approx 2\, l_0) \\ &= \left(\frac{l-l_0}{l_0}\right) = \epsilon_{11}^{\text{eng}} \end{aligned}$$

(d) Let's take $l_o = 10$ mm and $l = 15$ mm, and find the Engineering and Cauchy-Green Strain.

Engineering strain component,

$$\epsilon_{11}^{\text{eng}} = \left(\frac{l - l_o}{l_o}\right) = \frac{15 - 10}{10} = 0.5$$

.

Cauchy-Green finite strain component,

$$\epsilon_{11}^{\text{Cauchy}} = \frac{1}{2}\left(\frac{l^2 - l_o^2}{l_o^2}\right) = \frac{1}{2}\left(\frac{(15)^2 - (10)^2}{(10)^2}\right) = 0.625$$

.

Observation- In case of large deformation (Δ is significantly large), $\epsilon_{11}^{\text{eng}}$ will be significantly different than $\epsilon_{11}^{\text{Cauchy}}$. As seen from Fig. 3, both the strain components may provide identical results only when the deformation is minimal.

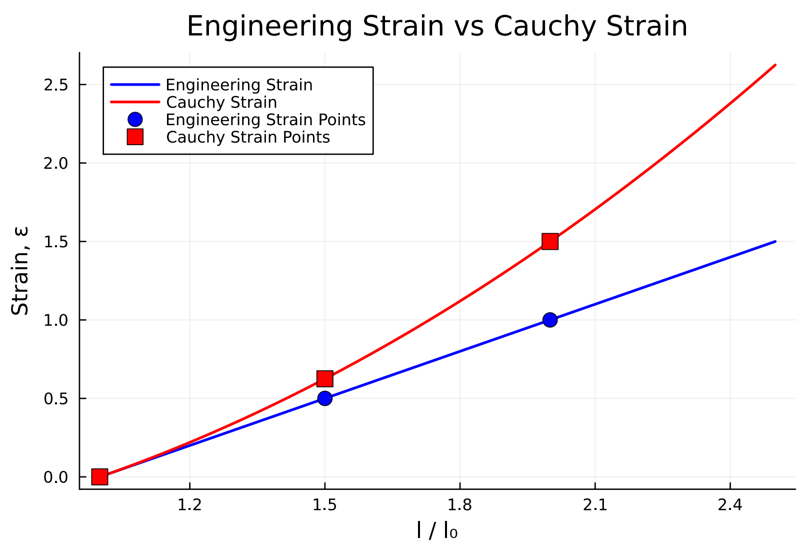

Figure 3: Relationship between engineering strain and Cauchy-Green strain component.

3. Consider a two-dimensional (2D) square infinitesimal element in the $x_1 - x_2$ plane as shown in Fig. 4. The displacement field within the element is defined as,

$$[\boldsymbol{u}] = \begin{bmatrix} 0.1x_1 + 0.2x_2 \\ 0.2x_2 \end{bmatrix}.$$

(a) Plot the displacement field. (b) Compute the divergence of the displacement field $\boldsymbol{u}$. (c) Determine the strain $\boldsymbol{E}$ and linearized strain $\boldsymbol{E}_s$.

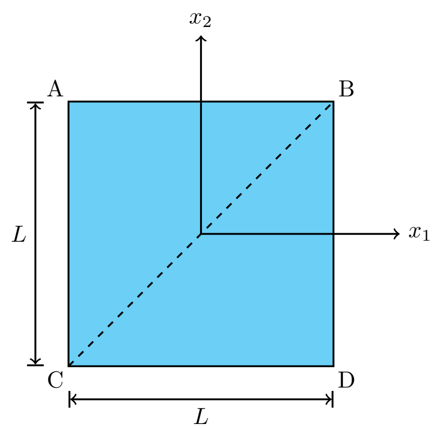

Figure 4: Inhomogeneous strain field

Solution: (a) One can plot the displacement field $\boldsymbol{u}$ considering different points such as $[1,1],[-1,1],[-1,-1],[-1,1],[1,0],[-1,0],[0,1],[0,-1]$ etc.

