Indian Institute of Technology Bhubaneswar

School of Infrastructure

Session: Autumn 2024

Solid Mechanics (CE2L001) Solution of Assignment No. 1

<u>Notations :</u>
Zeroth-order tensors or scalars are represented by small letters. For eg. a
First-order tensors or vectors are represented by bold small letters. For eg. $\boldsymbol{a}$.
Second-order tensors are represented by bold capital letters. For eg. $\boldsymbol{A}$

1. Simplify the following expressions using the contraction property of δ and the $\epsilon - \delta$ relation.
(a) $\delta_{ij}\delta_{jk}\delta_{kl}\delta_{lm}\delta_{mn}$ (b) $\epsilon_{jkq}\epsilon_{jkq}$.

Solution: (a) The Kronecker delta function δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1}$$

To simplify the expression $\delta_{ij}\delta_{jk}\delta_{kl}\delta_{lm}\delta_{mn}$, use the contraction properties of the Kronecker delta δ_{ij}.

$$\begin{aligned}
\delta_{ij}\delta_{jk}\delta_{kl}\delta_{lm}\delta_{mn} &= \delta_{ik}\delta_{kl}\delta_{lm}\delta_{mn} \qquad (\text{since } \delta_{ij}\delta_{jk} = \delta_{ik}) \\
&= \delta_{il}\delta_{lm}\delta_{mn} \qquad (\text{since } \delta_{ik}\delta_{kl} = \delta_{il}) \\
&= \delta_{im}\delta_{mn} \qquad (\text{since } \delta_{il}\delta_{lm} = \delta_{im}) \\
&= \delta_{in} = 0 \quad (\text{since, } i \neq n\ ) \quad (\text{using Eq.(1)}).
\end{aligned}$$

(b) The $\epsilon - \delta$ relation is given by

$$\epsilon_{jkq}\epsilon_{jkq} = \delta_{jj}\delta_{kk} - \delta_{jk}\delta_{jk} = (3) \times (3) - \delta_{jj} \tag{2}$$

$$= 9 - 3 = 6 \ \ (\text{since } \ \delta_{jj}, \delta_{kk} = 3) \tag{3}$$

2. Consider two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ whose matrix of components relative to an orthonormal basis $\{\boldsymbol{e_i}\}$ are

$$[\boldsymbol{a}] = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \quad \text{and} \quad [\boldsymbol{b}] = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

Compute (a) $|\boldsymbol{a}|$, magnitude of $\boldsymbol{a}$ vector, (b) the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ vectors (c) the area of the parallelogram bounded by $\boldsymbol{a}$ and $\boldsymbol{b}$ vectors (d) $\boldsymbol{b} \times \boldsymbol{a}$.

Solution:

(a) The norm or magnitude of a vector can be determined as

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14} \approx 3.741.$$

(b) The angle (θ) between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be determined as

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right), \tag{4}$$

where $\boldsymbol{a}\cdot\boldsymbol{b}$ can be found as

$$\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{b} &= a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= 3 \times 1 + -2 \times 4 + 1 \times -2 = 3 - 8 - 2 = -7. \end{aligned} \tag{5}$$

One can determine $|\boldsymbol{b}|$ as

$$|\boldsymbol{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{1+16+4} = \sqrt{21} = 4.582. \tag{6}$$

From Eq. (4), the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be determined as

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right) = \cos^{-1}\left(\frac{-7}{\sqrt{14}\sqrt{21}}\right) \\ \implies \theta &\approx 114.09^\circ. \end{aligned}$$

(c) The area of the parallelogram, A bounded by $\boldsymbol{a}$ and $\boldsymbol{b}$ can be determined

$$A = |\boldsymbol{a} \times \boldsymbol{b}|, \tag{7}$$

Let, $\boldsymbol{v} = \boldsymbol{a} \times \boldsymbol{b}$. The components of the vector $\boldsymbol{v} = v_i \boldsymbol{e}_i$ can be determined as $v_i = \epsilon_{ijk} a_j b_k$. Thus, the components along each basis vector can be determined as

$$\begin{aligned} v_1 &= \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2 = 4 - 4 = 0, \\ v_2 &= \epsilon_{231} a_3 b_1 + \epsilon_{213} a_1 b_3 = a_3 b_1 - a_1 b_3 = 1 + 6 = 7, \\ v_3 &= \epsilon_{312} a_1 b_2 + \epsilon_{321} a_2 b_1 = a_1 b_2 - a_2 b_1 = 12 + 2 = 14. \end{aligned}$$

$$\implies \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{v} = v_i \boldsymbol{e}_i = 0\boldsymbol{e}_1 + 7\boldsymbol{e}_2 + 14\boldsymbol{e}_3.$$

