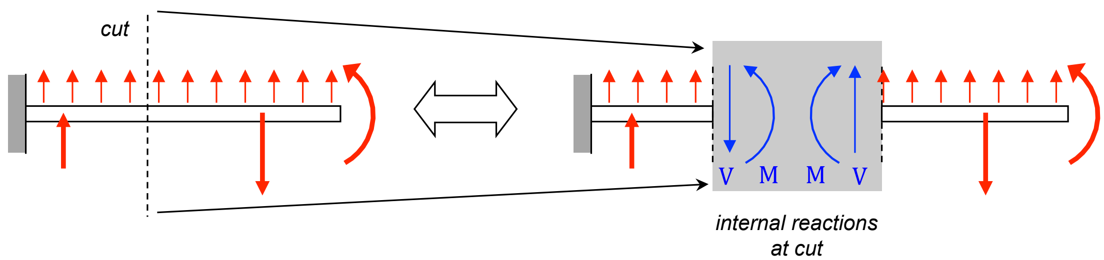
9. Equilibrium in beams: bending moments and shear forces

Objectives:
To understand the distribution of internal bending moments and internal shear forces at cross sections along the length of a beam with externally-applied transverse loads and couples.

Background:

- If a cut is made through the cross section of a beam, a bending moment M and shear force V must be applied at the cut in order to maintain equilibrium of the beam:



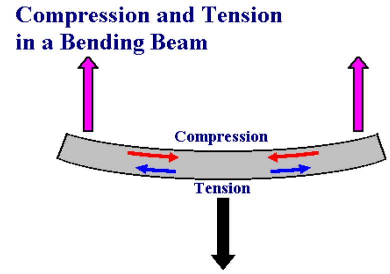
- V and M are force/couple *resultants* of the *normal* and *shear* stresses acting on the face of a cross section.

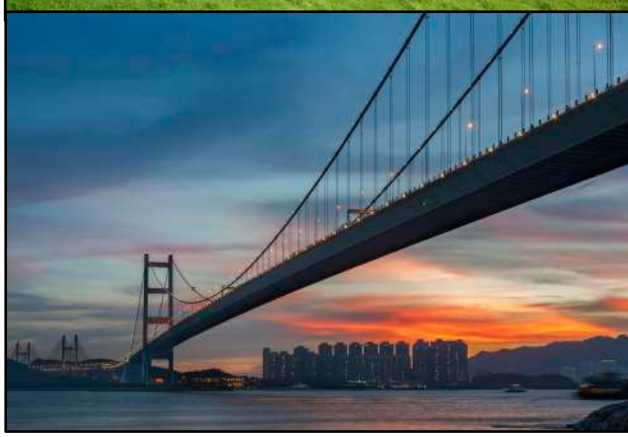
Lecture topics:

a) Sign conventions for bending moments and shear forces.
b) Equilibrium relations for bending moments and shear forces.
c) Bending-moment and shear-force diagrams.

Applications

Beams are structural members that are designed to support transverse loads, that is, loads that are perpendicular to the longitudinal axis of the beam. A beam resists the applied loads by a combination internal transverse shear force and bending moment.



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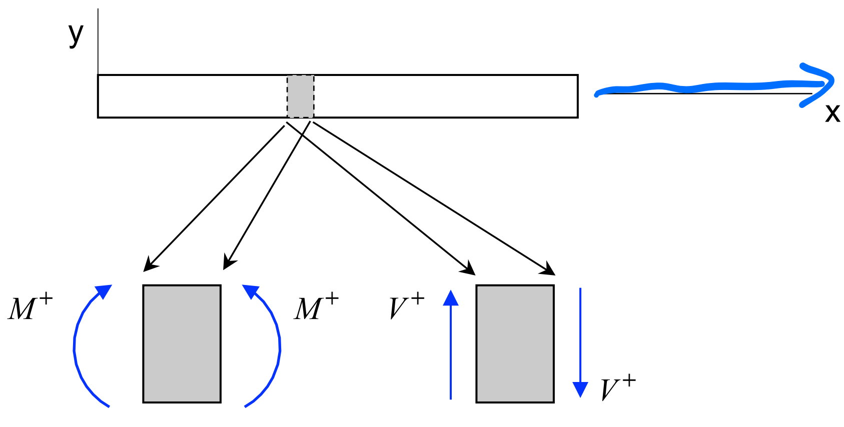
Lecture Notes

a) <u>Sign conventions for bending moments and shear forces</u>

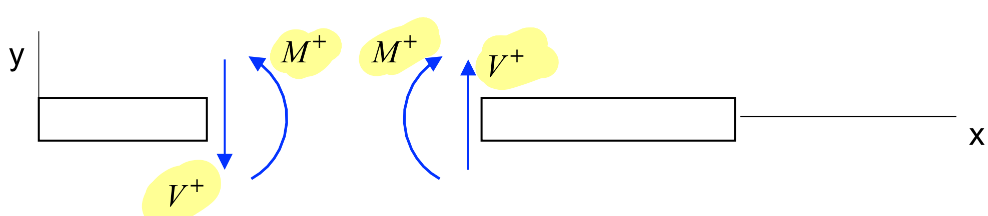
Sign conventions to be used in this course for internal bending moments and shear forces (see following figure):

- A positive bending moment M on the left face (negative x-face) of a section is CW. A positive bending moment M on the right face (positive x-face) of a section is CCW. Such a positive bending moment creates a concave curvature in the deflection of the beam.
- A positive shear force V on the left face (negative x-face) of a section is in positive y-direction. A positive shear force V on the right face (positive x-face) of a section is in negative y-direction.



positive bending moment **positive shear force**

When making a cut through a cross section of the beam, the positive sign conventions for the bending moment and shear force are as shown below:

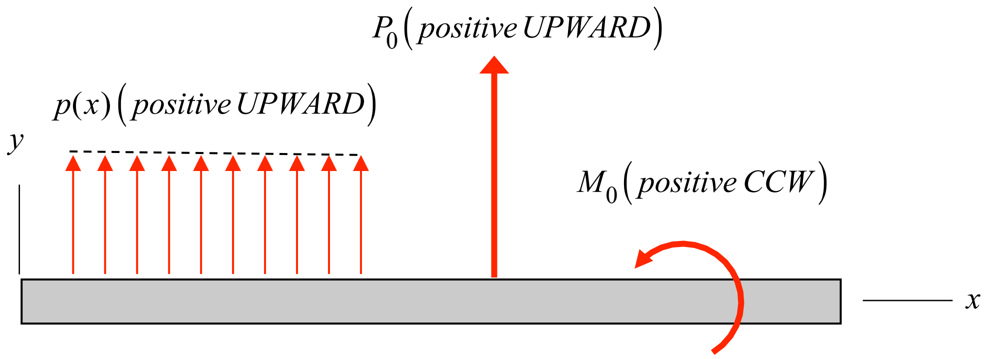


positive bending moment and shear force at cut in beam

Sign conventions for *external* loadings on beams:

- Positive *EXTERNAL distributed loads* p(x) and *EXTERNAL concentrated loads* P_0 act in the "+" *y*-direction:

- Positive *EXTERNAL couples* are in the "+" *z*-direction (CCW by the right hand rule):



Example 9.2

For the simply-supported beam loaded as shown:

