Indian Institute of Technology Bhubaneswar

School of Infrastructure

Subject Name : Solid Mechanics	Subject Code: CE2L001

Tutorial No. 2

1. Prove that $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$ and show it schematically.

Proof:

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= a_i \boldsymbol{e}_i \times b_j \boldsymbol{e}_j \\ &= a_i b_j (\boldsymbol{e}_i \times \boldsymbol{e}_j) \;\; \text{(Since } a_i \text{ and } b_j \text{ are scalar quantities)} \\ &= a_i b_j \epsilon_{ijk} \boldsymbol{e}_k \;\; \text{(Since } \boldsymbol{e}_i \times \boldsymbol{e}_j = \epsilon_{ijk} \boldsymbol{e}_k\text{)} \end{aligned}$$

$$\begin{aligned} \boldsymbol{u} \times \boldsymbol{v} &= u_i v_j \epsilon_{ijk} \boldsymbol{e}_k \;\; \text{(Since } \boldsymbol{a} \times \boldsymbol{b} = a_i b_j \epsilon_{ijk} \boldsymbol{e}_k\text{)} \\ &= -u_i v_j \epsilon_{jik} \boldsymbol{e}_k \;\; \text{(Since } \epsilon_{ijk} = -\epsilon_{jik}\text{)} \\ &= -u_j v_i \epsilon_{ijk} \boldsymbol{e}_k \;\; \text{(We can interchange } i \text{ and } j \text{ as they are dummy indices)} \\ &= -v_i u_j \epsilon_{ijk} \boldsymbol{e}_k \;\; \text{(Rearranging the terms)} \\ &= -\boldsymbol{v} \times \boldsymbol{u} \;\; \text{(Since } \boldsymbol{a} \times \boldsymbol{b} = a_i b_j \epsilon_{ijk} \boldsymbol{e}_k\text{)} \end{aligned}$$

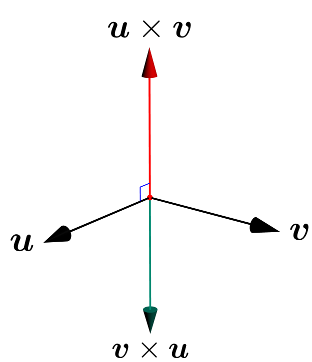

Figure 1: Graphical and numerical demonstration of the anti-commutative property of the cross product: $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$.

2. Prove that:

$$\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0$$

and visualize it schematically.

<u>Solution:</u>

Let $\boldsymbol{u} = u_i\, \boldsymbol{e}_i$ and $\boldsymbol{v} = v_j\, \boldsymbol{e}_j$. The cross product in index notation is:

$$(\boldsymbol{u} \times \boldsymbol{v})_k = \epsilon_{kij}\, u_i v_j$$

The dot product $\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v})$ becomes:

$$\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = u_k(\epsilon_{kij} u_i v_j)$$

Rearranging:

$$= \epsilon_{kij}\, u_k u_i v_j$$

Since $u_k u_i$ is **symmetric** in (k, i) but ϵ_{kij} is **antisymmetric** in (k, i), their contraction is zero, i.e.,

$$\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0$$

Geometric interpretation: The vector $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$, so its dot product with $\boldsymbol{u}$ is always zero.

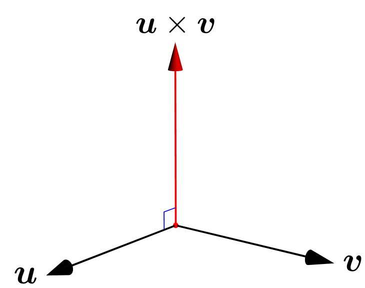

Figure 2: Graphical and numerical demonstration of the vector identity $\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0$.

3. Prove that $\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{v} = \boldsymbol{A}^T\boldsymbol{u} \cdot \boldsymbol{v}$, where $\boldsymbol{u}$, $\boldsymbol{v}$ are vectors and $\boldsymbol{A}$ is a second order tensor with $(\cdot)^T$ denoting the transpose of a tensor. Draw the figure corresponding to the proof and illustrate it schematically.
   <u>Proof:</u>

Let $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$. Then $\boldsymbol{A}^T = (\boldsymbol{a} \otimes \boldsymbol{b})^T = \boldsymbol{b} \otimes \boldsymbol{a}$.

$$\begin{aligned}
\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{v} &= \boldsymbol{u} \cdot (\boldsymbol{a} \otimes \boldsymbol{b})\, \boldsymbol{v} \ \ \text{(Since } \boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}\text{)} \\
&= \boldsymbol{u} \cdot (\boldsymbol{b} \cdot \boldsymbol{v})\, \boldsymbol{a} \ \ \text{(Since } (\boldsymbol{a} \otimes \boldsymbol{b})\, \boldsymbol{v} = (\boldsymbol{b} \cdot \boldsymbol{v})\, \boldsymbol{a}\text{)} \\
&= (\boldsymbol{b} \cdot \boldsymbol{v})\, (\boldsymbol{u} \cdot \boldsymbol{a}) \ \ \text{(Since } \boldsymbol{b} \cdot \boldsymbol{v} \text{ is a scalar quantity)} \\
&= (\boldsymbol{a} \cdot \boldsymbol{u})\, (\boldsymbol{b} \cdot \boldsymbol{v}) \ \ \text{(Since } \boldsymbol{u} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{u}\text{)} \\
&= (\boldsymbol{a} \cdot \boldsymbol{u})\, \boldsymbol{b} \cdot \boldsymbol{v} \ \ \text{(Since } \boldsymbol{a} \cdot \boldsymbol{u} \text{ is a scalar quantity)} \\
&= (\boldsymbol{b} \otimes \boldsymbol{a})\, \boldsymbol{u} \cdot \boldsymbol{v} \ \ \text{(Since } (\boldsymbol{b} \otimes \boldsymbol{a})\, \boldsymbol{u} = (\boldsymbol{a} \cdot \boldsymbol{u})\, \boldsymbol{b}\text{)} \\
&= \boldsymbol{A}^T \boldsymbol{u} \cdot \boldsymbol{v} \ \ \text{(proved)}
\end{aligned}$$