Thus the area, A can be found as

$$|\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{7^2 + 14^2} = \sqrt{49 + 196} \approx 15.65.$$

(d) The vector $\boldsymbol{b} \times \boldsymbol{a}$ can be determined as

$$\boldsymbol{b} \times \boldsymbol{a} = -(\boldsymbol{a} \times \boldsymbol{b}) = -7\boldsymbol{e}_2 - 14\boldsymbol{e}_3.$$

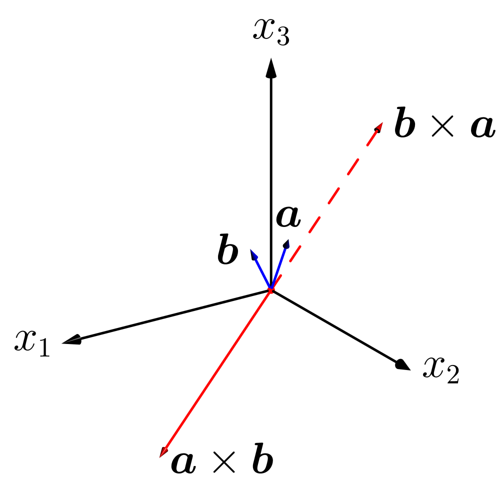

Figure 1: Schematic representation of the cross product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

3. Rewrite the expression $\epsilon_{mni} a_i b_j c_m d_n \boldsymbol{e_j}$ in direct notation using the scalar and cross products of vectors.

Solution: To simplify the expression $\epsilon_{mni} a_i b_j c_m d_n \boldsymbol{e_j}$ using direct notation as follows:

$$\begin{aligned} \epsilon_{mni} a_i b_j c_m d_n \boldsymbol{e_j} &= a_i b_j (\boldsymbol{c} \times \boldsymbol{d})_i \boldsymbol{e_j} \qquad (\text{since } \epsilon_{mni} c_m d_n = (\boldsymbol{c} \times \boldsymbol{d})_i) \\ &= (a_i (\boldsymbol{c} \times \boldsymbol{d})_i)\,(b_j \boldsymbol{e}_j) \qquad (\text{Rearranging the terms}) \\ &= (\boldsymbol{a} \cdot (\boldsymbol{c} \times \boldsymbol{d}))\boldsymbol{b} \qquad (\text{since } a_i (\boldsymbol{c} \times \boldsymbol{d})_i = \boldsymbol{a} \cdot (\boldsymbol{c} \times \boldsymbol{d})) \end{aligned}$$

4. For a two-dimensional (2D) problem, let the components of a second-order tensor $\boldsymbol{A}$ be $A_{11} = 2, A_{12} = 4 = A_{21}, A_{22} = 5$. Let the components of a vector $\boldsymbol{v}$ be $v_1 = 3, v_2 = -1$, all in the same orthogonal basis. Compute the components of the vector, $\boldsymbol{w} = \boldsymbol{A}\,\boldsymbol{v}$ using the relation $w_i = A_{ij} v_j$.