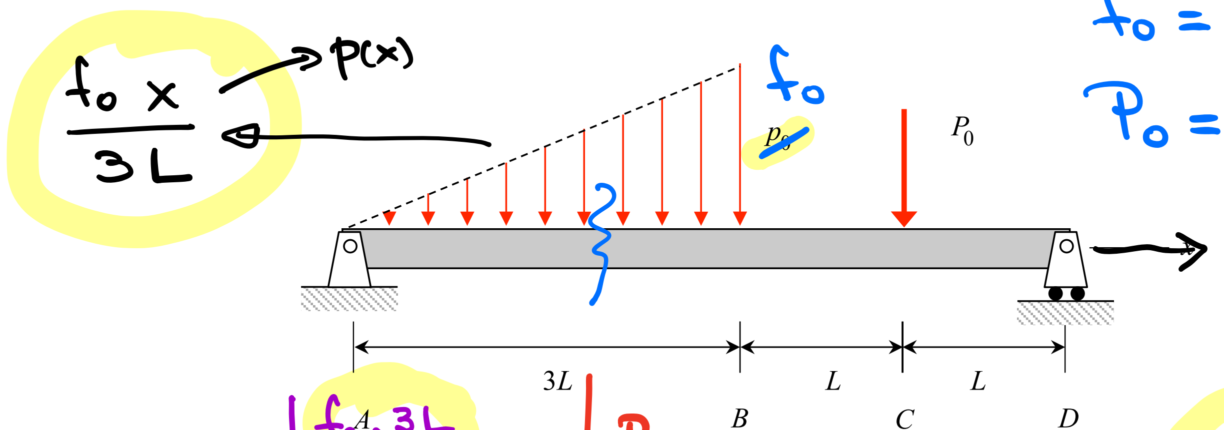
a) Determine the reactions on the beam at A and D.

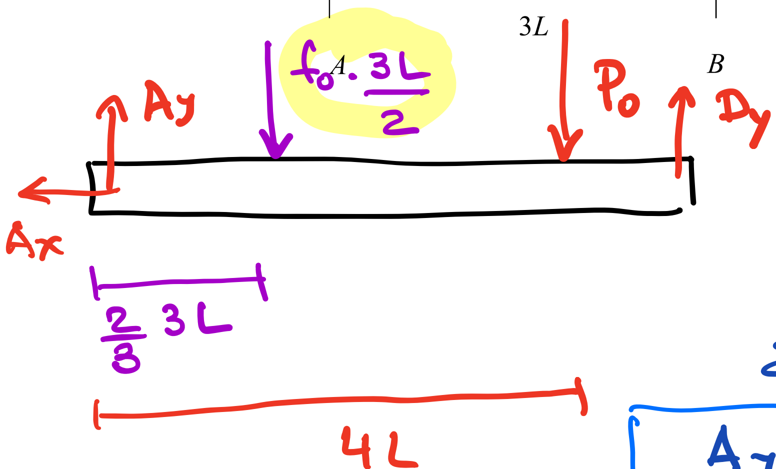
b) Determine the internal shear force and bending moment, $V(x)$ and $M(x)$, for an arbitrary location x between A and B.

$f_o = 2\,N/mm$

$P_o = 200\,N$

$L = 50\,mm$





$F_o = \frac{f_o\, 3L}{2}$

$A_x = 0$

$\sum F_y = 0$

$$\boxed{A_y - F_o - P_o + D_y = 0}$$

$A_y = F_o + P_o - D_y$

$\sum M_A = 0$

$$\boxed{F_o \cdot 2\cancel{L} + P_o\, 4\cancel{L} - D_y\, 5\cancel{L} = 0}$$

$$D_y = \frac{2F_o + 4P_o}{5}$$

$$F_o = f_o \cdot \frac{3L}{2} = 2\,\frac{N}{mm}\,\frac{3 \cdot 50\,mm}{2}$$

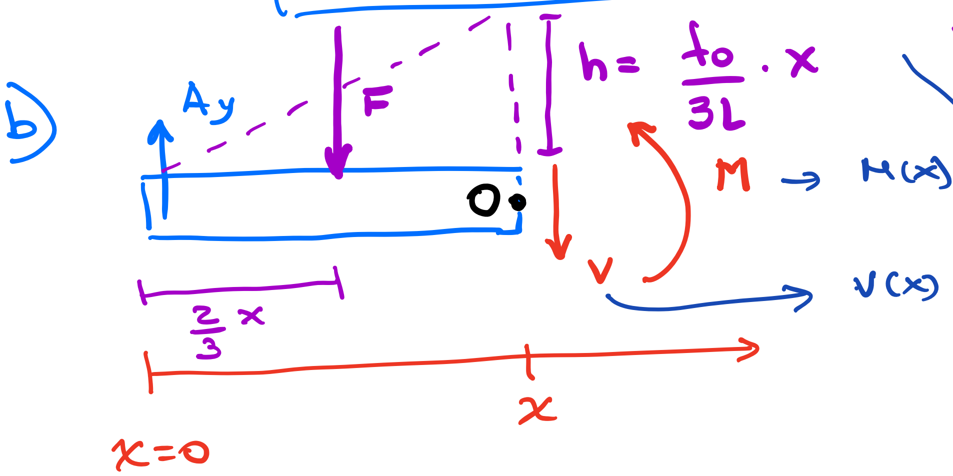
$$F_o = 150\ N \checkmark$$

$$\boxed{D_y = \frac{2 \cdot 150 + 4 \cdot 200}{5} = 220\ N}$$

$$A_y = F_o + P_o - D_y$$

$$\boxed{A_y = 130\ N}$$

b)



$$h = \frac{f_o}{3L} \cdot x \qquad F = \frac{f_o x}{3L} \cdot \frac{x}{2}$$

$$\Sigma F_y = 0 \qquad A_y - F - V(x) = 0$$

$$V(x) = A_y - F$$

$$= 130\ N - 2\,\frac{N}{mm} \cdot \frac{x^2}{6 \cdot 50\ mm}$$

in mm

$$\boxed{V(x) = \left(130 - \frac{1}{150}x^2\right) N}$$

x in mm

$$\Sigma M_o = 0$$

$$M(x) + F \cdot \frac{x}{3} - A_y \cdot x = 0$$

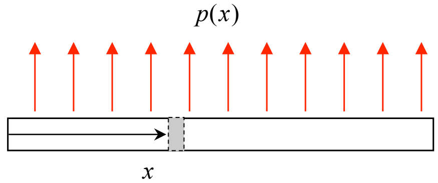
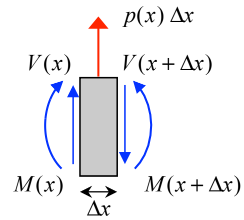
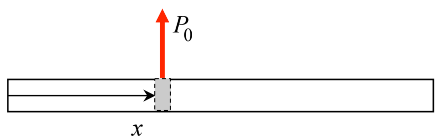
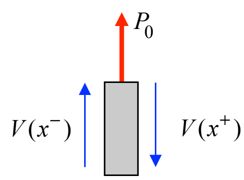
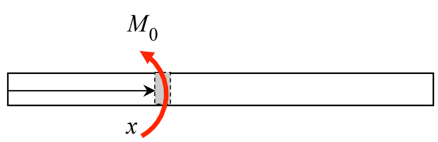
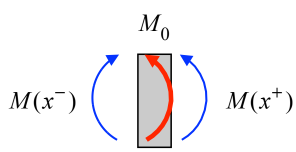
$$\boxed{M(x) = \left(130\,x - \frac{x^3}{450}\right) \cdot N.mm}$$

$\rightarrow$ x in mm.

