Indian Institute of Technology Bhubaneswar

School of Infrastructure

Session: Autumn 2025

Solid Mechanics (CE2L001) — Class Test 2

<u>Notations :</u>
Zeroth-order tensors or scalars are represented by small letters. For eg. a
First-order tensors or vectors are represented by bold small letters. For eg. $\boldsymbol{a}$.
Second-order tensors are represented by bold capital letters. For eg. $\boldsymbol{A}$

1. A square material element with side length 1 unit undergoes a volume-preserving deformation given by:

$$x_1' = \alpha x_1, \qquad x_2' = \frac{x_2}{\alpha},$$

where $\alpha = 1.5$.

(a) Determine the deformation gradient tensor $\boldsymbol{F}$.
(b) Plot the reference and current configurations.
(c) Calculate the stretch ratio in the x_1 direction and the contraction ratio in the x_2 direction.

Solution: (a) Components of the deformation gradient tensor $\boldsymbol{F}$ are given by:

$$F_{ij} = \frac{\partial x_i'}{\partial x_j}$$

Given the deformation:

$$x_1' = \alpha x_1, \qquad x_2' = \frac{x_2}{\alpha},$$

We compute the components of $\boldsymbol{F}$:

$$F_{11} = \frac{\partial x_1'}{\partial x_1} = \alpha, \qquad F_{12} = \frac{\partial x_1'}{\partial x_2} = 0, \qquad F_{21} = \frac{\partial x_2'}{\partial x_1} = 0, \qquad F_{22} = \frac{\partial x_2'}{\partial x_2} = \frac{1}{\alpha}.$$

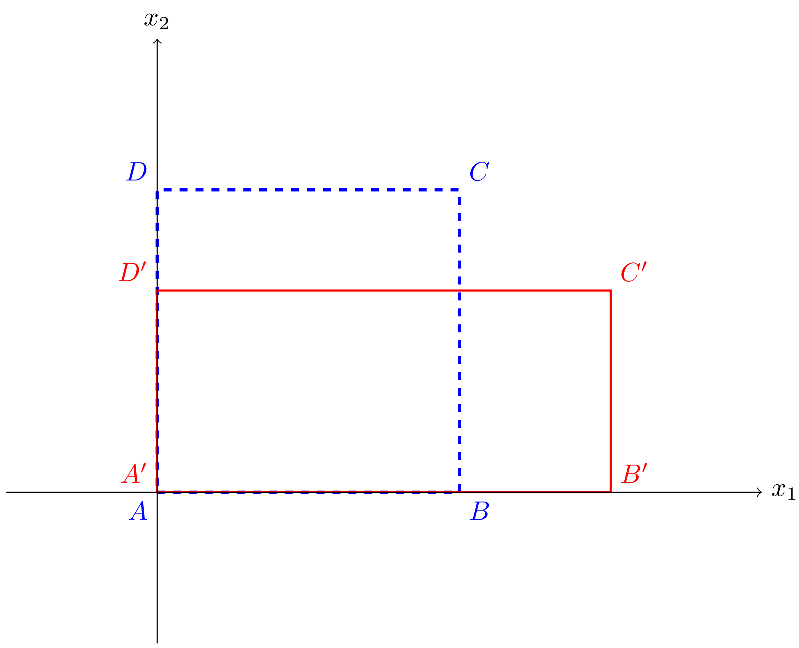
Thus,

$$[\boldsymbol{F}] = \begin{bmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix} = \begin{bmatrix} 1.5 & 0 \\ 0 & \frac{1}{1.5} \end{bmatrix} = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.667 \end{bmatrix}.$$

(b)

Reference Configuration (x_1, x_2)	**Current Configuration** (x_1', x_2')
(0,0)	(0,0)
(1,0)	(1.5,0)
(1,1)	(1.5, 0.667)
(0,1)	(0, 0.667)

Table 1: Reference and Current Configurations



(c) As $\boldsymbol{F}$ is diagonal, the stretch tensor, $\boldsymbol{U} = \boldsymbol{F}$.

Stretch ratio in x_1 direction: $\lambda_1 = F_{11} = \alpha = 1.5$.

Contraction ratio in x_2 direction: $\lambda_2 = F_{22} = \frac{1}{\alpha} = \frac{1}{1.5} = 0.667$.

2. The stress components in a material are given by:

$$\sigma_{11} = a_1\, x_1 x_2, \qquad \sigma_{22} \;= a_2\, x_2 x_3, \qquad \sigma_{12} = a_3\, x_1 x_3,$$

where a_1, a_2, and a_3 are small constants. Determine:

(a) The body force components.

(b) The strain components using Hooke's law.

(c) Whether the compatibility condition is satisfied.

