Indian Institute of Technology Bhubaneswar

School of Infrastructure

Subject Name : Solid Mechanics	Subject Code: CE2L001

Tutorial No. 6

1. A 45-degree strain rosette is used to measure strains on a steel surface. The readings are $\epsilon_a = 100\,\mu$m/m, $\epsilon_b = 400\,\mu$m/m, and $\epsilon_c = 900\,\mu$m/m. Determine the principal strains and stresses.

Solution: Calculate strain components ϵ_{11}, ϵ_{22}, and γ_{12}:

$$\begin{aligned}
\epsilon_{11} &= \epsilon_a = 100\,\mu\text{m/m} \\
\epsilon_{22} &= \epsilon_c = 900\,\mu\text{m/m} \\
\gamma_{12} &= 2\epsilon_b - (\epsilon_a + \epsilon_c) = 2 \times 400 - (100 + 900) = -200\,\mu\text{m/m}
\end{aligned}$$

Calculate principal strains:

$$\begin{aligned}
\epsilon_{I,II} &= \frac{\epsilon_{11} + \epsilon_{22}}{2} \pm \sqrt{\left(\frac{\epsilon_{11} - \epsilon_{22}}{2}\right)^2 + \left(\frac{\gamma_{12}}{2}\right)^2} \\
\epsilon_{I,II} &= \frac{100 + 900}{2} \pm \sqrt{\left(\frac{100 - 900}{2}\right)^2 + \left(\frac{-200}{2}\right)^2} \\
\epsilon_I &= 912.31\,\mu\text{m/m}, \quad \epsilon_{II} = 87.69\,\mu\text{m/m}
\end{aligned}$$

Calculate principal stresses:

$$\begin{aligned}
\sigma_I &= \frac{E}{1 - \nu^2}(\epsilon_I + \nu\epsilon_{II}) \\
\sigma_{II} &= \frac{E}{1 - \nu^2}(\epsilon_{II} + \nu\epsilon_I)
\end{aligned}$$

For a given steel with $E = 205\,$GPa and $\nu = 0.29$, we can calculate the principal stresses as $\sigma_I = 209.85$ MPa and $\sigma_{II} = 78.83$ MPa.

2. Consider a 60° strain gauge rosette to be mounted on the surface of a specimen as shown in Fig. 1.

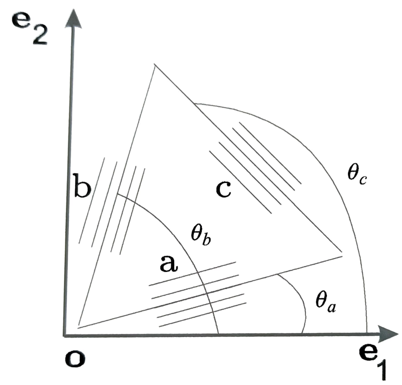

Figure 1: Schematic representation of a 60° strain-gauge rosette.

(a) Let $\{\boldsymbol{e}_a, \boldsymbol{e}_b, \boldsymbol{e}_c\}$ denote three non-collinear unit vectors which represent the directions in which the three strain gauges in a rosette are arranged, and let

$$E_{aa} = \boldsymbol{e}_a \cdot \boldsymbol{E}\boldsymbol{e}_a, \qquad E_{bb} = \boldsymbol{e}_b \cdot \boldsymbol{E}\boldsymbol{e}_b, \qquad E_{cc} = \boldsymbol{e}_c \cdot \boldsymbol{E}\boldsymbol{e}_c,$$

denote the components of the strain measured in the directions $\{\boldsymbol{e}_a, \boldsymbol{e}_b, \boldsymbol{e}_c\}$. Determine a general expression for the components of strain, $\boldsymbol{E}$, (i.e E_{11}, E_{22}, $E_{12} = E_{21}$) with respect to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, as functions of ($E_{aa}$, E_{bb}, E_{cc}) and the orientations $(\theta_a, \theta_b, \theta_c)$.

(b) Evaluate E_{11}, E_{22} and E_{12} for $\theta_a = 0°$, $\theta_b = 60°$, and $\theta_c = 120°$.

Solution: (a) Given that the components of the strain measured in the directions $\{\boldsymbol{e}_a, \boldsymbol{e}_b, \boldsymbol{e}_c\}$ is

$$E_{aa} = \boldsymbol{e}_a \cdot \boldsymbol{E}\boldsymbol{e}_a, \qquad E_{bb} = \boldsymbol{e}_b \cdot \boldsymbol{E}\boldsymbol{e}_b, \qquad E_{cc} = \boldsymbol{e}_c \cdot \boldsymbol{E}\boldsymbol{e}_c \tag{1}$$

where the strain $\boldsymbol{E}$ with respect to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ can be defined as $\boldsymbol{E} = E_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Thus Eq. (1) can be expressed as

$$\begin{aligned}
E_{aa} &= \boldsymbol{e}_a \cdot (E_{ij}\,(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{e}_a) \\
&= \boldsymbol{e}_a \cdot (E_{ij}(\boldsymbol{e}_j \cdot \boldsymbol{e}_a)\boldsymbol{e}_i) \qquad \text{since} \quad (\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{e}_a = (\boldsymbol{e}_j \cdot \boldsymbol{e}_a)\boldsymbol{e}_i \\
&= E_{ij}(\boldsymbol{e}_i \cdot \boldsymbol{e}_a)(\boldsymbol{e}_j \cdot \boldsymbol{e}_a) \\
\implies E_{aa} &= E_{11}(\boldsymbol{e}_1 \cdot \boldsymbol{e}_a)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_a) + E_{12}(\boldsymbol{e}_1 \cdot \boldsymbol{e}_a)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_a) + E_{21}(\boldsymbol{e}_2 \cdot \boldsymbol{e}_a)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_a) + E_{22}(\boldsymbol{e}_2 \cdot \boldsymbol{e}_a)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_a)
\end{aligned}$$

