Indian Institute of Technology Bhubaneswar

School of Infrastructure

Session: Autumn 2025

Solid Mechanics (CE2L001) Solution of Assignment No. 2

<u>Notations :</u>
Zeroth-order tensors or scalars are represented by small letters. For eg. a
First-order tensors or vectors are represented by bold small letters. For eg. $\boldsymbol{a}$.
Second-order tensors are represented by bold capital letters. For eg. $\boldsymbol{A}$

1. The three principal invariants of a second-order tensor $\boldsymbol{T} = T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ are defined as $I_1 = \text{trace}\,(\boldsymbol{T}) = T_{ii}$, $I_2 = (1/2)\left((\text{trace}\,(\boldsymbol{T}))^2 - \text{trace}\left(\boldsymbol{T}^2\right)\right) = (1/2)\left(T_{ii}T_{jj} - T_{ij}T_{ij}\right)$ and $I_3 = \det(\boldsymbol{T}) = (1/6)\,\epsilon_{lmn}\,\epsilon_{ijk}\,T_{li}\,T_{mj}\,T_{nk}$.
(a) Define the eigenvalue problem associated with the second-order tensor, $\boldsymbol{T}$.
(b) Show that the correspondence between the principal invariants and the characteristic polynomial of the second-order tensor, $\boldsymbol{T}$, can be given by

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0,$$

where λ's are the eigenvalues of the second-order tensor, $\boldsymbol{T}$.
(c) Explain the physical meaning of the three principal invariants of the second-order tensor, $\boldsymbol{T}$.

Solution: (a) The eigenvalue problem for a second-order tensor $\boldsymbol{T}$ is defined as:

$$\boldsymbol{T}\,\boldsymbol{u} = \lambda\boldsymbol{u},$$

where λ is the eigenvalue and $\boldsymbol{u}$ is the corresponding eigenvector.

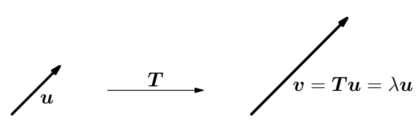

Figure 1: The schematic representation of a linear transformation of a vector $\boldsymbol{u}$ into $\boldsymbol{v}$ whose magnitutde is increased by λ, but the direction is not changed. The vectors that do not rotate upon transformation with the second-order tensor is called as eigen vectors. Here, the constant λ is the eigen value and $\boldsymbol{u}$ is an eigen vector.

In component form, this can be written as:

$$T_{ij}u_j = \lambda u_i$$

(b) The characteristic polynomial of a second-order tensor $\boldsymbol{T}$ is given by:

$$\det(\boldsymbol{T} - \lambda \boldsymbol{I}) = 0 \tag{1}$$

Expanding the determinant, we get:

$$\begin{vmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{vmatrix} = 0 \tag{2}$$

Simplifying and rearranging terms, we get:

$$-\lambda^3 + (T_{11} + T_{22} + T_{33})\lambda^2 - ((T_{11}T_{22} - T_{12}T_{21}) + (T_{22}T_{33} - T_{23}T_{32}) + (T_{11}T_{33} - T_{13}T_{31}))\lambda + \det(\boldsymbol{T}) = 0 \tag{3}$$

The principal invariants are defined as:

$$I_1 = T_{11} + T_{22} + T_{33} = \text{trace}(\boldsymbol{T}) \tag{4}$$

$$I_2 = \frac{1}{2}((\text{trace}(\boldsymbol{T}))^2 - \text{trace}(\boldsymbol{T}^2)) \tag{5}$$

$$I_3 = \det(\boldsymbol{T}) \tag{6}$$

Substituting these expressions into the characteristic polynomial, we get:

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{7}$$

(c)

- $I_1 = \text{trace}(\boldsymbol{T})$ represents the sum of the eigenvalues and can be thought of as a measure of the "size" or "scale" of the tensor's effect.
- $I_2 = \frac{1}{2}((\text{trace}(\boldsymbol{T}))^2 - \text{trace}(\boldsymbol{T}^2))$ can be related to the amount of "shape change" or "deviatoric" behavior of the tensor, though its interpretation can vary depending on the context (e.g., stress, strain).
- $I_3 = \det(\boldsymbol{T})$ represents the product of the eigenvalues and can indicate the amount of "volume change" or the scaling effect of the tensor on a material or space.

2. (a) Derive the relationship between the invariants of the stress tensor and the invariants of the deviatoric stress tensor.
(b) Show that the deviatoric stress tensor has zero first invariant.
(c) A material is subjected to a stress state with principal stresses $\sigma_I = 100\,\text{MPa}$, $\sigma_{II} = 50\,\text{MPa}$, and $\sigma_{III} = -20\,\text{MPa}$. Calculate the invariants of the stress tensor.

Solution: (a) Let's denote the stress tensor as σ_{ij} and the deviatoric stress tensor as s_{ij}. The deviatoric stress tensor is defined as:

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{8}$$

where δ_{ij} is the Kronecker delta.

The invariants of the stress tensor are:

$$I_1 = \sigma_{ii} \tag{9}$$
$$I_2 = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) \tag{10}$$
$$I_3 = \det(\sigma_{ij}) \tag{11}$$

The invariants of the deviatoric stress tensor are:

$$J_1 = s_{ii} \tag{12}$$
$$J_2 = \frac{1}{2}s_{ij}s_{ij} \tag{13}$$
$$J_3 = \det(s_{ij}) \tag{14}$$

Now, let's derive the relationships between the invariants:

$$J_1 = s_{ii} = \sigma_{ii} - \frac{1}{3}\sigma_{kk}\delta_{ii} = \sigma_{ii} - \sigma_{ii} = 0 \tag{15}$$

So, $J_1 = 0$.

$$\begin{aligned}
J_2 = \frac{1}{2}s_{ij}s_{ij} &= \frac{1}{2}(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij})(\sigma_{ij} - \frac{1}{3}\sigma_{mm}\delta_{ij}) && (16)\\
&= \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \frac{1}{3}\sigma_{ii}\sigma_{jj}) && (17)\\
&= \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \frac{1}{3}I_1^2) && (18)\\
&= \frac{1}{3}I_1^2 - I_2 && (19)
\end{aligned}$$

After some algebra, we can show that:

$$J_3 = \frac{1}{3}I_1 I_2 - I_3 - \frac{2}{27}I_1^3 \tag{20}$$

So, the relationships between the invariants are:

$$J_1 = 0 \tag{21}$$
$$J_2 = \frac{1}{3}I_1^2 - I_2 \tag{22}$$
$$J_3 = \frac{1}{3}I_1 I_2 - I_3 - \frac{2}{27}I_1^3 \tag{23}$$

(b) The deviatoric stress tensor is defined as:

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{24}$$

The first invariant of the deviatoric stress tensor is:

$$J_1 = s_{ii} \tag{25}$$
$$= \sigma_{ii} - \frac{1}{3}\sigma_{kk}\delta_{ii} \tag{26}$$
$$= \sigma_{ii} - \frac{1}{3}\sigma_{kk} \cdot 3 \tag{27}$$
$$= \sigma_{ii} - \sigma_{kk} \tag{28}$$
$$= 0 \tag{29}$$

Therefore, the first invariant of the deviatoric stress tensor is zero.

