Indian Institute of Technology Bhubaneswar

School of Infrastructure

Subject Name : Solid Mechanics	Subject Code: CE2L001

Tutorial No. 1

1. The vector field is represented by:

   (a) $\boldsymbol{f} = -y\boldsymbol{e}_1 + x\boldsymbol{e}_2$

   (b) $\boldsymbol{f} = (e^x y^2)\boldsymbol{e}_1 + (x + 2y)\boldsymbol{e}_2$

   Draw the vector field diagrammatically and determine the divergence and curl of the vector field.

   <u>Solution</u>

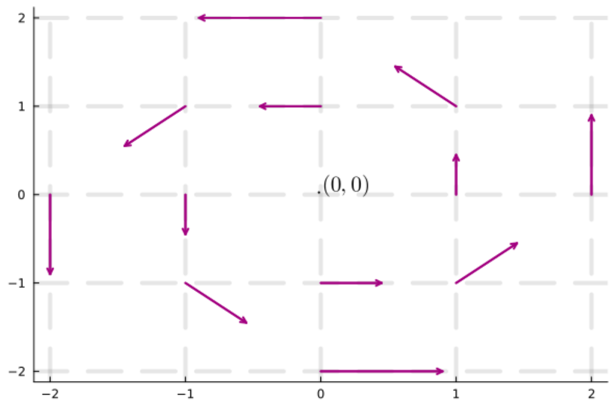
   (a) Consider the vector field $\boldsymbol{f} = -y\boldsymbol{e}_1 + x\boldsymbol{e}_2$

   - We can represent the vector field in some representative points in the xy plane as $\boldsymbol{f} = -y\boldsymbol{e}_1 + x\boldsymbol{e}_2$.

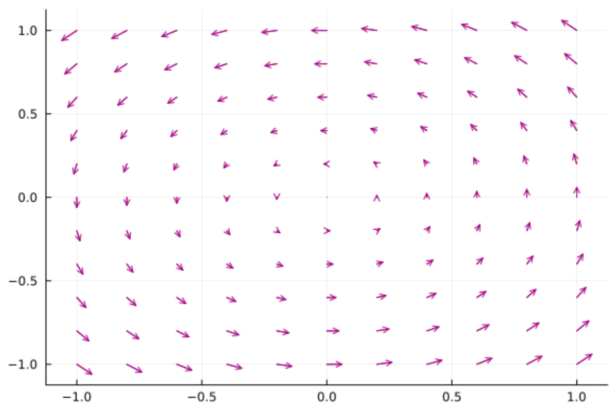
   - Write down the vectors at some representative points.

(x, y)	$\boldsymbol{f}$	$[\boldsymbol{f}]_{(e_1,e_2)}$	(x, y)	$\boldsymbol{f}$	$[\boldsymbol{f}]_{(e_1,e_2)}$
(0,1)	-$\boldsymbol{e}_1$	$\begin{pmatrix}-1\\0\end{pmatrix}$	(0,2)	-2 $\boldsymbol{e}_1$	$\begin{pmatrix}-2\\0\end{pmatrix}$
(1,0)	$\boldsymbol{e}_2$	$\begin{pmatrix}0\\1\end{pmatrix}$	(2,0)	2$\boldsymbol{e}_2$	$\begin{pmatrix}0\\2\end{pmatrix}$
(1,1)	-$\boldsymbol{e}_1$ + $\boldsymbol{e}_2$	$\begin{pmatrix}-1\\1\end{pmatrix}$	(0,-2)	2$\boldsymbol{e}_1$	$\begin{pmatrix}2\\0\end{pmatrix}$
(0,-1)	$\boldsymbol{e}_1$	$\begin{pmatrix}1\\0\end{pmatrix}$	(-2,0)	-2$\boldsymbol{e}_2$	$\begin{pmatrix}0\\-2\end{pmatrix}$
(-1,0)	-$\boldsymbol{e}_2$	$\begin{pmatrix}0\\-1\end{pmatrix}$	(1,-1)	$\boldsymbol{e}_1$ + $\boldsymbol{e}_2$	$\begin{pmatrix}1\\1\end{pmatrix}$
(-1,-1)	$\boldsymbol{e}_1$ - $\boldsymbol{e}_2$	$\begin{pmatrix}1\\-1\end{pmatrix}$	(-1,1)	-$\boldsymbol{e}_1$ - $\boldsymbol{e}_2$	$\begin{pmatrix}1\\-1\end{pmatrix}$

   - Plot the vector fields from the above step as



- The vector field for the entire domain can be represented by the figure shown below.



