Indian Institute of Technology Bhubaneswar

School of Infrastructure

Subject Name : Solid Mechanics	Subject Code: CE2L001

Tutorial No. 5

1. Derive the expressions for the strain compatibility conditions.

Solution: The compatibility of strain is intricately related to the continuity of infinitesimal rotations. In two dimensions, this can be shown from the $\dfrac{\partial^2 W_{12}}{\partial x_1 x_2} = \dfrac{\partial^2 W_{21}}{\partial x_2 x_1}$. The component W_{12} of the rotation tensor can be written in terms of the displacement components as

$$W_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \tag{1}$$

The first derivative of this strain component with respect to x_1 can be written as

$$\frac{\partial W_{12}}{\partial x_1} = \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_1 x_2} - \frac{\partial^2 u_2}{\partial x_1^2}\right) \tag{2}$$

Adding and subtracting the term $\dfrac{\partial^2 u_1}{\partial x_1 x_2}$, one can rearrange the above equation as

$$\begin{aligned}\frac{\partial W_{12}}{\partial x_1} &= \frac{1}{2}\left(\left(\frac{\partial^2 u_1}{\partial x_1 x_2} + \frac{\partial^2 u_1}{\partial x_1 x_2}\right) - \left(\frac{\partial^2 u_1}{\partial x_1 x_2} - \frac{\partial^2 u_2}{\partial x_1^2}\right)\right)\\ &= \frac{1}{2}\left(2\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}\right)\right) - \frac{1}{2}\left(\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\right)\end{aligned} \tag{3}$$

From the strain displacement relations given below,

$$\begin{aligned}E_{11} &= \frac{\partial u_1}{\partial x_1}, \qquad E_{22} = \frac{\partial u_2}{\partial x_2}\\ E_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right),\end{aligned} \tag{4}$$

one can rewrite the Eq.(3) as

$$\frac{\partial W_{12}}{\partial x_1} = \frac{\partial E_{11}}{\partial x_2} - \frac{\partial E_{12}}{\partial x_1} \tag{5}$$

and $\dfrac{\partial^2 W_{12}}{\partial x_1 x_2}$ will become

$$\frac{\partial^2 W_{12}}{\partial x_1 x_2} = \frac{\partial^2 E_{11}}{\partial x_2^2} - \frac{\partial^2 E_{12}}{\partial x_1 x_2} \tag{6}$$

Similarly, the first derivative of the strain component W_{12} with respect to x_2 can be written as

$$\frac{\partial W_{12}}{\partial x_2} = \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_2^2} - \frac{\partial^2 u_2}{\partial x_1 x_2}\right) \tag{7}$$

Adding and subtracting the term $\dfrac{\partial^2 u_1}{\partial x_1 x_2}$, one can rearrange the above equation as

$$\begin{aligned}\frac{\partial W_{12}}{\partial x_2} &= \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1 x_2} - \frac{\partial^2 u_2}{\partial x_1 x_2} - \frac{\partial^2 u_2}{\partial x_1 x_2}\right) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\right) - \frac{1}{2}\left(\frac{\partial}{\partial x_2}\left(2\frac{\partial u_2}{\partial x_2}\right)\right)\end{aligned} \tag{8}$$

From Eq.(4), the expression in Eq.(3) can be rewritten as

$$\frac{\partial W_{12}}{\partial x_2} = -\frac{\partial E_{22}}{\partial x_1} + \frac{\partial E_{12}}{\partial x_2} \tag{9}$$

Thus $\dfrac{\partial^2 W_{12}}{\partial x_1 x_2}$ can be expressed as

$$\frac{\partial^2 W_{12}}{\partial x_1 x_2} = -\frac{\partial^2 E_{22}}{\partial x_1^2} + \frac{\partial^2 E_{12}}{\partial x_2 x_1} \tag{10}$$

From the continuity of infinitesimal rotations, we know $\dfrac{\partial^2 W_{12}}{\partial x_1 x_2} = \dfrac{\partial^2 W_{12}}{\partial x_2 x_1}$. Thus from Eq. (6) and Eq. (10), we can write

$$\frac{\partial^2 E_{11}}{\partial x_2^2} - \frac{\partial^2 E_{12}}{\partial x_1 x_2} = -\frac{\partial^2 E_{22}}{\partial x_1^2} + \frac{\partial^2 E_{12}}{\partial x_2 x_1} \tag{11}$$