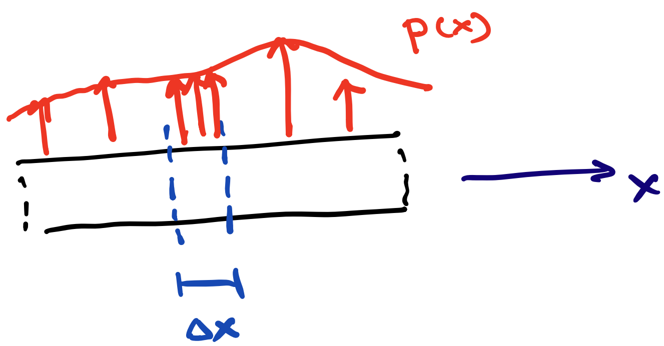
$$\frac{dM}{dx} = V(x)$$

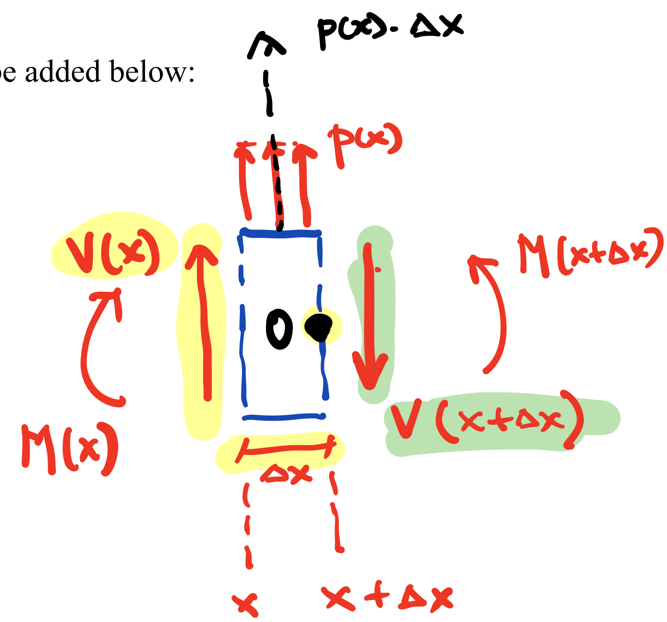
$$\frac{dV}{dx} = P(x)$$

b) Equilibrium relations for bending moments and shear forces

applied loading	*FBD*	*key relationship(s)*
		$\frac{dV}{dx} = p(x)$ $\frac{dM}{dx} = V(x)$
		$V\left(x^+\right) = V\left(x^-\right) + P_0$
		$M\left(x^+\right) = M\left(x^-\right) - M_0$

The *derivations* of the above key relationships are to be added below:





$$\sum F_y = 0$$

$$V(x) + p(x)\cdot\Delta x - V(x+\Delta x) = 0$$

$$\frac{V(x+\Delta x) - V(x)}{\Delta x} = p(x)$$

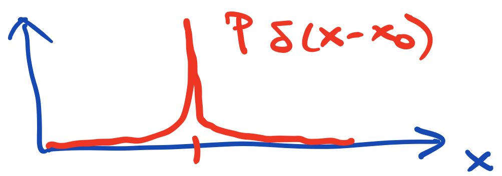
$$\Delta x \to 0$$

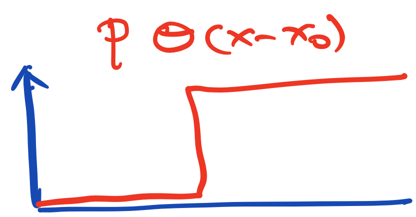
$$\frac{dV}{dx} = p(x)$$

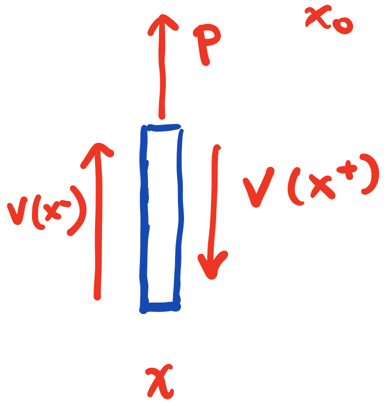
$$\int_{x_1}^{x_2} \frac{dV}{dx}\, dx = \int_{x_1}^{x_2} p(x)\, dx$$

$$\boxed{V(x_2) - V(x_1) = \int_{x_1}^{x_2} p(x)\, dx}$$

Concentrated Load







$$V(x^-) + P - V(x^+) = 0$$

$$\boxed{V(x^+) - V(x^-) = P}$$

$$\Sigma M_0 = 0$$

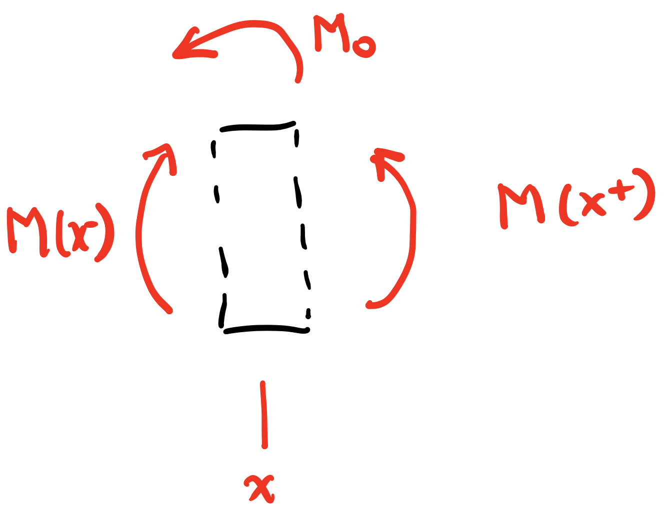
$$M(x+\Delta x) - M(x) - \cancelto{\sim 0}{\overbrace{p(x)\cdot\Delta x}^{\text{force}}\,\overbrace{\frac{\Delta x}{2}}^{\text{dist}}} - V(x)\,\Delta x = 0$$

$\Delta x \to 0$

$\Delta x^2 << \Delta x$

$$\boxed{\frac{dM}{dx} = V(x)}$$

Concentrated Moment



$\Sigma M = 0$

$M(x^+) + M_0 - M(x^-) = 0$

$M(x^+) - M(x^-) = -M_0$

Geometric meaning of the equilibrium relationships for beams

- $\dfrac{dV}{dx} = p(x)$

  The slope of the shear force diagram at any location x equals the value of the distributed external loading p at that location.

- $V(x_2) = V(x_1) + \int_{x_1}^{x_2} p(\xi)\,d\xi$ (integral form of the above)

  The shear force at point x_2 is equal to the shear force at x_1 plus the area under the external loading curve between these two points.

- $\dfrac{dM}{dx} = V(x)$

  The slope of the bending moment diagram at any location x equals the value of the shear force at that location.