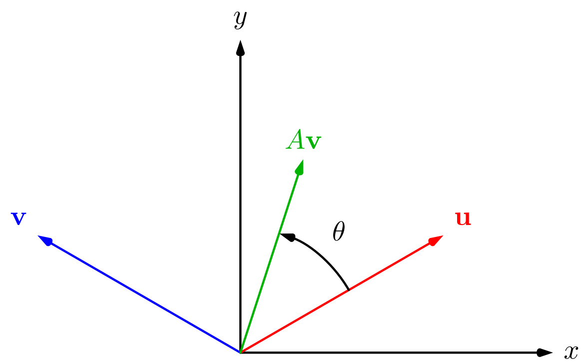

Figure 3

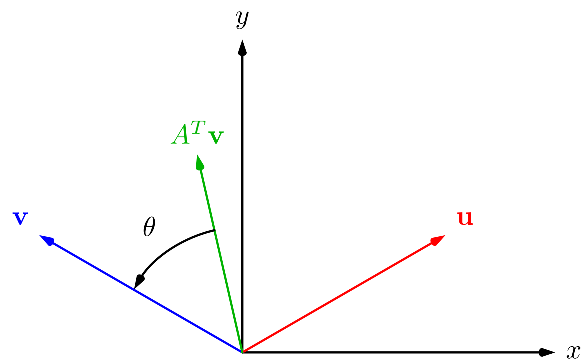

Figure 4

4. Prove the following and visualize it schematically:

$$\boldsymbol{e}_i = \frac{1}{2} \epsilon_{ijk} \boldsymbol{e}_j \times \boldsymbol{e}_k.$$

<u>Proof:</u> Let $\boldsymbol{e}_i$, $\boldsymbol{e}_j$, and $\boldsymbol{e}_k$ be the basis vectors. The cross product between the two basis vectors $\boldsymbol{e}_j$ and $\boldsymbol{e}_k$ can be written as:

$$\boldsymbol{e}_j \times \boldsymbol{e}_k = \epsilon_{jkm} \boldsymbol{e}_m \ \ \text{(By the definition of the Levi-Civita symbol, } \epsilon_{jkm}\text{)}$$

Thus, the given expression can be simplified as:

$$\begin{aligned}
\frac{1}{2} \epsilon_{ijk} \boldsymbol{e}_j \times \boldsymbol{e}_k &= \frac{1}{2} \epsilon_{ijk} \epsilon_{jkm} \boldsymbol{e}_m \\
&= -\frac{1}{2} \epsilon_{jik} \epsilon_{jkm} \boldsymbol{e}_m \ \ \text{(Since } \epsilon_{ijk} = -\epsilon_{jik}\text{)}
\end{aligned}$$

The expression can be expanded using the $\epsilon - \delta$ identity:

$$\epsilon_{jik} \epsilon_{jkm} = \delta_{ik} \delta_{km} - \delta_{im} \delta_{kk}$$

Substituting this identity into the equation, we get:

$$
\begin{aligned}
\frac{1}{2}\epsilon_{ijk}\boldsymbol{e}_j \times \boldsymbol{e}_k &= -\frac{1}{2}\epsilon_{jik}\epsilon_{jkm}\boldsymbol{e}_m \\
&= -\frac{1}{2}\left(\delta_{ik}\delta_{km} - \delta_{im}\delta_{kk}\right)\boldsymbol{e}_m \\
&= -\frac{1}{2}\left(\delta_{im} - 3\delta_{im}\right)\boldsymbol{e}_m \quad (\text{Using } \delta_{kk} = 3 \text{ and } \delta_{ik}\delta_{km} = \delta_{im}) \\
&= \delta_{im}\boldsymbol{e}_m = \boldsymbol{e}_i \quad (\text{Using contraction property })
\end{aligned}
$$