(c) Given principal stresses:

$$\sigma_I = 100\,\text{MPa}$$
$$\sigma_{II} = 50\,\text{MPa}$$
$$\sigma_{III} = -20\,\text{MPa}$$

The first invariant I_1 is the sum of the principal stresses:

$$I_1 = \sigma_I + \sigma_{II} + \sigma_{III}$$
$$= 100 + 50 - 20$$
$$= 130\,\text{MPa}$$

The second invariant I_2 can be calculated using the formula:

$$
\begin{aligned}
I_2 &= \sigma_I \sigma_{II} + \sigma_{II} \sigma_{III} + \sigma_{III} \sigma_I \\
&= (100)(50) + (50)(-20) + (-20)(100) \\
&= 5000 - 1000 - 2000 \\
&= 2000\,\text{MPa}^2
\end{aligned}
$$

The third invariant I_3 is the product of the principal stresses:

$$
\begin{aligned}
I_3 &= \sigma_I \sigma_{II} \sigma_{III} \\
&= (100)(50)(-20) \\
&= -100000\,\text{MPa}^3
\end{aligned}
$$

Therefore, the invariants of the stress tensor are:

$$\boxed{I_1 = 130\,\text{MPa}, \qquad I_2 = 2000\text{MPa}^2, \qquad I_3 = -100000\,\text{MPa}^3.}$$

3. At a particular point in a wooden member, the state of stress is as shown in Fig. 13. The direction of the grain in the wood makes an angle of $+30°$ with the x-axis (i.e, horizontal axis). The allowable shear stress parallel to the grain is 150 psi for this wood. Is this state of stress permissible? Verify your answer by calculations.

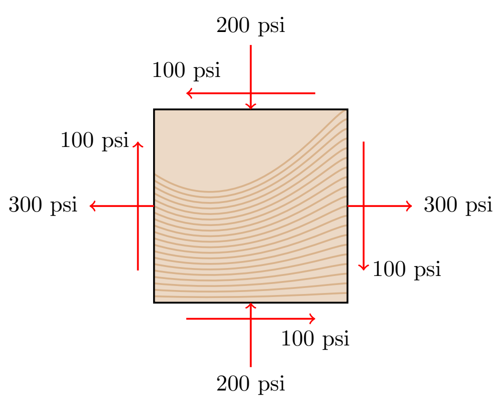

Figure 2

Solution: The given stress state is

$$\sigma_{xx} = 300 \text{ psi}, \qquad \sigma_{yy} = -200 \text{ psi}, \qquad \sigma_{xy} = -100 \text{ psi},$$

and the grain direction makes an angle $\theta = 30°$ with the x-axis. The shear stress parallel to the grain is obtained by transforming stresses to the rotated (x', y')-system:

$$\sigma_{x'y'} = -\frac{\sigma_{xx} - \sigma_{yy}}{2} \sin(2\theta) + \sigma_{xy} \cos(2\theta).$$

Substituting values:

$$\frac{\sigma_{xx} - \sigma_{yy}}{2} = \frac{300 - (-200)}{2} = 250 \text{ psi},$$

$$\sigma_{x'y'} = -250 \sin 60° - 100 \cos 60° = -250 \cdot \frac{\sqrt{3}}{2} - 100 \cdot \frac{1}{2},$$

$$\sigma_{x'y'} = -216.5 - 50 = -266.5 \text{ psi}.$$

Hence,

$$\boxed{|\tau_{\text{parallel to grain}}| \approx 266.5 \text{ psi}}$$

The allowable shear stress parallel to the grain is 150 psi. Since $266.5 > 150$, the given stress state is not permissible.

4. Consider a stress field whose matrix of scalar components in the vector basis $\{\boldsymbol{e}_i \mid i = 1, 2, 3\}$ is

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 4x_1x_3 & 0 & -2x_3^2 \\ 0 & 1 & 2 \\ -2x_3^2 & 2 & 3x_1^2 \end{bmatrix} \text{ MPa},$$

where the constants are given with appropriate units so as to be compatible with Cartesian coordinates x_i in meters.

(i) For the static case (no inertial forces) plus assuming no body forces, is this stress field in equilibrium?
(ii) Determine the traction vector acting at a point $\boldsymbol{x} = 2\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ on the plane $x_1 + x_2 - x_3 = 2$. Note the unit normal to a plane defined by $a_i x_i = b$ is,

$$\boldsymbol{n} = \pm \frac{a_i \boldsymbol{e}_i}{\sqrt{a_j a_j}}$$

(iii) Find the magnitude of the normal and shear traction on this plane at the given point.
(iv) Determine the principal stresses and directions at the given point.

Solution: The stress field (components with respect to the orthonormal basis $\{\boldsymbol{e}_i\}$) is

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 4x_1x_3 & 0 & -2x_3^2 \\ 0 & 1 & 2 \\ -2x_3^2 & 2 & 3x_1^2 \end{bmatrix} \text{ MPa}.$$

(i) Equilibrium (static, no body forces).

For equilibrium with no body forces we require

$$\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}, \qquad \text{i.e.} \quad \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \ \ (i = 1, 2, 3).$$

Compute the divergence components.

$$\begin{aligned}
(\operatorname{div}\boldsymbol{\sigma})_1 &= \frac{\partial}{\partial x_1}(4x_1x_3) + \frac{\partial}{\partial x_2}(0) + \frac{\partial}{\partial x_3}(-2x_3^2) = 4x_3 - 4x_3 = 0,\\
(\operatorname{div}\boldsymbol{\sigma})_2 &= \frac{\partial}{\partial x_1}(0) + \frac{\partial}{\partial x_2}(1) + \frac{\partial}{\partial x_3}(2) = 0 + 0 + 0 = 0,\\
(\operatorname{div}\boldsymbol{\sigma})_3 &= \frac{\partial}{\partial x_1}(-2x_3^2) + \frac{\partial}{\partial x_2}(2) + \frac{\partial}{\partial x_3}(3x_1^2) = 0 + 0 + 0 = 0.
\end{aligned}$$

Hence $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$ everywhere, so the stress field is in equilibrium.

(ii) Traction vector on the plane at the given point.