Similarly E_{bb} and E_{cc} can be determined as

$$\begin{aligned}
E_{bb} &= \boldsymbol{e}_b \cdot (E_{ij}\,(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{e}_b) \\
&= \boldsymbol{e}_b \cdot (E_{ij}(\boldsymbol{e}_j \cdot \boldsymbol{e}_b)\boldsymbol{e}_i) \qquad \text{since} \quad (\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{e}_b = (\boldsymbol{e}_j \cdot \boldsymbol{e}_b)\boldsymbol{e}_i \\
&= E_{ij}(\boldsymbol{e}_i \cdot \boldsymbol{e}_b)(\boldsymbol{e}_j \cdot \boldsymbol{e}_b)
\end{aligned}$$

$$\implies E_{bb} = E_{11}(\boldsymbol{e}_1\cdot\boldsymbol{e}_b)(\boldsymbol{e}_1\cdot\boldsymbol{e}_b) + E_{12}(\boldsymbol{e}_1\cdot\boldsymbol{e}_b)(\boldsymbol{e}_2\cdot\boldsymbol{e}_b) + E_{21}(\boldsymbol{e}_2\cdot\boldsymbol{e}_b)(\boldsymbol{e}_1\cdot\boldsymbol{e}_b) + E_{22}(\boldsymbol{e}_2\cdot\boldsymbol{e}_b)(\boldsymbol{e}_2\cdot\boldsymbol{e}_b),$$

and

$$\begin{aligned}
E_{cc} &= \boldsymbol{e}_c\cdot\left(E_{ij}\,(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\boldsymbol{e}_c\right)\\
&= \boldsymbol{e}_c\cdot\left(E_{ij}(\boldsymbol{e}_j\cdot\boldsymbol{e}_c)\boldsymbol{e}_i\right) \qquad \text{since} \quad (\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\boldsymbol{e}_c = (\boldsymbol{e}_j\cdot\boldsymbol{e}_c)\boldsymbol{e}_i\\
&= E_{ij}(\boldsymbol{e}_i\cdot\boldsymbol{e}_c)(\boldsymbol{e}_j\cdot\boldsymbol{e}_c)
\end{aligned}$$

$$\implies E_{cc} = E_{11}(\boldsymbol{e}_1\cdot\boldsymbol{e}_c)(\boldsymbol{e}_1\cdot\boldsymbol{e}_c) + E_{12}(\boldsymbol{e}_1\cdot\boldsymbol{e}_c)(\boldsymbol{e}_2\cdot\boldsymbol{e}_c) + E_{21}(\boldsymbol{e}_2\cdot\boldsymbol{e}_c)(\boldsymbol{e}_1\cdot\boldsymbol{e}_c) + E_{22}(\boldsymbol{e}_2\cdot\boldsymbol{e}_c)(\boldsymbol{e}_2\cdot\boldsymbol{e}_c)$$

From Fig. 1, the dot product between the basis vectors can be determined as follows:

$$\begin{aligned}
\boldsymbol{e}_1\cdot\boldsymbol{e}_a &= \cos\theta_a, \quad \boldsymbol{e}_1\cdot\boldsymbol{e}_b = \cos\theta_b, \quad \boldsymbol{e}_1\cdot\boldsymbol{e}_c = \cos\theta_c\\
\boldsymbol{e}_2\cdot\boldsymbol{e}_a &= \sin\theta_a, \quad \boldsymbol{e}_2\cdot\boldsymbol{e}_b = \sin\theta_b, \quad \boldsymbol{e}_2\cdot\boldsymbol{e}_c = \sin\theta_c
\end{aligned}$$

Thus the components of the strain measured in the directions $\{\boldsymbol{e}_a, \boldsymbol{e}_b, \boldsymbol{e}_c\}$ can be expressed as (by considering a symmetric tensor i.e, $E_{12} = E_{21}$) as

$$\begin{aligned}
E_{aa} &= E_{11}\cos^2\theta_a + 2E_{12}\cos\theta_a\sin\theta_a + E_{22}\sin^2\theta_a\\
E_{bb} &= E_{11}\cos^2\theta_b + 2E_{12}\cos\theta_b\sin\theta_b + E_{22}\sin^2\theta_b\\
E_{cc} &= E_{11}\cos^2\theta_c + 2E_{12}\cos\theta_c\sin\theta_c + E_{22}\sin^2\theta_c,
\end{aligned} \tag{2}$$

which can be expressed in the matrix form as

$$\begin{bmatrix} E_{aa}\\ E_{bb}\\ E_{cc}\end{bmatrix} = \begin{bmatrix}\cos^2\theta_a & 2\cos\theta_a\sin\theta_a & \sin^2\theta_a\\ \cos^2\theta_b & 2\cos\theta_b\sin\theta_b & \sin^2\theta_b\\ \cos^2\theta_c & 2\cos\theta_c\sin\theta_c & \sin^2\theta_c\end{bmatrix}\begin{bmatrix}E_{11}\\ E_{12}\\ E_{22}\end{bmatrix}$$