- $M(x_2) = M(x_1) + \int_{x_1}^{x_2} V(\xi)\,d\xi$ (integral form of the above)

  The bending moment at point x_2 is equal to the bending moment at x_1 plus the area under the external loading curve between these two points.

- $V(x^+) = V(x^-) + P_0$

  The shear force diagram has an *upward* step jump at location x where an external point force is applied. The value of the shear force jump *increase* equals the value of the external point force.

- $M(x^+) = M(x^-) - M_0$

  The bending moment diagram has a *downward* step jump at location x where an external point moment is applied. The value of the bending moment jump *decrease* equals value of the external point moment.

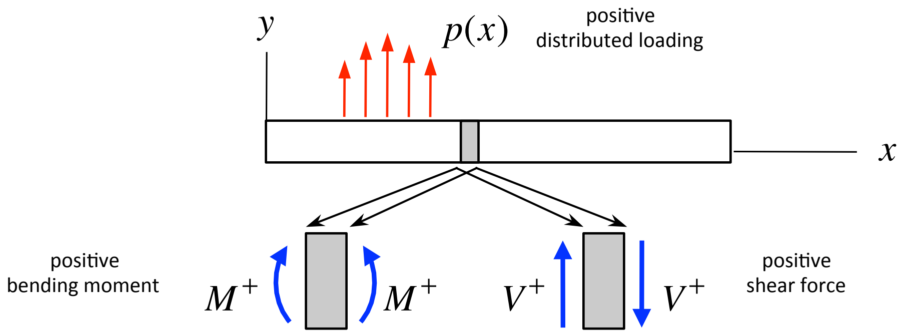
c) *Bending-moment and shear-force diagrams*

Three methods for determining the internal shear force and bending moment resultants:

- Using free body diagrams with cut sections, as demonstrated in the earlier examples of this section of notes.
- Using equilibrium relationships among applied loads, shear force and bending moments derived earlier and summarized above (integration and discontinuities).
- Using a graphical method based on the integration and discontinuity equations from the equilibrium method. The description of this method follows.

Graphical method for constructing shear force and bending moment diagrams

Sign conventions:



Basic relationships (as derived via equilibrium relations):

$$\frac{dV}{dx} = p(x) \qquad \Rightarrow \qquad V_2 = V_1 + \int_{x_1}^{x_2} p(x)dx$$

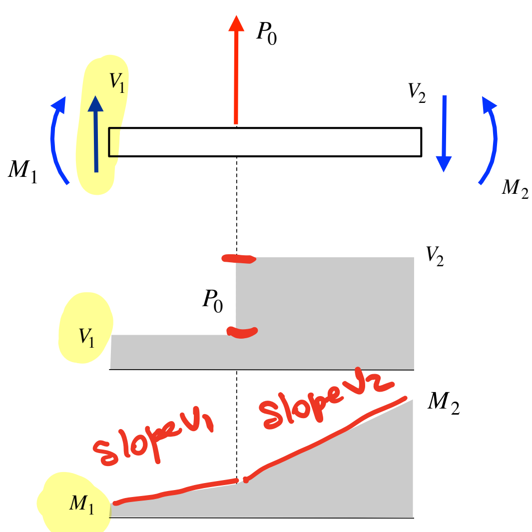
$$\frac{dM}{dx} = V(x) \qquad \Rightarrow \qquad M_2 = M_1 + \int_{x_1}^{x_2} V(x)dx$$

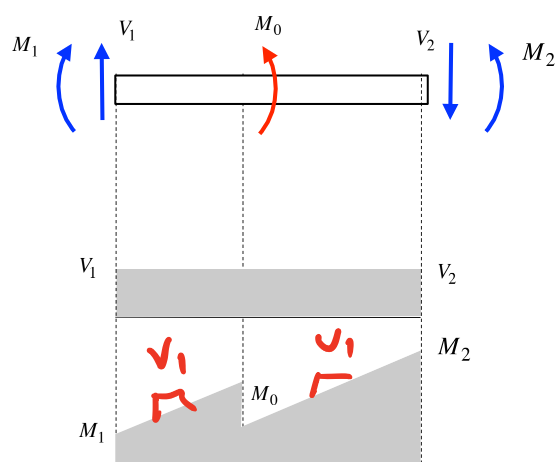
Concentrated shear force V_0 applied at location x:

$V(x^+) = V(x^-) + V_0$ (jump UP in shear force)

Concentrated moment M_0 applied at location x:

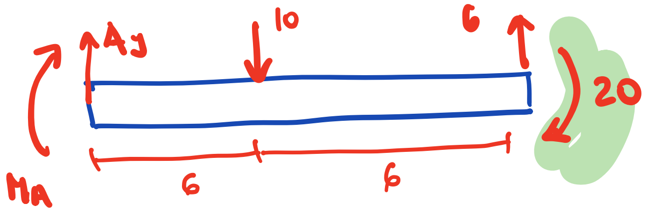
$M(x^+) = M(x^-) - M_0$ (jump DOWN in moment)

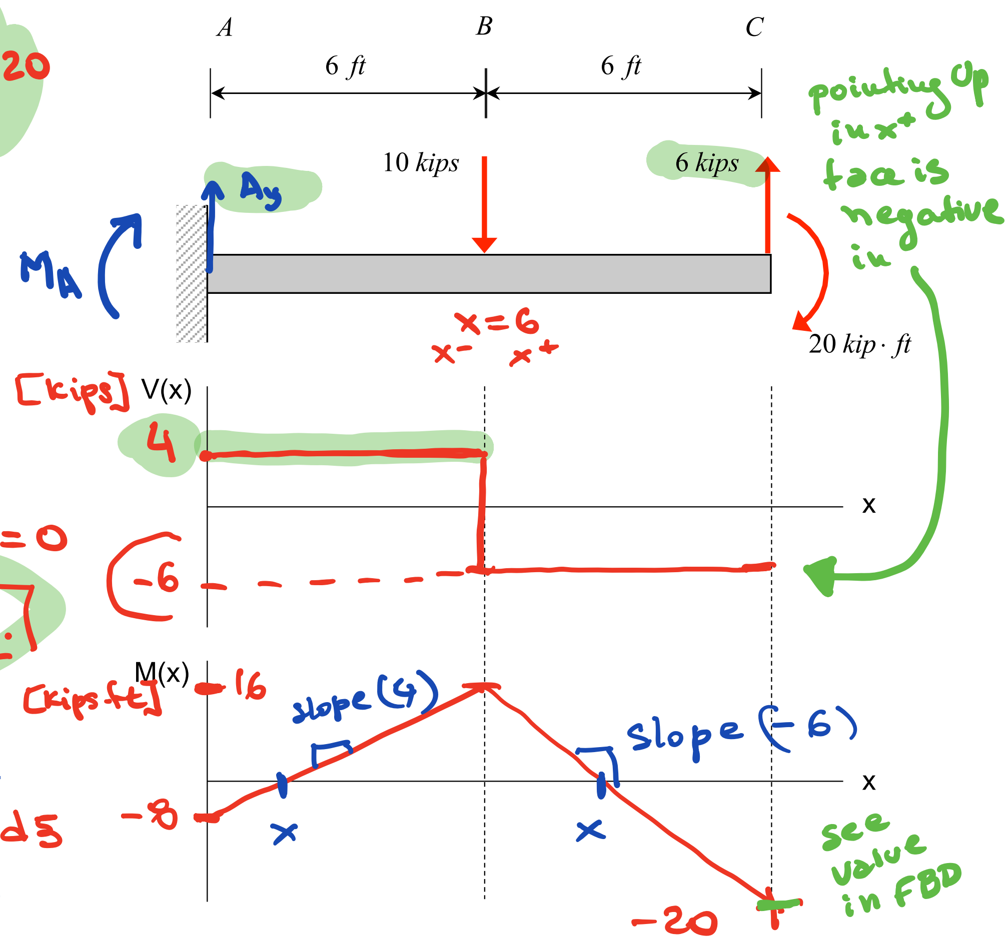




Example 9.3

Two transverse forces and a couple are applied as external loads to the cantilevered beam AC. Draw the shear force and bending moment diagrams in the plot axes below.