Solution: (a) The equilibrium equations are:

$$\sigma_{ij,j} + b_i = 0$$

Given stress components:

$$\sigma_{11} = a_1\, x_1 x_2, \qquad \sigma_{22} \quad = a_2\, x_2 x_3, \qquad \sigma_{12} = a_3\, x_1 x_3,$$

So, the components of body force vectors can be calculated as given below.

$$b_1 = -\sigma_{11,1} - \sigma_{12,2} = -a_1 x_2 - 0 = -a_1 x_2,$$

$$b_2 = -\sigma_{12,1} - \sigma_{22,2} = -a_3 x_3 - a_2 x_3 = -(a_3 + a_2)x_3,$$

$$b_3 = -\sigma_{13,3} - \sigma_{23,3} = 0 - 0 = 0.$$

(b) Strain Components using Hooke's Law can be computed for an isotropic material as:

$$\epsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}]$$

Now,

$$\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = a_1 x_1 x_2 + a_2 x_2 x_3 + 0 = a_1 x_1 x_2 + a_2 x_2 x_3.$$

So,

$$\epsilon_{11} = \frac{1}{E}[(1+\nu)(a_1 x_1 x_2) - \nu(a_1 x_1 x_2 + a_2 x_2 x_3)] = \frac{1}{E}[a_1 x_1 x_2 - \nu a_2 x_2 x_3],$$

$$\epsilon_{22} = \frac{1}{E}[(1+\nu)(a_2 x_2 x_3) - \nu(a_1 x_1 x_2 + a_2 x_2 x_3)] = \frac{1}{E}[a_2 x_2 x_3 - \nu a_1 x_1 x_2],$$

$$\epsilon_{12} = \frac{1+\nu}{E}\sigma_{12} = \frac{1+\nu}{E} a_3 x_1 x_3.$$

(c) For 2D, Compatibility Condition:

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2}$$

Computing derivatives:

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} = \frac{1}{E}[a_1 x_1 - \nu a_2 x_3], \qquad \frac{\partial^2 \epsilon_{11}}{\partial x_2^2} \quad = \frac{1}{E}[a_1 x_1 - \nu a_2 x_3], \qquad \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} = \frac{1+\nu}{E} a_3 x_3.$$

Substituting:

$$\frac{2}{E}[a_1 x_1 - \nu a_2 x_3] \neq 2\frac{1+\nu}{E} a_3 x_3$$

The compatibility condition is not satisfied in general.

3. The displacement field is given by:

$$u_1 = \alpha\, x_1 x_2, \qquad u_2 \;= \beta\, x_1^2,$$

where α and β are constants. Calculate the strain tensor and derive the principal strains and principal directions.

Solution: For a given displacement components, strain tensor $\boldsymbol{E}_s = E_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ can be computed from $\boxed{E_{ij} = (1/2)\,(u_{i,j} + u_{j,i})}$. For the given displacement components,, strain tensor components:

$$[\boldsymbol{E}_s] = \begin{bmatrix} \alpha x_2 & \frac{1}{2}(\alpha x_1 + 2\beta x_1) \\ \frac{1}{2}(\alpha x_1 + 2\beta x_1) & 0 \end{bmatrix}.$$

From the Mohr's circle, one can derive the principal strains expression as:

$$\epsilon_{I,II} = \frac{E_{11} + E_{22}}{2} \pm \sqrt{\left(\frac{E_{11} - E_{22}}{2}\right)^2 + E_{12}^2},$$

and the principal direction of strain for the angle with respect to $\boldsymbol{e_1}$ satisfies:

$$\tan 2\theta_p = \frac{2E_{12}}{E_{11} - E_{22}}.$$

For the given problem, the principal strains are:

$$\epsilon_{I,II} = \frac{\alpha x_2}{2} \pm \sqrt{(\frac{\alpha x_2}{2})^2 + (\frac{1}{2}(\alpha x_1 + 2\beta x_1))^2}.$$

and the principal directions:

$$\tan 2\theta_p = \frac{(\alpha x_1 + 2\beta x_1)}{\alpha x_2}.$$

4. Consider a homogeneous deformation of a cube with initial side length a. The cube is sheared in the x-y plane by an angle γ. Determine:
(a) Deformation gradient tensor $\boldsymbol{F}$.
(b) Finite strain tensor $\boldsymbol{E}$.
(c) Linearized strain tensor $\boldsymbol{E}_s$ for small deformations.
(d) Provide a comparison between the finite strain components and the linearized strain components through a numerical example.