Thus, the general expression for the components of strain, $\boldsymbol{E}$, (i.e E_{11}, E_{22}, $E_{12} = E_{21}$) with respect to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ can be found as

$$\begin{bmatrix}E_{11}\\ E_{12}\\ E_{22}\end{bmatrix} = \begin{bmatrix}\cos^2\theta_a & 2\cos\theta_a\sin\theta_a & \sin^2\theta_a\\ \cos^2\theta_b & 2\cos\theta_b\sin\theta_b & \sin^2\theta_b\\ \cos^2\theta_c & 2\cos\theta_c\sin\theta_c & \sin^2\theta_c\end{bmatrix}^{-1}\begin{bmatrix} E_{aa}\\ E_{bb}\\ E_{cc}\end{bmatrix} \tag{3}$$

(b) Given that $\theta_a = 0°$, $\theta_b = 60°$, and $\theta_c = 120°$, the components of strain, E_{11}, E_{22}, E_{12} can be determined by substituting the values of θ_a, θ_b, θ_c in Eq. (3) as

$$\begin{bmatrix}E_{11}\\ E_{12}\\ E_{22}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ \frac{1}{4} & \frac{\sqrt{3}}{2} & \frac{3}{4}\\ \frac{1}{4} & -\frac{\sqrt{3}}{2} & \frac{3}{4}\end{bmatrix}^{-1}\begin{bmatrix} E_{aa}\\ E_{bb}\\ E_{cc}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}}\\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{bmatrix}\begin{bmatrix} E_{aa}\\ E_{bb}\\ E_{cc}\end{bmatrix}. \tag{4}$$

The values of E_{11}, E_{22}, E_{12} can be found from Eq. (4) as

$$E_{11} = E_{aa}, \qquad E_{12} = \frac{E_{bb} - E_{cc}}{\sqrt{3}}, \qquad E_{22} = \frac{-E_{aa} + 2E_{bb} + 2E_{cc}}{3}.$$

3. Stress is not a directly measurable quantity for most materials and is usually computed from the strain measurements in a complex engineering system. A common method for measuring the state of strain is to use *strain gauges* which are simple electrical devices that can measure only the normal strain along its length.

   A strain rosette having three strain gauges a, b and c is installed on a block as shown in Fig.2. During a static test of the block in plane strain ($\epsilon_{zz} = 0, \gamma_{xz} = 0$ and $\gamma_{yz} = 0$), the strain rosettes read $\epsilon_a = 0.003$, $\epsilon_b = 0.001$ and $\epsilon_c = 0.001$.

   1) Calculate the shear strain γ_{xy} for an element oriented along the xy plane (Round your answer to 4 decimal points). Note that for a strain gauge oriented at an angle of θ to the x-axis , the gauge reading ϵ_θ can be expressed as:

   $$\epsilon_\theta = \epsilon_{xx}\cos^2(\theta) + \epsilon_{yy}\sin^2(\theta) + \gamma_{xy}\sin(\theta)\cos(\theta)$$

   2) If the block is made of a material with elastic modulus $E = 100$ GPa and Poisson's ratio $\nu = 0.3$, use Hooke's law to find the stress components in the $x - y$ plane.

   3) Determine the principal stresses in the block.

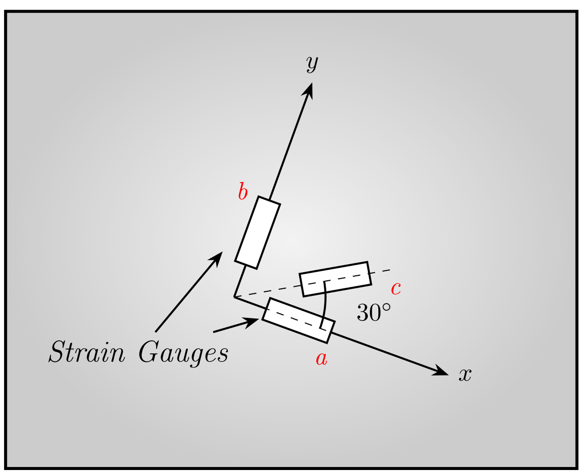

Figure 2: Strain Gauges

Solution: We use the given strain transformation equation for each gauge:

- Gauge a ($\theta_a = 0°$):

$$\begin{aligned}
\epsilon_a &= \epsilon_{xx}\cos^2(0°) + \epsilon_{yy}\sin^2(0°) + \gamma_{xy}\sin(0°)\cos(0°) \\
0.003 &= \epsilon_{xx}(1) + \epsilon_{xy}(0) + \gamma_{xy}(0) \\
\implies \epsilon_{xx} &= 0.003
\end{aligned}$$