Solution: The components of $\boldsymbol{w} = \boldsymbol{A}\,\boldsymbol{v}$ are related by $w_i = A_{ij}v_j$, where $\boldsymbol{w} = w_i\boldsymbol{e}_i$, $\boldsymbol{v} = v_j\boldsymbol{e}_j$ and $\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Thus the individual components of $\boldsymbol{w}$ along each basis vector can be determined as

$$\begin{aligned} w_1 &= A_{11}v_1 + A_{12}v_2 = 2 \times 3 + 4 \times (-1) = -2 \\ w_2 &= A_{21}v_1 + A_{22}v_2 = 4 \times 3 + 5 \times (-1) = 7. \end{aligned}$$

5. Show that $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$.

Solution:

$$\begin{aligned} (\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}))_i &= \epsilon_{ijk}a_j(\boldsymbol{b} \times \boldsymbol{c})_k \\ &= \epsilon_{ijk}a_j\epsilon_{klm}b_lc_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})a_jb_lc_m \\ &= a_jb_ic_j - a_jb_jc_i \\ &= (\boldsymbol{a} \cdot \boldsymbol{c})b_i - (\boldsymbol{a} \cdot \boldsymbol{b})c_i \end{aligned}$$

$$\Rightarrow \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$$

6. Show that $\mathbf{a} \times (\nabla \times \mathbf{a}) = \frac{1}{2}\nabla(\mathbf{a} \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla \mathbf{a})$

Solution:

$$
\begin{aligned}
\mathbf{a} \times (\nabla \times \mathbf{a}) &= a_i \mathbf{e}_i \times \left(\frac{\partial a_k}{\partial x_j} \mathbf{e}_j \times \mathbf{e}_k \right) && \left(\mathbf{a} = a_i \mathbf{e}_i,\ \nabla \times \mathbf{a} = \tfrac{\partial a_k}{\partial x_j} \mathbf{e}_j \times \mathbf{e}_k\right) \\
&= \epsilon_{jkm} a_i \frac{\partial a_k}{\partial x_j} (\mathbf{e}_i \times \mathbf{e}_m) && (\mathbf{e}_j \times \mathbf{e}_k = \epsilon_{jkm} \mathbf{e}_m) \\
&= \epsilon_{jkm} \epsilon_{imn} a_i \frac{\partial a_k}{\partial x_j} \mathbf{e}_n && (\mathbf{e}_i \times \mathbf{e}_m = \epsilon_{imn} \mathbf{e}_n) \\
&= a_i \frac{\partial a_k}{\partial x_j} (\delta_{jn}\delta_{ki} - \delta_{ji}\delta_{kn}) \mathbf{e}_n && (\epsilon_{jkm}\epsilon_{imn} = \delta_{jn}\delta_{ki} - \delta_{ji}\delta_{kn}) \\
&= a_i \frac{\partial a_i}{\partial x_j} \mathbf{e}_j - a_i \frac{\partial a_k}{\partial x_i} \mathbf{e}_k && \left(\delta_{jn}\mathbf{e}_n = \mathbf{e}_j, \delta_{ji} \tfrac{\partial}{\partial x_j} = \tfrac{\partial}{\partial x_i}\right) \\
&= \frac{1}{2} \frac{\partial (a_i a_i)}{\partial x_j} \mathbf{e}_j - a_i \frac{\partial a_k}{\partial x_i} \mathbf{e}_k && \left(a_i \tfrac{\partial a_i}{\partial x_j} = \tfrac{1}{2} \tfrac{\partial (a_i a_i)}{\partial x_j}\right) \\
&= \frac{1}{2} \frac{\partial (a_i a_i)}{\partial x_j} \mathbf{e}_j - a_i \frac{\partial \mathbf{a}}{\partial x_i} && (a_k \mathbf{e}_k = \mathbf{a}) \\
&= \frac{1}{2} \nabla (\mathbf{a} \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla \mathbf{a}) && \left(\nabla f = \mathbf{e}_j \tfrac{\partial f}{\partial x_j}, a_i \tfrac{\partial \mathbf{a}}{\partial x_i} = \mathbf{a} \cdot \nabla \mathbf{a}\right)
\end{aligned}
$$

7. Prove the following identities:

a) $\nabla \cdot (\nabla \phi) = \nabla^2 \phi$, where ϕ is a scalar field.