Rearranging the terms in Eq.(11), the compatibility equation can be expressed as

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} = 2\,\frac{\partial^2 E_{12}}{\partial x_2 x_1} \tag{12}$$

Solution:2

The strain compatibility condition in two dimensional condition is defined as

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} = 2\frac{\partial^2 E_{12}}{\partial x_1 x_2} \tag{13}$$

The above equation can proven as follows. The strain-displacement relations in a two dimensional condition can be written as

$$\boldsymbol{E}_s = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T) \tag{14}$$

The components of the strain tensor can be obtained as

$$E_{11} = \frac{\partial u_1}{\partial x_1} \qquad E_{22} = \frac{\partial u_2}{\partial x_2} \qquad E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

Double differentiating the expressions for E_{11}, E_{22} and E_{12} with x_2, x_1 and (x_1, x_2), respectively,

$$\begin{aligned}
\frac{\partial^2 E_{11}}{\partial x_2^2} &= \frac{\partial^3 u_1}{\partial x_1 x_2^2} \\
\frac{\partial^2 E_{22}}{\partial x_1^2} &= \frac{\partial^3 u_2}{\partial x_2 x_1^2} \\
\frac{\partial^2 E_{12}}{\partial x_1 x_2} &= \frac{1}{2}\left(\frac{\partial^3 u_1}{\partial x_1 x_2^2} + \frac{\partial^3 u_2}{\partial x_2 x_1^2}\right)
\end{aligned} \tag{15}$$

From Eq.(15),

$$\begin{aligned}
\frac{\partial^2 E_{22}}{\partial x_1^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} - 2\frac{\partial^2 E_{12}}{\partial x_1 x_2} &= \frac{\partial^3 u_1}{\partial x_1 x_2^2} + \frac{\partial^3 u_2}{\partial x_2 x_1^2} - \left(\frac{\partial^3 u_1}{\partial x_1 x_2^2} + \frac{\partial^3 u_2}{\partial x_2 x_1^2}\right) \\
&= 0 \\
\implies \frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} - 2\frac{\partial^2 E_{12}}{\partial x_1 x_2} &= 0
\end{aligned}$$

2. a) Given the following two-dimensional, infinitesimal strain field:

$$\begin{aligned}
E_{11} &= c_1\, x_1\, (x_1^2 + x_2^2), \\
E_{22} &= \frac{1}{3} c_2\, x_1^3, \\
E_{12} &= E_{21} = c_3\, x_1^2\, x_2,
\end{aligned}$$

where c_1, c_2, and c_3 are constants, determine if the strain field is compatible.

b) The strain field is given as:

$$\begin{aligned}
E_{11} &= 2\,\alpha x_1\, x_2, \\
E_{22} &= 2\,\beta x_1\, x_2, \\
E_{12} &= E_{21} = \frac{1}{2}(\alpha\, x_1^2 + \beta x_2^2),
\end{aligned}$$

Check whether the strain field is compatible or not, where α and β are constants.

Solution:

a) For the strain field to be compatible, it should satisfy the Saint-Venant Compatibility Equations:

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} = 2\frac{\partial^2 E_{12}}{\partial x_1 \partial x_2}$$

After solving each part separately, we get:

$$\frac{\partial^2 E_{11}}{\partial x_2^2} = 2c_1 x_1$$

$$\frac{\partial^2 E_{22}}{\partial x_1^2} = 2c_2 x_1$$

$$2\frac{\partial^2 E_{12}}{\partial x_1 \partial x_2} = 4c_3 x_1$$

After writing all the terms we get

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} - 2\frac{\partial^2 E_{12}}{\partial x_1 \partial x_2} = 2c_1 x_1 + 2c_2 x_1 - 4c_3 x_1$$

Thus the strain field is not compatible, unless c_1+ c_2 - $2c_3 = 0$

b) For the second part also we can satisfy the saint -venant compatibility equation

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} = 2\frac{\partial^2 E_{12}}{\partial x_1 \partial x_2}$$

solving each part separately, we get:

$$\frac{\partial^2 E_{11}}{\partial x_2^2} = 0$$

$$\frac{\partial^2 E_{22}}{\partial x_1^2} = 0$$

$$2\frac{\partial^2 E_{12}}{\partial x_1 \partial x_2} = 0$$

After writing all the terms we get

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} - 2\frac{\partial^2 E_{12}}{\partial x_1 \partial x_2} = 0$$

Thus the given strain field is compatible

3. In two dimensions, let us consider two basis vectors $\boldsymbol{e_i}$ and $\boldsymbol{e_j^*}$ such that $\boldsymbol{e_1^*}$ is oriented at an angle θ with respect to $\boldsymbol{e_1}$. E_{ij} and E_{ij}^* are, respectively, the components of a strain tensor E expressed in the $\boldsymbol{e_i}$ and $\boldsymbol{e_j^*}$ bases (corresponding to the same state of deformation).