- Gauge b ($\theta_b = 90°$):

$$\begin{aligned}
\epsilon_b &= \epsilon_{xx}\cos^2(90°) + \epsilon_{yy}\sin^2(90°) + \gamma_{xy}\sin(90°)\cos(90°) \\
0.001 &= \epsilon_{xx}(0) + \epsilon_{yy}(1) + \gamma_{xy}(0) \\
\implies \epsilon_{yy} &= 0.001
\end{aligned}$$

- Gauge c ($\theta_c = 30°$):

$$\begin{aligned}
\epsilon_c &= \epsilon_{xx}\cos^2(30°) + \epsilon_{yy}\sin^2(30°) + \gamma_{xy}\sin(30°)\cos(30°) \\
0.001 &= (0.003)(\cos 30°)^2 + (0.001)(\sin 30°)^2 + \gamma_{xy}(\sin 30°)(\cos 30°) \\
0.001 &= (0.003)(0.75) + (0.001)(0.25) + \gamma_{xy}(0.5)(0.866) \\
0.001 &= 0.00225 + 0.00025 + 0.433\gamma_{xy} \\
0.001 &= 0.0025 + 0.433\gamma_{xy} \\
-0.0015 &= 0.433\gamma_{xy} \\
\implies \gamma_{xy} &\approx -0.0035
\end{aligned}$$

First, calculate the shear modulus G:

$$G = \frac{E}{2(1+\nu)} = \frac{100 \text{ GPa}}{2(1+0.3)} = 38.46 \text{ GPa}$$

The shear stress τ_{xy} is:

$$\tau_{xy} = G\gamma_{xy} = (38.46 \text{ GPa})(-0.0035) = -0.1346 \text{ GPa}$$

For plane strain ($\epsilon_{zz} = 0$), we use the 3D Hooke's Law equations:

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \\
\epsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] \\
\epsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] = 0
\end{aligned}$$

Substituting known values (with E=100 GPa):

$$
\begin{aligned}
0.003 &= \frac{1}{100}[\sigma_{xx} - 0.3(\sigma_{yy} + \sigma_{zz})] \implies 300 = \sigma_{xx} - 0.3\sigma_{yy} - 0.3\sigma_{zz} \\
0.001 &= \frac{1}{100}[\sigma_{yy} - 0.3(\sigma_{xx} + \sigma_{zz})] \implies 100 = \sigma_{yy} - 0.3\sigma_{xx} - 0.3\sigma_{zz} \\
0 &= \frac{1}{100}[\sigma_{zz} - 0.3(\sigma_{xx} + \sigma_{yy})] \implies \sigma_{zz} = 0.3(\sigma_{xx} + \sigma_{yy})
\end{aligned}
$$

Solving this system of 3 equations (all stresses in GPa):

$$
\sigma_{xx} = 0.4615 \text{ GPa}, \quad \sigma_{yy} = 0.3077 \text{ GPa}, \quad \sigma_{zz} = 0.2308 \text{ GPa}.
$$

Since the problem is in plane strain ($\gamma_{xz} = 0, \gamma_{yz} = 0$), there are no shear stresses on the z-face. This means the z-axis is a principal direction. Therefore, one of the principal stresses is σ_{zz}:

$$
\sigma_{II} = \sigma_{zz} = 0.2308 \text{ GPa}
$$

The other two principal stresses, σ_I and σ_{III}, lie in the xy-plane. We find them using the center and radius of the Mohr's circle for the xy-plane:

$$
\sigma_{ave} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{0.4615 + 0.3077}{2} = 0.3846 \text{ GPa}
$$

$$
R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{0.4615 - 0.3077}{2}\right)^2 + (-0.1346)^2} = \sqrt{(0.0769)^2 + (-0.1346)^2} = 0.1550 \text{ GPa}
$$

The principal stresses are:

$$
\begin{aligned}
\sigma_I &= \sigma_{ave} + R = 0.3846 + 0.1550 = 0.5396 \text{ GPa} \quad \text{(Max)} \\
\sigma_{III} &= \sigma_{ave} - R = 0.3846 - 0.1550 = 0.2296 \text{ GPa} \quad \text{(Min)}
\end{aligned}
$$

So $\sigma_I = 0.5396$ GPa, $\sigma_{II} = 0.2308$ GPa, $\sigma_{III} = 0.2296$ GPa.

4. A rectangular strain rosette (0-90 degrees) is used to measure strains on an aluminum surface. The readings are $\epsilon_{11} = 200\,\mu$m/m and $\epsilon_{22} = 100\,\mu$m/m. Determine the strain measured by a 45-degree gauge.