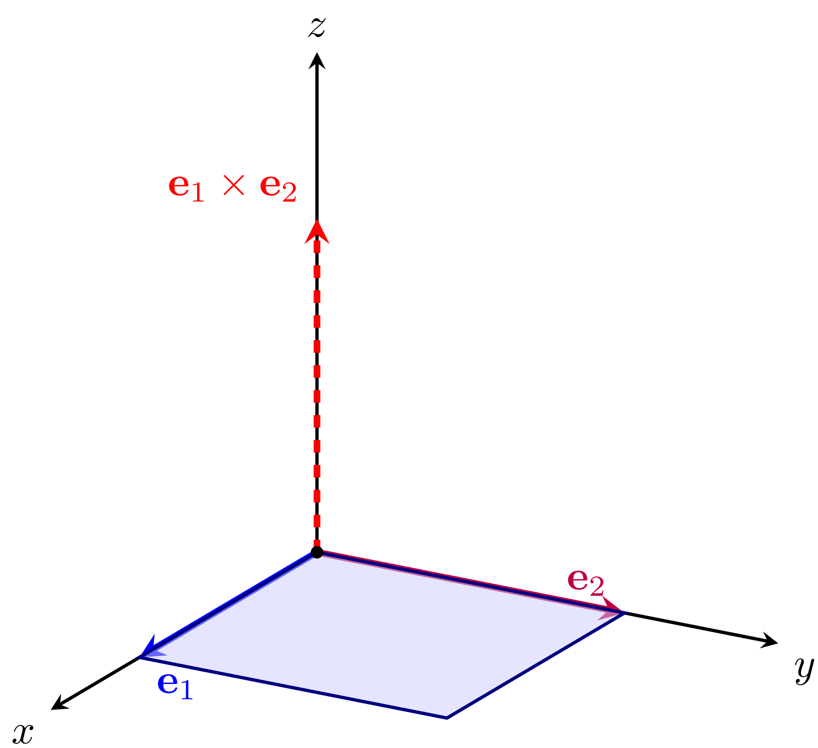

Figure 5: Graphical demonstration of the vector identity $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$.

5. Consider two scalar functions f and g. Establish the following identities and explain them through illustrative examples.

   (a) $\nabla \times (\nabla f) = \mathbf{0}$,

   (b) $\nabla \cdot (\nabla f \times \nabla g) = 0$.

Solution (a)

$$
\begin{aligned}
\nabla \times (\nabla f) &= \left(\boldsymbol{e}_i \frac{\partial()}{\partial x_i}\right) \times \left(\boldsymbol{e}_j \frac{\partial f}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i \partial x_j} (\boldsymbol{e}_i \times \boldsymbol{e}_j) \\
&= \frac{\partial^2 f}{\partial x_i \partial x_j} \epsilon_{ijk} \boldsymbol{e}_k \;\; (\text{Using}\, \boldsymbol{e}_i \times \boldsymbol{e}_j = \epsilon_{ijk} \boldsymbol{e}_k) \\
&= \frac{\partial^2 f}{\partial x_j \partial x_i} \epsilon_{ijk} \boldsymbol{e}_k \;\; \left(\text{Since} \;\; \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}\right) \\
&= -\frac{\partial^2 f}{\partial x_j \partial x_i} \epsilon_{jik} \boldsymbol{e}_k \;\; (\text{Since } \epsilon_{jik} = -\epsilon_{ijk}) \\
&= -\frac{\partial^2 f}{\partial x_i \partial x_j} \epsilon_{ijk} \boldsymbol{e}_k \;\; (\text{Interchanging}\, i \text{ and } j \text{ as they are dummy indices}) \\
&= -\nabla \times (\nabla f)
\end{aligned}
$$