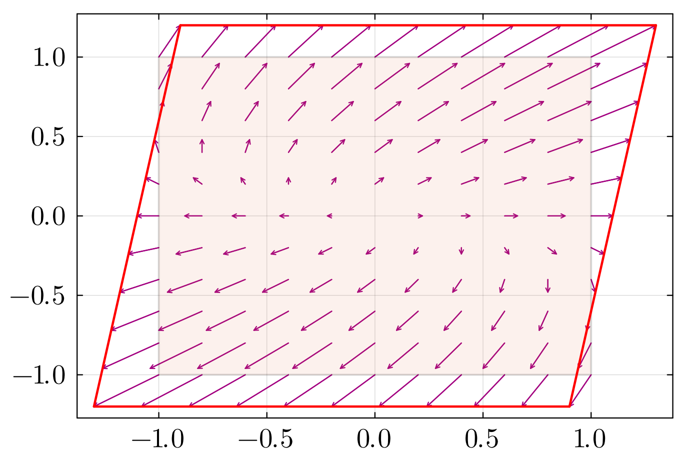

Figure 5: Resulting displacement field is shown.

(b) The divergence of the given vector field can be determined as

$$\begin{aligned}\nabla\cdot\boldsymbol{u} &= \boldsymbol{e}_i\left(\frac{\partial}{\partial x_i}\right)\cdot(u_j\boldsymbol{e}_j)\\ &= \frac{\partial u_j}{\partial x_i}\delta_{ij} \qquad (\text{since } \boldsymbol{e}_i\cdot\boldsymbol{e}_j=\delta_{ij})\\ \implies \nabla\cdot\boldsymbol{u} &= \frac{\partial u_i}{\partial x_i}.\end{aligned}$$

Thus, for the given problem, the divergence of a vector field can be determined as

$$\nabla\cdot\boldsymbol{u} = \frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}$$

$$= 0.1 + 0.2$$
$$\implies \nabla \cdot \boldsymbol{u} = 0.3.$$

(c) The finite strain tensor, $\boldsymbol{E}$, can be expressed as

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I}),$$

where $\boldsymbol{F} = \nabla \boldsymbol{u} + \boldsymbol{I}$, which can be determined as

$$\begin{aligned} \boldsymbol{F} &= \nabla \boldsymbol{u} + \boldsymbol{I} \\ &= \left(\frac{\partial u_i}{\partial x_j} + \delta_{ij}\right) \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ \implies F_{ij} &= \frac{\partial u_i}{\partial x_j} + \delta_{ij}. \end{aligned}$$

Thus the components of $\boldsymbol{F}$ can be determined as

$$[\boldsymbol{F}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} + 1 & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} + 1 \end{bmatrix} = \begin{bmatrix} 1.1 & 0.2 \\ 0.0 & 1.2 \end{bmatrix}.$$

The strain, $\boldsymbol{E}$, can be computed using its component form as

$$\begin{aligned} [\boldsymbol{E}] &= \frac{1}{2}\left([\boldsymbol{F}]^T [\boldsymbol{F}] - [\boldsymbol{I}]\right) \\ &= \frac{1}{2}\left(\begin{bmatrix} 1.1 & 0.0 \\ 0.2 & 1.2 \end{bmatrix} \begin{bmatrix} 1.1 & 0.2 \\ 0.0 & 1.2 \end{bmatrix} - \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}\right) = \begin{bmatrix} 0.105 & 0.11 \\ 0.11 & 0.24 \end{bmatrix}. \end{aligned}$$

The linearized strain, $\boldsymbol{E}_s = \frac{1}{2}(\boldsymbol{F}^T + \boldsymbol{F} - 2\boldsymbol{I})$, can be computed using its component form as

$$\begin{aligned} [\boldsymbol{E}_s] &= \frac{1}{2}([\boldsymbol{F}^T] + [\boldsymbol{F}] - 2[\boldsymbol{I}]) \\ &= \frac{1}{2}\left(\begin{bmatrix} 1.1 & 0.0 \\ 0.2 & 1.2 \end{bmatrix} + \begin{bmatrix} 1.1 & 0.2 \\ 0.0 & 1.2 \end{bmatrix} - \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 2.0 \end{bmatrix}\right) = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}. \end{aligned}$$

4. Consider the motion given by $[\boldsymbol{F}] = \begin{bmatrix} \lambda(t) & 0 & 0 \\ 0 & \lambda(t) & 0 \\ 0 & 0 & \lambda(t) \end{bmatrix}$, where $\lambda(t)$ is a time-dependent function. Determine the values of $\lambda(t)$ for which the motion is a rigid body motion.

