Indian Institute of Technology Bhubaneswar

School of Infrastructure

Session: Autumn 2025

Solid Mechanics (CE2L001) Solution of Assignment No. 3

Notations :
Zeroth-order tensors or scalars are represented by small letters. For eg. a
First-order tensors or vectors are represented by bold small letters. For eg. $\boldsymbol{a}$.
Second-order tensors are represented by bold capital letters. For eg. $\boldsymbol{A}$

1. Consider the following two-dimensional transformation

$$x_1^{'} = 4 - 2x_1 - x_2$$
$$x_2^{'} = 2 + \frac{3x_1}{2} - \frac{x_2}{2}$$

(a) Is the transformation linear?

(b) Calculate the components of the deformation gradient $\boldsymbol{F}$. Compute det $(\boldsymbol{F})$, and $\boldsymbol{F}^{-1}$, where det $(\cdot)$ and $(\cdot)^{-1}$ denote the determinant and the inverse of a second order tensor, respectively.

(c) Study the transformation over a unit square defined through the following corner points (0,0), (1,0), (1,1), (0,1).

Solution:

(a) The given transformation is an **affine** linear.

Affine Linear

Affine linear, or affine transformation, is a mathematical concept that combines linear transformations (like scaling, rotating, or flipping) with translations (shifts). In essence, it's a way to transform geometric objects while preserving certain properties.

Key Properties

- **Linearity**: Affine transformations preserve straight lines and ratios of distances between points.

- **Translation**: Affine transformations can shift or move objects in space.

Mathematical Representation

An affine transformation can be represented as a matrix multiplication followed by a vector addition: $[\boldsymbol{x}'] = [\boldsymbol{A}]\,[\boldsymbol{x}] + [\boldsymbol{c}]$ where $[\boldsymbol{A}]$ is a linear transformation matrix, $[\boldsymbol{x}]$ is the input vector, $[\boldsymbol{c}]$ is the translation vector, $[\boldsymbol{x}']$ is the output vector.

Examples

- Rotating an image and then shifting it
- Scaling a shape and then translating it
- Reflecting an object across a line and then moving it

(b) The deformation gradient tensor field $\boldsymbol{F} = F_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ is defined as

$$F_{ij} = \frac{\partial x_i'}{\partial x_j}.$$

For two-dimensional case, $[\boldsymbol{F}] := (F_{ij})_{i,j\in(1,2)}$ can be expressed as

$$[\boldsymbol{F}] = \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} \end{bmatrix}$$

For the given problem, $[\boldsymbol{F}]$ can be computed as

$$[\boldsymbol{F}] = \begin{bmatrix} -2 & -1 \\ 3/2 & -1/2 \end{bmatrix}.$$

The determinant of $\boldsymbol{F}$ can be determined as

$$\det(\boldsymbol{F}) = \begin{vmatrix} -2 & -1 \\ 3/2 & -1/2 \end{vmatrix} = \frac{5}{2}.$$

The inverse of the $\boldsymbol{F}$ can be written as

$$[\boldsymbol{F}^{-1}] = \frac{2}{5}\begin{bmatrix} -1/2 & 1 \\ -3/2 & -2 \end{bmatrix} = \begin{bmatrix} -1/5 & 2/5 \\ -3/5 & -4/5 \end{bmatrix}.$$

(c) The mapping between the reference and deformed configurations is given in Table 1,

(x_1, x_2)	(x'_1, x'_2)
(0,0)	(4,2)
(1,0)	(2,3.5)
(1,1)	(1,3)
(0,1)	(3,1.5)

Table 1: Reference and deformed coordinates

The reference and deformed configuration for the given mapping function is depicted in Fig.1 below.