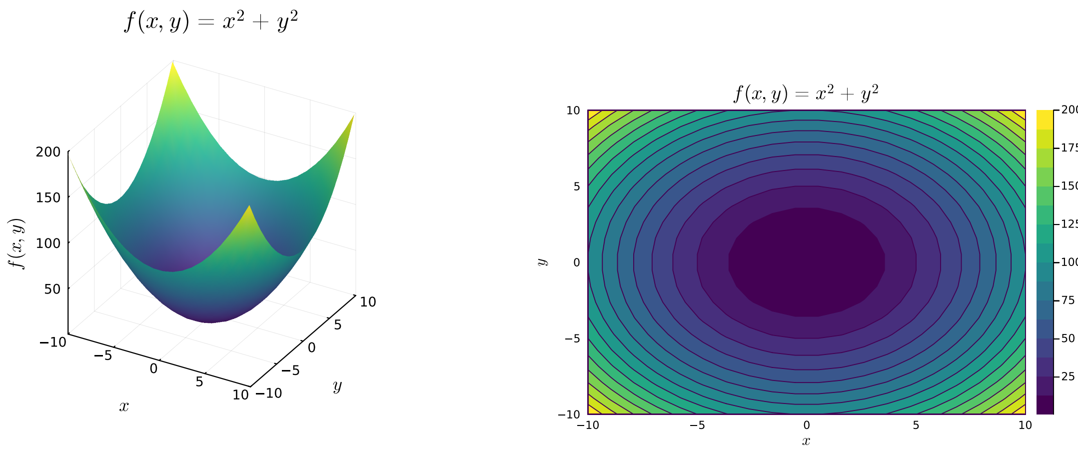

Figure 6: Representation of the Scalar field $f(x, y)$

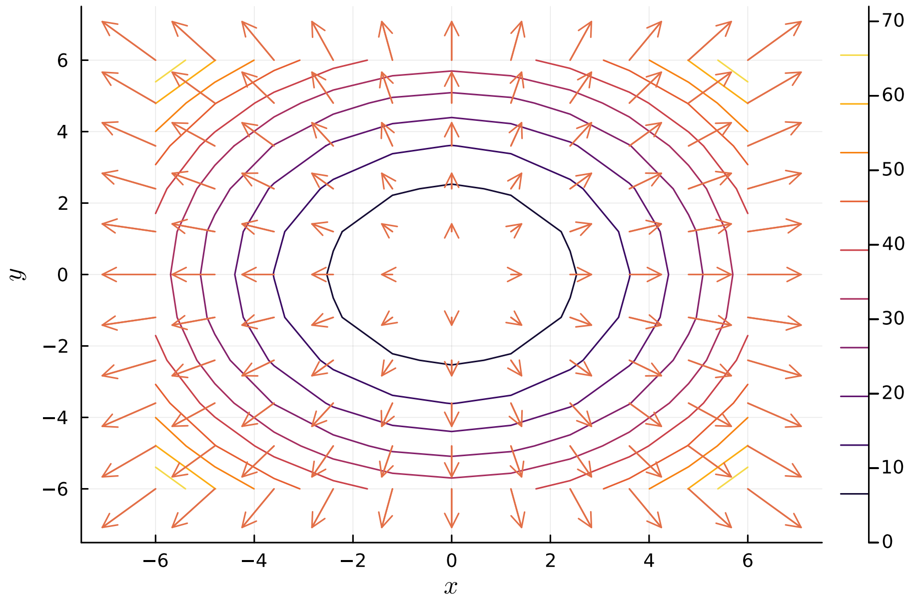

Figure 7: Gradient of the function $f(x,y) = x^2 + y^2$. The vector field ∇f is visualized.

Solution (b)

$$
\begin{aligned}
\nabla \cdot (\nabla f \times \nabla g) &= \nabla \cdot \left(\frac{\partial f}{\partial x_i} \boldsymbol{e}_i \times \frac{\partial g}{\partial x_j} \boldsymbol{e}_j \right) \\
&= \nabla \cdot \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \boldsymbol{e}_i \times \boldsymbol{e}_j \right) \ \left(\text{Since } \frac{\partial g}{\partial x_j} \text{ is a scalar quantity} \right) \\
&= \nabla \cdot \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \epsilon_{ijk} \boldsymbol{e}_k \right) \ (\text{Using } \boldsymbol{e}_i \times \boldsymbol{e}_j = \epsilon_{ijk} \boldsymbol{e}_k)) \\
&= \left(\boldsymbol{e}_l \frac{\partial ()}{\partial x_l} \right) \cdot \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \epsilon_{ijk} \boldsymbol{e}_k \right) \ \left(\text{Since } \ \nabla := \boldsymbol{e}_l \frac{\partial ()}{\partial x_l} \right) \\
&= \frac{\partial}{\partial x_l} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \epsilon_{ijk} \right) (\boldsymbol{e}_l \cdot \boldsymbol{e}_k) \\
&= \epsilon_{ijk} \delta_{lk} \frac{\partial}{\partial x_l} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \right) \ (\text{Since } \ \boldsymbol{e}_l \cdot \boldsymbol{e}_k = \delta_{lk})
\end{aligned}
$$

Expanding the partial derivatives:

$$
\begin{aligned}
\nabla \cdot (\nabla f \times \nabla g) &= \epsilon_{ijk} \delta_{lk} \left(\frac{\partial^2 f}{\partial x_i \partial x_l} \frac{\partial g}{\partial x_j} + \frac{\partial f}{\partial x_i} \frac{\partial^2 g}{\partial x_j \partial x_l} \right) \\
&= \epsilon_{ijl} \left(\frac{\partial^2 f}{\partial x_i \partial x_l} \frac{\partial g}{\partial x_j} + \frac{\partial f}{\partial x_i} \frac{\partial^2 g}{\partial x_j \partial x_l} \right) \ (\text{Since } \epsilon_{ijk} \delta_{lk} = \epsilon_{ijl})
\end{aligned}
$$

Using the result from Part (a), one can show that

$$
\epsilon_{ijl} \frac{\partial^2 f}{\partial x_i \partial x_l} \frac{\partial g}{\partial x_j} = 0,
$$

and

$$\epsilon_{ijl} \frac{\partial^2 g}{\partial x_j \partial x_l} \frac{\partial f}{\partial x_i} = 0.$$

Thus, $\nabla \cdot (\nabla f \times \nabla g) = 0$.

