



Indian Institute of Technology Bhubaneswar

School of Infrastructure

Session: Autumn 2025

Solid Mechanics (CE2L001)

Class Test 2

Notations :

Zeroth-order tensors or scalars are represented by small letters. For eg. a

First-order tensors or vectors are represented by bold small letters. For eg. \mathbf{a} .

Second-order tensors are represented by bold capital letters. For eg. \mathbf{A}

1. A square material element with side length 1 unit undergoes a volume-preserving deformation given by:

$$x'_1 = \alpha x_1, \quad x'_2 = \frac{x_2}{\alpha},$$

where $\alpha = 1.5$.

- (a) Determine the deformation gradient tensor \mathbf{F} .
- (b) Plot the reference and current configurations.
- (c) Calculate the stretch ratio in the x_1 direction and the contraction ratio in the x_2 direction.

Solution: (a) Components of the deformation gradient tensor \mathbf{F} are given by:

$$F_{ij} = \frac{\partial x'_i}{\partial x_j}$$

Given the deformation:

$$x'_1 = \alpha x_1, \quad x'_2 = \frac{x_2}{\alpha},$$

We compute the components of \mathbf{F} :

$$F_{11} = \frac{\partial x'_1}{\partial x_1} = \alpha, \quad F_{12} = \frac{\partial x'_1}{\partial x_2} = 0, \quad F_{21} = \frac{\partial x'_2}{\partial x_1} = 0, \quad F_{22} = \frac{\partial x'_2}{\partial x_2} = \frac{1}{\alpha}.$$

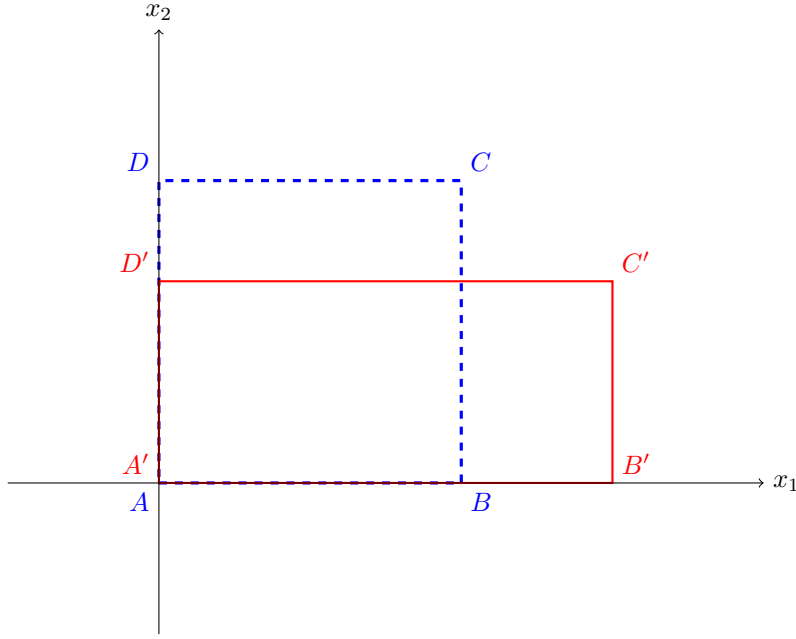
Thus,

$$[\mathbf{F}] = \begin{bmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix} = \begin{bmatrix} 1.5 & 0 \\ 0 & \frac{1}{1.5} \end{bmatrix} = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.667 \end{bmatrix}.$$

(b)

Reference Configuration (x_1, x_2)	Current Configuration (x'_1, x'_2)
(0,0)	(0,0)
(1,0)	(1.5,0)
(1,1)	(1.5, 0.667)
(0,1)	(0, 0.667)

Table 1: Reference and Current Configurations



(c) As \mathbf{F} is diagonal, the stretch tensor, $\mathbf{U} = \mathbf{F}$.

Stretch ratio in x_1 direction: $\lambda_1 = F_{11} = \alpha = 1.5$.

Contraction ratio in x_2 direction: $\lambda_2 = F_{22} = \frac{1}{\alpha} = \frac{1}{1.5} = 0.667$.

2. The stress components in a material are given by:

$$\sigma_{11} = a_1 x_1 x_2, \quad \sigma_{22} = a_2 x_2 x_3, \quad \sigma_{12} = a_3 x_1 x_3,$$

where a_1 , a_2 , and a_3 are small constants. Determine:

- The body force components.
- The strain components using Hooke's law.
- Whether the compatibility condition is satisfied.