- The divergence of the vector field can be found as $\nabla \cdot \boldsymbol{f}$

- If the divergence of a two-dimensional vector field is represented as $\boldsymbol{f} = f_1\boldsymbol{e}_1 + f_2\boldsymbol{e}_2$, then

$$\nabla \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$$

- In this problem, $f_1 = -y$ and $f_2 = x$. Thus

$$\frac{\partial f_1}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f_2}{\partial y} = 0$$

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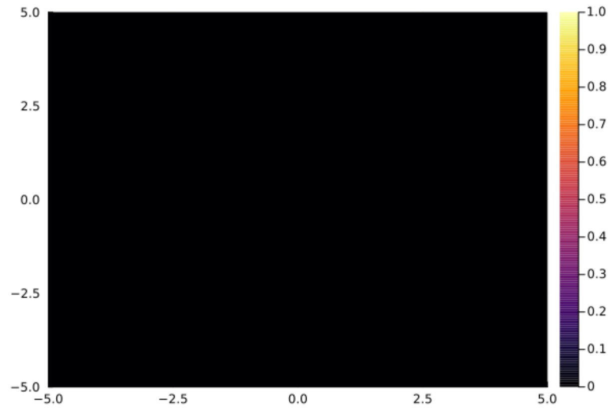
- Thus the divergence of the vector field $\nabla \cdot \boldsymbol{f} = 0$ and can be schematically shown as

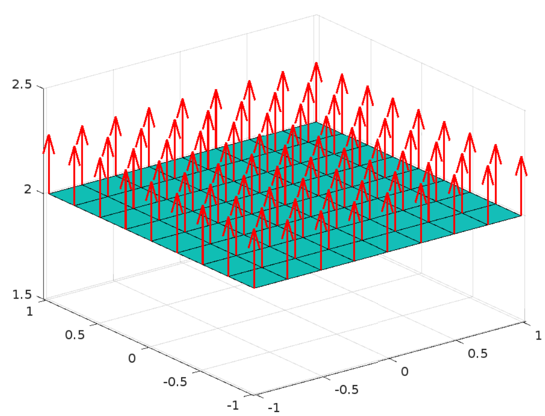


- The curl of the vector field can be found as $\nabla \times \boldsymbol{f}$

- For a vector field $\boldsymbol{f} = f_1\boldsymbol{e}_1 + f_2\boldsymbol{e}_2$, the curl can be found as

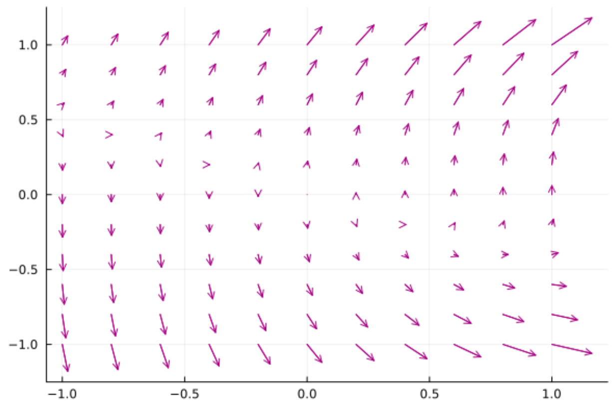
$$\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

- Thus the curl vector can be found as $\nabla \times \boldsymbol{f} = 2\boldsymbol{e}_3$. It can be represented as



b) Consider the vector field $\boldsymbol{f} = (e^x y^2)\boldsymbol{e}_1 + (x + 2y)\boldsymbol{e}_2$

- We can represent the vector field in some representative points in the xy plane as $\boldsymbol{f} = (e^x y^2)\boldsymbol{e}_1 + (x + 2y)\boldsymbol{e}_2$. The vector field for the entire domain can be shown as in Figure below



- The divergence of the vector field can be obtained as $e^x y^2 + 2$.
- The curl of the vector field can be obtained as $1 - 2e^x y$
- The divergence and curl are plotted as shown below

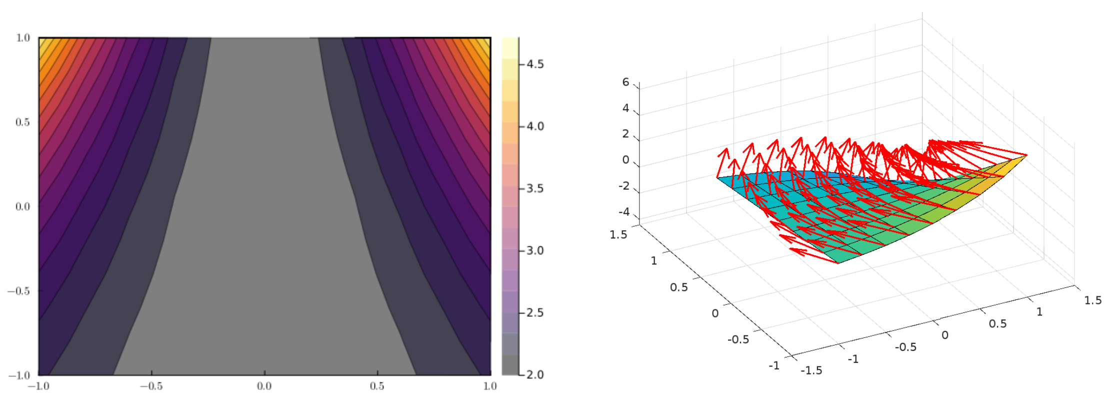


Figure 1: Plot of divergence (on the left), and curl (on the right)

2. A hill is described by the scalar function $h(x, y) = 200 - x^2 - 2y^2$, where h is the height of the hill at any point (x, y) on the $x - y$ plane. If you are standing at the point $(1, 2)$, in which direction should you walk to climb the hill most rapidly, and what is the rate of increase of height in that direction?