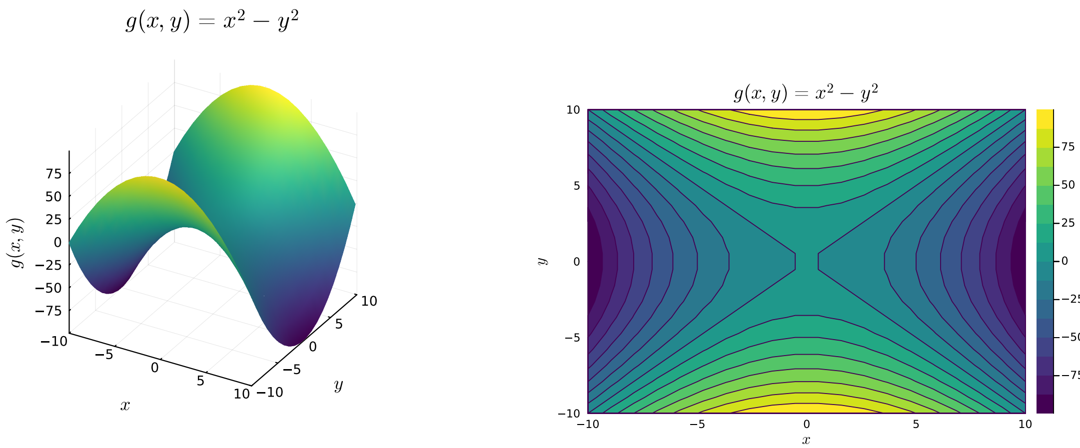

Figure 8: Representation of the Scalar field $g(x, y)$

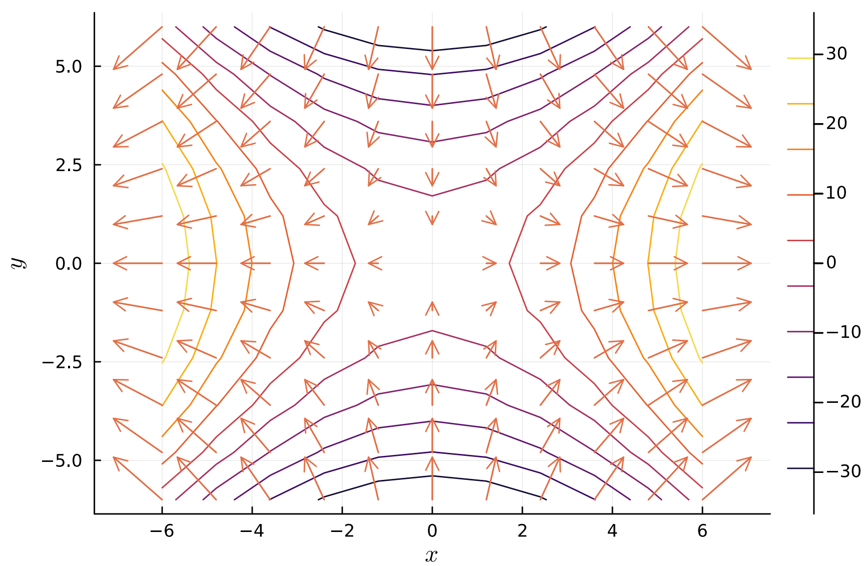

Figure 9: Gradient of the function $g(x, y) = x^2 - y^2$. The vector field ∇g is visualized

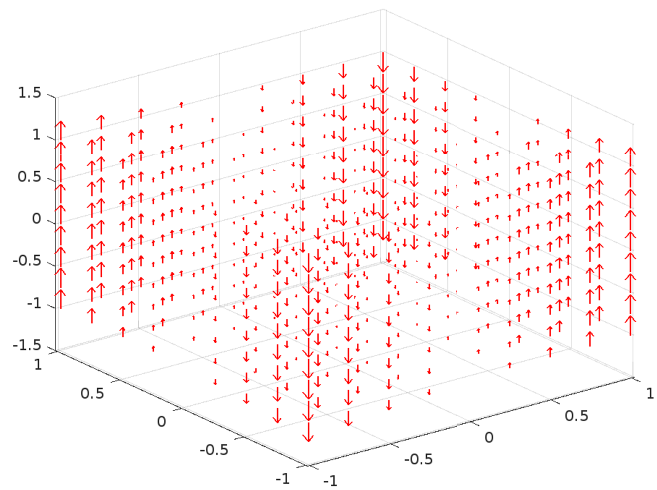

Figure 10: Visualization of the vector identity $\nabla f \times \nabla g$. The vector field $\nabla f \times \nabla g$ is plotted for $f(x, y) = x^2 + y^2$ and $g(x, y) = x^2 - y^2$. The divergence of this vector field is uniformly zero, as expected.

6. Prove that $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}$ where $\boldsymbol{Q}$ is orthogonal tensor and $\boldsymbol{I}$ is the second-order identity tensor. Draw the figure corresponding to the proof and explain it schematically.

   Solution:

   An orthogonal tensor $\boldsymbol{T}$ is a second-order tensor which follows the following conditions

$$\boldsymbol{T}\boldsymbol{u} \cdot \boldsymbol{T}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V \tag{1}$$

   It can be schematically shown in Fig. below

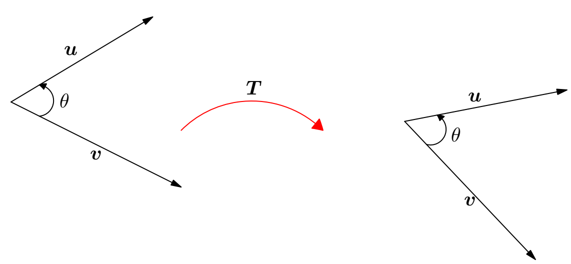

Figure 11: Schematic representation of Orthogonal tensor

   Thus, the magnitude of the vectors and the angle between the vectors are preserved.