Solution: A motion is a rigid body motion if it preserves the distance between any two material points. This is true if and only if the deformation gradient $\mathbf{F}$ is an orthogonal tensor, which requires the right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ to be equal to the identity tensor $\mathbf{I}$.

The deformation gradient tensor $\mathbf{F}$ is given in matrix form as:

$$[\boldsymbol{F}] = \begin{bmatrix} \lambda(t) & 0 & 0 \\ 0 & \lambda(t) & 0 \\ 0 & 0 & \lambda(t) \end{bmatrix} = \lambda(t)[\mathbf{I}]$$

The transpose of $\mathbf{F}$ is identical:

$$[\boldsymbol{F}]^T = \begin{bmatrix} \lambda(t) & 0 & 0 \\ 0 & \lambda(t) & 0 \\ 0 & 0 & \lambda(t) \end{bmatrix}$$

The right Cauchy-Green deformation tensor $\mathbf{C}$ is $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. In matrix form:

$$\begin{aligned} [\mathbf{C}] = [\boldsymbol{F}]^T[\boldsymbol{F}] &= \begin{bmatrix} \lambda(t) & 0 & 0 \\ 0 & \lambda(t) & 0 \\ 0 & 0 & \lambda(t) \end{bmatrix} \begin{bmatrix} \lambda(t) & 0 & 0 \\ 0 & \lambda(t) & 0 \\ 0 & 0 & \lambda(t) \end{bmatrix} \\ &= \begin{bmatrix} \lambda^2(t) & 0 & 0 \\ 0 & \lambda^2(t) & 0 \\ 0 & 0 & \lambda^2(t) \end{bmatrix} = \lambda^2(t)[\mathbf{I}] \end{aligned}$$

For a rigid body motion, we must have $\mathbf{C} = \mathbf{I}$.

$$[\mathbf{C}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By comparing the components, we get:

$$\lambda^2(t) = 1 \implies \lambda(t) = \pm 1$$

However, a motion is typically required to be a proper rigid body motion, which means it must also satisfy the condition $\det(\mathbf{F}) = +1$.

We calculate the determinant of $\mathbf{F}$:

$$\det(\mathbf{F}) = \det \begin{pmatrix} \lambda(t) & 0 & 0 \\ 0 & \lambda(t) & 0 \\ 0 & 0 & \lambda(t) \end{pmatrix} = \lambda^3(t)$$

We check our two possible values for $\lambda(t)$:

- If $\lambda(t) = 1$, then $\det(\mathbf{F}) = (1)^3 = 1$. This satisfies the condition. This corresponds to the identity transformation (no motion).

- If $\lambda(t) = -1$, then $\det(\mathbf{F}) = (-1)^3 = -1$. This does not satisfy the condition. This motion is an inversion (a reflection through the origin), which is an improper rigid body motion.

Therefore, for the motion to be a proper rigid body motion, the only possible value is:

$$\lambda(t) = 1$$

5. Consider the motion given by $\boldsymbol{x}' = \boldsymbol{x} + t^2\,\boldsymbol{e}_1 + \sin(t)\,\boldsymbol{e}_2$. Determine if the motion is a rigid body motion by computing the strain tensor.

Solution: The given motion is a pure translation. The position components are:

$$\begin{aligned} x_1' &= x_1 + t^2 \\ x_2' &= x_2 + \sin(t) \end{aligned}$$

A motion is a rigid body motion if the strain tensor is zero. We first compute the deformation gradient tensor $\mathbf{F}$, where $F_{ij} = \frac{\partial x_i'}{\partial x_j}$.

$$[\boldsymbol{F}] = \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\mathbf{I}]$$

The deformation gradient is the identity tensor. Now, we compute the Green-Lagrange strain tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$.

$$\begin{aligned} [\boldsymbol{F}]^T[\boldsymbol{F}] &= [\mathbf{I}]^T[\mathbf{I}] = [\mathbf{I}] \\ [\boldsymbol{E}] &= \frac{1}{2}\left([\mathbf{I}] - [\mathbf{I}]\right) = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = [\mathbf{0}] \end{aligned}$$

Since the strain tensor $\boldsymbol{E}$ is the zero tensor, the motion is a rigid body motion (specifically, a pure translation).