Using the following expression:

$$E_{ij}^* = E_{lk}(\boldsymbol{e_l} \cdot \boldsymbol{e_i^*})(\boldsymbol{e_j^*} \cdot \boldsymbol{e_k}),$$

derive the following relations:

$$E_{11}^* = E_{11}\cos^2\theta + E_{22}\sin^2\theta + E_{12}\sin 2\theta,$$

$$
\begin{aligned}
E_{22}^* &= E_{11}\sin^2\theta + E_{22}\cos^2\theta - E_{12}\sin 2\theta, \\
E_{12}^* &= -\frac{E_{11}-E_{22}}{2}\sin 2\theta + E_{12}\cos 2\theta.
\end{aligned}
$$

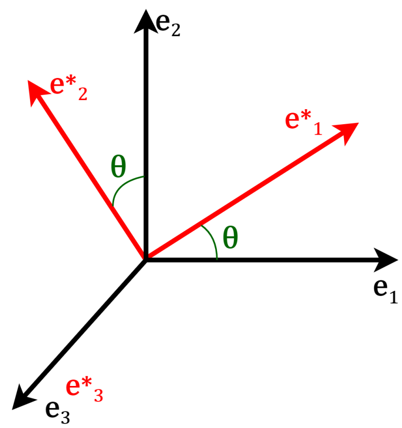

Figure 1: Representation of the two basis vectors

Solution: First, let us recall the following trigonometric relations between the vectors of $\boldsymbol{e_i}$ and $\boldsymbol{e_j^*}$:

$$
\boldsymbol{e_1}\cdot\boldsymbol{e_1}^* = \cos\theta, \quad \boldsymbol{e_2}\cdot\boldsymbol{e_2}^* = \cos\theta, \quad \boldsymbol{e_1}\cdot\boldsymbol{e_2}^* = -\sin\theta, \quad \boldsymbol{e_2}\cdot\boldsymbol{e_1}^* = \sin\theta
$$

By using the above trigonometric relations, we can prove the following:

$$
\begin{aligned}
E_{11}^* &= E_{11}(\boldsymbol{e_1}\cdot\boldsymbol{e_1}^*)^2 + E_{22}(\boldsymbol{e_2}\cdot\boldsymbol{e_2}^*)^2 + 2E_{12}(\boldsymbol{e_1}\cdot\boldsymbol{e_1}^*)(\boldsymbol{e_2}\cdot\boldsymbol{e_2}^*) \\
&= E_{11}\cos^2\theta + E_{22}\sin^2\theta + E_{12}\sin 2\theta \\
E_{22}^* &= E_{11}(\boldsymbol{e_1}\cdot\boldsymbol{e_2}^*)^2 + E_{22}(\boldsymbol{e_2}\cdot\boldsymbol{e_1}^*)^2 + 2E_{12}(\boldsymbol{e_1}\cdot\boldsymbol{e_2}^*)(\boldsymbol{e_2}\cdot\boldsymbol{e_1}^*) \\
&= E_{11}\sin^2\theta + E_{22}\cos^2\theta - E_{12}\sin 2\theta \\
E_{12}^* &= E_{11}(\boldsymbol{e_1}\cdot\boldsymbol{e_1}^*)(\boldsymbol{e_2}\cdot\boldsymbol{e_1}) + E_{22}(\boldsymbol{e_2}\cdot\boldsymbol{e_1})(\boldsymbol{e_2}\cdot\boldsymbol{e_2}^*) \\
&\quad + E_{12}(\boldsymbol{e_1}\cdot\boldsymbol{e_1}^*)(\boldsymbol{e_2}\cdot\boldsymbol{e_2}^*) + E_{21}(\boldsymbol{e_2}\cdot\boldsymbol{e_1})(\boldsymbol{e_2}\cdot\boldsymbol{e_1}^*) \\
&= -\frac{E_{11}}{2}\sin 2\theta + \frac{E_{22}}{2}\sin 2\theta + E_{12}(\cos^2\theta - \sin^2\theta) \\
&= -\frac{E_{11}-E_{22}}{2}\sin 2\theta + E_{12}\cos 2\theta
\end{aligned}
$$