The only way this equality holds for all $\boldsymbol{a}, \boldsymbol{b}$ is if

$$
\begin{aligned}
\boldsymbol{u} \cdot \boldsymbol{Q}^T \boldsymbol{Q} \boldsymbol{v} &= \boldsymbol{Q}\boldsymbol{u} \cdot \boldsymbol{Q}\boldsymbol{v} && \text{(definition of orthogonality)} \\
\boldsymbol{Q}\boldsymbol{u} \cdot \boldsymbol{Q}\boldsymbol{v} &= \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{I}\boldsymbol{v} && \text{(dot product preserved)} \\
\boldsymbol{u} \cdot (\boldsymbol{Q}^T \boldsymbol{Q})\boldsymbol{v} &= \boldsymbol{u} \cdot \boldsymbol{I}\boldsymbol{v} && \text{(So combining them)} \\
\boldsymbol{u} \cdot (\boldsymbol{Q}^T \boldsymbol{Q} - \boldsymbol{I})\boldsymbol{v} &= 0 && \text{(subtracting } \boldsymbol{u} \cdot \boldsymbol{I}\boldsymbol{v}\text{)} \\
\boldsymbol{Q}^T \boldsymbol{Q} - \boldsymbol{I} &= 0 && \text{(holds } \forall \boldsymbol{u}, \boldsymbol{v}\text{)} \\
\boldsymbol{Q}^T \boldsymbol{Q} &= \boldsymbol{I} && \text{(orthogonality condition)}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{u} \cdot \boldsymbol{Q}\boldsymbol{Q}^T \boldsymbol{v} &= \boldsymbol{Q}\boldsymbol{u} \cdot \boldsymbol{Q}\boldsymbol{v} && \text{(symmetry argument)} \\
\boldsymbol{Q}^T\boldsymbol{u} \cdot \boldsymbol{Q}^T\boldsymbol{v} &= \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{I}\boldsymbol{v} && \text{(dot product preserved)} \\
\boldsymbol{u} \cdot (\boldsymbol{Q}\boldsymbol{Q}^T)\boldsymbol{v} &= \boldsymbol{u} \cdot \boldsymbol{I}\boldsymbol{v} && \text{(So combining them)} \\
\boldsymbol{u} \cdot (\boldsymbol{Q}\boldsymbol{Q}^T - \boldsymbol{I})\boldsymbol{v} &= 0 && \text{(subtracting } \boldsymbol{u} \cdot \boldsymbol{I}\boldsymbol{v}\text{)} \\
\boldsymbol{Q}\boldsymbol{Q}^T - \boldsymbol{I} &= 0 && \text{(holds } \forall \boldsymbol{u}, \boldsymbol{v}\text{)} \\
\boldsymbol{Q}\boldsymbol{Q}^T &= \boldsymbol{I} && \text{(orthogonality condition)}
\end{aligned}
$$

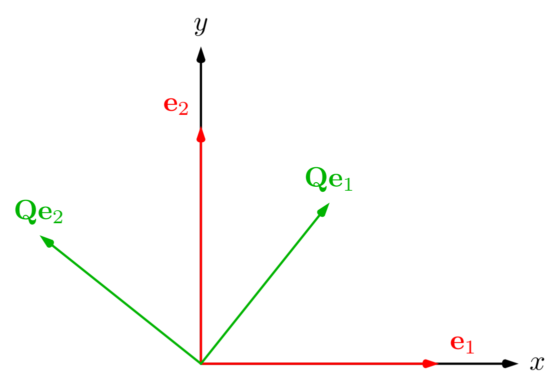

Figure 12: $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}$, showing that an orthogonal tensor $\boldsymbol{Q}$ preserves lengths and angles (rotation of basis vectors)

7. For the vector field $\mathbf{u} = 2x_1\,\mathbf{e}_1 + x_1x_2\,\mathbf{e}_2$, calculate the quantities $\nabla \cdot \mathbf{u}$, $\nabla \times \mathbf{u}$, $\nabla^2\mathbf{u}$, $\nabla\mathbf{u}$, and $\mathrm{tr}(\nabla\mathbf{u})$.

   Solution:

Calculation of $\nabla \cdot \mathbf{u}$

For any vector field $\mathbf{u} = u_i \, \mathbf{e}_i$, the quantity $\nabla \cdot \mathbf{u}$ is defined as

$$
\begin{aligned}
\nabla \cdot \mathbf{u} &= \boldsymbol{e}_i \frac{\partial()}{\partial x_i} \cdot (u_j \, \mathbf{e}_j) \quad \left(\text{Since } \nabla := \boldsymbol{e}_i \frac{\partial()}{\partial x_i}\right) \\
&= \frac{\partial u_j}{\partial x_i} (\boldsymbol{e}_i \cdot \boldsymbol{e}_j) \quad \left(\text{Note } \frac{\partial u_j}{\partial x_i} \text{ is a scalar quantity}\right) \\
&= \frac{\partial u_j}{\partial x_i} \delta_{ij} \quad (\text{Since } \delta_{ij} := \boldsymbol{e}_i \cdot \boldsymbol{e}_j) \\
&= \frac{\partial u_i}{\partial x_i} \quad (\text{Using contraction property of } \delta_{ij}) \\
&= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \quad (\text{Since } i = 1, 2 \text{ for 2D case}) \, .
\end{aligned}
$$

For given $\mathbf{u} = 2x_1 \, \mathbf{e}_1 + x_1 x_2 \, \mathbf{e}_2$, one can identify from $\mathbf{u} = u_i \, \mathbf{e}_i$ that $u_1 = 2x_1$ and $u_2 = x_1 x_2$. So,

$$
\frac{\partial u_1}{\partial x_1} = \frac{\partial \, (2x_1)}{\partial x_1} = 2 \text{ and } \frac{\partial u_2}{\partial x_2} = \frac{\partial \, (x_1 x_2)}{\partial x_2} = x_1.
$$

Hence, $\nabla \cdot \mathbf{u} = 2 + x_1$.

