Indian Institute of Technology Bhubaneswar

School of Infrastructure

Subject Name : Solid Mechanics	Subject Code: CE2L001

Problem Sheet No. 2 — Date: September 18, 2025

<u>Instructions:</u>

Provide neatly labelled diagrams whenever necessary.

<u>Notations :</u>

Zeroth-order tensors or scalars are represented by small letters. For eg. a

First-order tensors or vectors are represented by bold small letters. For eg. $\boldsymbol{a}$.

Second-order tensors are represented by bold capital letters. For eg. $\boldsymbol{A}$.

1. Derive the expression for the stress tensor in terms of the principal stresses and principal directions. Show that the stress tensor can be diagonalized using the principal directions as the basis.

2. Derive the Mohr's circle equations for a plane stress state. Show that the Mohr's circle represents a graphical method for determining the principal stresses and principal directions.

3. (a) Derive the three invariants (I_1, I_2, I_3) of the stress tensor $\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and explain their physical significance.

$$I_1 = \sigma_{ii}, \quad I_2 = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}), \quad I_3 = \det(\sigma_{ij})$$

(b) Prove that the first invariant (I_1) of the stress tensor is equal to the sum of the principal stresses.

$$I_1 = \sigma_I + \sigma_{II} + \sigma_{III}$$

(c) Show that the second invariant (I_2) of the stress tensor can be expressed in terms of the principal stresses.

$$I_2 = \sigma_I\sigma_{II} + \sigma_{II}\sigma_{III} + \sigma_{III}\sigma_I$$

4. (a) Derive the relationship between the stress tensor $\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and the deviatoric stress tensor $\boldsymbol{\sigma}^{\text{dev}} = s_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$.

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

(b) Prove that the components s_{ij} of deviatoric stress tensor $\boldsymbol{\sigma}^{\text{dev}} = s_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ represent the shear stress components of the stress tensor $\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$.

(c) Show that the invariants of the deviatoric stress tensor (J_2, J_3) can be used to predict material failure under complex loading conditions.

5. (a) Derive the invariants (J_1, J_2, J_3) of the deviatoric stress tensor $\boldsymbol{\sigma}^{\text{dev}} = s_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and explain their relationship to the invariants of the stress tensor.

$$J_1 = s_{ii} = 0, \quad J_2 = \frac{1}{2}s_{ij}s_{ji}, \quad J_3 = \det(s_{ij})$$

(b) Prove that the first invariant (J_1) of the deviatoric stress tensor is zero.

$$J_1 = s_{ii} = \sigma_{ii} - \frac{1}{3}\sigma_{kk}\delta_{ii} = 0$$

(c) Show that the second invariant (J_2) of the deviatoric stress tensor is related to the von Mises stress.

$$\sigma_{vm} = \sqrt{3J_2}$$

6. Given the stress components $\sigma_{xx} = x^2$, $\sigma_{yy} = y^2$, and $\sigma_{xy} = xy$ for a plane stress state of a solid body, compute the required body force to satisfy the linear momentum balance equation for a static equilibrium of the solid body.

7. Given the stress components $\sigma_{xx} = x^2+y^2$, $\sigma_{yy} = x^2-y^2$, and $\sigma_{xy} = 2xy$ for a plane stress state of a solid body, compute the required body force to satisfy the linear momentum balance equation for a static equilibrium of the solid body.

8. Given the components of stress tensor:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} x^2+y^2 & 2xy & 0 \\ 2xy & x^2-y^2 & 0 \\ 0 & 0 & z^2 \end{bmatrix}$$

check if the stress tensor satisfies the linear momentum balance equation for a static equilibrium.

9. Given the components of stress tensor:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 200 + 10x^2 & 40xy & 0 \\ 40xy & 100 + 20y^2 & 0 \\ 0 & 0 & 300 \end{bmatrix}$$

check if the stress tensor satisfies the linear momentum balance equation for a static equilibrium.