4. Using the relations $E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, show that, given the components E_{ij} of a 2D strain tensor in a basis $\boldsymbol{e_i}$:

a) Derive the principal strains expression:

$$E_{1,2} = \frac{E_{11} + E_{22}}{2} \pm \sqrt{\left(\frac{E_{11} - E_{22}}{2}\right)^2 + E_{12}^2},$$

and the principal direction of strain for the angle with respect to $\boldsymbol{e_1}$ satisfies:

$$\tan 2\theta^p = \frac{2E_{12}}{E_{11} - E_{22}}.$$

b) Derive the maximum shear strain expression:

$$E_{12}^{\max} = \sqrt{\left(\frac{E_{11} - E_{22}}{2}\right)^2 + E_{12}^2},$$

and the normals of the planes of maximum shear form angles with respect to $\boldsymbol{e_1}$:

$$\tan 2\theta^s = -\frac{E_{11} - E_{22}}{2E_{12}}.$$

Conclude that the direction of maximum shear is always oriented at an angle equal to 45° with respect to the principal directions of strain.

Solution:

b) The maximum shear strain can be found by determining the value of θ that makes the derivative of E_{12} with respect to θ vanish.

$$E_{12}^{\max} = -\frac{E_{11} - E_{22}}{2} \sin 2\theta^s + E_{12} \cos 2\theta^s$$

The derivative is given by:

$$\frac{\partial E_{12}}{\partial \theta} = -2\left(\frac{E_{11} - E_{22}}{2} \cos 2\theta^s + E_{12} \sin 2\theta^s\right) = 0$$

Taking the square of the two previous equations and summing them, we get:

$$(E_{12}^{\max})^2 = \left(\frac{E_{11} - E_{22}}{2}\right)^2 + E_{12}^2$$

The second equation leads directly to the angular relation:

$$\tan 2\theta^s = \frac{E_{22} - E_{11}}{2E_{12}}$$

From the trigonometric relation: $\tan(\alpha + \frac{\pi}{4}) = -\frac{1}{\tan \alpha}$ it is also easy to see that:

$$\tan(2(\theta^p + \frac{\pi}{4})) = -\frac{1}{\tan 2\theta^p} = -\frac{E_{11} - E_{22}}{2E_{12}} = \tan 2\theta^s$$

Thus proving that , $\theta^s = \theta^p + \frac{\pi}{4}$ which means that:
The direction of maximum shear is always oriented at an angle equal to 45° with respect to the principal directions of strain.

5. Consider the problem of the isotropic cantilever beam bent by a load P at the free end, as shown in Fig. 2. From the elementary beam theory, we have the following strains:

$$\begin{aligned} S_{11} &= -\frac{P x_1 x_2}{EI}, \\ S_{22} &= \nu \frac{P x_1 x_2}{EI}, \\ S_{12} &= -\frac{(1+\nu)P}{2EI}(h^2 - x_2^2). \end{aligned} \tag{16}$$

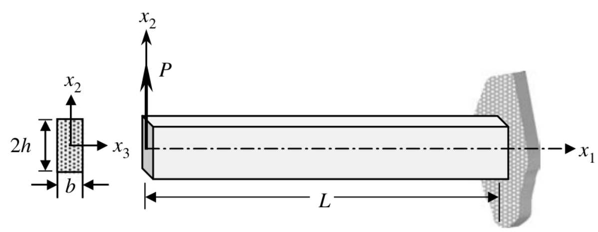

Figure 2: Cantilever beam bent by a point load, P

where I is the second moment of area about the x_3-axis, ν is the Poisson ratio, E is Young's modulus, and $2h$ is the height of the beam.

(a) Determine whether the strains are compatible.
(b) If compatible, find the displacement field using the linearized strain-displacement relations.
(c) Determine the constants of integration using suitable boundary conditions.