Calculation of $\nabla \times \mathbf{u}$:

$$
\begin{aligned}
(\mathbf{a} \times \mathbf{b})_i &= \epsilon_{ijk} a_j b_k, \\
(\nabla \times \mathbf{u})_i &= \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}
\end{aligned}
$$

Only the third component is nonzero:

$$
\begin{aligned}
(\nabla \times \mathbf{u})_3 &= \epsilon_{312} \frac{\partial u_2}{\partial x_1} + \epsilon_{321} \frac{\partial u_1}{\partial x_2} \\
&= (1) \cdot x_2 + (-1) \cdot 0 \qquad (\text{Since } \epsilon_{ijk} = 1 \text{ and } \epsilon_{ikj} = -1) \\
&= x_2
\end{aligned}
$$

So,

$$
\nabla \times \mathbf{u} = x_2 \, \mathbf{e}_3
$$

Calculation of $\nabla^2 \mathbf{u}$

Vector Laplacian: The Laplacian of a scalar field is the sum of second partial derivatives with respect to all coordinates,

$$
(\nabla f)_i = \frac{\partial f}{\partial x_i}.
$$

$$
\nabla^2 f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}.
$$

$$(\nabla^2 \mathbf{u})_i = \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$u_1 = 2x_1 : \frac{\partial^2}{\partial x_1^2}(2x_1) = 0, \quad \frac{\partial^2}{\partial x_2^2}(2x_1) = 0$$

$$u_2 = x_1 x_2 : \frac{\partial^2}{\partial x_1^2}(x_1 x_2) = 0, \quad \frac{\partial^2}{\partial x_2^2}(x_1 x_2) = 0$$

Both components vanish:

$$\nabla^2 \mathbf{u} = 0 \quad \text{(each component)}$$

<u>Calculation of ∇u</u>

$$(\nabla f)_i = \frac{\partial f}{\partial x_i}.$$

$$(\nabla \mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j}$$

Explicitly:

$$(\nabla \mathbf{u})_{11} = \frac{\partial u_1}{\partial x_1} = 2$$
$$(\nabla \mathbf{u})_{12} = \frac{\partial u_1}{\partial x_2} = 0$$
$$(\nabla \mathbf{u})_{21} = \frac{\partial u_2}{\partial x_1} = x_2$$
$$(\nabla \mathbf{u})_{22} = \frac{\partial u_2}{\partial x_2} = x_1$$

<u>Calculation of $\mathrm{tr}(\nabla \mathbf{u})$</u>

$$\mathrm{tr}(\nabla \mathbf{u}) = (\nabla \mathbf{u})_{ii} = (\nabla \mathbf{u})_{11} + (\nabla \mathbf{u})_{22} = 2 + x_1$$

Physical Interpretation

- **Divergence** ($u_{i,i}$): Net outflow at a point, indicating expansion or contraction.
- **Curl** ($\epsilon_{ijk}\partial_j u_k$): Tendency of the field to circulate, here only in e_3 direction.
- **Laplacian** ($\partial_j \partial_j u_i$): Measures diffusion or smoothness, zero means harmonic components.
- **Gradient** ($\partial_j u_i$): Local spatial variation of the field, shown above.
- **Trace** ($\partial_i u_i$): Sum of changes along all axes, closely linked to divergence.

8. Show that for any two differentiable vector fields **a** and **b**, the following vector identity holds:

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}).$$

Explain the identity with illustrative figure.

Solution

We want to prove the identity

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}).$$

Proof:

Write the divergence of the cross product in index notation:

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \partial_i\big(\varepsilon_{ijk} a_j b_k\big),$$

where ε_{ijk} is the Levi-Civita symbol.

Expanding,

$$\partial_i\big(\varepsilon_{ijk} a_j b_k\big) = \varepsilon_{ijk}(\partial_i a_j) b_k + \varepsilon_{ijk} a_j (\partial_i b_k).$$

For the first term:

$$\varepsilon_{ijk}(\partial_i a_j) b_k = b_k \varepsilon_{kij} \partial_i a_j = b_k (\nabla \times \mathbf{a})_k = \mathbf{b} \cdot (\nabla \times \mathbf{a}).$$

For the second term:

$$\varepsilon_{ijk} a_j (\partial_i b_k) = -\, a_j \varepsilon_{ikj} \partial_i b_k = -\, a_j (\nabla \times \mathbf{b})_j = -\,\mathbf{a} \cdot (\nabla \times \mathbf{b}).$$

Therefore, combining both results:

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}),$$

which proves the identity.

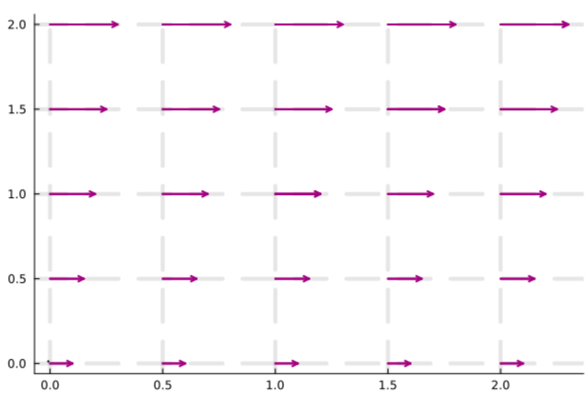

Figure 13: Scalar field **a**.