Solution: The normal strain measured by a gauge oriented at an angle θ from the x_1-axis is given by the strain transformation equation:

$$
\epsilon_\theta = \epsilon_{11}\cos^2\theta + \epsilon_{22}\sin^2\theta + \frac{\gamma_{12}}{2}\sin 2\theta. \tag{5}
$$

For a 45-degree gauge, we have:

$$\cos^2 45° = \sin^2 45° = \frac{1}{2}, \qquad \sin 2(45°) = 1.$$

Thus the strain at 45 degrees becomes:

$$\epsilon_{45°} = \frac{\epsilon_{11} + \epsilon_{22}}{2} + \frac{\gamma_{12}}{2}. \tag{6}$$

Since a rectangular rosette does not measure the in-plane shear strain γ_{12}, it is common to assume

$$\gamma_{12} = 0,$$

unless additional data are provided. With this assumption:

$$\epsilon_{45°} = \frac{200 + 100}{2} = 150\,\mu\text{m/m}.$$

So,

$$\boxed{\epsilon_{45°} = 150\,\mu\text{m/m}}$$

5. A rectangular strain rosette is attached to a steel plate with gauge angles of 0°, 45°, and 90°. If the measured strains are $1000\,\mu\varepsilon$, $800\,\mu\varepsilon$, and $1200\,\mu\varepsilon$, respectively, calculate the principal strains and stresses.

Solution: Given a rectangular strain rosette with gauges at 0°, 45°, and 90° measures:

$$\epsilon_{0°} = \epsilon_{11} = 1000\ \mu\varepsilon, \qquad \epsilon_{45°} = 800\ \mu\varepsilon, \qquad \epsilon_{90°} = \epsilon_{22} = 1200\ \mu\varepsilon.$$

The strain at 45° can be written as:

$$\epsilon_{45°} = \frac{\epsilon_{11} + \epsilon_{22}}{2} + \frac{\gamma_{12}}{2}. \tag{7}$$

Solving for the engineering shear strain γ_{12}:

$$\gamma_{12} = 2\epsilon_{45°} - (\epsilon_{11} + \epsilon_{22}) = 2(800) - (1000 + 1200) = 1600 - 2200 = -600\ \mu\varepsilon.$$

The in-plane strain tensor is:

$$[\mathbf{E}] = \begin{bmatrix} \epsilon_{11} & \gamma_{12}/2 \\ \gamma_{12}/2 & \epsilon_{22} \end{bmatrix} = \begin{bmatrix} 1000 & -300 \\ -300 & 1200 \end{bmatrix}\ \mu\varepsilon.$$

The average strain is:

$$\bar{\epsilon} = \frac{\epsilon_{11} + \epsilon_{22}}{2} = \frac{1000 + 1200}{2} = 1100\ \mu\varepsilon.$$

The strain radius is

$$R = \sqrt{\left(\frac{\epsilon_{11} - \epsilon_{22}}{2}\right)^2 + \left(\frac{\gamma_{12}}{2}\right)^2} = \sqrt{(-100)^2 + (-300)^2} = \sqrt{10000 + 90000} = 316.2278\ \mu\varepsilon.$$

Thus the principal strains are:

$$\epsilon_I = \bar{\epsilon} + R = 1100 + 316.2278 = 1416.23\ \mu\varepsilon,$$

$$\epsilon_{II} = \bar{\epsilon} - R = 1100 - 316.2278 = 783.77\ \mu\varepsilon.$$

The principal direction is:

$$\tan(2\theta_p) = \frac{\gamma_{12}}{\epsilon_{11} - \epsilon_{22}} = \frac{-600}{1000 - 1200} = \frac{-600}{-200} = 3.$$

So

$$2\theta_p = \arctan(3), \qquad \theta_p \approx 35.78^\circ.$$

Assume, plane stress with:

$$E = 210\ \text{GPa}, \qquad \nu = 0.30.$$

Hooke's law in principal directions:

$$\sigma_I = \frac{E}{1-\nu^2}(\epsilon_I + \nu\epsilon_{II}), \qquad \sigma_{II} = \frac{E}{1-\nu^2}(\epsilon_{II} + \nu\epsilon_I).$$

Converting $\mu\varepsilon$ to strain:

$$\epsilon_I = 1.41623 \times 10^{-3}, \qquad \epsilon_{II} = 0.78377 \times 10^{-3}.$$

Then:

$$\sigma_I = \frac{210 \times 10^3}{1 - 0.3^2}\left(1.41623 \times 10^{-3} + 0.3(0.78377 \times 10^{-3})\right) \approx 381.08\ \text{MPa},$$

$$\sigma_{II} = \frac{210 \times 10^3}{1 - 0.3^2}\left(0.78377 \times 10^{-3} + 0.3(1.41623 \times 10^{-3})\right) \approx 278.92\ \text{MPa}.$$

Hence,

$$\boxed{\gamma_{12} = -600\ \mu\varepsilon}$$

$$\boxed{\epsilon_I = 1416.23\ \mu\varepsilon, \qquad \epsilon_{II} = 783.77\ \mu\varepsilon}$$

$$\boxed{\theta_p \approx 35.78^\circ}$$

$$\boxed{\sigma_I \approx 381.08\ \text{MPa}, \qquad \sigma_{II} \approx 278.92\ \text{MPa}}$$

6. A delta strain rosette with gauge angles of 0°, 60°, and 120° measures strains of 500 $\mu\varepsilon$, 700 $\mu\varepsilon$, and 900 $\mu\varepsilon$. Determine the principal strains, principal stresses, and maximum shear stress.