10. Given the components of stress tensor:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 200\sin(x) & 50\cos(y) & 0 \\ 50\cos(y) & 150\sin(x) & 0 \\ 0 & 0 & 100 \end{bmatrix}$$

check if the stress tensor satisfies the linear momentum balance equation for a static equilibrium.

11. A material point is subjected to a stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 100 & 20 & 0 \\ 20 & 50 & 0 \\ 0 & 0 & 30 \end{bmatrix} \text{ MPa}$$

Find the principal stresses and principal directions.

12. A cube of side length 1 m is subjected to a uniform stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 100 \end{bmatrix} \text{ MPa}$$

Find the stress vector on a plane with normal vector $[\mathbf{n}] = \frac{1}{\sqrt{3}}(1, 1, 1)$.

13. A material point is subjected to a stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 50 & 10 & 0 \\ 10 & 20 & 0 \\ 0 & 0 & 40 \end{bmatrix} \text{ MPa}$$

Find the maximum shear stress and the plane on which it acts.

14. A material point of a plate is subjected to a stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 200 & 50 & 0 \\ 50 & 150 & 0 \\ 0 & 0 & 100 \end{bmatrix} \text{ MPa}$$

Find the stress invariants I_1, I_2, and I_3.

15. A material point is subjected to a stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 150 & 30 & 0 \\ 30 & 80 & 0 \\ 0 & 0 & 40 \end{bmatrix} \text{ MPa}$$

Find the von Mises stress and the Tresca stress.

16. A material point is subjected to a stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 80 & 20 & 0 \\ 20 & 120 & 0 \\ 0 & 0 & 60 \end{bmatrix} \text{ MPa}$$

Find the maximum normal stress and the plane on which it acts.

17. A material point of a beam is subjected to a stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 120 & 30 & 0 \\ 30 & 80 & 0 \\ 0 & 0 & 40 \end{bmatrix} \text{ MPa}$$

Find the stress vector on a plane with normal vector $[\mathbf{n}] = (1, 0, 0)$.

18. A material point of a plate is subjected to a stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 250 & 50 & 0 \\ 50 & 180 & 0 \\ 0 & 0 & 120 \end{bmatrix} \text{ MPa}$$

Find the principal stresses and principal directions.

19. A material point is subjected to a stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 50 & 10 & 0 \\ 10 & 20 & 0 \\ 0 & 0 & 40 \end{bmatrix} \text{ MPa}$$

Compute the deviatoric stress tensor. Find the first and second invariants of the deviatoric stress tensor, J_1 and J_2, respectively.

20. A material point of a beam is subjected to a stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 50 & 20 \\ 0 & 20 & 30 \end{bmatrix} \text{ MPa}$$

Compute the deviatoric stress tensor. Find the third invariant of the deviatoric stress tensor, J_3 and discuss its significance.

21. Let $\mathbf{v} = 2x\,\boldsymbol{e}_1 + 3y\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3$ and S be the surface of the cube bounded by $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$. Use the divergence theorem to evaluate $\int_S \mathbf{v} \cdot \mathbf{n}\, dS$.

22. Let $\mathbf{v} = x^2\,\boldsymbol{e}_1 + y^2\,\boldsymbol{e}_2 + z^2\,\boldsymbol{e}_3$ and S be the surface of the sphere $x^2 + y^2 + z^2 = 4$. Use the divergence theorem to evaluate $\int_S \mathbf{v} \cdot \mathbf{n}\, dS$.

23. A fluid flows through a pipe with velocity field $\mathbf{v} = 2x\,\boldsymbol{e}_1 + 3y\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3$ m/s. The pipe has a circular cross-section with radius 0.5 m and is oriented along the z-axis. Use the divergence theorem to calculate the volume flow rate through the pipe.

24. Let $\mathbf{v} = 3x\,\boldsymbol{e}_1 + 2y\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3$ and S be the surface of the cylinder $x^2 + y^2 = 1$, $0 \leq z \leq 2$. Use the divergence theorem to evaluate $\int_S \mathbf{v} \cdot \mathbf{n}\, dS$.