Solution: The strain can be determined as

$$\begin{aligned} S_{11} &= \frac{\partial u_1}{\partial x_1} = -\frac{P x_1 x_2}{EI} \\ S_{22} &= \frac{\partial u_2}{\partial x_2} = \nu \frac{P x_1 x_2}{EI} \\ 2S_{12} &= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \frac{-(1+\nu)P}{2EI}(h^2 - x_2^2) \end{aligned} \tag{17}$$

By integration,

$$\begin{aligned} u_1 &= -\frac{P x_1^2 x_2}{2EI} + f(x_2) \\ u_2 &= \nu \frac{P x_1 x_2^2}{2EI} + g(x_1) \end{aligned} \tag{18}$$

From $2S_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$, we can write

$$\frac{-(1+\nu)P}{EI}(h^2 - x_2^2) = -\frac{Px_1^2}{2EI} + \frac{df}{dx_2} + \nu\frac{Px_2^2}{2EI} + \frac{dg}{dx_1} \tag{19}$$

Rearranging the terms

$$\begin{aligned}
\frac{Px_1^2}{2EI} - \frac{dg}{dx_1} - \frac{(1+\nu)P}{EI}h^2 &= \frac{df}{dx_2} + \nu\frac{Px_2^2}{2EI} + \frac{-(1+\nu)P}{EI}x_2^2 \\
\frac{Px_1^2}{2EI} - \frac{dg}{dx_1} - \frac{(1+\nu)P}{EI}h^2 &= \frac{df}{dx_2} - \frac{(2+\nu)P}{2EI}x_2^2
\end{aligned} \tag{20}$$

Equating both sides to a constant c_0,

$$\begin{aligned}
\frac{Px_1^2}{2EI} - \frac{dg}{dx_1} - \frac{(1+\nu)P}{EI}h^2 &= c_0 \\
\frac{df}{dx_2} - \frac{(2+\nu)P}{2EI}x_2^2 &= c_0
\end{aligned} \tag{21}$$

Integrating the above equations,

$$\begin{aligned}
f &= \frac{(2+\nu)P}{6EI}x_2^3 + c_0x_2 + c_1 \\
g &= \frac{Px_1^3}{6EI} - \frac{(1+\nu)P}{EI}h^2x_1 - c_0x_1 + c_2
\end{aligned} \tag{22}$$

Substituting above equations in the expression fro displacement u_1 and u_2,

$$\begin{aligned}
u_1 &= -\frac{Px_1^2x_2}{2EI} + \frac{(2+\nu)P}{6EI}x_2^3 + c_0x_2 + c_1 \\
u_2 &= \nu\frac{Px_1x_2^2}{2EI} + \frac{Px_1^3}{6EI} - \frac{(1+\nu)P}{EI}h^2x_1 - c_0x_1 + c_2
\end{aligned} \tag{23}$$

Apply the boundary conditions $u_1(L,0) = 0$, $u_2(L,0) = 0$, $\frac{\partial u_2}{\partial x_1}(L,0) = 0$ on the above equations

$$\begin{aligned}
u_1(L,0) = 0 &\implies c_1 = 0 \\
u_2(L,0) = 0 &\implies \frac{PL^3}{6EI} - \frac{(1+\nu)P}{EI}h^2L - c_0L + c_2 = 0 \\
\frac{\partial u_2}{\partial x_1}(L,0) = 0 &\implies \frac{PL^2}{2EI} - \frac{(1+\nu)Ph^2}{EI} = c_0
\end{aligned} \tag{24}$$

Thus the integration constants can be written as

$$\begin{aligned}
c_0 &= \frac{PL^2}{2EI} - \frac{(1+\nu)Ph^2}{EI} \\
c_1 &= 0 \\
c_2 &= \frac{PL^3}{3EI}
\end{aligned} \tag{25}$$

Substituting these equations into the expressions of displacement,

$$\begin{aligned}
u_1 &= -\frac{Px_1^2x_2}{2EI} + \frac{(2+\nu)P}{6EI}x_2^3 + \frac{PL^2}{2EI}x_2 - \frac{(1+\nu)Ph^2}{EI}x_2 \\
u_2 &= \nu\frac{Px_1x_2^2}{2EI} + \frac{Px_1^3}{6EI} - \frac{PL^2}{2EI}x_1 + \frac{PL^3}{3EI}
\end{aligned} \tag{26}$$

6. A rectangular loaded plate is clamped along the x_1 and x_2 axes (see Fig. 3). On the basis of measurements, the strain components are determined as:

$$E_{11} = a\,(x_1^2\,x_2 + x_2^3), \qquad E_{22} = b\,x_1\,x_2^2.$$

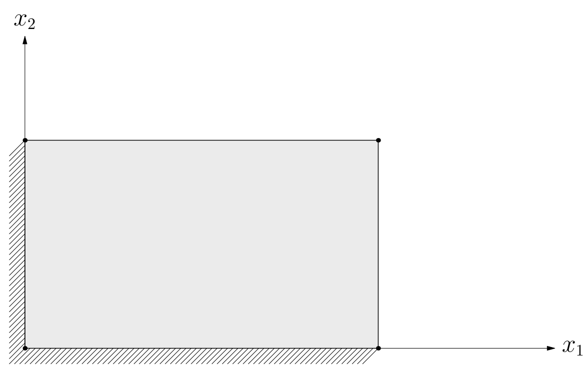

Figure 3: Rectangular plate

Determine the following:

(a) Find the displacement field.