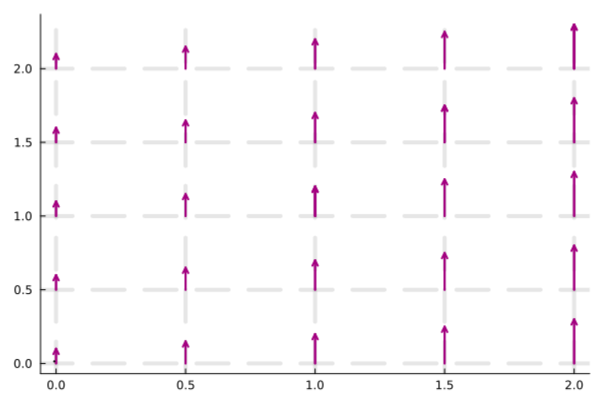

Figure 14: Scalar field **b**.

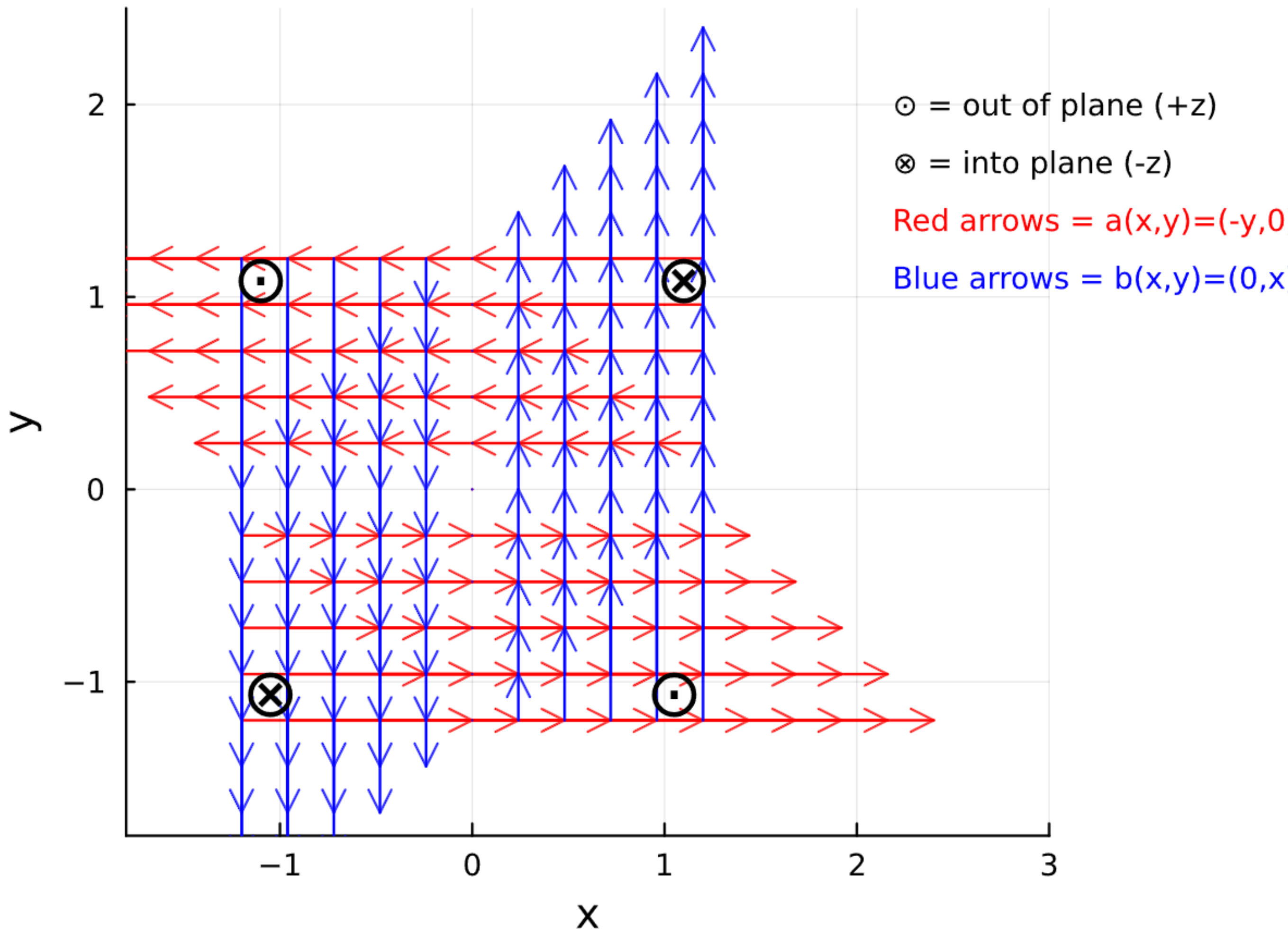

Figure 15: Shows the two vector fields $\boldsymbol{a}$ and $\boldsymbol{b}$ and their cross product $(\boldsymbol{a} \times \boldsymbol{b})$.