25. A heat flux vector field is given by $\mathbf{q} = -k\nabla T$, where $T = 100 - 50x^2 - 50y^2 - 50z^2$ and $k = 10$ W/mK. Use the divergence theorem to calculate the total heat flux through the

surface of a sphere of radius 0.5 m centered at the origin.

26. Let $\mathbf{v} = x^3\,\boldsymbol{e}_1 + y^3\,\boldsymbol{e}_2 + z^3\,\boldsymbol{e}_3$ and S be the surface of the cube bounded by $-1 \le x \le 1$, $-1 \le y \le 1$, and $-1 \le z \le 1$. Use the divergence theorem to evaluate $\int_S \mathbf{v}\cdot\mathbf{n}\,dS$.

27. A vector field is given by $\mathbf{v} = 2xy\,\boldsymbol{e}_1 + x^2\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3$. Use the divergence theorem to calculate the flux of $\mathbf{v}$ through the surface of the hemisphere $x^2 + y^2 + z^2 = 4$, $z \ge 0$.

28. Let $\mathbf{v} = (2x + y)\,\boldsymbol{e}_1 + (x - z)\,\boldsymbol{e}_2 + (y + z)\,\boldsymbol{e}_3$ and S be the surface of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$. Use the divergence theorem to evaluate $\int_S \mathbf{v}\cdot\mathbf{n}\,dS$.

29. A fluid flows through a pipe with velocity field $\mathbf{v} = x\,\boldsymbol{e}_1 + 2y\,\boldsymbol{e}_2 + 3z\,\boldsymbol{e}_3$ m/s. The pipe has a rectangular cross-section with dimensions 1 m $\times$ 2 m and is oriented along the z-axis. Use the divergence theorem to calculate the volume flow rate through the pipe.

30. Let $\mathbf{v} = x^2y\,\boldsymbol{e}_1 + xy^2\,\boldsymbol{e}_2 + xyz\,\boldsymbol{e}_3$ and S be the surface of the ellipsoid $\boxed{\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1}$. Use the divergence theorem to evaluate $\int_S \mathbf{v}\cdot\mathbf{n}\,dS$.

31. Let $\mathbf{v} = (x^2 + y^2)\,\boldsymbol{e}_1 + (y^2 + z^2)\,\boldsymbol{e}_2 + (z^2 + x^2)\,\boldsymbol{e}_3$ and S be the surface of the sphere $x^2 + y^2 + z^2 = 9$. Use the divergence theorem to evaluate $\int_S \mathbf{v}\cdot\mathbf{n}\,dS$.

32. A vector field is given by $\mathbf{v} = (2x + 3y)\,\boldsymbol{e}_1 + (x - 2y)\,\boldsymbol{e}_2 + z\,\boldsymbol{e}_3$. Use the divergence theorem to calculate the flux of $\mathbf{v}$ through the surface of the cube bounded by $0 \le x \le 2$, $0 \le y \le 2$, and $0 \le z \le 2$.

33. Let $\mathbf{v} = x^2\,\boldsymbol{e}_1 + y^2\,\boldsymbol{e}_2 + z^2\,\boldsymbol{e}_3$ and S be the surface of the cylinder $x^2 + y^2 = 4$, $0 \le z \le 3$. Use the divergence theorem to evaluate $\int_S \mathbf{v}\cdot\mathbf{n}\,dS$.

34. A fluid flows through a pipe with velocity field $\mathbf{v} = (x + y)\,\boldsymbol{e}_1 + (y + z)\,\boldsymbol{e}_2 + (z + x)\,\boldsymbol{e}_3$ m/s. The pipe has a circular cross-section with radius 1 m and is oriented along the z-axis. Use the divergence theorem to calculate the volume flow rate through the pipe.