The stress tensor at the point $\boldsymbol{x} = (2, 1, 1)$:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 4(2)(1) & 0 & -2(1)^2 \\ 0 & 1 & 2 \\ -2(1)^2 & 2 & 3(2)^2 \end{bmatrix} = \begin{bmatrix} 8 & 0 & -2 \\ 0 & 1 & 2 \\ -2 & 2 & 12 \end{bmatrix} \text{ MPa}.$$

Unit normal to the plane.

$$\boldsymbol{n} = \frac{1}{\sqrt{1^2 + 1^2 + (-1)^2}}(1, 1, -1) = \frac{1}{\sqrt{3}}(1, 1, -1).$$

The traction vector acting on the plane is

$$\boldsymbol{t} = [\boldsymbol{\sigma}]\,\boldsymbol{n} = \begin{bmatrix} 8 & 0 & -2 \\ 0 & 1 & 2 \\ -2 & 2 & 12 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Thus,

$$\boxed{\boldsymbol{t} = \frac{1}{\sqrt{3}} \begin{bmatrix} 10 \\ -1 \\ -12 \end{bmatrix} \text{ MPa.}}$$

(iii) Normal and shear components of the traction. The normal component (scalar) is

$$t_n = \boldsymbol{t}\cdot\boldsymbol{n} = \frac{1}{3}\big(10\cdot 1 + (-1)\cdot 1 + (-12)\cdot(-1)\big) = \frac{21}{3} = 7 \text{ MPa}.$$

The magnitude of the total traction is

$$\|\boldsymbol{t}\|^2 = \frac{1}{3}(10^2 + (-1)^2 + (-12)^2) = \frac{245}{3}, \qquad \|\boldsymbol{t}\| = \sqrt{\frac{245}{3}}.$$

Hence the shear (tangential) traction magnitude is

$$t_s = \sqrt{\|\boldsymbol{t}\|^2 - t_n^2} = \sqrt{\frac{245}{3} - 7^2} = \frac{7\sqrt{6}}{3} \text{ MPa}.$$

$$\boxed{t_n = 7 \text{ MPa}, \qquad t_s = \frac{7\sqrt{6}}{3} \text{ MPa}}$$

(iv) Principal stresses and directions at the given point. The given stress tensor is

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 8 & 0 & -2 \\ 0 & 1 & 2 \\ -2 & 2 & 12 \end{bmatrix} \text{ MPa}.$$

We use the stress invariants I_1, I_2, I_3. For a 3×3 stress tensor

$$I_1 = \operatorname{tr}(\boldsymbol{\sigma}), \qquad I_2 = \tfrac{1}{2}\left[(\operatorname{tr}\boldsymbol{\sigma})^2 - \operatorname{tr}(\boldsymbol{\sigma}^2)\right], \qquad I_3 = \det(\boldsymbol{\sigma}).$$

Compute them for the given $\boldsymbol{\sigma}$:

$I_1 = 8 + 1 + 12 = 21,$

$I_2 = (8)(1) + (1)(12) + (12)(8) - \left(0^2 + 2^2 + (-2)^2\right) = 8 + 12 + 96 - (0 + 4 + 4) = 116 - 8 = 108,$

$$I_3 = \det[\boldsymbol{\sigma}] = \begin{vmatrix} 8 & 0 & -2 \\ 0 & 1 & 2 \\ -2 & 2 & 12 \end{vmatrix} = 8(1 \cdot 12 - 2 \cdot 2) - 0 + (-2)(0 \cdot 2 - 1 \cdot (-2)) = 8(12 - 4) - 2(2) = 64 - 4 = 60.$$

The characteristic equation for the principal stresses λ is

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0,$$

so here

$$\lambda^3 - 21\lambda^2 + 108\lambda - 60 = 0.$$

The roots (principal stresses) of this cubic are (numerically)

$$\boxed{\sigma_1 \approx 13.12 \text{ MPa}, \qquad \sigma_2 \approx 7.26 \text{ MPa}, \qquad \sigma_3 \approx 0.631 \text{ MPa}.}$$

Therefore the principal stresses are

$$\sigma_I \approx 13.12 \text{ MPa}, \qquad \sigma_{II} \approx 7.26 \text{ MPa}, \qquad \sigma_{III} \approx 0.631 \text{ MPa}.$$

Now find the corresponding principal directions (eigenvectors).

For $\quad \sigma_1 \approx 13.12$ MPa :

$$(\boldsymbol{\sigma} - \sigma_1 \mathbf{I})\boldsymbol{n}^{(1)} = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{n}^{(1)} \approx \begin{bmatrix} 0.360 \\ -0.152 \\ -0.920 \end{bmatrix}.$$

For $\quad \sigma_2 \approx 7.26$ MPa :

$$\boldsymbol{n}^{(2)} \approx \begin{bmatrix} 0.931 \\ 0.111 \\ 0.346 \end{bmatrix}.$$

For $\quad \sigma_3 \approx 0.631$ MPa :

$$\boldsymbol{n}^{(3)} \approx \begin{bmatrix} -0.049 \\ 0.982 \\ -0.181 \end{bmatrix}.$$

5. Provide qualitative stress state and the corresponding Mohr's circle considering an example for the following practical applications:
   (a) Analysis of Structural Members.
   (b) Design of Mechanical Components.
   (c) Material Failure Analysis.
   (d) Rock Mechanics.
   (e) Biomechanical Applications.

Solution: (a) **Analysis of Structural Members:** For a beam in bending, the stress state can be represented as a combination of normal stresses (tensile or compressive) and shear stresses. Let's consider a point on the beam where the stress state is:

$$\begin{aligned} \sigma_x &= 100\,\text{MPa} \quad \text{(tensile)} \\ \sigma_y &= 0\,\text{MPa} \\ \tau_{xy} &= 50\,\text{MPa} \end{aligned}$$

The Mohr's circle for this stress state would have a center at:

$$\sigma_c = \frac{\sigma_x + \sigma_y}{2} = \frac{100 + 0}{2} = 50\,\text{MPa}$$

and a radius of:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{100 - 0}{2}\right)^2 + (50)^2} = \sqrt{(50)^2 + (50)^2} = 70.71\,\text{MPa}$$