Solution: Given a delta strain rosette has gauges at angles 0°, 60°, and 120° and records:

$$\epsilon_{0^\circ} = 500\ \mu\varepsilon, \qquad \epsilon_{60^\circ} = 700\ \mu\varepsilon, \qquad \epsilon_{120^\circ} = 900\ \mu\varepsilon.$$

We assume plane stress and take typical steel elastic constants:

$$E = 210\ \text{GPa}, \qquad \nu = 0.30.$$

The general strain transformation for a gauge at angle θ is

$$\epsilon_\theta = \epsilon_{xx}\cos^2\theta + \epsilon_{yy}\sin^2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta.$$

Applying this for $\theta = 0°, 60°, 120°$ gives the linear system

$$\begin{aligned}
\epsilon_{0^\circ} &= \epsilon_{xx}, \\
\epsilon_{60^\circ} &= \epsilon_{xx}\cos^2 60^\circ + \epsilon_{yy}\sin^2 60^\circ + \frac{\gamma_{xy}}{2}\sin 120^\circ, \\
\epsilon_{120^\circ} &= \epsilon_{xx}\cos^2 120^\circ + \epsilon_{yy}\sin^2 120^\circ + \frac{\gamma_{xy}}{2}\sin 240^\circ.
\end{aligned}$$

Solving the above three equations yields

$$\epsilon_{xx} = 500\ \mu\varepsilon, \qquad \epsilon_{yy} = 900\ \mu\varepsilon, \qquad \gamma_{xy} \approx -230.9401\ \mu\varepsilon.$$

So, the in-plane strain tensor:

$$[\mathbf{E}] = \begin{bmatrix} \epsilon_{xx} & \frac{\gamma_{xy}}{2} \\ \frac{\gamma_{xy}}{2} & \epsilon_{yy} \end{bmatrix} = \begin{bmatrix} 500 & -115.47005 \\ -115.47005 & 900 \end{bmatrix} \mu\varepsilon.$$

The average strain and radius are

$$\bar{\epsilon} = \frac{\epsilon_{xx} + \epsilon_{yy}}{2} = 700\ \mu\varepsilon, \qquad R = \sqrt{\left(\frac{\epsilon_{xx} - \epsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \approx 230.9401\ \mu\varepsilon.$$

Hence the principal strains are

$$\begin{aligned}
\epsilon_I &= \bar{\epsilon} + R \approx 930.9401\ \mu\varepsilon, \\
\epsilon_{II} &= \bar{\epsilon} - R \approx 469.0599\ \mu\varepsilon.
\end{aligned}$$

The principal direction (angle from the x-axis to the ϵ_I principal axis) satisfies

$$\tan(2\theta_p) = \frac{\gamma_{xy}}{\epsilon_{xx} - \epsilon_{yy}} = \frac{-230.9401}{500 - 900} = \frac{-230.9401}{-400} = 0.5773503.$$

So

$$2\theta_p = \arctan(0.5773503) \quad \Rightarrow \quad \theta_p \approx -75^\circ.$$

Using Hooke's law for plane stress in principal directions:

$$\sigma_I = \frac{E}{1-\nu^2}\big(\epsilon_I + \nu\epsilon_{II}\big), \qquad \sigma_{II} = \frac{E}{1-\nu^2}\big(\epsilon_{II} + \nu\epsilon_I\big).$$

Convert principal strains to dimensionless form ($1\ \mu\varepsilon = 10^{-6}$) and compute:

$$\epsilon_I = 930.9401 \times 10^{-6}, \qquad \epsilon_{II} = 469.0599 \times 10^{-6}.$$

Numerically (with $E = 210$ GPa $= 210000$ MPa):

$$\sigma_I \approx 247.306 \text{ MPa}, \qquad \sigma_{II} \approx 172.694 \text{ MPa}.$$

For plane stress the maximum in-plane shear stress equals half the difference of the principal stresses:

$$\tau_{\max} = \frac{\sigma_I - \sigma_{II}}{2} \approx \frac{247.306 - 172.694}{2} \approx 37.3057 \text{ MPa}.$$

Hence,

$$\boxed{\epsilon_{xx} = 500\ \mu\varepsilon, \quad \epsilon_{yy} = 900\ \mu\varepsilon, \quad \gamma_{xy} = -230.9401\ \mu\varepsilon}$$

$$\boxed{\epsilon_I \approx 930.94\ \mu\varepsilon, \quad \epsilon_{II} \approx 469.06\ \mu\varepsilon}$$

$$\boxed{\theta_p \approx -75^\circ}$$

$$\boxed{\sigma_I \approx 247.31 \text{ MPa}, \quad \sigma_{II} \approx 172.69 \text{ MPa}}$$

$$\boxed{\tau_{\max} \approx 37.31 \text{ MPa}}$$

(**Note:** You should give the above expressions in terms of E and ν if you want to make the expression generic for any material.).

7. A strain gauge rosette is used to measure the strain on a machine component. If the measured strains are $2000\,\mu\varepsilon$, $1500\,\mu\varepsilon$, and $2500\,\mu\varepsilon$ at angles of 0°, 45°, and 90°, calculate the normal strain in the x-direction, normal strain in the y-direction, and shear strain in $x-y$ plane.