35. Let $\mathbf{v} = (x + y + z)\,\boldsymbol{e}_1 + (x - y + z)\,\boldsymbol{e}_2 + (x + y - z)\,\boldsymbol{e}_3$ and S be the surface of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 2$. Use the divergence theorem to evaluate $\int_S \mathbf{v}\cdot\mathbf{n}\,dS$.

36. Use the divergence theorem to find the surface area of a cube with side length a.

37. A solid object is defined by the region $x^2 + y^2 + z^2 \le 9$. Use the divergence theorem to find the volume and surface area of the object.

38. Use the divergence theorem to find the volume of a cylinder of radius R and height h.

39. A rock sample is subjected to a plane stress state with $\sigma_{xx} = 300$ MPa, $\sigma_{yy} = 150$ MPa, $\sigma_{xy} = 50$ MPa. Using Mohr's circle, find the principal stresses and the orientation of the principal planes.

40. A mechanical component is subjected to a plane stress state with $\sigma_{xx} = 250$ MPa, $\sigma_{yy} = 120$ MPa, and $\sigma_{yy} = 60$ MPa. Draw Mohr's circle and find the maximum and minimum normal stresses and the planes on which they act.

41. A structural element is subjected to a plane stress state with $\sigma_{xx} = 150$ MPa,$\sigma_{yy} = 80$ MPa, and $\sigma_{xy} = 40$ MPa. Draw Mohr's circle and find the normal and shear stresses on a plane oriented at 30° to the x-axis.

42. A soil sample is subjected to a plane stress state with $\sigma_{xx} = 200$ kPa, $\sigma_{yy} = 100$ kPa, and$\sigma_{xy} = 30$ kPa. Using Mohr's circle, find the maximum shear stress and the orientation of the plane on which it acts.

43. A material is subjected to a plane stress state with $\sigma_{xx} = 100$ MPa, $\sigma_{yy} = 50$ MPa, and $\sigma_{xy} = 20$ MPa. Draw Mohr's circle and find the principal stresses and principal directions.

44. A material point of a plate is subjected to a stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 200 & 50 & 0 \\ 50 & 150 & 0 \\ 0 & 0 & 100 \end{bmatrix} \text{ MPa}$$

Draw Mohr's circle for the stress tensor and the deviatoric stress tensor. Compare the maximum and minimum normal stresses.

45. A material point of a beam is subjected to a stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 50 & 20 \\ 0 & 20 & 30 \end{bmatrix} \text{ MPa}$$

Draw Mohr's circle for the stress tensor and the deviatoric stress tensor. Compare the stress invariants.

46. A material point is subjected to a stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 50 & 10 & 0 \\ 10 & 20 & 0 \\ 0 & 0 & 40 \end{bmatrix} \text{ MPa}$$

Draw Mohr's circle for the stress tensor and the deviatoric stress tensor. Compare the principal stress directions.

47. A cube of side length 1 m is subjected to a uniform stress state given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 100 \end{bmatrix} \text{ MPa}$$

Draw Mohr's circle for the stress tensor and the deviatoric stress tensor. Compare the maximum shear stresses.

48. A material point is subjected to a stress tensor given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 100 & 20 & 0 \\ 20 & 50 & 0 \\ 0 & 0 & 30 \end{bmatrix} \text{ MPa}$$

Draw Mohr's circle for the stress tensor and the deviatoric stress tensor. Compare the principal stresses and principal directions.

49. Given a traction, $[\boldsymbol{t}] = \begin{bmatrix} 50 \\ 30 \\ 0 \end{bmatrix}$ MPa on a surface with normal $[\mathbf{n}] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, find the components of the stress tensor $\boldsymbol{\sigma}$ on the surface with normal vector $\mathbf{n}$.

50. Given a stress state $[\boldsymbol{\sigma}] = \begin{bmatrix} 100 & 20 & 0 \\ 20 & 50 & 0 \\ 0 & 0 & 30 \end{bmatrix}$ MPa and a surface with normal $[\mathbf{n}] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, find the traction vector and its shear components.