Solution:

$$
\begin{aligned}
&\nabla \cdot (\nabla \phi) \\
&\quad = \mathbf{e}_i \frac{\partial ()}{\partial x_i} \cdot \mathbf{e}_j \frac{\partial \phi}{\partial x_j} && \left(\text{since } \nabla = \mathbf{e}_i \frac{\partial ()}{\partial x_i},\ \nabla \phi = \mathbf{e}_i \frac{\partial \phi}{\partial x_i}\right) \\
&\quad = \frac{\partial}{\partial x_i} \left(\frac{\partial \phi}{\partial x_j} \right) (\mathbf{e}_i \cdot \mathbf{e}_j) && \left(\text{note } \frac{\partial}{\partial x_i} \left(\frac{\partial \phi}{\partial x_j} \right) \text{ is a scalar quantity}\right) \\
&\quad = \frac{\partial^2 \phi}{\partial x_i \partial x_j} \delta_{ij} && \left(\text{since } \frac{\partial}{\partial x_i} \left(\frac{\partial \phi}{\partial x_j} \right) = \frac{\partial^2 \phi}{\partial x_i \partial x_j} \text{ and } \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}\right) \\
&\quad = \frac{\partial^2 \phi}{\partial x_i^2} && \left(\text{using contraction property of } \delta_{ij}\right) \\
&\quad = \nabla^2 \phi && \left(\text{since } \nabla^2 \phi := \frac{\partial^2 \phi}{\partial x_i^2}\right)
\end{aligned}
$$

b) $\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$, where $\mathbf{a}$ is a vector field.

Solution:

$$
\begin{aligned}
[\nabla \times (\nabla \times \mathbf{a})]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times \mathbf{a})_k && \left([\nabla \times \mathbf{v}]_i = \epsilon_{ijk} \tfrac{\partial v_k}{\partial x_j}\right) \\
&= \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\epsilon_{klm} \frac{\partial a_m}{\partial x_l} \right) && \left((\nabla \times \mathbf{a})_k = \epsilon_{klm} \tfrac{\partial a_m}{\partial x_l}\right) \\
&= \epsilon_{ijk} \epsilon_{klm} \frac{\partial^2 a_m}{\partial x_j \partial x_l} && \left(\tfrac{\partial}{\partial x_j}(\epsilon_{klm} \tfrac{\partial a_m}{\partial x_l}) = \epsilon_{klm} \tfrac{\partial^2 a_m}{\partial x_j \partial x_l}\right) \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial^2 a_m}{\partial x_j \partial x_l} && \left(\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) \\
&= \frac{\partial^2 a_j}{\partial x_i \partial x_j} - \frac{\partial^2 a_i}{\partial x_j^2} && \left(\delta\text{-contractions, } \tfrac{\partial^2 a_j}{\partial x_j \partial x_i} = \tfrac{\partial^2 a_j}{\partial x_i \partial x_j}\right) \\
&= \frac{\partial}{\partial x_i} \left(\frac{\partial a_j}{\partial x_j} \right) - \nabla^2 a_i && \left([\nabla(\nabla \cdot \mathbf{a})]_i = \tfrac{\partial}{\partial x_i}(\tfrac{\partial a_j}{\partial x_j}),\ [\nabla^2 \mathbf{a}]_i = \tfrac{\partial^2 a_i}{\partial x_j^2}\right) \\
&= \big[\nabla(\nabla \cdot \mathbf{a})\big]_i - \big[\nabla^2 \mathbf{a}\big]_i && \left(\nabla f = \mathbf{e}_i \tfrac{\partial f}{\partial x_i},\ \nabla \cdot \mathbf{a} = \tfrac{\partial a_i}{\partial x_i}\right)
\end{aligned}
$$

$$\Rightarrow \nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

8. Consider a cyclone in the northern hemisphere described by the velocity vector field of the wind:

$$\mathbf{v}(x, y) = x\,\mathbf{e}_1 - y^2\,\mathbf{e}_2$$

(a) Calculate the divergence and curl of the vector field $\mathbf{v}(x, y)$.