Solution:

To find the direction in which you should walk to climb the hill most rapidly and the rate of increase of height in that direction, we need to calculate the gradient of the scalar function $h(x, y)$ at the point $(1, 2)$.

The gradient of $h(x, y)$ is given by:

$$\nabla h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right)$$

where $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are the partial derivatives of h with respect to x and y, respectively.

First, let's find the gradient at the point $(1, 2)$:

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\left(200 - x^2 - 2y^2\right) = -2x$$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(200 - x^2 - 2y^2\right) = -4y$$

Now, evaluate the gradient at the point $(1, 2)$:

$$\nabla h(1, 2) = (-2(1), -4(2)) = (-2, -8)$$

The direction in which you should walk to climb the hill most rapidly is the direction of the gradient vector $(-2, -8)$. This direction points towards the steepest increase in height.

The rate of increase of height in that direction is equal to the magnitude of the gradient vector. The magnitude of $(-2, -8)$ is calculated as follows:

$$\|\nabla h(1, 2)\| = \sqrt{(-2)^2 + (-8)^2} = \sqrt{4 + 64} = \sqrt{68}$$

Thus, the rate of increase of height in the direction of $(-2, -8)$ is $\sqrt{68}$ (approximately 8.246).

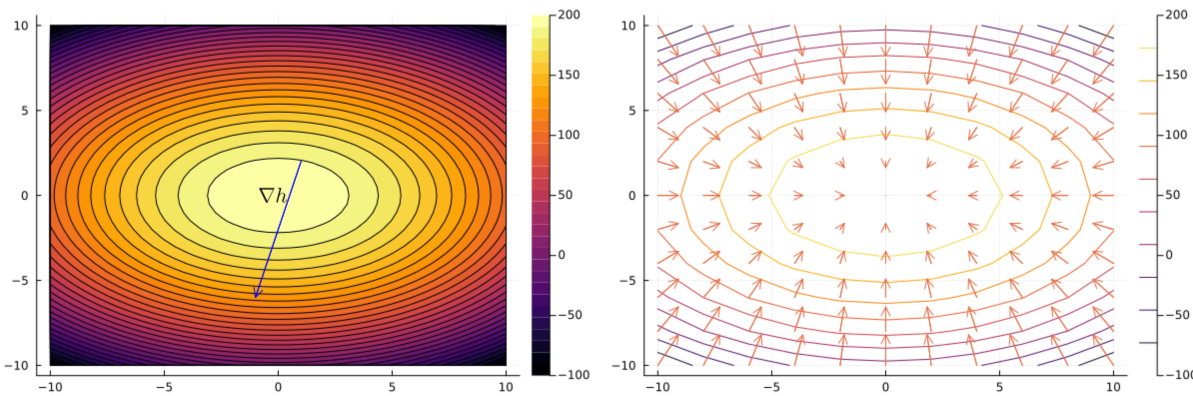


Figure 2: Plot of scalar field h (on the left), and its gradient (on the right)

3. a) Plot the following vector fields and determine which of them acts as source, sink, or solenoidal.

1. $\boldsymbol{f} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$
2. $\boldsymbol{f} = -x\boldsymbol{e}_1 - y\boldsymbol{e}_2$
3. $\boldsymbol{f} = \boldsymbol{e}_1 + \boldsymbol{e}_2$

Consider the following coordinates for x and y for plotting vector field

$[x, y]$	[0,1]	[1,0]	[1,1]	[0,-1]	[-1,0]	[-1,-1]	[1,-1]	[-1,1]	[0,0]

b) Determine which of the following vector fields acts as source, sink, or solenoidal @ co-ordinate (1,2): (**No need to plot**)

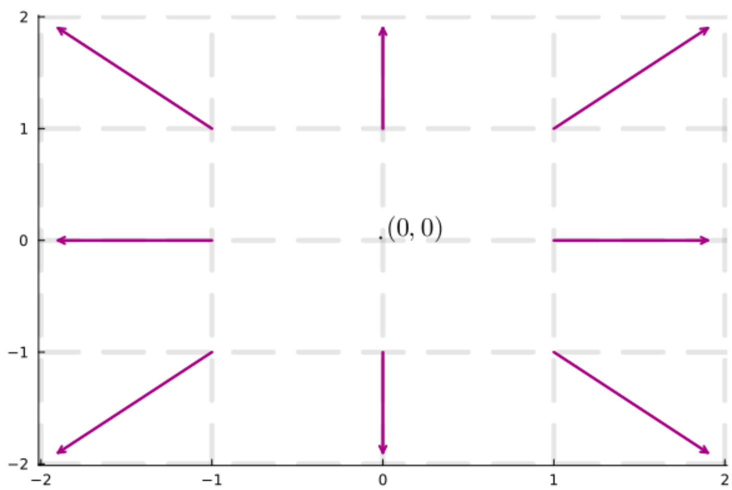
1. $\boldsymbol{f} = (2x + y^3)\boldsymbol{e}_1 + (x^3 + 2 * y)\boldsymbol{e}_2$
2. $\boldsymbol{f} = (e^x)\boldsymbol{e}_1 + (e^y)\boldsymbol{e}_2$