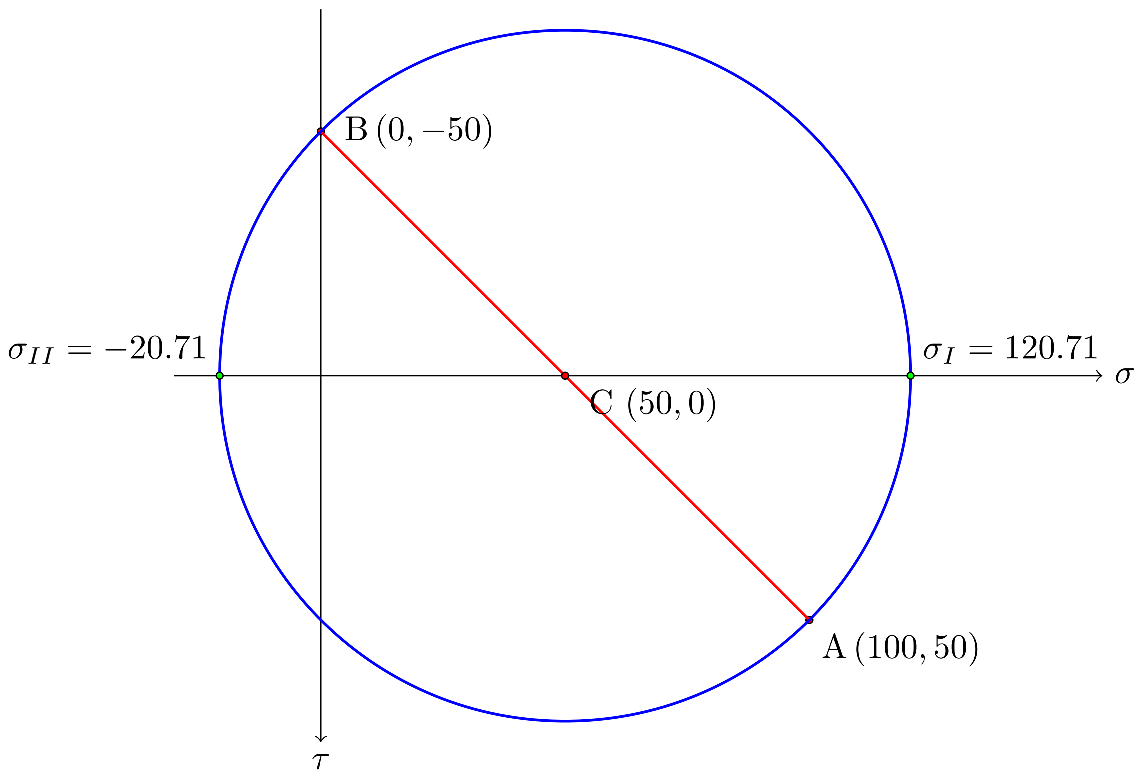

Figure 3

(b) **Design of Mechanical Components:** For a shaft under torsion, the stress state can be represented as pure shear stress. Let's consider a point on the shaft where the stress state is:

$$\sigma_x = 0\,\text{MPa}$$
$$\sigma_y = 0\,\text{MPa}$$
$$\tau_{xy} = 100\,\text{MPa}$$

The Mohr's circle for this stress state would have a center at:

$$\sigma_c = \frac{\sigma_x + \sigma_y}{2} = \frac{0 + 0}{2} = 0\,\text{MPa}$$

and a radius of:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{0-0}{2}\right)^2 + (100)^2} = 100\,\text{MPa}$$

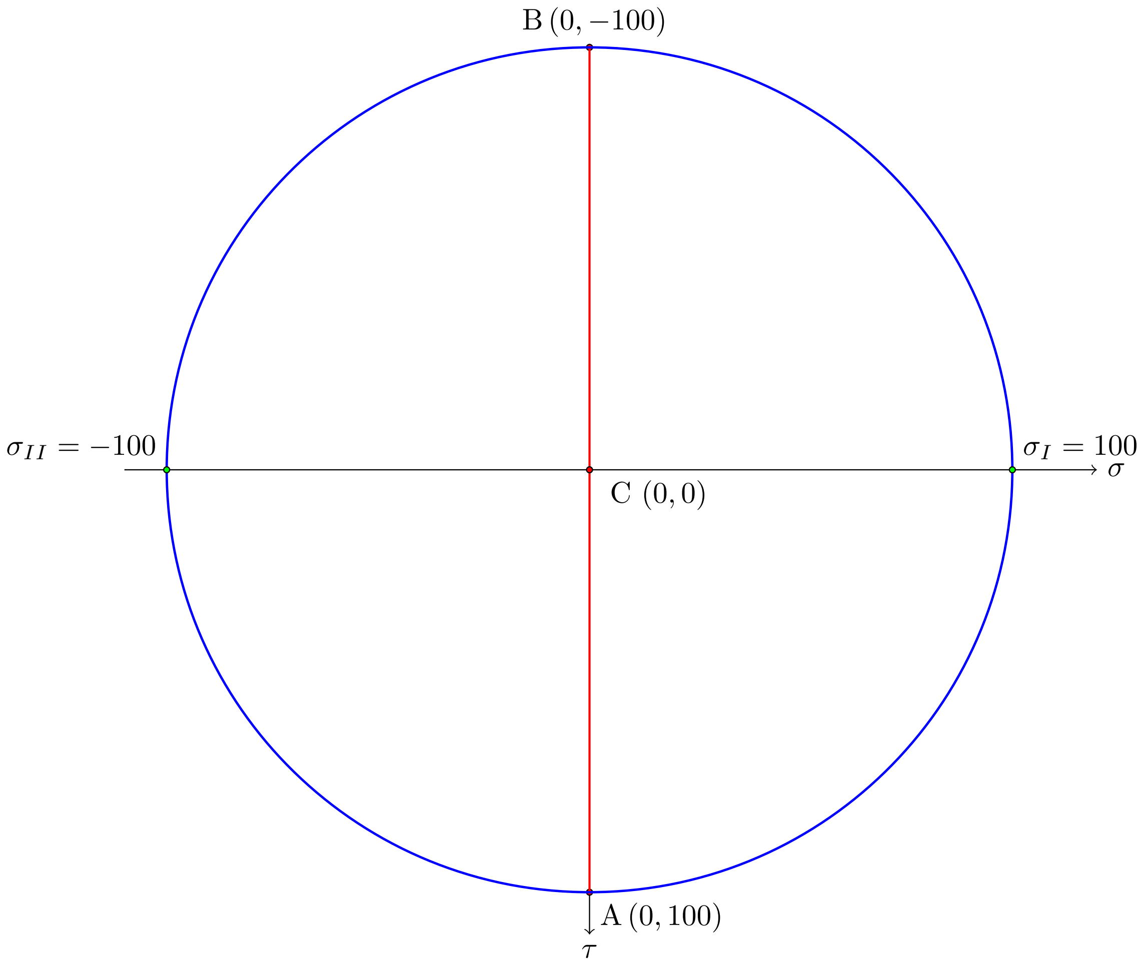

Figure 4

(c) **Material Failure Analysis:** For a material under uniaxial tension, the stress state can be represented as:

$$\sigma_x = 500\,\text{MPa} \quad \text{(tensile)}$$
$$\sigma_y = 0\,\text{MPa}$$
$$\tau_{xy} = 0\,\text{MPa}$$

The Mohr's circle for this stress state would have a center at:

$$\sigma_c = \frac{\sigma_x + \sigma_y}{2} = \frac{500 + 0}{2} = 250\,\text{MPa}$$

and a radius of:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{500 - 0}{2}\right)^2 + (0)^2} = 250\,\text{MPa}$$