6. Consider the motion given by $\boldsymbol{x}' = \boldsymbol{Q}(t)\,\boldsymbol{x} + \boldsymbol{c}(t)$, where $\boldsymbol{Q}(t)$ is a time-dependent rotation matrix and $\boldsymbol{c}(t)$ is a time-dependent translation vector. Determine if the motion is a rigid body motion.

Solution: A motion is a rigid body motion if the associated strain tensor is zero. The given motion in component form is $x_i' = Q_{ij}(t)x_j + c_i(t)$.

First, we compute the deformation gradient tensor $\mathbf{F}$, where $F_{ik} = \frac{\partial x'_i}{\partial x_k}$.

$$
\begin{aligned}
F_{ik} &= \frac{\partial}{\partial x_k}\left(Q_{ij}(t)x_j + c_i(t)\right) \\
&= Q_{ij}(t)\frac{\partial x_j}{\partial x_k} + \frac{\partial c_i(t)}{\partial x_k} \\
&= Q_{ij}(t)\delta_{jk} + 0 \\
&= Q_{ik}(t)
\end{aligned}
$$

Thus, the deformation gradient tensor is the rotation tensor, $[\mathbf{F}] = [\mathbf{Q}(t)]$.

Next, we compute the Green-Lagrange strain tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$.

$$
[\boldsymbol{E}] = \frac{1}{2}\left([\mathbf{Q}(t)]^T[\mathbf{Q}(t)] - [\mathbf{I}]\right)
$$

By definition, a rotation matrix $\mathbf{Q}$ is an orthogonal matrix, which satisfies the property $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$.

Substituting this property into the strain equation:

$$
[\boldsymbol{E}] = \frac{1}{2}\left([\mathbf{I}] - [\mathbf{I}]\right) = [\mathbf{0}]
$$

Since the Green-Lagrange strain tensor $\boldsymbol{E}$ is the zero tensor, the motion is a rigid body motion.

7. Consider the motion given by $\boldsymbol{u} = a\,x_1\,\boldsymbol{e}_1 + b\,x_2\,\boldsymbol{e}_2 + c\,x_3\,\boldsymbol{e}_3$. Determine the values of a, b, and c for which the motion is a rigid body motion by computing the strain tensor.

Solution: The displacement vector $\boldsymbol{u}$ is given. The relationship between the deformed position $\boldsymbol{x}'$ and the original position $\boldsymbol{x}$ is $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}$.

The components of the deformed position are:

$$
\begin{aligned}
x'_1 &= x_1 + u_1 = x_1 + a\,x_1 = (1+a)x_1 \\
x'_2 &= x_2 + u_2 = x_2 + b\,x_2 = (1+b)x_2 \\
x'_3 &= x_3 + u_3 = x_3 + c\,x_3 = (1+c)x_3
\end{aligned}
$$

First, we compute the deformation gradient tensor $\mathbf{F}$, where $F_{ij} = \frac{\partial x'_i}{\partial x_j}$.

$$
[\boldsymbol{F}] = \begin{bmatrix} \frac{\partial x'_1}{\partial x_1} & \frac{\partial x'_1}{\partial x_2} & \frac{\partial x'_1}{\partial x_3} \\ \frac{\partial x'_2}{\partial x_1} & \frac{\partial x'_2}{\partial x_2} & \frac{\partial x'_2}{\partial x_3} \\ \frac{\partial x'_3}{\partial x_1} & \frac{\partial x'_3}{\partial x_2} & \frac{\partial x'_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} (1+a) & 0 & 0 \\ 0 & (1+b) & 0 \\ 0 & 0 & (1+c) \end{bmatrix}
$$

Next, we compute the Green-Lagrange strain tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$.

$$
\begin{aligned}
[\boldsymbol{F}]^T[\boldsymbol{F}] &= \begin{bmatrix} (1+a) & 0 & 0 \\ 0 & (1+b) & 0 \\ 0 & 0 & (1+c) \end{bmatrix} \begin{bmatrix} (1+a) & 0 & 0 \\ 0 & (1+b) & 0 \\ 0 & 0 & (1+c) \end{bmatrix} \\
&= \begin{bmatrix} (1+a)^2 & 0 & 0 \\ 0 & (1+b)^2 & 0 \\ 0 & 0 & (1+c)^2 \end{bmatrix}
\end{aligned}
$$