Solution Hints:

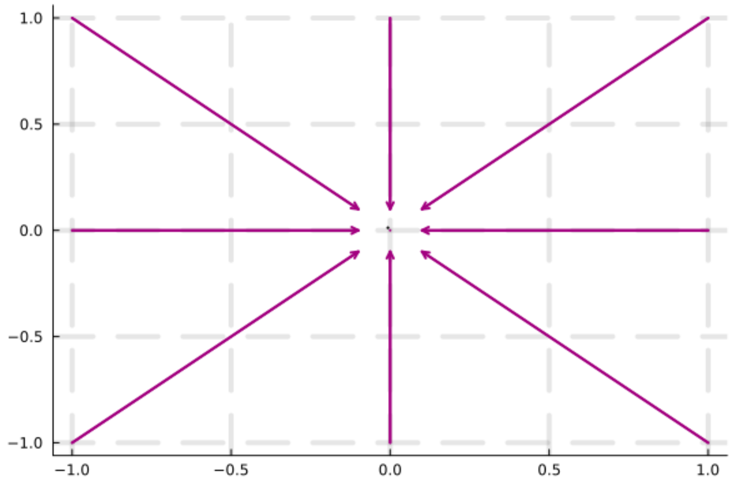
- Divergence of a vector is positive: Source (diverging).
- Divergence of a vector is zero: Solenoidal (divergence-free).
- Divergence of a vector is negative: Source (converging).

a) The vector field for

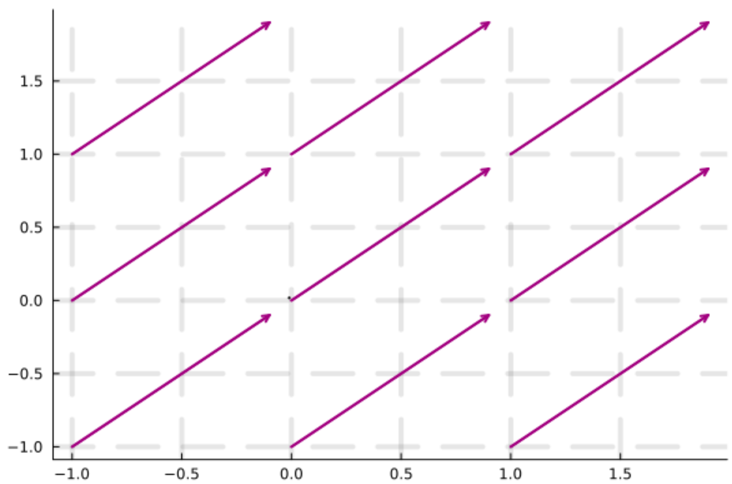
- 1) $\boldsymbol{f} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$



- 2) $\boldsymbol{f} = -x\boldsymbol{e}_1 - y\boldsymbol{e}_2$



- 3) $\boldsymbol{f} = \boldsymbol{e}_1 + \boldsymbol{e}_2$



- The divergence of the vector field $\boldsymbol{f} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$ is:

$$\nabla \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 1$$

  Since the divergence is greater than zero, the vector field is a source.

- The divergence of the vector field $\boldsymbol{f} = -x\boldsymbol{e}_1 - y\boldsymbol{e}_2$ is:

$$\nabla \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = -1$$

  Since the divergence is less than zero, the vector field is a sink.

- The divergence of the vector field $\boldsymbol{f} = 1\boldsymbol{e}_1 + 1\boldsymbol{e}_2$ is:

$$\nabla \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 0$$

Since the divergence equals zero, the vector field is solenoidal.

4.) Consider that a paddle wheel is being placed on flowing water with the following velocity vector field of water particles on the pond's surface. Determine whether the paddle wheel will undergo any rotation or not. If rotation occurs, specify the direction of rotation. Consider a X-Y cartesian coordinate system.