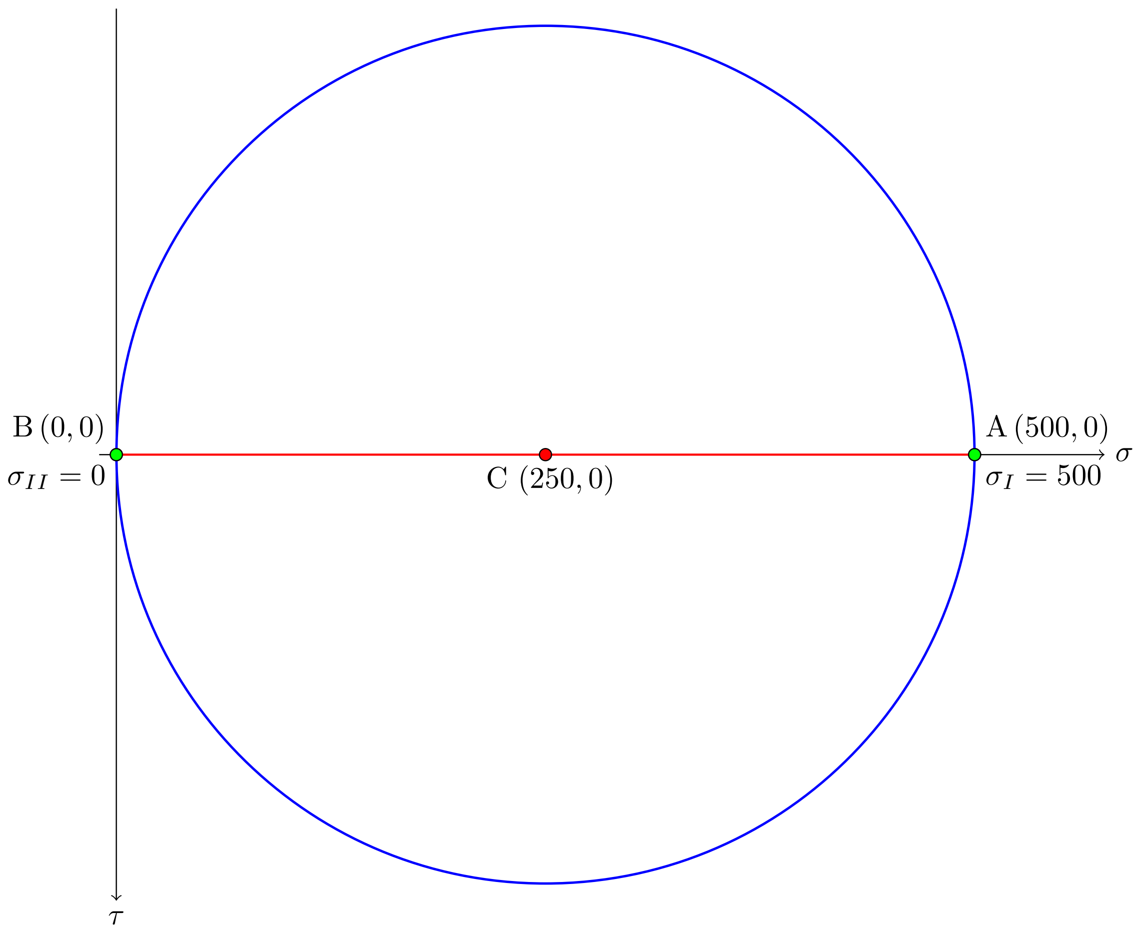

Figure 5

(d) **Rock Mechanics:** For a rock mass under biaxial compression, the stress state can be represented as:

$$\sigma_x = -100\,\text{MPa} \quad \text{(compressive)}$$
$$\sigma_y = -200\,\text{MPa} \quad \text{(compressive)}$$
$$\tau_{xy} = 0\,\text{MPa}$$

The Mohr's circle for this stress state would have a center at:

$$\sigma_c = \frac{\sigma_x + \sigma_y}{2} = \frac{-100 - 200}{2} = -150\,\text{MPa}$$

and a radius of:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{-100 + 200}{2}\right)^2 + (0)^2} = 50\,\text{MPa}$$

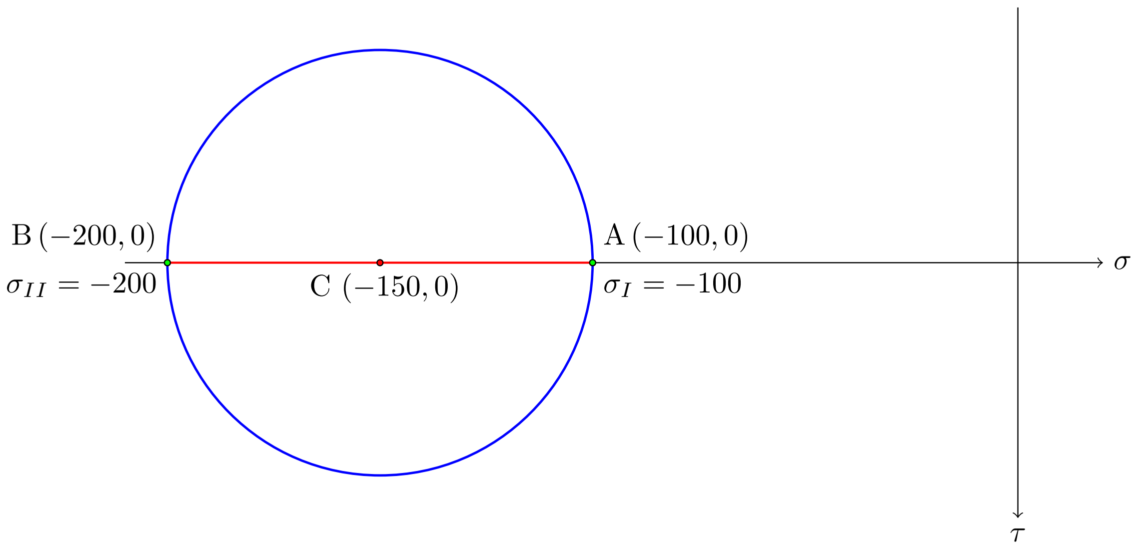

Figure 6

(e) **Biomechanical Applications:** For a blood vessel under internal pressure, the stress state can be represented as:

$$\begin{aligned}
\sigma_x &= 100\,\text{MPa} \quad \text{(hoop stress)} \\
\sigma_y &= 50\,\text{MPa} \quad \text{(axial stress)} \\
\tau_{xy} &= 0\,\text{MPa}
\end{aligned}$$

The Mohr's circle for this stress state would have a center at:

$$\sigma_c = \frac{\sigma_x + \sigma_y}{2} = \frac{100 + 50}{2} = 75\,\text{MPa}$$

and a radius of:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{100 - 50}{2}\right)^2 + (0)^2} = 25\,\text{MPa}$$