$\Sigma F_y = 0$

$A_y - 10 + 6 = 0$

$\boxed{A_y = 4 \text{ kips.}}$

$\Sigma M = 0$

$-M_A - 10 \cdot 6 + 12 \cdot 6 - 20 = 0$

$\boxed{M_A = -8 \text{ kip ft.}}$

$V(0) = A_y = 4$

$V(x) = V(0) + \cancel{\int_0^x p(\xi)\, d\xi}$ $\quad p = 0$

$\underline{V(x) = 4}$ $\quad 0 \le x \le 6$

$V(6^+) - \underbrace{V(6^-)}_{4} = -10$

$V(6^+) = -10 + 4$
$= -6$

$V(12) = -6$

$M(0) = -8$ $\qquad M(x) = +4x - 8$

$0 \le x \le 6$

$M(6) - M(0) = \int_0^6 V(x)\, dx$

$M(6) = -8 + \int_0^6 4\, dx = -8 + 24$
$= 16$

$6 \le x \le 12$

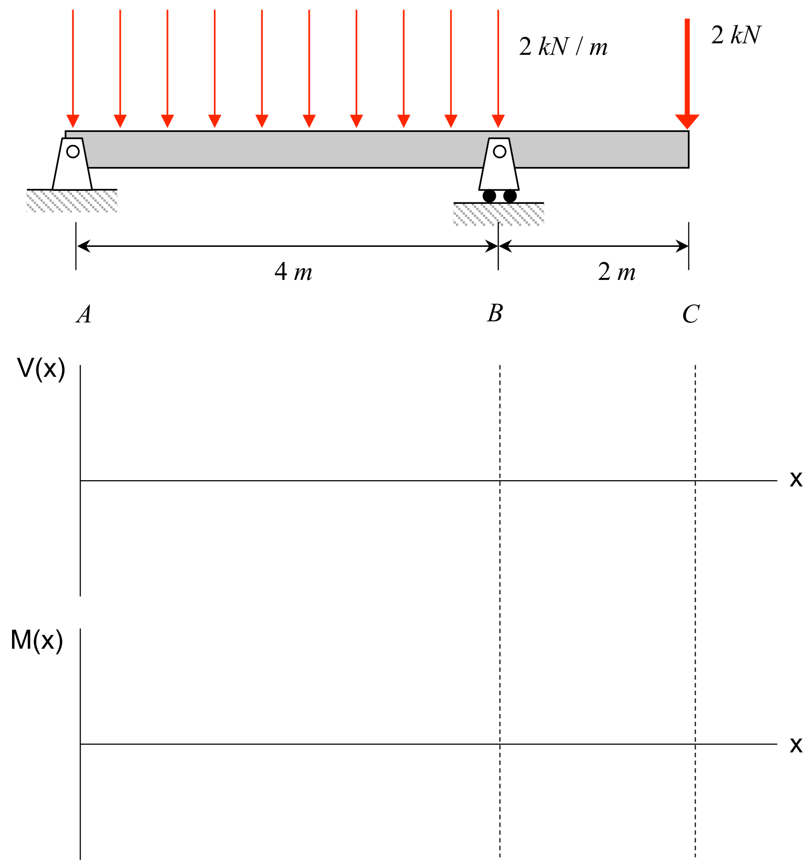
$M(12) - M(6) = \int_6^{12} V(x)\, dx$

$= \int_6^{12} (-6)\, dx \Rightarrow$

$M(12) = 16 + (-6) \cdot 6 = -20$ kip·ft

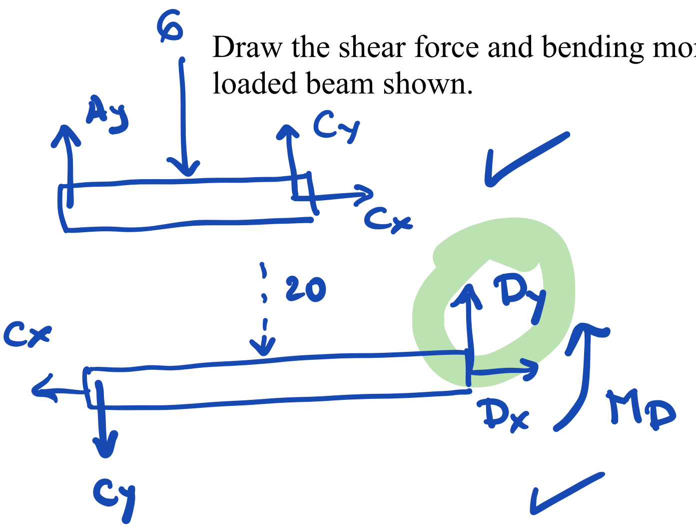
Example 9.4

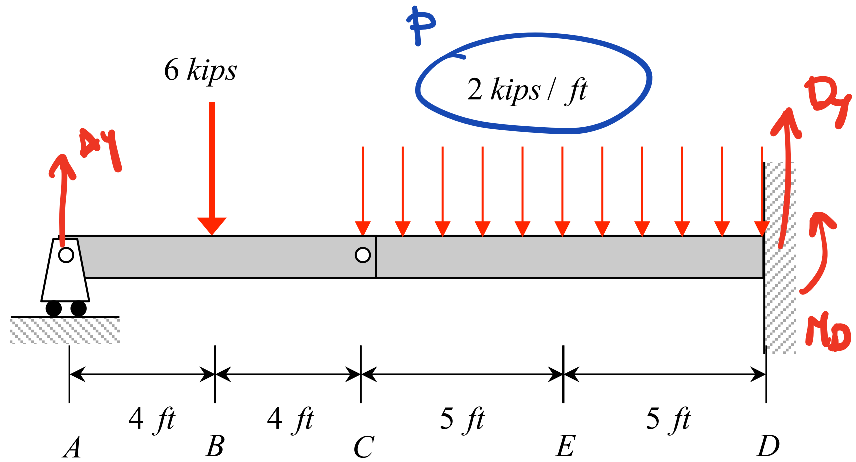
Draw the shear force and bending moment diagrams in the plot axes below for the loaded beam shown.
`



Example 9.6

Draw the shear force and bending moment diagrams in the plot axes below for the loaded beam shown.