a.) $\boldsymbol{f} = (y+1)\boldsymbol{e}_1$

b.) $\boldsymbol{f} = (x+1)\boldsymbol{e}_2$

c.) $\boldsymbol{f} = \boldsymbol{e}_1 + \boldsymbol{e}_2$

Consider the following coordinates for x and y for plotting vector field

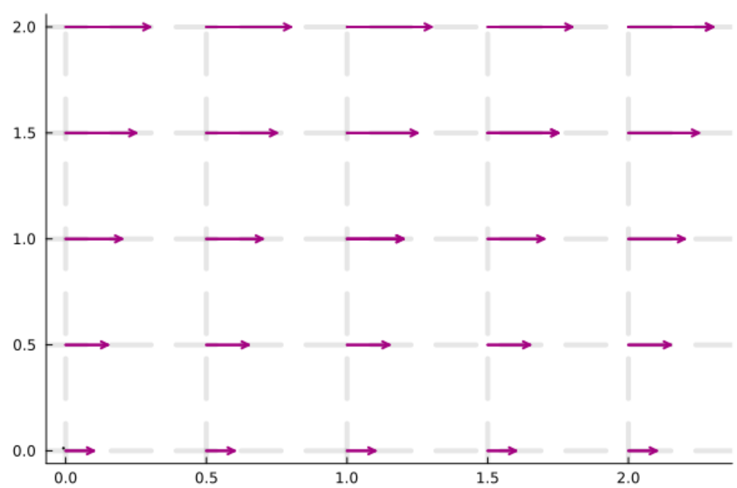
$[x, y]$	[0,0]	[0,0.5]	[0.5,0]	[0.5,1.5]	[1,1.5]	[1.5,1.5]	[2,1.5]
$[x, y]$	[0.5,0.5]	[1,0.5]	[1.5,0.5]	[2,0.5]	[0,1]	[0.5,1]	[1,1]
$[x, y]$	[1.5,1]	[2,1]	[1,0]	[1,1]	[0,2]	[0.5,2]	[1,2]
$[x, y]$	[1.5,2]	[2,2]	[2,0]	[2,2]	[0,1.5]	[1.5,0]	[1.5,1.5]

Hints:

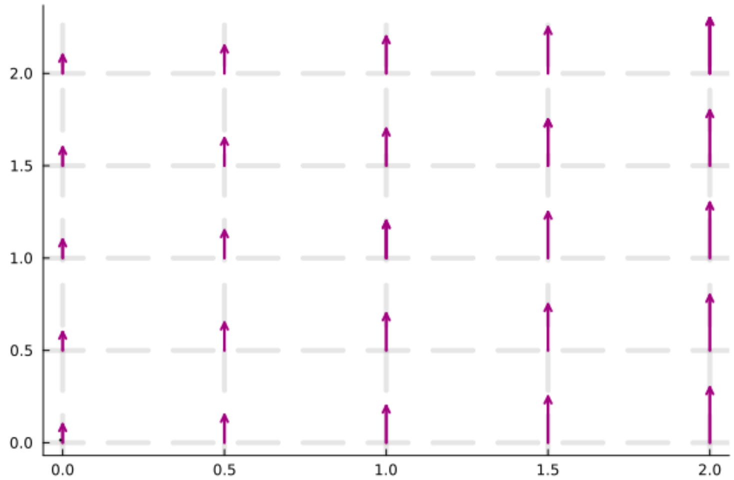
- Curl of a vector is positive: Anticlockwise.
- Curl of a vector is negative: Clockwise.
- Curl of a vector is zero: No rotation.

a) The vector field for

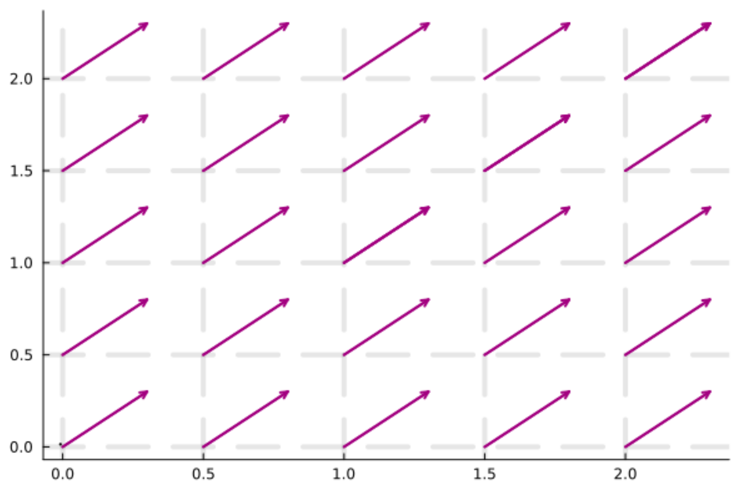
- 1) $\boldsymbol{f} = (y+1)\boldsymbol{e}_1$



- 2) $\boldsymbol{f} = (x+1)\boldsymbol{e}_2$



- 3) $\boldsymbol{f} = \boldsymbol{e}_1 + \boldsymbol{e}_2$



- The curl of the vector field $\boldsymbol{f} = (y+1)\boldsymbol{e}_1$ is:

$$\nabla \times \boldsymbol{f} = -2\boldsymbol{e}_3$$

  Since the curl is less than zero, the rotation of the paddle wheel will be in the clockwise direction.

- The curl of the vector field $\boldsymbol{f} = (x+1)\boldsymbol{e}_2$ is:

$$\nabla \times \boldsymbol{f} = 2\boldsymbol{e}_3$$

  Since the curl is greater than zero, the rotation of the paddle wheel will be in the anti-clockwise direction.