Solution: (a) Deformation Gradient Tensor $\boldsymbol{F} = F_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, where F_{ij} is defined as

$$F_{ij} = \frac{\partial x_i'}{\partial x_j}.$$

For three-dimensional case, $[\boldsymbol{F}] := (F_{ij})_{i,j\in(1,2)}$ can be expressed as

$$[\boldsymbol{F}] = \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} & \frac{\partial x_1'}{\partial x_3} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} & \frac{\partial x_2'}{\partial x_3} \\ \frac{\partial x_3'}{\partial x_1} & \frac{\partial x_3'}{\partial x_2} & \frac{\partial x_3'}{\partial x_3} \end{bmatrix}.$$

The deformation is given by:

$$x_1' = x_1 + \gamma x_2, \qquad x_2' = x_2, \qquad x_3' = x_3.$$

So,

$$[\boldsymbol{F}] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Finite Strain Tensor $\boldsymbol{E}$:

$$[\boldsymbol{E}] = \frac{1}{2}\left([\boldsymbol{F}^T][\boldsymbol{F}] - [\boldsymbol{I}]\right) = \begin{bmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & \frac{\gamma^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) Linearized Strain Tensor $\boldsymbol{E}_s$:

$$[\boldsymbol{E}_s] = \frac{1}{2}\left([\boldsymbol{F}^T] + [\boldsymbol{F}] - 2\,[\boldsymbol{I}]\right) = \begin{bmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

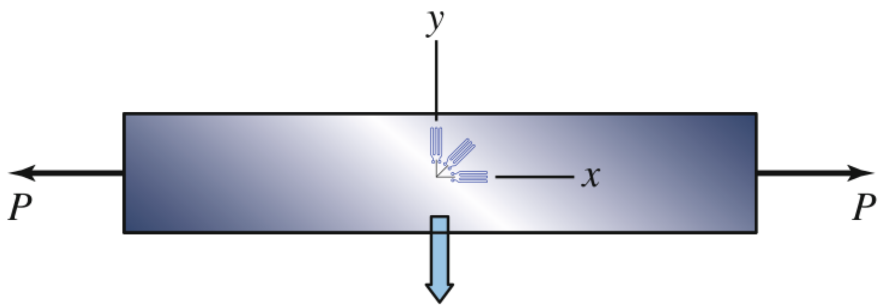
(d) Numerical Example:

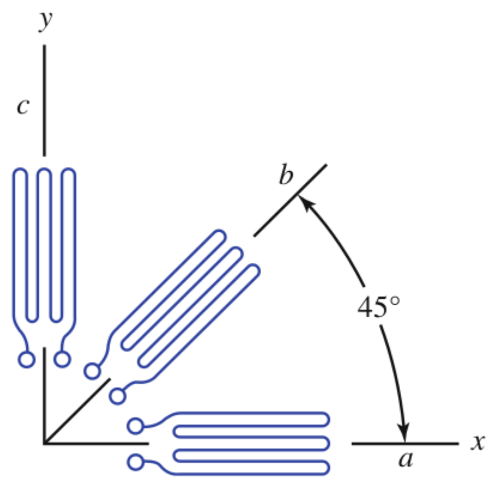
Component	$\gamma = 0.1$	$\gamma = 0.001$
E_{12}	0.05	0.0005
E_{22}	0.005	5×10^{-7}
$E_{s,12}$	0.05	0.0005
$E_{s,22}$	0	0

Table 2: Finite and Linearized Strain Components

As can be seen from the above table, for small value of γ, the linearized strain tensor will be almost the same as the finite strain tensor.

5. A bar is subjected to axial forces. The strains measured by a strain gauge rosette oriented as shown in the below figure are $E_{aa} = 0.003$, $E_{bb} = 0.001$, and $E_{cc} = -0.001$. What are the strain components E_{xx}, E_{yy}, and E_{xy}?





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Solution: Given strains:

$$E_{aa} = 0.003, \quad E_{bb} = 0.001, \quad E_{cc} = -0.001.$$

The strain transformation equations are:

$$\begin{aligned}
E_{aa} &= E_{xx}\cos^2\theta_a + E_{yy}\sin^2\theta_a + 2E_{xy}\sin\theta_a\cos\theta_a,\\
E_{bb} &= E_{xx}\cos^2\theta_b + E_{yy}\sin^2\theta_b + 2E_{xy}\sin\theta_b\cos\theta_b,\\
E_{cc} &= E_{xx}\cos^2\theta_c + E_{yy}\sin^2\theta_c + 2E_{xy}\sin\theta_c\cos\theta_c.
\end{aligned}$$

From the figure, $\theta_a = 0°$, $\theta_b = 45°$, $\theta_c = 90°$.

$$\begin{aligned}
E_{aa} &= E_{xx} = 0.003,\\
E_{bb} &= \frac{1}{2}(E_{xx} + E_{yy}) + E_{xy} = 0.001,\\
E_{cc} &= E_{yy} = -0.001.
\end{aligned}$$

Solving:

$$
\begin{aligned}
E_{xx} &= 0.003, \\
E_{yy} &= -0.001, \\
0.001 &= \frac{1}{2}(0.003 - 0.001) + E_{xy} \\
E_{xy} &= 0.001 - \frac{1}{2}(0.002) = 0.
\end{aligned}
$$

Therefore, the strain components are:

$$
\begin{aligned}
E_{xx} &= 0.003, \\
E_{yy} &= -0.001, \\
E_{xy} &= 0.
\end{aligned}
$$