(b) Compute the shear strain component, E_{12}.

(c) Check for strain compatibility condition.

Solution: a) The strain tensor components E_{11} and E_{22} are given as

$$\begin{aligned}
E_{11} &= a(x_1^2x_2 + x_2^3) \\
E_{22} &= bx_1x_2
\end{aligned} \tag{27}$$

The strain displacement relationship can be written as

$$\begin{aligned}
E_{11} &= \frac{\partial u_1}{\partial x_1} \\
E_{22} &= \frac{\partial u_2}{\partial x_2}
\end{aligned}$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \tag{28}$$

By integrating the strain-displacement relation subjected to the given boundary conditions, one can obtain the displacement fields as

$$\begin{aligned} u_1 &= \int E_{11}\, dx_1 \\ u_2 &= \int E_{22}\, dx_2, \end{aligned} \tag{29}$$

subjected to the following boundary conditions from Fig. 3 as

$$\begin{aligned} u_1(x_1 = 0, x_2) &= 0 \\ u_2(x_1, x_2 = 0) &= 0 \end{aligned} \tag{30}$$

From Eq. (29), the displacement can be obtained as

$$\begin{aligned} u_1 &= \frac{1}{3}ax_1^3x_2 + ax_1x_2^3 + f(x_2) \\ u_2 &= \frac{1}{3}bx_1x_2^3 + g(x_1) \end{aligned} \tag{31}$$

Applying the boundary conditions in Eq.(30),

$$\begin{aligned} f(x_2) &= 0 \\ g(x_1) &= 0 \end{aligned} \tag{32}$$

Thus displacement components in Eq.(31) can be written as

$$\begin{aligned} u_1 &= \frac{1}{3}ax_1^3x_2 + ax_1x_2^3 \\ u_2 &= \frac{1}{3}bx_1x_2^3 \end{aligned} \tag{33}$$

b) From Eq. (28), the shear strain E_{12} can be expressed as

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{1}{3}ax_1^3 + 3ax_1x_2^2 + \frac{1}{3}bx_2^2\right)$$

c) The compatibility conditions in 2D can be written as

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} - 2\frac{\partial^2 E_{12}}{\partial x_1 x_2} = 0 \tag{34}$$

Substituting the expressions for the strain given in Eq.(27) and Eq.(34) into Eq.(**??**),

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} - 2\frac{\partial^2 E_{12}}{\partial x_1 x_2} = 6ax_2 + 0 - 6ax_2 = 0$$

7. Find the linearized strain field associated with the following displacements:

$$u_1 = x_1^3 x_2 + 2c_1 c_2^3 x_1 + 3c_1 c_2^2 x_1 x_2 - c_1 x_1 x_2^3, \tag{35}$$

$$u_2 = -2c_2^3 x_2 - \frac{3}{2} c_2^2 x_2^2 + \frac{1}{4} x_2^4 - \frac{3}{2} c_1 x_1^2 x_2^2, \tag{36}$$

where c_1, c_2, and c_3 are constants.

Solution: We will define the linear part of the Green-Lagrange strain tensor as the small strain tensor

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

so the strain tensor matrix can be written as

$$[\boldsymbol{E}] = \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}$$

Now we can solve all the component of strain tensor

$$E_{11} = \frac{\partial u_1}{\partial x_1}$$
$$E_{22} = \frac{\partial u_2}{\partial x_2}$$
$$E_{12} = E_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

so we can write the whole strain tensor component as follows

$$E_{11} = 3x_1^2 x_2 + 2c_1 c_2^3 + 3c_1 c_2^2 x_2 - c_1 x_2^3$$
$$E_{22} = -3c_1 x_1^2 x_2 + x_2^3 - 3c_2^2 x_2 - 2c_2^3$$
$$E_{12} = E_{21} = \frac{1}{2} x_1^3 + \frac{3}{2} c_1 c_2^2 x_1 - 3c_1 x_1 x_2^2$$