Solution: (a) **Divergence:**

$$
\begin{aligned}
\mathbf{v}(x, y) &= x\,\mathbf{e}_1 - y^2\,\mathbf{e}_2 && \text{(given)} \\
\nabla \cdot \mathbf{v} &= \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} && \left(\nabla \cdot \mathbf{v} = \partial v_i / \partial x_i,\ \mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2\right) \\
&= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y^2) && (v_1 = x,\ v_2 = -y^2 \text{ substituted}) \\
&= 1 - 2y && \left(\frac{\partial}{\partial x} x = 1,\ \frac{\partial}{\partial y}(-y^2) = -2y \right)
\end{aligned}
$$

$$\boxed{\nabla \cdot \mathbf{v} = 1 - 2y}$$

(b) **Curl:**

$$
\begin{aligned}
\mathbf{v}(x,y) &= x\,\mathbf{e}_1 - y^2\,\mathbf{e}_2 && \text{(given)} \\
(\nabla \times \mathbf{v})_3 &= \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} && \big(\text{note other components will be zero}\big) \\
&= \frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(x) && (v_1 = x,\ v_2 = -y^2) \\
&= 0 - 0 && \left(\frac{\partial(-y^2)}{\partial x} = 0,\ \frac{\partial x}{\partial y} = 0\right) \\
&= 0 && (\text{so the scalar } z\text{-component is } 0) \\
\nabla \times \mathbf{v} &= 0\,\mathbf{e}_1 + 0\,\mathbf{e}_2 + 0\,\mathbf{e}_3 && \text{(extend to 3D vector)} \\
&= \mathbf{0} && \text{(zero vector)}
\end{aligned}
$$

$$\boxed{\nabla \times \mathbf{v} = \mathbf{0}}$$

(b) Explain the physical significance of the divergence and curl in the context of a cyclone.

Solution: (b) **Physical significance:**
Divergence of vector field tells us how the air is spreading or compressing. - Positive when $y < 0.5$: indicates source (air spreading). - Negative when $y > 0.5$: indicates sink (air converging).

Curl of the vector field is zero: the field is irrotational, meaning there's no local spinning or swirl in the wind.

(c) Based on the curl, determine the direction of rotation of the cyclone.

Solution: (c) Since the curl is zero, the cyclone exhibits no rotational behavior — the wind field is irrotational.

9. In Geo technical engineering, understanding the flow of water in a dam's vicinity is crucial. The potential function $\phi(x, y)$ of a water flow around a dam is given by:

$$\phi(x, y) = xy$$

(a) Calculate the velocity vector field $\mathbf{v}(x, y)$ from the potential function $\phi(x, y)$.

Solution: (a) Velocity field is the gradient of potential:

$$\begin{aligned}
\mathbf{v} := \nabla\phi &= \mathbf{e}_i \left(\frac{\partial\phi}{\partial x_i}\right) \\
&= \mathbf{e}_1 \left(\frac{\partial\phi}{\partial x_1}\right) + \mathbf{e}_2 \left(\frac{\partial\phi}{\partial x_2}\right) && \big(\text{for 2D case, } i = 1, 2\big) \\
&= \left(\frac{\partial\phi}{\partial x}\right)\mathbf{e}_1 + \left(\frac{\partial\phi}{\partial y}\right)\mathbf{e}_2 && \big(\text{here, } x_1 = x \text{ and } x_2 = y\big) \\
&= y\,\mathbf{e}_1 + x\,\mathbf{e}_2. && \left(\text{since } \frac{\partial\phi}{\partial x} = y \text{ and } \frac{\partial\phi}{\partial y} = x\right)
\end{aligned}$$

(b) Determine the divergence and curl of the velocity vector field.

Solution: (b) **Divergence:**

$$\nabla \cdot \mathbf{v} = \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} = 0 + 0 = 0$$

Curl:

$$(\nabla \times \mathbf{v})_3 = \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} = 1 - 1 = 0$$

(c) Draw the vector field and discuss the water flow behavior around the dam.

Solution: (c) **Flow interpretation:** - The flow is both **incompressible** (divergence = 0) and **irrotational** (curl = 0). - The streamlines are symmetric diagonal lines. - Water flows smoothly without swirling or compressing, ideal for modelling laminar groundwater flow near dams.

10. Given a vector $\mathbf{a} = a_i \boldsymbol{e}_i = a_i^* \boldsymbol{e}_i^*$ defined with respect to the basis $\mathbf{e}_i$ by

$$\mathbf{a} = 3\mathbf{e}_1 - 2\mathbf{e}_2 + 4\mathbf{e}_3$$

Find the components a_i^* of $\mathbf{a}$ with respect to the basis $\mathbf{e}_i^*$ defined in Fig. 2.