Substituting this into the strain equation:

$$
\begin{aligned}
[\boldsymbol{E}] &= \frac{1}{2}\left(\begin{bmatrix} (1+a)^2 & 0 & 0 \\ 0 & (1+b)^2 & 0 \\ 0 & 0 & (1+c)^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \\
&= \frac{1}{2} \begin{bmatrix} (1+a)^2 - 1 & 0 & 0 \\ 0 & (1+b)^2 - 1 & 0 \\ 0 & 0 & (1+c)^2 - 1 \end{bmatrix}
\end{aligned}
$$

A motion is a rigid body motion if the strain tensor $\boldsymbol{E}$ is the zero tensor, $[\boldsymbol{E}] = [\mathbf{0}]$. This requires all diagonal components to be zero.

- $(1+a)^2 - 1 = 0 \implies (1+a)^2 = 1 \implies 1 + a = \pm 1$. This gives $a = 0$ or $a = -2$.
- $(1+b)^2 - 1 = 0 \implies (1+b)^2 = 1 \implies 1 + b = \pm 1$. This gives $b = 0$ or $b = -2$.
- $(1+c)^2 - 1 = 0 \implies (1+c)^2 = 1 \implies 1 + c = \pm 1$. This gives $c = 0$ or $c = -2$.

Therefore, the motion is a rigid body motion (i.e., preserves distances) if and only if $a, b,$ and c are chosen from the set $\{0, -2\}$.

8. Consider the motion given by $\boldsymbol{x}' = \boldsymbol{x} + a\,\sin(\omega t)\,\boldsymbol{e}_1 + b\,\cos(\omega t)\,\boldsymbol{e}_2$. Determine if the motion is a rigid body motion by computing the strain tensor.

Solution: The given motion is a pure translation, where the translation vector $\boldsymbol{c}(t) = a\,\sin(\omega t)\,\boldsymbol{e}_1 + b\,\cos(\omega t)\,\boldsymbol{e}_2$ depends only on time.

The components of the deformed position $\boldsymbol{x}'$ are:

$$
\begin{aligned}
x'_1 &= x_1 + a\,\sin(\omega t) \\
x'_2 &= x_2 + b\,\cos(\omega t)
\end{aligned}
$$

First, we compute the deformation gradient tensor $\mathbf{F}$, where $F_{ij} = \frac{\partial x'_i}{\partial x_j}$.

$$[\boldsymbol{F}] = \begin{bmatrix} \frac{\partial x'_1}{\partial x_1} & \frac{\partial x'_1}{\partial x_2} \\ \frac{\partial x'_2}{\partial x_1} & \frac{\partial x'_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\mathbf{I}]$$

Next, we compute the Green-Lagrange strain tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$.

$$[\boldsymbol{F}]^T[\boldsymbol{F}] = [\mathbf{I}]^T[\mathbf{I}] = [\mathbf{I}]$$
$$[\boldsymbol{E}] = \frac{1}{2}([\mathbf{I}] - [\mathbf{I}]) = [\mathbf{0}]$$

Since the strain tensor $\boldsymbol{E}$ is the zero tensor, the motion is a rigid body motion. This is expected, as a pure translation (even time-dependent) does not cause any deformation.

9. Consider the motion given by $\boldsymbol{F} = \boldsymbol{I} + \gamma(t)\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_2$. Determine the values of $\gamma(t)$ for which the motion is a rigid body motion by computing the strain tensor.

Solution: The deformation gradient tensor $\boldsymbol{F}$ is given. In matrix form, it can be written as:

$$[\boldsymbol{F}] = [\boldsymbol{I}] + \gamma(t)[\boldsymbol{e}_1 \otimes \boldsymbol{e}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \gamma(t)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \gamma(t) \\ 0 & 1 \end{bmatrix}$$

This motion is a simple shear.

Next, we compute the Green-Lagrange strain tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$.