Solution: (a) The equilibrium equations are:

$$\sigma_{ij,j} + b_i = 0$$

Given stress components:

$$\sigma_{11} = a_1 x_1 x_2, \quad \sigma_{22} = a_2 x_2 x_3, \quad \sigma_{12} = a_3 x_1 x_3,$$

So, the components of body force vectors can be calculated as given below.

$$b_1 = -\sigma_{11,1} - \sigma_{12,2} = -a_1 x_2 - 0 = -a_1 x_2,$$

$$b_2 = -\sigma_{12,1} - \sigma_{22,2} = -a_3 x_3 - a_2 x_3 = -(a_3 + a_2) x_3,$$

$$b_3 = -\sigma_{13,3} - \sigma_{23,3} = 0 - 0 = 0.$$

(b) Strain Components using Hooke's Law can be computed for an isotropic material as:

$$\epsilon_{ij} = \frac{1}{E} [(1 + \nu) \sigma_{ij} - \nu \sigma_{kk} \delta_{ij}]$$

Now,

$$\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = a_1 x_1 x_2 + a_2 x_2 x_3 + 0 = a_1 x_1 x_2 + a_2 x_2 x_3.$$

So,

$$\epsilon_{11} = \frac{1}{E} [(1 + \nu)(a_1 x_1 x_2) - \nu(a_1 x_1 x_2 + a_2 x_2 x_3)] = \frac{1}{E} [a_1 x_1 x_2 - \nu a_2 x_2 x_3],$$

$$\epsilon_{22} = \frac{1}{E} [(1 + \nu)(a_2 x_2 x_3) - \nu(a_1 x_1 x_2 + a_2 x_2 x_3)] = \frac{1}{E} [a_2 x_2 x_3 - \nu a_1 x_1 x_2],$$

$$\epsilon_{12} = \frac{1 + \nu}{E} \sigma_{12} = \frac{1 + \nu}{E} a_3 x_1 x_3.$$

(c) For 2D, Compatibility Condition:

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2 \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2}$$

Computing derivatives:

$$\frac{\partial \epsilon_{11}}{\partial x_2} = \frac{1}{E} [a_1 x_1 - \nu a_2 x_3], \quad \frac{\partial \epsilon_{22}}{\partial x_1} = \frac{1}{E} [-\nu a_1 x_2], \quad \frac{\partial \epsilon_{12}}{\partial x_1} = \frac{1 + \nu}{E} a_3 x_3.$$

So, the second derivatives will vanish and the compatibility condition will be satisfied.

3. The displacement field is given by:

$$u_1 = \alpha x_1 x_2, \quad u_2 = \beta x_1^2,$$

where α and β are constants. Calculate the strain tensor and derive the principal strains and principal directions.

Solution: For a given displacement components, strain tensor $\mathbf{E}_s = E_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ can be computed from $E_{ij} = (1/2)(u_{i,j} + u_{j,i})$. For the given displacement components, strain tensor components:

$$[\mathbf{E}_s] = \begin{bmatrix} \alpha x_2 & \frac{1}{2}(\alpha x_1 + 2\beta x_1) \\ \frac{1}{2}(\alpha x_1 + 2\beta x_1) & 0 \end{bmatrix}.$$

From the Mohr's circle, one can derive the principal strains expression as:

$$\epsilon_{I,II} = \frac{E_{11} + E_{22}}{2} \pm \sqrt{\left(\frac{E_{11} - E_{22}}{2}\right)^2 + E_{12}^2},$$

and the principal direction of strain for the angle with respect to \mathbf{e}_1 satisfies:

$$\tan 2\theta_p = \frac{2E_{12}}{E_{11} - E_{22}}.$$

For the given problem, the principal strains are:

$$\epsilon_{I,II} = \frac{\alpha x_2}{2} \pm \sqrt{\left(\frac{\alpha x_2}{2}\right)^2 + \left(\frac{1}{2}(\alpha x_1 + 2\beta x_1)\right)^2}.$$

and the principal directions:

$$\tan 2\theta_p = \frac{(\alpha x_1 + 2\beta x_1)}{\alpha x_2}.$$

4. Consider a homogeneous deformation of a cube with initial side length a . The cube is sheared in the x - y plane by an angle γ . Determine:

- Deformation gradient tensor \mathbf{F} .
- Finite strain tensor \mathbf{E} .
- Linearized strain tensor \mathbf{E}_s for small deformations.
- Provide a comparison between the finite strain components and the linearized strain components through a numerical example.

Solution: (a) Deformation Gradient Tensor $\mathbf{F} = F_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$, where F_{ij} is defined as

$$F_{ij} = \frac{\partial x'_i}{\partial x_j}.$$

For three-dimensional case, $[\mathbf{F}] := (F_{ij})_{i,j \in (1,2)}$ can be expressed as

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial x'_1}{\partial x_1} & \frac{\partial x'_1}{\partial x_2} & \frac{\partial x'_1}{\partial x_3} \\ \frac{\partial x'_2}{\partial x_1} & \frac{\partial x'_2}{\partial x_2} & \frac{\partial x'_2}{\partial x_3} \\ \frac{\partial x'_3}{\partial x_1} & \frac{\partial x'_3}{\partial x_2} & \frac{\partial x'_3}{\partial x_3} \end{bmatrix}.$$