- The curl of the vector field $\boldsymbol{f} = \boldsymbol{e}_1 + \boldsymbol{e}_2$ is:

$$\nabla \times \boldsymbol{f} = 0$$

  Since the curl is equal to zero, there will be no rotation for the paddle wheel.

5. Consider a cyclone in the northern hemisphere described by the velocity vector field of the wind:

$$\mathbf{v}(x, y) = -y\boldsymbol{e}_1 + (x^2 + y^2)\boldsymbol{e}_2$$

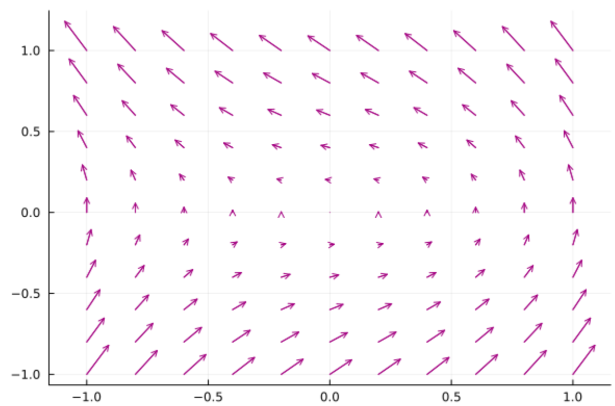
where x and y are the coordinates in the horizontal plane, and $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors in the x- and y-directions, respectively.

a) Calculate the divergence and curl of the vector field $\mathbf{v}(x, y)$.

b) Explain the physical significance of the divergence and curl in the context of a cyclone.

c) Based on the curl, determine the direction of rotation of the cyclone.

Solution:



a) Divergence and Curl:

To find the divergence of the vector field $\mathbf{v}(x, y)$, use the formula:

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \\
\text{where } \mathbf{v}(x, y) &= (-y, x^2 + y^2) \\
\nabla \cdot \mathbf{v} &= \frac{\partial(-y)}{\partial x} + \frac{\partial(x^2 + y^2)}{\partial y} \\
&= 0 + 2y \\
&= 2y
\end{aligned}$$

To find the curl of the vector field $\mathbf{v}(x, y)$, use the formula:

$$\begin{aligned}
\nabla \times \mathbf{v} &= \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\boldsymbol{e}_3 \\
\text{where } \mathbf{v}(x, y) &= (-y, x^2 - y^2) \\
\nabla \times \mathbf{v} &= \left(\frac{\partial(x^2 - y^2)}{\partial x} - \frac{\partial(-y)}{\partial y}\right)\boldsymbol{e}_3 \\
&= (2x - 1)\boldsymbol{e}_3
\end{aligned}$$

b) Physical Significance:

Divergence: The divergence $2y$ indicates how the flow density is changing with respect to y. For a cyclone, this suggests that there is a net outflow or inflow depending on the y-coordinate. Specifically, the flow is expanding or contracting in the vertical direction.

Curl: The curl of $(2x-1)\boldsymbol{e}_3$ suggests rotational motion. The value depends on the x-coordinate and is positive when $x > 0.5$, indicating counterclockwise rotation in the northern hemisphere. The varying magnitude of the curl indicates the strength of rotation changes with x.

c) Direction of Rotation:

The positive curl value indicates counterclockwise rotation, which is typical for cyclones in the northern hemisphere. The varying magnitude shows that the strength of the cyclone's rotation changes with the horizontal position x.

Question 6. In Geotechnical engineering, understanding the flow of water in a dam's vicinity is crucial. The potential function $\phi(x, y)$ of a water flow around a dam is given by:

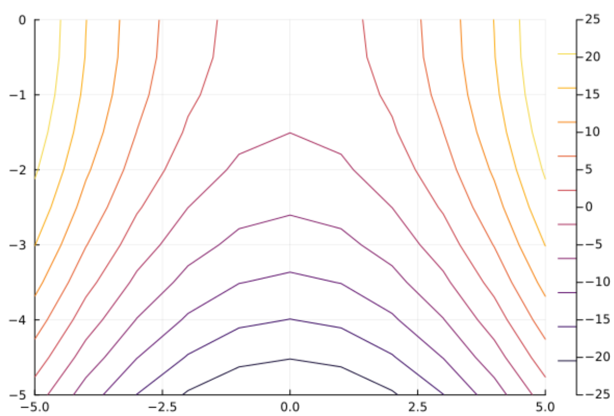
$$\phi(x, y) = x^2 - y^2$$

a) Calculate the velocity vector field $\mathbf{v}(x, y)$ from the potential function $\phi(x, y)$.

b) Determine the divergence and curl of the velocity vector field.

c) Draw the vector field and discuss the behavior of water flow around the dam.