$C_y = 3$ kip

$A_y = 3$ kip

$M_D = -130$ kip ft

$D_y = 23$ kip

$V(0) = 3 = A_y$

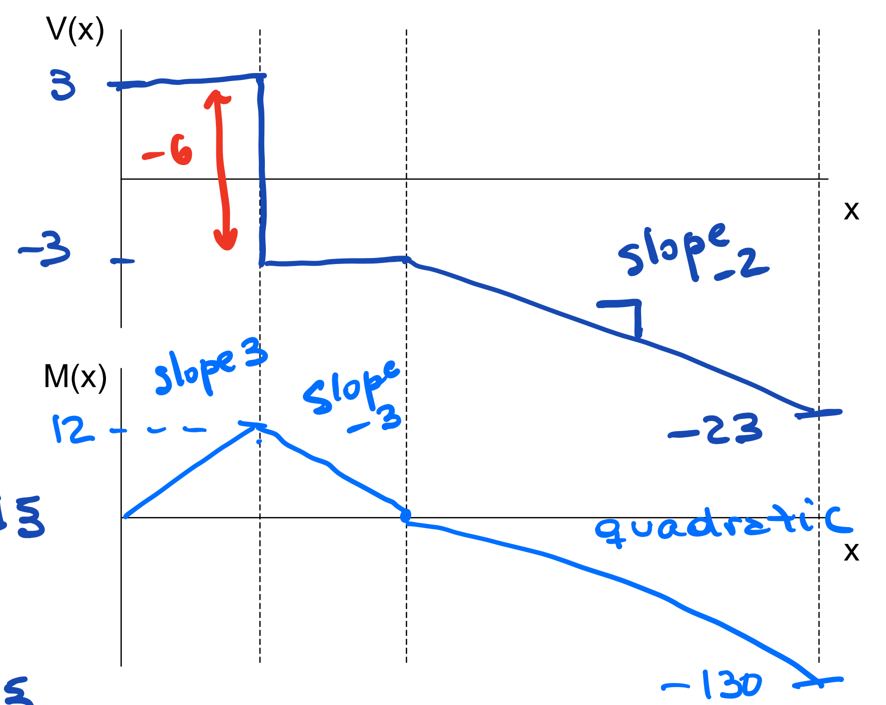
$8 < x \leq 18$

$$V(x) - V(8) = \int_8^x p(\xi)\, d\xi$$

$$V(x) - V(8) = \int_8^x (-2)\, d\xi$$

$$= -2\xi \Big|_8^x$$

$$V(x) = -3 + (-2x) + 16 = 13 - 2x$$



$$V(18) = -23$$

Then $V(x) = 13 - 2x$ for $8 < x \leq 18$

Another approach to solve the shear force in this segment follows:

We know $V(8) = -3$

$$V(18) - V(8) = \int_{8}^{18} p(x)\, dx$$

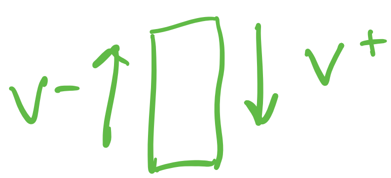
$$V(18) = V(8) + \int_{8}^{18} (-2)\, dx$$

$$V(18) = -3 + (-2) \cdot 10 = -23 \quad ✓$$

Then you know that $V(x)$ is linear with slope $\underline{-2}$

NOTE $D_y = 23$ kips ✓

$V(18) = -23$ kips ✓



Calculate Moment diagram

$M(0) = 0$ (pin)

$0 \le x \le 4$

$$M(4) - M(0) = \int_0^4 V(x)\,dx$$

$$M(4) = \cancel{M(0)}_{=0} + \int_0^4 3\,dx = 12$$

You can also proceed as follows:

$$M(x) - M(0) = \int_0^x V(\xi)\,d\xi \qquad 0 \le x \le 4$$

$$M(x) = M(0) + \int_0^x 3\,d\xi$$

$$= 0 + 3\xi \Big|_0^x$$

$$M(x) = 3x \longrightarrow M(4) = 12 \checkmark$$

$4 \le x \le 8$

$$M(8) = M(4) + \int_4^8 V(x)\,dx$$

$$= 12 + (-3)(8-4)$$

$$M(8) = 12 - 12 = 0 \checkmark$$

or you can proceed as follows

$4 \le x \le 8$

$$M(x) - M(4) = \int_4^x V(\xi)\, d\xi$$

$$M(x) = M(4) + \int_4^x (-3)\, d\xi$$

$$= 12 + (-3x) + 4 \cdot 3$$

$$M(x) = 24 - 3x$$

$$M(8) = 0$$

$$8 \le x \le 18$$

$$M(18) = M(8) + \int_8^{18} V(x)\, dx$$

$$= 0 + \overbrace{\frac{10 \cdot (-20)}{2} - 3 \cdot 10}^{\text{area under } V(x)}$$

$$= -130$$

You could also integrate the expression for $V(x)$ in $8 \le x \le 18$

$$M(x) = M(8) + \int_8^x V(x)\, dx$$

$$= M(8) + \int_8^x (13 - 2\xi)\, d\xi$$

$$= 0 + \left(13\xi - \xi^2\right)\Big|_8^x$$

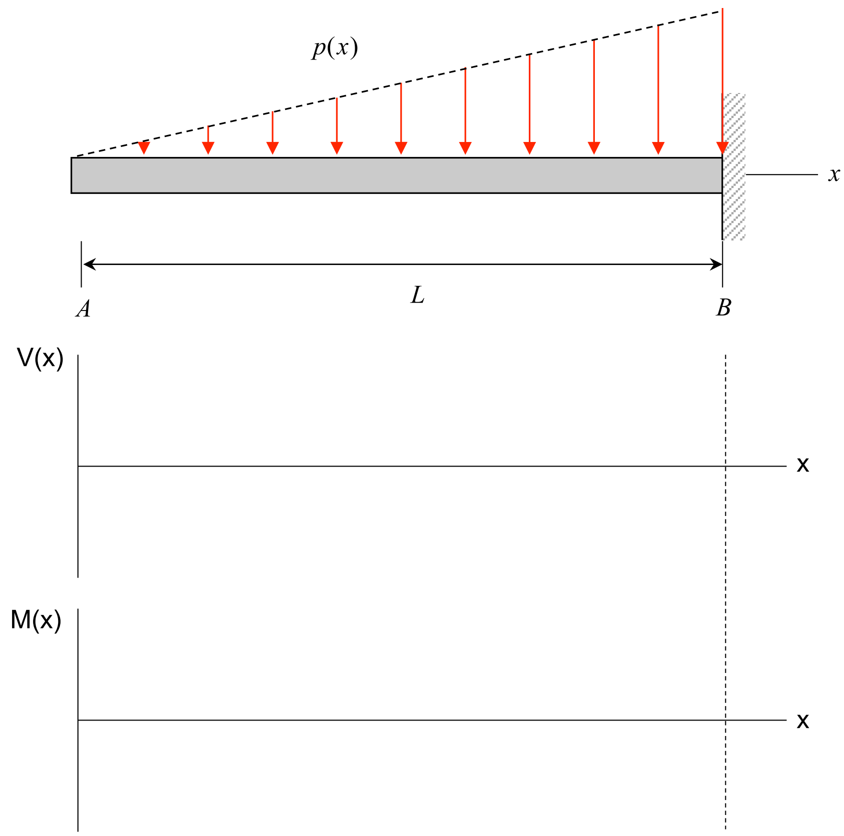
$$= 13(x-8) - x^2 + 8^2$$

$$M(x) = -x^2 + 13x - 40$$

$$M(18) = -130 \checkmark$$

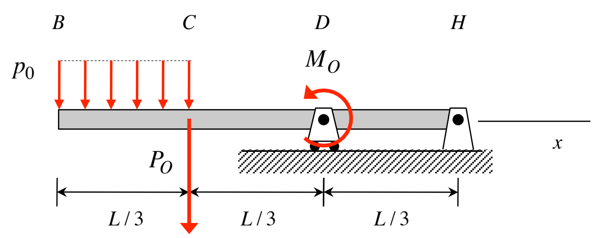
Example 9.7

Draw the shear force and bending moment diagrams in the plot axes below for the loaded beam shown.