Solution: Given a rectangular strain rosette with gauges at 0°, 45°, and 90° measures:

$$\epsilon_{0^\circ} = 2000\ \mu\varepsilon, \qquad \epsilon_{45^\circ} = 1500\ \mu\varepsilon, \qquad \epsilon_{90^\circ} = 2500\ \mu\varepsilon.$$

Normal Strain in the x-Direction

$$\epsilon_{xx} = \epsilon_{0^\circ} = 2000\ \mu\varepsilon.$$

Normal Strain in the y-Direction

$$\epsilon_{yy} = \epsilon_{90^\circ} = 2500\ \mu\varepsilon.$$

For a 0°–45°–90° rosette, the strain at 45° is

$$\epsilon_{45^\circ} = \frac{\epsilon_{xx} + \epsilon_{yy}}{2} + \frac{\gamma_{xy}}{2}.$$

Solving for the engineering shear strain:

$$\gamma_{xy} = 2\epsilon_{45^\circ} - (\epsilon_x + \epsilon_y) = 2(1500) - (2000 + 2500) = 3000 - 4500 = -1500\ \mu\varepsilon.$$

Hence,

$$\boxed{\epsilon_{xx} = 2000\ \mu\varepsilon, \quad \epsilon_{yy} = 2500\ \mu\varepsilon, \quad \gamma_{xy} = -1500\ \mu\varepsilon}$$

8. A material with a Young's modulus of 200 GPa and Poisson's ratio of 0.3 is subjected to a strain gauge rosette measurement. If the principal strains are 1500 $\mu\varepsilon$ and 800 $\mu\varepsilon$, calculate the principal stresses and maximum shear stress.

Solution: Given the material properties are:

$$E = 200 \text{ GPa} = 200000 \text{ MPa}, \qquad \nu = 0.3.$$

The principal strains are:

$$\varepsilon_I = 1500\ \mu\varepsilon = 1500 \times 10^{-6}, \qquad \varepsilon_{II} = 800\ \mu\varepsilon = 800 \times 10^{-6}.$$

Assume plane stress conditions. For plane stress, Hooke's law in the principal directions gives:

$$\sigma_I = \frac{E}{1-\nu^2}\left(\varepsilon_I + \nu\varepsilon_{II}\right), \qquad \sigma_{II} = \frac{E}{1-\nu^2}\left(\varepsilon_{II} + \nu\varepsilon_I\right).$$

Compute the elastic constant:

$$\frac{E}{1-\nu^2} = \frac{200000}{1-0.3^2} = \frac{200000}{0.91} \approx 219780.22 \text{ MPa}.$$

Now evaluate the stresses:

$$\sigma_I \approx 219780.22\left(1500 \times 10^{-6} + 0.3(800 \times 10^{-6})\right) \approx 382.42 \text{ MPa},$$

$$\sigma_{II} \approx 219780.22\left(800 \times 10^{-6} + 0.3(1500 \times 10^{-6})\right) \approx 274.73 \text{ MPa}.$$

For plane stress, the maximum in-plane shear stress is:

$$\tau_{\max} = \frac{|\sigma_I - \sigma_{II}|}{2}.$$

Thus:

$$\tau_{\max} = \frac{382.42 - 274.73}{2} \approx 53.85 \text{ MPa}.$$

Hence,

$$\boxed{\sigma_I \approx 382.42 \text{ MPa}, \quad \sigma_{II} \approx 274.73 \text{ MPa}, \quad \tau_{\max} \approx 53.85 \text{ MPa}.}$$

9. A steel plate with a Young's modulus of 210 GPa and Poisson's ratio of 0.29 is subjected to a load. If the measured strains using a strain gauge rosette are $1000\,\mu\varepsilon$, $600\,\mu\varepsilon$, and $1200\,\mu\varepsilon$ at angles of 0°, 60°, and 120°, calculate the principal stresses and maximum shear stress.

Solution: Given a steel plate is instrumented with a $(0°, 60°, 120°)$ delta strain rosette. Measured strains:

$$\epsilon_{0°} = 1000\ \mu\varepsilon, \qquad \epsilon_{60°} = 600\ \mu\varepsilon, \qquad \epsilon_{120°} = 1200\ \mu\varepsilon.$$

Material properties:

$$E = 210 \text{ GPa}, \qquad \nu = 0.29.$$

For a delta rosette, the in-plane strains are

$$\epsilon_{xx} = \epsilon_{0°},$$

$$\gamma_{xy} = \frac{2(\epsilon_{60°} - \epsilon_{120°})}{\sqrt{3}},$$

$$\epsilon_{yy} = \frac{2}{3}\left(\epsilon_{60°} + \epsilon_{120°}\right) - \frac{1}{3}\epsilon_{0°}.$$

Thus,

$$\epsilon_{xx} = 1000\ \mu\varepsilon,$$

$$\gamma_{xy} = \frac{2(600 - 1200)}{\sqrt{3}} = -692.8203\ \mu\varepsilon,$$

$$\epsilon_{yy} = \frac{2}{3}(600 + 1200) - \frac{1}{3}(1000) = 866.6667\ \mu\varepsilon.$$

Average normal strain:

$$\bar{\epsilon} = \frac{\epsilon_{xx} + \epsilon_{yy}}{2} = 933.3333\ \mu\varepsilon.$$