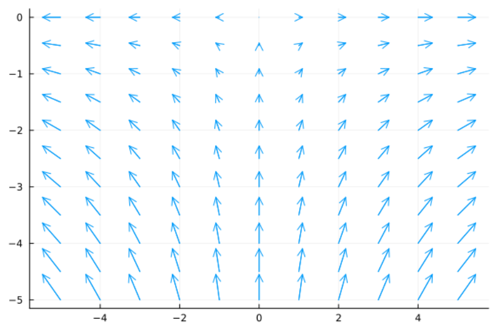
Plotting of ϕ:



a) Velocity:

Given the potential function $\phi(x, y)$, the velocity vector field is:

$$\mathbf{v}(x, y) = \nabla\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)$$
$$= (2x, -2y)$$



b) Divergence:

To find the divergence:

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{\partial(2x)}{\partial x} + \frac{\partial(-2y)}{\partial y} \\ &= 2 - 2 \\ &= 0\end{aligned}$$

Curl:

To find the curl:

$$\begin{aligned}\nabla \times \mathbf{v} &= \left(\frac{\partial(-2y)}{\partial x} - \frac{\partial(2x)}{\partial y}\right)\mathbf{e}_3 \\ &= (0 - 0)\mathbf{e}_3 \\ &= \mathbf{0}\end{aligned}$$

c) Vector Field Drawing:

Field Behavior: Plotting the vector field $\mathbf{v}(x, y) = (2x, -2y)$ will show hyperbolic streamlines, indicating saddle points at the origin $(0, 0)$. Water flows away from the origin along the x-axis and towards the origin along the y-axis.

Real-Life Significance:

Divergence (zero): Indicates incompressible flow, which is realistic for water flow scenarios.

Curl (zero vector): Absence of rotational motion, suggesting that water particles follow a straight path along the streamline without swirling.

7. Consider the velocity vector field of water flow in an open channel given by:

$$\mathbf{v}(x, y) = (y, x)$$

where x and y are the coordinates in the horizontal plane, and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x- and y-directions, respectively.

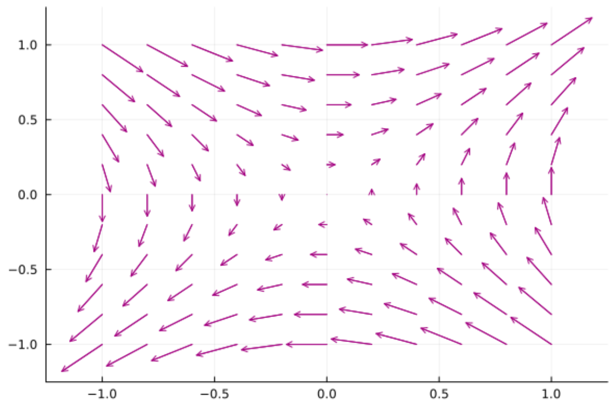
a) Calculate the divergence and curl of the vector field $\mathbf{v}(x, y)$.

b) Explain the physical significance of the divergence and curl in the context of open channel flow.

c) Plot the vector field and discuss the flow characteristics in the channel based on the vector field and curl.

Solution

The vector field is



a) **Divergence**:

To find the divergence of the vector field $\mathbf{v}(x, y)$, use the formula:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$$

Given $\mathbf{v}(x, y) = (y, x)$,

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} \\ &= 0 + 0 \\ &= 0\end{aligned}$$

Curl:

To find the curl of the vector field $\mathbf{v}(x, y)$, use the formula:

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \mathbf{e}_3$$

Given $\mathbf{v}(x, y) = (y, x)$,

$$\begin{aligned}\nabla \times \mathbf{v} &= \left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}\right)\mathbf{e}_3 \\ &= (1-1)\mathbf{e}_3 \\ &= 0\end{aligned}$$

b) **Physical Significance**:

- **Divergence** (0): A zero divergence indicates that there is no net change in the density of the flow in the horizontal plane. This implies that the flow is incompressible, with no sources or sinks of water in the channel.
- **Curl** (0): A zero curl indicates that there is no rotational motion in the flow. The water flow is irrotational, meaning there are no vortices or circular motion within the channel.

c) **Plot and Discussion**:

The vector field can be plotted to visualize the direction and magnitude of the water flow in the channel. The zero curl indicates that the flow has no rotational components, suggesting a straight and uniform flow pattern.

The plot shows the vector field with arrows indicating the direction and magnitude of the water flow. The flow pattern is straight and uniform, with no rotational motion, as indicated by the zero curl.

8. Consider the airflow around an aircraft wing described by the velocity vector field:

$$\mathbf{v}(x, y) = \left(-ye^{-x^2-y^2}, (x-y)e^{-x^2-y^2}\right)$$

where x and y represent coordinates in the horizontal plane, and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x- and y-directions, respectively.

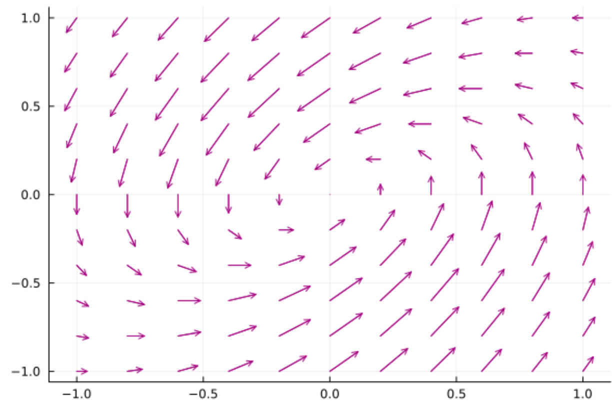
a) Calculate the divergence and curl of the vector field $\mathbf{v}(x, y)$.

b) Explain the physical significance of the divergence and curl in the context of aerodynamics around the aircraft wing.

c) Plot the vector field and discuss the flow characteristics around the wing based on the vector field and curl.