Example 9.10

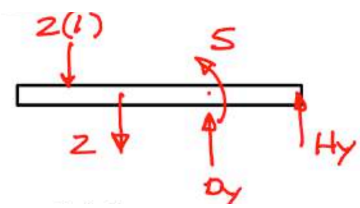
Use the following three methods to draw the shear force and bending moment diagrams in the plot axes below for the loaded beam shown: i) direct use of FBDs and equilibrium, ii) formal integration, and iii) integration by inspection. Clearly indicate the values of V and M at the labeled points as well as any maximum/minimum values. Please provide details on your work. Use $p_0 = 2kN/m$, $L = 3m$, $P_0 = 2kN$ and $M_0 = 5kN \cdot m$.



<u>*External reactions*</u>

$$\sum M_H = (2)(2.5) + (2)(2) + 5 - D_y(1) = 0 \;\; \Rightarrow \;\; D_y = 14\, kN$$

$$\sum F_y = -2 - 2 + D_y + H_y = 0 \;\; \Rightarrow \;\; H_y = -10\, kN$$



<u>*Section BC*</u>

$$V(0) = 0$$

$$V(1) = V(0) - (2)(1) = -2kN$$

$$M(0) = 0$$

$$M(1) = M(0) - 0.5(2)(1) = -1\, kN \cdot m$$

<u>*Section CD*</u>

$$V(1^+) = V(1^-) - 2 = -4kN$$

$$V(2) = V(1^+) = -4kN$$

$$M(2) = M(1) - (4)(1) = -5\, kN \cdot m$$

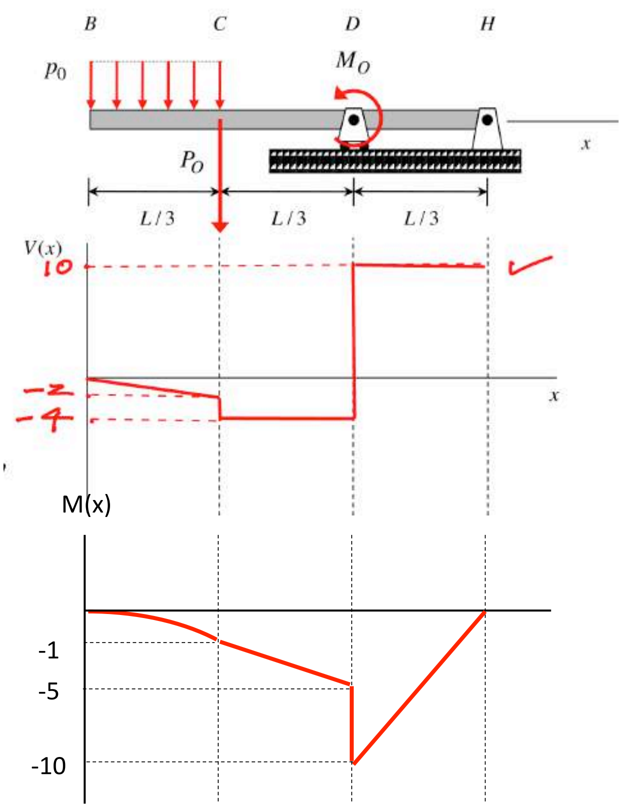
<u>*Section DH*</u>

$$V(2^+) = V(2^-) - 2 = -4 + D_y = 10kN$$

$$V(3) = V(2^+) = 10\, kN\, (checks\ with\ H_y\ found)$$

$$M(2^+) = M(2^-) - 5 = -10\, kN \cdot m$$

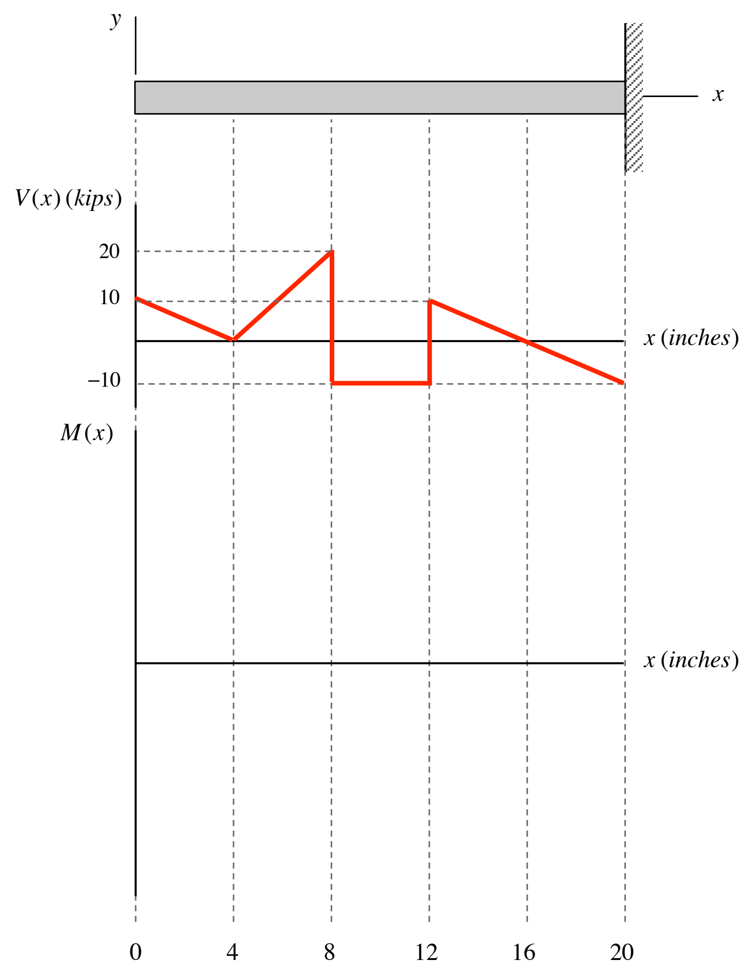
$$M(3) = M(2^+) + (10)(1) = 0\, (checks, pin\ joint\ @\, H)$$



Example 9.11

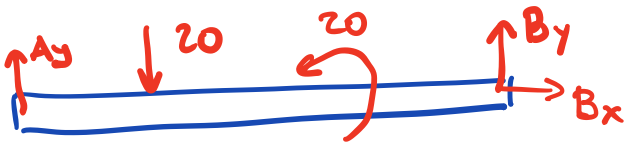
Consider the cantilevered beam shown below that is loaded only by concentrated and distributed forces (no external couples applied). The loading is not shown in the figure of the beam. The internal shear force distribution in the beam is shown below. For this beam:

a) Determine the internal bending moment M(x) in the beam and show M(x) in the plot below.
b) Determine the external loading (both concentrated and distributed forces) acting on the beam and show these on the figure of the beam below.