Radius of Mohr's circle:

$$R = \sqrt{\left(\frac{\epsilon_{xx} - \epsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 352.7668\ \mu\varepsilon.$$

Principal strains:

$$\epsilon_I = \bar{\epsilon} + R = 1286.10\ \mu\varepsilon,$$

$$\epsilon_{II} = \bar{\epsilon} - R = 580.57\ \mu\varepsilon.$$

Assuming plane stress, Hooke's law gives

$$\sigma_I = \frac{E}{1-\nu^2}(\epsilon_I + \nu\epsilon_{II}), \qquad \sigma_{II} = \frac{E}{1-\nu^2}(\epsilon_{II} + \nu\epsilon_I),$$

where strains are converted to dimensionless values. Thus,

$$\sigma_1 \approx 333.48 \text{ MPa}, \qquad \sigma_2 \approx 218.63 \text{ MPa}.$$

Maximum Shear Stress

$$\tau_{\max} = \frac{|\sigma_I - \sigma_{II}|}{2} = 57.43 \text{ MPa}.$$

Hence,

$$\boxed{\epsilon_I \approx 1286.10\ \mu\varepsilon, \quad \epsilon_{II} \approx 580.57\ \mu\varepsilon}$$

$$\boxed{\sigma_I \approx 333.48 \text{ MPa}, \quad \sigma_{II} \approx 218.63 \text{ MPa}}$$

$$\boxed{\tau_{\max} \approx 57.43 \text{ MPa}}$$

10. A machine component is subjected to a complex loading condition. If the strain gauge rosette measurements are $2500\,\mu\varepsilon$, $1800\,\mu\varepsilon$, and $3000\,\mu\varepsilon$ at angles of 0°, 45°, and 90°, calculate the principal strains, principal stresses, and von Mises stress.

Solution: Assume plane stress and material properties typical for steel:

$$E = 210 \text{ GPa}, \qquad \nu = 0.30.$$

To compute the in-plane strain components for a 0°–45°–90° rosette, we have

$$\epsilon_{xx} = \epsilon_{0^\circ}, \qquad \epsilon_{yy} = \epsilon_{90^\circ},$$

and the engineering shear strain

$$\gamma_{xy} = 2\epsilon_{45^\circ} - (\epsilon_{0^\circ} + \epsilon_{90^\circ}).$$

Numerically,

$$\epsilon_{xx} = 2500\ \mu\varepsilon, \qquad \epsilon_{yy} = 3000\ \mu\varepsilon,$$

$$\gamma_{xy} = 2(1800) - (2500 + 3000) = -1900\ \mu\varepsilon.$$

Thus, $\epsilon_{xy} = \gamma_{xy}/2 = -950\ \mu\varepsilon$ is the tensor shear component. The in-plane strain tensor

$$[\mathbf{E}] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{bmatrix} = \begin{bmatrix} 2500 & -950 \\ -950 & 3000 \end{bmatrix}\ \mu\varepsilon.$$

Compute the average strain and radius (Mohr's circle for strain):

$$\bar{\epsilon} = \frac{\epsilon_{xx} + \epsilon_{yy}}{2} = \frac{2500 + 3000}{2} = 2750 \ \mu\varepsilon,$$

$$R = \sqrt{\left(\frac{\epsilon_{xx} - \epsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = \sqrt{(-250)^2 + (-950)^2} \approx 982.34 \ \mu\varepsilon.$$

Hence the principal strains are

$$\epsilon_I = \bar{\epsilon} + R \approx 2750 + 982.34 = 3732.34 \ \mu\varepsilon,$$

$$\epsilon_{II} = \bar{\epsilon} - R \approx 2750 - 982.34 = 1767.66 \ \mu\varepsilon.$$

Equivalently in dimensionless form:$\epsilon_I \approx 3.73234 \times 10^{-3}, \ \epsilon_{II} \approx 1.76766 \times 10^{-3}$.

Using Hooke's law for plane stress in principal directions:

$$\sigma_I = \frac{E}{1 - \nu^2}\big(\epsilon_I + \nu\epsilon_{II}\big), \qquad \sigma_{II} = \frac{E}{1 - \nu^2}\big(\epsilon_{II} + \nu\epsilon_I\big).$$

With $E = 210$ GPa and $\nu = 0.30$, and converting microstrain to dimensionless strain,

$$\sigma_I \approx 983.69 \text{ MPa}, \qquad \sigma_{II} \approx 666.31 \text{ MPa}.$$

For plane stress, the von Mises equivalent stress computed from principal stresses is

$$\sigma_{\text{vm}} = \sqrt{\sigma_I^2 - \sigma_I\sigma_{II} + \sigma_{II}^2}.$$

Numerically,

$$\sigma_{\text{vm}} \approx 869.58 \text{ MPa}.$$

So,

$$\boxed{\epsilon_I \approx 3732.34 \ \mu\varepsilon, \qquad \epsilon_{II} \approx 1767.66 \ \mu\varepsilon}$$

$$\boxed{\sigma_I \approx 983.69 \text{ MPa}, \qquad \sigma_{II} \approx 666.31 \text{ MPa}}$$

$$\boxed{\sigma_{\text{vm}} \approx 869.58 \text{ MPa}}.$$