Solution

The vector field is



a) **Divergence**: To find the divergence of the vector field $\mathbf{v}(x, y)$, use the formula:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$$

Given $\mathbf{v}(x, y) = \left(-ye^{-x^2-y^2}, (x-y)e^{-x^2-y^2}\right)$,

$$\begin{aligned} v_x &= -ye^{-x^2-y^2} \\ v_y &= (x-y)e^{-x^2-y^2} \end{aligned}$$

For the partial derivatives:

$$\begin{aligned} \frac{\partial v_x}{\partial x} &= -y\left(\frac{\partial}{\partial x}e^{-x^2-y^2}\right) \\ &= -y \cdot (-2xe^{-x^2-y^2}) \\ &= 2xye^{-x^2-y^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial v_y}{\partial y} &= (x-y)\left(\frac{\partial}{\partial y}e^{-x^2-y^2}\right) + e^{-x^2-y^2}\frac{\partial(x-y)}{\partial y} \\ &= (x-y) \cdot (-2ye^{-x^2-y^2}) + e^{-x^2-y^2} \cdot (-1) \\ &= -2y(x-y)e^{-x^2-y^2} - e^{-x^2-y^2} \\ &= -2xye^{-x^2-y^2} + 2y^2e^{-x^2-y^2} - e^{-x^2-y^2} \end{aligned}$$

Therefore, the divergence is:

$$\begin{aligned} \nabla \cdot \mathbf{v} &= 2xye^{-x^2-y^2} - 2xye^{-x^2-y^2} + 2y^2e^{-x^2-y^2} - e^{-x^2-y^2} \\ &= 2y^2e^{-x^2-y^2} - e^{-x^2-y^2} \\ &= (2y^2-1)e^{-x^2-y^2} \end{aligned}$$

Curl:

To find the curl of the vector field $\mathbf{v}(x, y)$, use the formula:

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{k}$$

The partial derivatives are calculated as follows:

$$\begin{aligned}
\frac{\partial v_y}{\partial x} &= (x - y) \left(\frac{\partial}{\partial x} e^{-x^2 - y^2} \right) + e^{-x^2 - y^2} \frac{\partial (x - y)}{\partial x} \\
&= (x - y) \cdot (-2x e^{-x^2 - y^2}) + e^{-x^2 - y^2} \cdot 1 \\
&= -2x(x - y) e^{-x^2 - y^2} + e^{-x^2 - y^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial v_x}{\partial y} &= -e^{-x^2 - y^2} - y \left(\frac{\partial}{\partial y} e^{-x^2 - y^2} \right) \\
&= -e^{-x^2 - y^2} - y \cdot (-2y e^{-x^2 - y^2}) \\
&= -e^{-x^2 - y^2} + 2y^2 e^{-x^2 - y^2}
\end{aligned}$$

Now, compute the curl:

$$\begin{aligned}
\nabla \times \mathbf{v} &= \left(-2x(x - y) e^{-x^2 - y^2} + e^{-x^2 - y^2} - (-e^{-x^2 - y^2} + 2y^2 e^{-x^2 - y^2}) \right) \mathbf{e}_3 \\
&= \left(-2x(x - y) e^{-x^2 - y^2} + e^{-x^2 - y^2} + e^{-x^2 - y^2} - 2y^2 e^{-x^2 - y^2} \right) \mathbf{e}_3 \\
&= \left(-2x(x - y) e^{-x^2 - y^2} + 2e^{-x^2 - y^2} - 2y^2 e^{-x^2 - y^2} \right) \mathbf{e}_3 \\
&= \left(-2x^2 + 2xy + 2 - 2y^2 \right) e^{-x^2 - y^2} \mathbf{e}_3 \\
&= \left(2 - 2x^2 - 2y^2 + 2xy \right) e^{-x^2 - y^2} \mathbf{e}_3
\end{aligned}$$

b) **Physical Significance**:

- **Divergence**: The non-zero divergence indicates the presence of sources or sinks in the flow. This means that the fluid density is not uniform, leading to areas where the fluid is converging or diverging.
- **Curl**: The non-zero curl indicates the presence of rotational motion in the flow. This suggests that the airflow around the wing has vortices or swirling motion, which is critical for understanding lift and drag forces acting on the wing.

c) **Plot and Discussion**:

The vector field can be plotted to visualize the direction and magnitude of the airflow around the wing.

The plot shows the vector field with arrows indicating the direction and magnitude of the airflow. The presence of rotational motion is evident from the curved arrows, indicating vortices in the flow, which are crucial for understanding lift generation and potential turbulence around the wing.