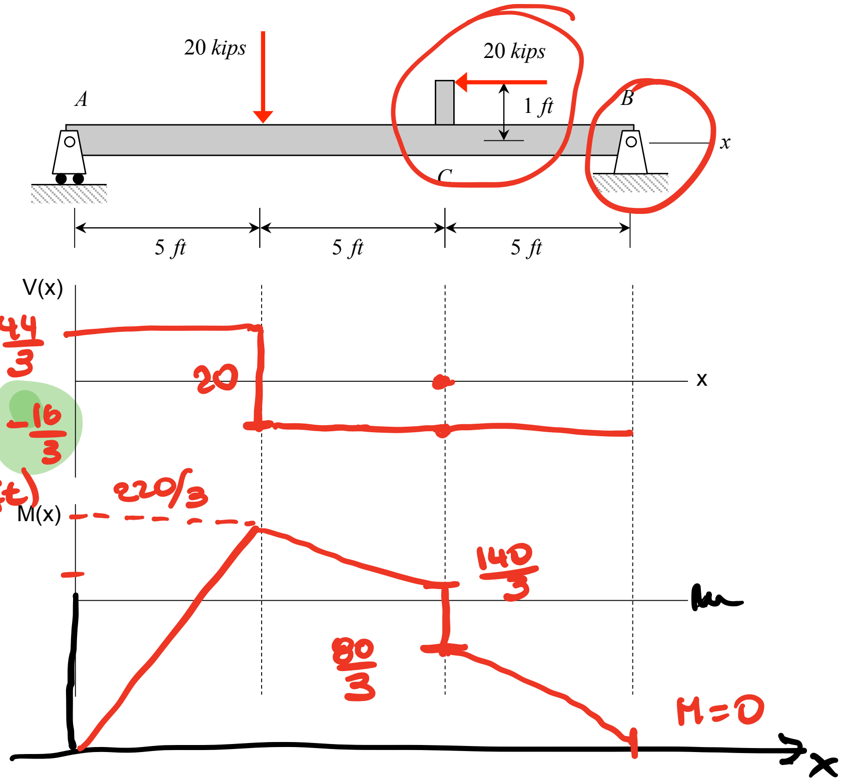


Example 9.13

Draw the shear force and bending moment diagrams in the plot axes below for the loaded beam shown.



$B_x = 20$



$$\left.\begin{array}{l}\Sigma M\\ \Sigma F_y\end{array}\right\}\quad B_y = \frac{16}{3}, \quad A_y = \frac{44}{3}$$

$$V(0) = A_y = \frac{44}{3}$$

$$V(5^+) - V(5^-) = -20$$

$$V(5^+) = V(5^-) - 20 = \frac{44}{3} - 20 = -\frac{16}{3}$$

$M(0) = 0$ (pin-roller)

$$M(5) = M(0) + \int_0^5 V(x)\,dx$$

$$= 0 + \int_0^5 \frac{44}{3}\,dx = \frac{44}{3}\cdot 5 = \frac{220}{3}$$

$$M(10) = M(5) + \int_5^{10} V(x)\,dx$$

$$= \frac{220}{3} + \int_5^{10} \left(-\frac{16}{3}\right) dx$$

$$= \frac{220}{3} + \left(-\frac{16}{3}\right)\cdot 5 = \frac{140}{3}$$

$$M(10^+) - M(10^-) = -M_c$$

$$M(10^+) = M(10^-) - 20$$

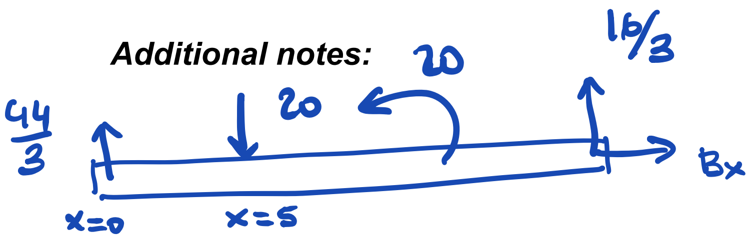
$$\frac{140}{3} - 20 = \frac{80}{3}$$

$$M(15) - M(10^+) = \int_{10}^{15} V(x)\,dx$$

$$M(15) - M(10^+) = -\frac{16}{3}\cdot 5$$

$$M(15) = \frac{80}{3} - \frac{16.5}{3} = 0$$

Additional notes:



$$V(0) = A_y = \frac{44}{3}$$

$0 \le x < 5$

$$V(x) - V(0) = \int_0^x p(\xi)\, d\xi$$

$$V(x) - \frac{44}{3} = 0$$

$$V(x) = \frac{44}{3}$$

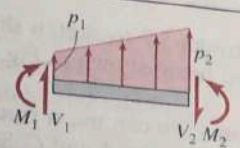
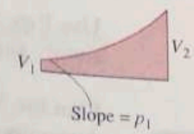
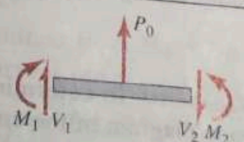
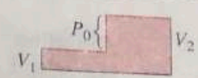
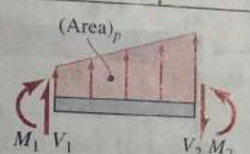
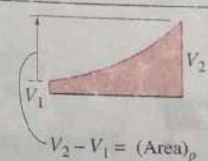
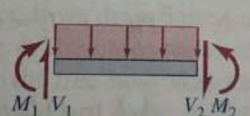
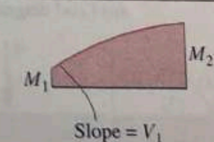
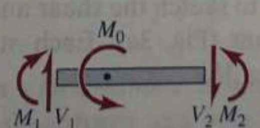
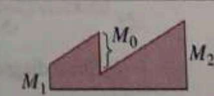
$$M(x) - M(0) = \int_0^x V(\xi)\, d\xi$$

$$M(x) = M(0) + \int_0^x \frac{44}{3}\, d\xi$$

$$= M(0) + \frac{44}{3}\xi \Big|_0^x$$

$$\boxed{M(x) = 0 + \frac{44}{3}x}$$

TABLE 5.1 Shear and Moment Diagram Features

Equation	Load Diagram p	Shear Diagram V	Moment Diagram M
1. Slope of shear diagram equals value of load			
$\frac{dV}{dx} = p(x)$ (Eq. 5.2)			
2. Jump in shear equals value of concentrated load			
$\Delta V = P_0$ (Eq. 5.4)		 Positive V-jump	
3. Change in shear equals area under distributed-load diagram			
$V_2 - V_1 = \int_{x_1}^{x_2} p(x)dx$ (Eq. 5.6)			
4. Slope of moment diagram equals value of shear			
$\frac{dM}{dx} = V(x)$ (Eq. 5.3)			
5. Jump in moment equals – (value of concentrated couple)			
$\Delta M = -M_0$ (Eq. 5.5)			 Negative M-jump
6. Change in moment equals area under shear diagram			
$M_2 - M_1 = \int_{x_1}^{x_2} V(x)dx$ (Eq. 5.7)