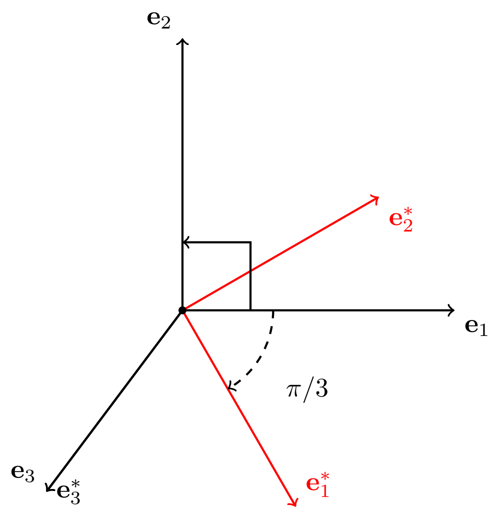

Figure 2: New ortho-normal basis $\mathbf{e}_i^*$ is obtained by a clockwise rotation of the ortho-normal basis $\mathbf{e}_i$ about the $\mathbf{e}_3$-axis.

Solution: The vector is independent of the coordinate system, that is, $\mathbf{a} = a_i\mathbf{e}_i = a_i^*\mathbf{e}_i^*$. One can show that the components of the vector in the new basis $\mathbf{e}_i^* = \boldsymbol{Q}\mathbf{e}_i$ (where $\boldsymbol{Q}$ is a orthogonal tensor) can be found

$$\begin{aligned} a_i^* &:= \mathbf{a} \cdot \mathbf{e}_i^* = \mathbf{a} \cdot \boldsymbol{Q}\mathbf{e}_i && \left(\text{since } \mathbf{e}_i^* = \boldsymbol{Q}\mathbf{e}_i\right) \\ &= \left(\boldsymbol{Q}^T\mathbf{a}\right) \cdot \mathbf{e}_i && \left(\text{since } \mathbf{u} \cdot \boldsymbol{A}\mathbf{v} = \left(\boldsymbol{A}^T\mathbf{u}\right) \cdot \mathbf{v}\right) \end{aligned}$$

So, the components in the rotated basis are

$$[\mathbf{a}]_{\{\mathbf{e}_i^*\}} = \left[\boldsymbol{Q}^T\mathbf{a}\right] = \left[\boldsymbol{Q}^T\right][\mathbf{a}]_{\{\mathbf{e}_i\}},$$

where, $[\mathbf{a}]_{\{\mathbf{e}_i^*\}} := (a_i^*)_{i\in(1,2,3)}$. For the given problem, rotation about the $\mathbf{e}_3$-axis equals $\theta = -\frac{\pi}{3}$ (since anticlockwise rotation is positive and clockwise rotation is negative).

The matrix representation of $\boldsymbol{Q}$ can be determined as,

$$\begin{aligned}
\left[\boldsymbol{Q}\right] &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} && \left(\text{using } Q_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j^*, \text{ for instance, } Q_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1^* = \cos\theta\right) \\
&= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} && \left(\text{since, } \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2},\ \sin\left(-\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\right).
\end{aligned}$$

Hence,

$$\left[\boldsymbol{Q}\right]^T = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For given $\mathbf{a} = 3\mathbf{e}_1 - 2\mathbf{e}_2 + 4\mathbf{e}_3$, the matrix representation of $\mathbf{a}$ can be given by

$$[\mathbf{a}]_{\{\mathbf{e}_i\}} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}.$$

Thus,

$$[\mathbf{a}]_{\{\mathbf{e}_i^*\}} = \left[\boldsymbol{Q}\right]^T [\mathbf{a}]_{\{\mathbf{e}_i\}} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$\boxed{[\mathbf{a}]_{\{\mathbf{e}_i^*\}} = \begin{bmatrix} \frac{3+2\sqrt{3}}{2} \\ \frac{3\sqrt{3}-2}{2} \\ 4 \end{bmatrix}}$$