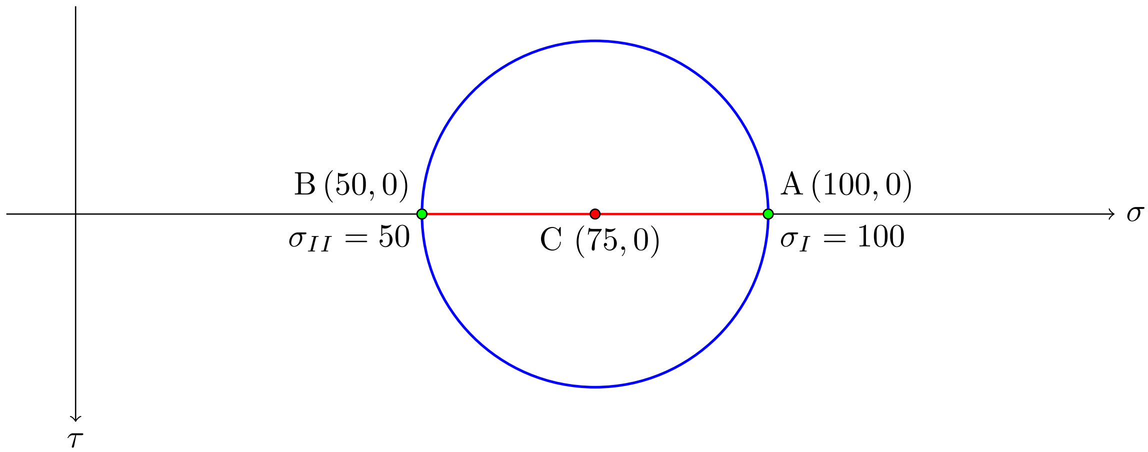

Figure 7

6. Find the traction-free planes (i.e., planes whose unit normal vectors make the traction vector vanish) passing through a point in a body subjected to the following stress state, expressed in the standard Cartesian basis:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & \sigma_0 & 0 \\ 1 & 0 & -3 \end{bmatrix} \text{ MPa.}$$

Also, determine the value of σ_0.

Solution: The given stress tensor is

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & \sigma_0 & 0 \\ 1 & 0 & -3 \end{bmatrix} \text{ MPa} \tag{30}$$

For a traction-free plane we require

$$[\boldsymbol{\sigma}]\,\boldsymbol{n} = \boldsymbol{0} \tag{31}$$

which implies

$$\det[\boldsymbol{\sigma}] = \begin{vmatrix} 1 & 2 & 1 \\ 2 & \sigma_0 & 0 \\ 1 & 0 & -3 \end{vmatrix} \tag{32}$$

$$\det[\boldsymbol{\sigma}] = -4(\sigma_0 - 3) \tag{33}$$

by eq. $\det[\boldsymbol{\sigma}] = 0$.

$$\boxed{\sigma_0 = 3} \tag{34}$$

Substituting $\sigma_0 = 3$ and solving $[\boldsymbol{\sigma}]\boldsymbol{n} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{35}$$

So,

$$n_1 = 3\,n_3, \qquad n_2 = -\frac{2}{3}\,n_1 = -2\,n_3, \qquad n_1 + 2\,n_2 = -\,n_3.$$

Hence, a nontrivial solution for the unit normals are

$$\boxed{\boldsymbol{n} = \pm\frac{3\boldsymbol{e}_1 - 2\boldsymbol{e}_2 + \boldsymbol{e}_3}{\sqrt{14}}} \tag{36}$$

The equation of the traction-free plane through $P_0(x_0, y_0, z_0)$ is $\boxed{\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x}_0) = 0}$, where $\boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3$ and $\boldsymbol{x}_0 = x_0\boldsymbol{e}_1 + y_0\boldsymbol{e}_2 + z_0\boldsymbol{e}_3$. Then, one can find the equation of the traction free plane as

$$3(x - x_0) - 2(y - y_0) + (z - z_0) = 0. \tag{37}$$

7. Suppose that at a point on the surface of a body the unit outward normal is

$$\boldsymbol{n} = \frac{\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3}{\sqrt{3}}$$

and the traction vector is

$$\boldsymbol{t} = P(\boldsymbol{e}_1 + 2\boldsymbol{e}_2),$$

where P is a constant.

(a) Determine the normal traction vector $\boldsymbol{t}_{nn}$ and the shear traction vector $\boldsymbol{t}_{ns}$ at this point on the surface of the body.

(b) Determine the conditions between the stress tensor components and the traction vector components.

Solution: Given

$$\boldsymbol{n} = \frac{\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3}{\sqrt{3}}, \qquad \boldsymbol{t} = P(\boldsymbol{e}_1 + 2\boldsymbol{e}_2),$$

with scalar P.

(a) The normal component of the traction is

$$\boldsymbol{t}_{nn} = (\boldsymbol{t} \cdot \boldsymbol{n})\,\boldsymbol{n}. \tag{38}$$

Compute the scalar product

$$\begin{aligned}\boldsymbol{t}\cdot\boldsymbol{n} &= P(\boldsymbol{e}_1+2\boldsymbol{e}_2)\cdot\frac{\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3}{\sqrt{3}}\\ &= \frac{P(1+2)}{\sqrt{3}} = \sqrt{3}\,P. \end{aligned} \tag{39}$$

Hence

$$\begin{aligned}\boldsymbol{t}_{nn} &= \sqrt{3}P\,\boldsymbol{n}\\ &= \sqrt{3}P\,\frac{\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3}{\sqrt{3}}\\ &= P(\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3).\end{aligned} \tag{40}$$

The shear traction is

$$\begin{aligned}\boldsymbol{t}_{ns} &= \boldsymbol{t}-\boldsymbol{t}_{nn}\\ &= P(\boldsymbol{e}_1+2\boldsymbol{e}_2)-P(\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3)\\ &= P(\boldsymbol{e}_2+\boldsymbol{e}_3).\end{aligned} \tag{41}$$

Thus,

$$[\boldsymbol{t}_{nn}] = P\begin{bmatrix}1\\1\\-1\end{bmatrix}, \qquad [\boldsymbol{t}_{ns}] = P\begin{bmatrix}0\\1\\1\end{bmatrix}. \tag{42}$$

(b) The traction vector is related to the stress tensor by

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} \tag{43}$$

$$\begin{bmatrix}P\\2P\\0\end{bmatrix} = [\boldsymbol{\sigma}]\,\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}. \tag{44}$$

Writing $\boldsymbol{\sigma}\boldsymbol{n}$, we have

$$[\boldsymbol{\sigma}]\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}\sigma_{11}+\sigma_{12}-\sigma_{13}\\ \sigma_{21}+\sigma_{22}-\sigma_{23}\\ \sigma_{31}+\sigma_{32}-\sigma_{33}\end{bmatrix}. \tag{45}$$

Equating and multiplying through by $\sqrt{3}$ gives

$$\boxed{\begin{aligned}&\sigma_{11}+\sigma_{12}-\sigma_{13} = \sqrt{3}\,P,\\ &\sigma_{21}+\sigma_{22}-\sigma_{23} = 2\sqrt{3}\,P,\\ &\sigma_{31}+\sigma_{32}-\sigma_{33} = 0.\end{aligned}} \tag{46}$$

Multiplying through by $\sqrt{3}$ gives the conditions:

$$\sigma_{11} + \sigma_{12} - \sigma_{13} = \sqrt{3}P, \quad \sigma_{21} + \sigma_{22} - \sigma_{23} = 2\sqrt{3}P, \quad \sigma_{31} + \sigma_{32} - \sigma_{33} = 0. \tag{47}$$

If $\boldsymbol{\sigma}$ is symmetric ($\sigma_{ij} = \sigma_{ji}$), the conditions reduce to

$$\sigma_{11} + \sigma_{12} - \sigma_{13} = \sqrt{3}P, \quad \sigma_{12} + \sigma_{22} - \sigma_{23} = 2\sqrt{3}P, \quad \sigma_{13} + \sigma_{23} - \sigma_{33} = 0. \tag{48}$$

8. At a certain point in a material under stress the intensity of the resultant stress on a vertical plane is 1000 N/cm^2, inclined at 30° to the normal to that plane, and the stress on a horizontal plane has a normal tensile component of intensity 600 N/cm^2 as shown in Fig. 8. Find the magnitude and direction of the resultant stress on the horizontal plane and the principal stresses.