The deformation is given by:

$$x'_1 = x_1 + \gamma x_2, \quad x'_2 = x_2, \quad x'_3 = x_3.$$

So,

$$[\mathbf{F}] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Finite Strain Tensor \mathbf{E} :

$$[\mathbf{E}] = \frac{1}{2} \left([\mathbf{F}^T][\mathbf{F}] - [\mathbf{I}] \right) = \begin{bmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & \frac{\gamma^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) Linearized Strain Tensor \mathbf{E}_s :

$$[\mathbf{E}_s] = \frac{1}{2} \left([\mathbf{F}^T] + [\mathbf{F}] - 2[\mathbf{I}] \right) = \begin{bmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

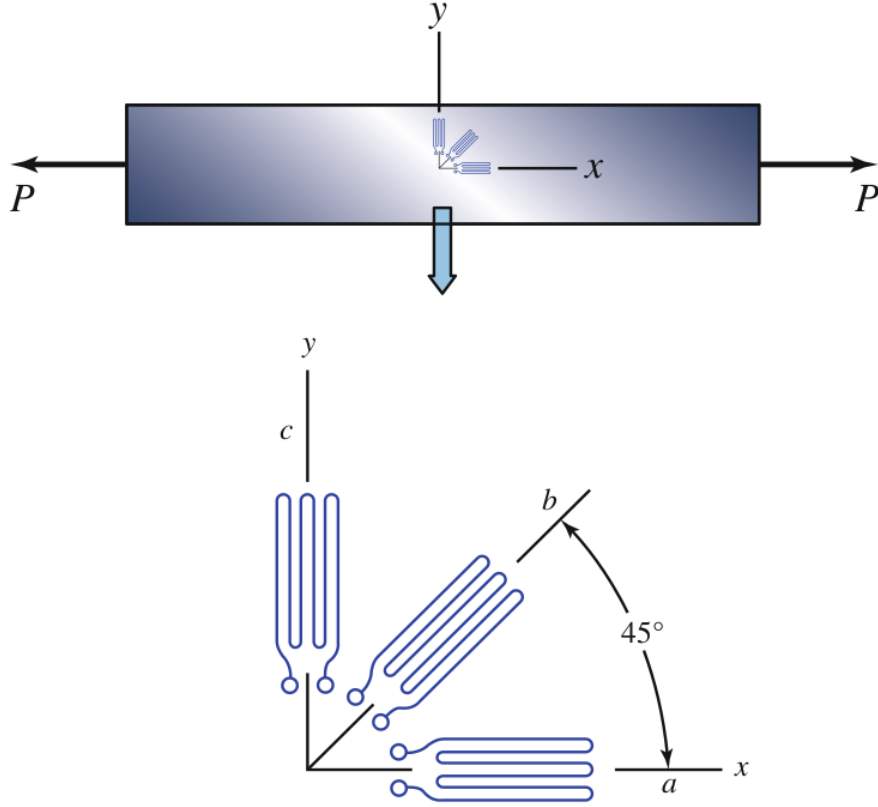
(d) Numerical Example:

Component	$\gamma = 0.1$	$\gamma = 0.001$
E_{12}	0.05	0.0005
E_{22}	0.005	5×10^{-7}
$E_{s,12}$	0.05	0.0005
$E_{s,22}$	0	0

Table 2: Finite and Linearized Strain Components

As can be seen from the above table, for small value of γ , the linearized strain tensor will be almost the same as the finite strain tensor.

5. A bar is subjected to axial forces. The strains measured by a strain gauge rosette oriented as shown in the below figure are $E_{aa} = 0.003$, $E_{bb} = 0.001$, and $E_{cc} = -0.001$. What are the strain components E_{xx} , E_{yy} , and E_{xy} ?



Solution: Given strains:

$$E_{aa} = 0.003, \quad E_{bb} = 0.001, \quad E_{cc} = -0.001.$$

The strain transformation equations are:

$$E_{aa} = E_{xx} \cos^2 \theta_a + E_{yy} \sin^2 \theta_a + 2E_{xy} \sin \theta_a \cos \theta_a,$$

$$E_{bb} = E_{xx} \cos^2 \theta_b + E_{yy} \sin^2 \theta_b + 2E_{xy} \sin \theta_b \cos \theta_b,$$

$$E_{cc} = E_{xx} \cos^2 \theta_c + E_{yy} \sin^2 \theta_c + 2E_{xy} \sin \theta_c \cos \theta_c.$$

From the figure, $\theta_a = 0^\circ$, $\theta_b = 45^\circ$, $\theta_c = 90^\circ$.

$$E_{aa} = E_{xx} = 0.003,$$

$$E_{bb} = \frac{1}{2}(E_{xx} + E_{yy}) + E_{xy} = 0.001,$$

$$E_{cc} = E_{yy} = -0.001.$$

Solving:

$$E_{xx} = 0.003,$$

$$E_{yy} = -0.001,$$

$$0.001 = \frac{1}{2}(0.003 - 0.001) + E_{xy}$$

$$E_{xy} = 0.001 - \frac{1}{2}(0.002) = 0.$$

Therefore, the strain components are:

$$E_{xx} = 0.003,$$

$$E_{yy} = -0.001,$$

$$E_{xy} = 0.$$