The linearized strain tensor for the displacement field u_1 and u_2 is

$$E = \begin{bmatrix} 3x_1^2 x_2 + 2c_1 c_2^3 + 3c_1 c_2^2 x_2 - c_1 x_2^3 & \frac{1}{2} x_1^3 + \frac{3}{2} c_1 c_2^2 x_1 - 3c_1 x_1 x_2^2 \\ \frac{1}{2} x_1^3 + \frac{3}{2} c_1 c_2^2 x_1 - 3c_1 x_1 x_2^2 & -3c_1 x_1^2 x_2 + x_2^3 - 3c_2^2 x_2 - 2c_2^3 \end{bmatrix}$$

8. The displacement field in a material is given by:

$$u_1 = a x_2 x_1^3, \qquad u_2 = b x_1 x_2^2,$$

where a is a small constant. Determine:

(a) The components of the linearized strain tensor.

(b) The components of the linearized rotation tensor.

(c) Whether the compatibility condition is satisfied.

Solution: a) Assuming small deformation, the linearized strain tensor can be written as

$$\boldsymbol{E}_s = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) \tag{37}$$

In the component form it can be written as

$$[\boldsymbol{E}_s] = \frac{1}{2}\begin{bmatrix} 2\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} & 2\frac{\partial u_2}{\partial x_2} \end{bmatrix} \tag{38}$$

Since $u_1 = ax_2x_1^3$, $u_2 = bx_1x_2^2$, and $u_3 = c(x_1^2 + x_2^2)$,

$$[\boldsymbol{E}_s] = \frac{1}{2}\begin{bmatrix} 3ax_2x_1^2 & ax_1^3 + bx_2^2 \\ ax_1^3 + bx_2^2 & 4bx_1x_2 \end{bmatrix} \tag{39}$$

b) The linearized rotation tensor can be written as

$$\boldsymbol{W}_s = \frac{1}{2}(\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^T) \tag{40}$$

In the component form, it can be written as

$$[\boldsymbol{W}_s] = \frac{1}{2}\begin{bmatrix} 0 & \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} & 0 \end{bmatrix} \tag{41}$$

Since $u_1 = ax_2x_1^3$, $u_2 = bx_1x_2^2$, and $u_3 = c(x_1^2 + x_2^2)$,

$$[\boldsymbol{W}_s] = \frac{1}{2}\begin{bmatrix} 0 & ax_1^3 - bx_2^2 \\ bx_2^2 - ax_1^3 & 0 \end{bmatrix} \tag{42}$$

c) The compatibility condition for the strain in two-dimension can be written as,

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} - 2\frac{\partial^2 E_{12}}{\partial x_1 x_2} = 0 \tag{43}$$

From Eq.(39),

$$\begin{aligned} E_{11} &= \frac{3}{2}ax_2x_1^2 \\ E_{22} &= 2bx_1x_2 \\ E_{12} &= \frac{1}{2}(ax_1^3 + bx_2^2) \end{aligned} \tag{44}$$

Thus

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} - 2\frac{\partial^2 E_{12}}{\partial x_1 x_2} = 0 + 0 - 0 = 0$$

Hence proved.

9. A continuous body undergoes a homogeneous plane deformation. A material point initially at position $\boldsymbol{x} = (x_1, x_2)$ moves to a new position $\boldsymbol{x}' = (x_1', x_2')$ according to the following mapping:

$$x_1' = 3x_1 + x_2, \qquad x_2' = 2x_1 + 2x_2$$

Calculate the following:

(a) The deformation gradient tensor, $\mathbf{F}$.

(b) The right Cauchy-Green deformation tensor, $\mathbf{C}$.

(c) The principal stretches, λ_1 and λ_2.

Solution: a) The deformation gradient tensor $\mathbf{F}$ is defined by its components $F_{ij} = \frac{\partial x_i'}{\partial x_j}$. For this 2D deformation, the tensor can be written in matrix form as:

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} \end{bmatrix} \tag{45}$$

Given the mapping $x_1' = 3x_1 + x_2$ and $x_2' = 2x_1 + 2x_2$, we can compute the partial derivatives:

$$\frac{\partial x_1'}{\partial x_1} = 3, \qquad \frac{\partial x_1'}{\partial x_2} = 1,$$
$$\frac{\partial x_2'}{\partial x_1} = 2, \qquad \frac{\partial x_2'}{\partial x_2} = 2$$