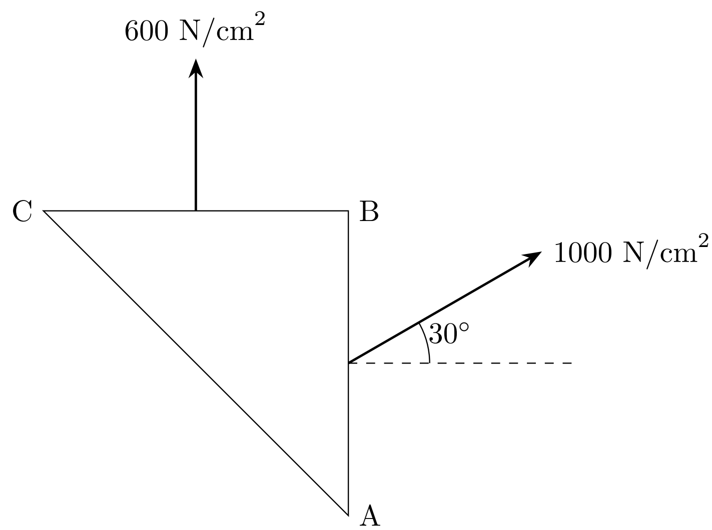

Figure 8

Solution: We are given that the intensity of the resultant stress on a vertical plane is

$$R_v = 1000 \text{ N/cm}^2, \quad \text{inclined at } 30^\circ \text{ to the normal.}$$

Also, the stress on a horizontal plane has a normal tensile component

$$\sigma_y = 600 \text{ N/cm}^2.$$

On the vertical plane, the normal is along $\boldsymbol{e}_x$:

$$\sigma_x = R_v \cos 30^\circ = 1000 \cos 30^\circ = 866.03 \text{ N/cm}^2, \tag{49}$$

$$\tau_{xy} = R_v \sin 30^\circ = 1000 \sin 30^\circ = 500 \text{ N/cm}^2. \tag{50}$$

The horizontal plane has normal stress $\sigma_y = 600$ N/cm^2 and shear $\tau_{xy} = 500$ N/cm^2. Thus the resultant traction vector $\boldsymbol{t}$ is

$$\|\boldsymbol{t}\| = \sqrt{\sigma_y^2 + \tau_{xy}^2} = \sqrt{600^2 + 500^2} = 781.02 \text{ N/cm}^2, \tag{51}$$

$$\varphi = \arctan\left(\tfrac{\tau_{xy}}{\sigma_y}\right) = \arctan\left(\tfrac{500}{600}\right) = 39.81^\circ, \tag{52}$$

where φ is the inclination of $\boldsymbol{t}$ to the normal $\boldsymbol{n}$.

The average normal stress is

$$\sigma = \frac{\sigma_x + \sigma_y}{2} = \frac{866.03 + 600}{2} = 733.01 \text{ N/cm}^2. \tag{53}$$

The radius of Mohr's circle is

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{54}$$

$$= \sqrt{\left(\frac{866.03 - 600}{2}\right)^2 + 500^2} = 517.39 \text{ N/cm}^2. \tag{55}$$

Thus the principal stresses are

$$\sigma_1 = \sigma + R = 733.01 + 517.39 = 1250.40 \text{ N/cm}^2, \tag{56}$$

$$\sigma_2 = \sigma - R = 733.01 - 517.39 = 215.62 \text{ N/cm}^2. \tag{57}$$

The orientation of the principal planes is.

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{1000}{266.03}, \tag{58}$$

$$2\theta_p = \arctan\left(\frac{1000}{266.03}\right), \tag{59}$$

$$\theta_p = 37.55^\circ. \tag{60}$$

Final Results:

$$\boxed{\begin{aligned} &\|\boldsymbol{t}\| = 781.02 \text{ N/cm}^2, \quad \varphi = 39.81^\circ, \\ &\sigma_1 = 1250.40 \text{ N/cm}^2, \quad \sigma_2 = 215.62 \text{ N/cm}^2, \\ &\theta_p = 37.55^\circ. \end{aligned}} \tag{61}$$

9. A thin walled pressure vessel shown in Fig. 9 (a) is subjected to an internal pressure which results in an axial normal stress $\sigma_a = 70$ MPa and a hoop stress $\sigma_h = 140$ MPa on its outer surface. An accidental event causes a torque to be applied on the entire structure causing a shear stress of magnitude equal to τ_{xy} as shown in Fig. 9 (b).

   (a) Use the stress element in Fig. 9 (b) to draw the Mohr's circle on the graph paper.

   (b) Use the Mohr's circle to calculate:

      i. The principal stresses at point A.

      ii. The maximum in-plane shear stress.

      iii. The absolute maximum shear stress.

      iv. The angle of rotation from the x-axis to the direction of the in-plane principal stress σ_{p1}.

      v. Draw a stress element to show the in-plane principal stresses correctly oriented with respect to the x-axis.

      vi. Draw a stress element to show the in-plane maximum shear stress correctly oriented with respect to the x-axis.

   (c) Use the Mohr's circle to calculate the normal and shear stresses in the x'–y' directions. Draw a stress element to show the calculated stresses, and mark the state of stress in the x'–y' directions on the Mohr's circle.
   Note: The x'–axis is oriented at 30° from the x–axis as shown in Fig. 9 (b).

   (d) What would the value of the principal stresses and maximum in-plane shear stresses acting at point A be if the accidental torque could be avoided? Explain your answer.