First, find $\boldsymbol{F}^T$:

$$[\boldsymbol{F}]^T = \begin{bmatrix} 1 & 0 \\ \gamma(t) & 1 \end{bmatrix}$$

Now, compute $\boldsymbol{F}^T\boldsymbol{F}$:

$$\begin{aligned} [\boldsymbol{F}]^T[\boldsymbol{F}] &= \begin{bmatrix} 1 & 0 \\ \gamma(t) & 1 \end{bmatrix}\begin{bmatrix} 1 & \gamma(t) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \gamma(t) \\ \gamma(t) & \gamma(t)^2 + 1 \end{bmatrix} \end{aligned}$$

Now, compute $\boldsymbol{E}$:

$$\begin{aligned} [\boldsymbol{E}] &= \frac{1}{2}\left(\begin{bmatrix} 1 & \gamma(t) \\ \gamma(t) & \gamma(t)^2 + 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 0 & \gamma(t) \\ \gamma(t) & \gamma(t)^2 \end{bmatrix} \end{aligned}$$

A motion is a rigid body motion if the strain tensor $\boldsymbol{E}$ is the zero tensor, $[\boldsymbol{E}] = [\mathbf{0}]$. This requires all components of $[\boldsymbol{E}]$ to be zero.

- $\frac{1}{2}\gamma(t) = 0 \implies \gamma(t) = 0$
- $\frac{1}{2}\gamma(t)^2 = 0 \implies \gamma(t) = 0$

Therefore, the motion is a rigid body motion if and only if $\gamma(t) = 0$ for all time t. A non-zero shear $\gamma(t)$ always causes deformation.

10. Consider the motion given by $\boldsymbol{u} = \beta(t)\,(x_2\,\boldsymbol{e}_1 - x_1\,\boldsymbol{e}_2)$, where $\beta(t)$ is a time-dependent function. Determine the values of $\beta(t)$ for which the motion is a rigid body motion.

Solution: The displacement vector $\boldsymbol{u}$ has components:

$$\begin{aligned} u_1 &= \beta(t)x_2 \\ u_2 &= -\beta(t)x_1 \end{aligned}$$

The relationship between the deformed position $\boldsymbol{x}'$ and the original position $\boldsymbol{x}$ is $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}$. The components of the deformed position are:

$$\begin{aligned} x_1' &= x_1 + u_1 = x_1 + \beta(t)x_2 \\ x_2' &= x_2 + u_2 = x_2 - \beta(t)x_1 \end{aligned}$$

First, we compute the deformation gradient tensor $\mathbf{F}$, where $F_{ij} = \frac{\partial x_i'}{\partial x_j}$.

$$[\boldsymbol{F}] = \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & \beta(t) \\ -\beta(t) & 1 \end{bmatrix}$$

Next, we compute the Green-Lagrange strain tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$.

First, find $\boldsymbol{F}^T$:

$$[\boldsymbol{F}]^T = \begin{bmatrix} 1 & -\beta(t) \\ \beta(t) & 1 \end{bmatrix}$$

Now, compute $\boldsymbol{F}^T\boldsymbol{F}$:

$$\begin{aligned} [\boldsymbol{F}]^T[\boldsymbol{F}] &= \begin{bmatrix} 1 & -\beta(t) \\ \beta(t) & 1 \end{bmatrix} \begin{bmatrix} 1 & \beta(t) \\ -\beta(t) & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+\beta(t)^2 & 0 \\ 0 & 1+\beta(t)^2 \end{bmatrix} \end{aligned}$$

Now, compute $\boldsymbol{E}$:

$$[\boldsymbol{E}] = \frac{1}{2}\left(\begin{bmatrix} 1+\beta(t)^2 & 0 \\ 0 & 1+\beta(t)^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$
$$= \frac{1}{2}\begin{bmatrix} \beta(t)^2 & 0 \\ 0 & \beta(t)^2 \end{bmatrix}$$

A motion is a rigid body motion if the strain tensor $\boldsymbol{E}$ is the zero tensor, $[\boldsymbol{E}] = [\mathbf{0}]$. This requires all components of $[\boldsymbol{E}]$ to be zero.

- $\frac{1}{2}\beta(t)^2 = 0 \implies \beta(t) = 0$

Therefore, the motion is a rigid body motion if and only if $\beta(t) = 0$ for all time t. This means no motion is occurring. Any non-zero $\beta(t)$ introduces a uniform stretch (since $\boldsymbol{F}^T\boldsymbol{F} \neq \boldsymbol{I}$), so it is not a rigid motion.