Substituting these values into Eq. (45), the deformation gradient tensor is:

$$[\mathbf{F}] = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \tag{46}$$

b) The right Cauchy-Green deformation tensor $\mathbf{C}$ is defined as $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. First, the transpose of $\mathbf{F}$ is:

$$[\mathbf{F}^T] = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \tag{47}$$

Now, we perform the matrix multiplication:

$$
\begin{aligned}
[\mathbf{C}] = [\mathbf{F}^T][\mathbf{F}] &= \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 13 & 7 \\ 7 & 5 \end{bmatrix}
\end{aligned} \tag{48}
$$

c) The principal stretches λ_1 and λ_2 are the square roots of the eigenvalues $\Lambda = \lambda^2$ of the right Cauchy-Green tensor $\mathbf{C}$. The eigenvalues are found by solving the characteristic equation $\det(\mathbf{C} - \Lambda\mathbf{I}) = 0$:

$$
\begin{aligned}
\det \begin{bmatrix} 13 - \Lambda & 7 \\ 7 & 5 - \Lambda \end{bmatrix} &= 0 \\
(13 - \Lambda)(5 - \Lambda) - 7^2 &= 0 \\
\Lambda^2 - 18\Lambda + 16 &= 0
\end{aligned} \tag{49}
$$

We solve this quadratic equation for Λ:

$$
\begin{aligned}
\Lambda &= \frac{18 \pm \sqrt{18^2 - 4(1)(16)}}{2} \\
&= \frac{18 \pm \sqrt{324 - 64}}{2} \\
&= \frac{18 \pm \sqrt{260}}{2} \\
&= \frac{18 \pm 2\sqrt{65}}{2} \\
&= 9 \pm \sqrt{65}
\end{aligned} \tag{50}
$$

The eigenvalues are $\Lambda_1 = 9 + \sqrt{65}$ and $\Lambda_2 = 9 - \sqrt{65}$. The principal stretches λ_1 and λ_2 are the square roots of these eigenvalues:

$$
\begin{aligned}
\lambda_1 &= \sqrt{\Lambda_1} = \sqrt{9 + \sqrt{65}} \approx 4.131, \\
\lambda_2 &= \sqrt{\Lambda_2} = \sqrt{9 - \sqrt{65}} \approx 0.968
\end{aligned} \tag{51}
$$

10. A body undergoes simple shear deformation, where γ is a constant representing the shear. The deformation mapping is given by:

$$x_1' = x_1 + \gamma\, x_2, \qquad x_2' = x_2$$

Determine the following:

(a) The deformation gradient tensor, $\mathbf{F}$.

(b) The right Cauchy-Green deformation tensor, $\mathbf{C}$.

(c) The principal stretches, λ_1 and λ_2, in terms of γ.

Solution: a) The deformation gradient has components $F_{ij} = \partial x_i'/\partial x_j$. Thus

$$[\mathbf{F}] = \begin{bmatrix} \partial x_1'/\partial x_1 & \partial x_1'/\partial x_2 \\ \partial x_2'/\partial x_1 & \partial x_2'/\partial x_2 \end{bmatrix} = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}. \tag{52}$$

b) The right Cauchy–Green tensor is $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Compute

$$[\mathbf{F}^T] = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}, \tag{53}$$

$$[\mathbf{C}] = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \gamma \\ \gamma & 1+\gamma^2 \end{bmatrix}. \tag{54}$$

c) The principal stretches are $\lambda_i = \sqrt{\Lambda_i}$ where Λ_i are the eigenvalues of $\mathbf{C}$. Solve $\det(\mathbf{C} - \Lambda\mathbf{I}) = 0$:

$$\det \begin{bmatrix} 1-\Lambda & \gamma \\ \gamma & 1+\gamma^2-\Lambda \end{bmatrix} = 0 \tag{55}$$

$$\Rightarrow \Lambda^2 - (2+\gamma^2)\Lambda + 1 = 0. \tag{56}$$

Hence

$$\Lambda_{1,2} = \frac{2+\gamma^2 \pm \sqrt{(2+\gamma^2)^2 - 4}}{2} = \frac{2+\gamma^2 \pm \sqrt{\gamma^4 + 4\gamma^2}}{2} \tag{57}$$

$$= \frac{2+\gamma^2 \pm |\gamma|\sqrt{\gamma^2+4}}{2}. \tag{58}$$

Therefore the principal stretches are

$$\lambda_{1,2} = \sqrt{\Lambda_{1,2}} = \sqrt{\frac{2+\gamma^2 \pm |\gamma|\sqrt{\gamma^2+4}}{2}}. \tag{59}$$