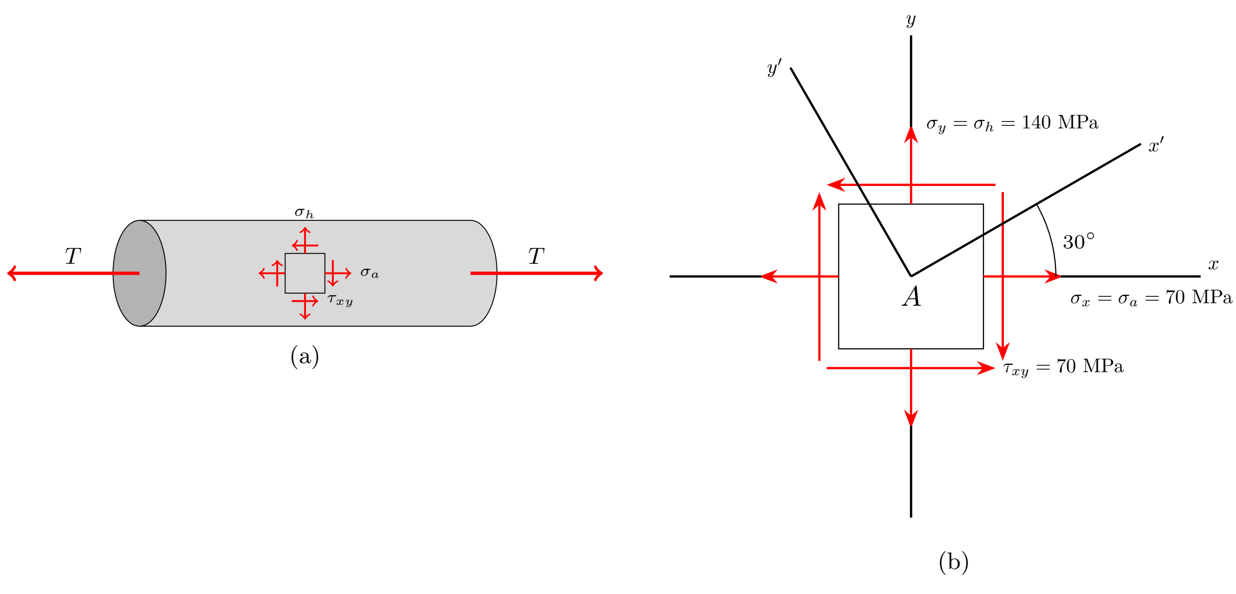

Figure 9

Solution: Given:

$$\begin{aligned}\sigma_x &= \sigma_a = 70\,\text{MPa}\\ \sigma_y &= \sigma_h = 140\,\text{MPa}\\ \tau_{xy} &= -70\,\text{MPa}\end{aligned}$$

Center and Radius of Mohr's Circle

The center of Mohr's circle is given by:

$$\begin{aligned}C &= \left(\frac{\sigma_x + \sigma_y}{2}, 0\right)\\ &= \left(\frac{70 + 140}{2}, 0\right)\\ &= (105, 0)\end{aligned}$$

The radius of Mohr's circle is given by:

$$\begin{aligned}R &= \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\\ &= \sqrt{\left(\frac{70 - 140}{2}\right)^2 + (-70)^2}\\ &= \sqrt{(-35)^2 + (-70)^2}\\ &= \sqrt{1225 + 4900}\\ &= \sqrt{6125}\\ &= 78.26\,\text{MPa}\end{aligned}$$

The principal stresses are given by:

$$\begin{aligned}\sigma_{1,2} &= C \pm R\\ &= 105 \pm 78.26\\ \sigma_1 &= 183.26\,\text{MPa}\\ \sigma_2 &= 26.74\,\text{MPa}\end{aligned}$$

The maximum in-plane shear stress is given by:

$$\begin{aligned}\tau_{max} &= R\\ &= 78.26\,\text{MPa}\end{aligned}$$

The absolute maximum shear stress is given by:

$$
\begin{aligned}
\tau_{abs} &= \max\left(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_1|}{2}, \frac{|\sigma_2|}{2}\right) \\
&= \max\left(\frac{|183.26 - 26.74|}{2}, \frac{|183.26|}{2}, \frac{|26.74|}{2}\right) \\
&= \max\left(78.26, 91.63, 13.37\right) \\
&= 91.63\,\mathrm{MPa}
\end{aligned}
$$

The angle of rotation from the x-axis to the direction of the in-plane principal stress σ_{p1} is given by:

$$
\begin{aligned}
\tan(2\theta) &= \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \\
&= \frac{2(-70)}{70 - 140} \\
&= \frac{-140}{-70} \\
&= 2 \\
2\theta &= \tan^{-1}(2) \\
&= 63.43^\circ \\
\theta &= 31.71^\circ
\end{aligned}
$$

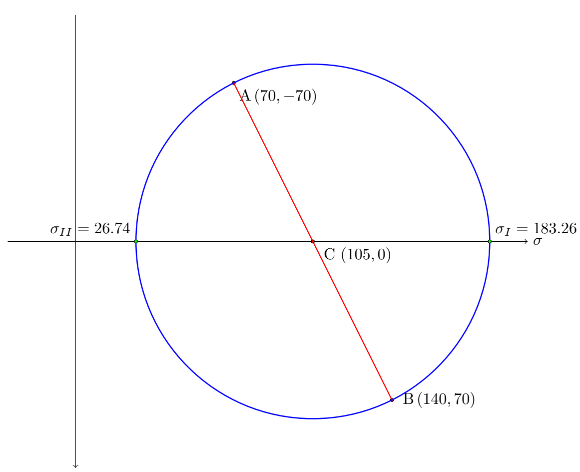

Figure 10

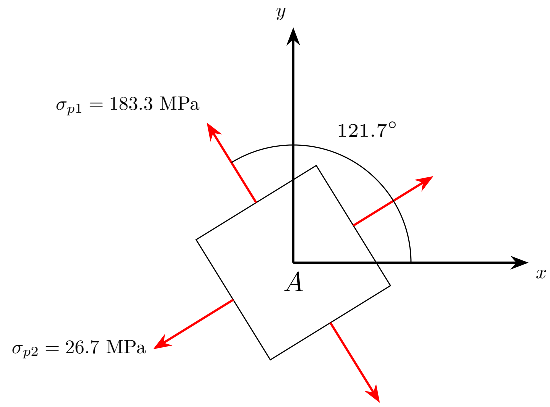

Figure 11

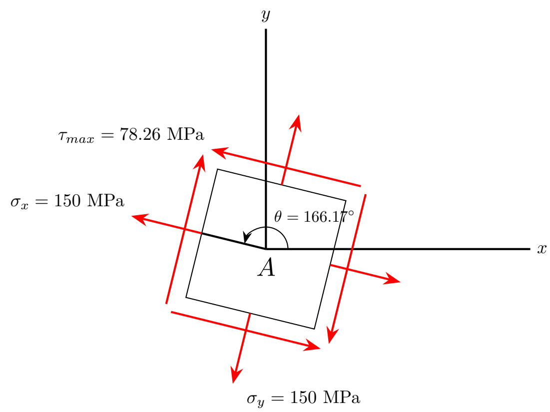

Figure 12

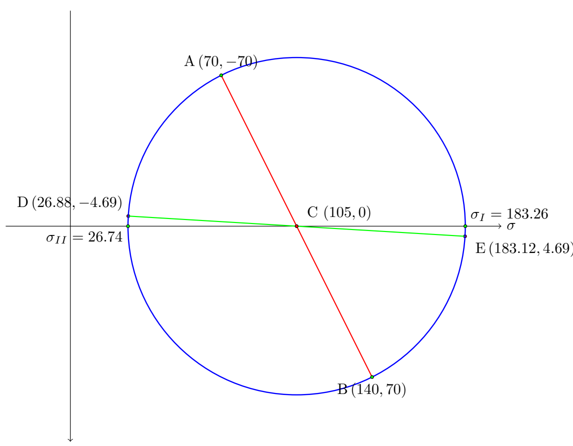

Figure 13