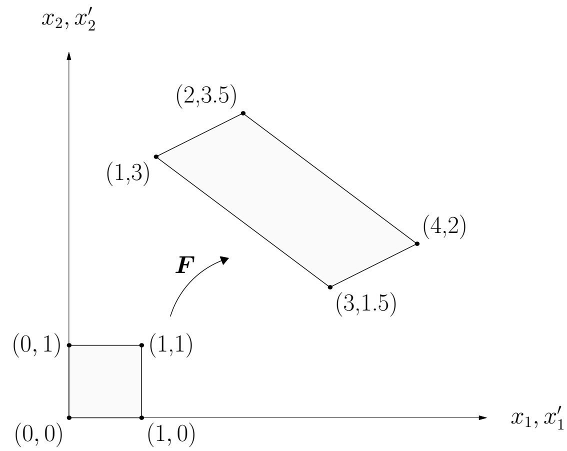

Figure 1: Deformation mapping

2. Determine the linear strain tensor $\boldsymbol{E}_s = (1/2)(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ for the following displacement fields, given in a Cartesian basis:

   (a) $\boldsymbol{u} = a\,(x_2 - x_1)\,\boldsymbol{e_1} + a\,(x_1 - x_2)\,\boldsymbol{e_2} + a\,x_1\,x_3\,\boldsymbol{e_3}$

   (b) $\boldsymbol{u} = a\,x_2\,x_3\,\boldsymbol{e_1} + b\,x_3\,x_1\,\boldsymbol{e_2} + c\,x_1\,x_2\,\boldsymbol{e_3}$

   (c) $\boldsymbol{u} = -a\,x_1\,x_2\,\boldsymbol{e_1} + (bx_1^2 + cx_2^2 - cx_3^2)\,\boldsymbol{e_2} + c\,x_2\,x_3\,\boldsymbol{e_3}$

   (d) $\boldsymbol{u} = a\,(3x_1^2 + x_2)\,\boldsymbol{e_1} + a\,(2x_2^2 + x_3)\,\boldsymbol{e_2} + a\,(4x_3^2 + x_1)\,\boldsymbol{e_3}$

   where a, b and c are positive constants. Also for each of the above displacement fields, determine the normal strain at the point $(x_1,\, x_2,\, x_3) = (1,\, 1,\, 1)$ for the direction $\boldsymbol{n} = \frac{1}{\sqrt{3}}(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)$.

Solution: The linear strain tensor is defined as $\boldsymbol{E}_s = (1/2)(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$, where $\nabla \boldsymbol{u}$ is the gradient of the displacement field. The gradient of the displacement field $\boldsymbol{u}$ is defined as

$$\nabla \boldsymbol{u} = \frac{\partial u_i}{\partial x_j}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)$$

In the component form, $\nabla \boldsymbol{u}$ can be expressed as

$$[\nabla \boldsymbol{u}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

a) Given that $u_1 = a\,(x_2 - x_1)$, $u_2 = a\,(x_1 - x_2)$ and $u_3 = a\,x_1\,x_3$. Thus $\nabla \boldsymbol{u}$ can be expressed as

$$[\nabla \boldsymbol{u}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -a & a & 0 \\ a & -a & 0 \\ ax_3 & 0 & ax_1 \end{bmatrix}$$

The transpose of $\nabla \boldsymbol{u}$ can be expressed as

$$[\nabla \boldsymbol{u}^T] = \begin{bmatrix} -a & a & ax_3 \\ a & -a & 0 \\ 0 & 0 & ax_1 \end{bmatrix}$$

Thus the linear strain tensor $\boldsymbol{E} = (1/2)(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ can be written as

$$[\boldsymbol{E}] = \frac{1}{2} \begin{bmatrix} -2a & 2a & ax_3 \\ 2a & -2a & 0 \\ ax_3 & 0 & 2ax_1 \end{bmatrix}$$

b) Given that $u_1 = a\,x_2\,x_3$, $u_2 = b\,x_3\,x_1$ and $u_3 = c\,x_1\,x_2$. Thus $\nabla \boldsymbol{u}$ can be expressed as

$$[\nabla \boldsymbol{u}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & ax_3 & ax_2 \\ bx_3 & 0 & bx_1 \\ cx_2 & cx_1 & 0 \end{bmatrix}$$

The transpose of $\nabla \boldsymbol{u}$ can be expressed as

$$[\nabla \boldsymbol{u}^T] = \begin{bmatrix} 0 & bx_3 & cx_2 \\ ax_3 & 0 & cx_1 \\ ax_2 & bx_1 & 0 \end{bmatrix}$$

Thus the linear strain tensor $\boldsymbol{E} = (1/2)(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ can be written as

$$[\boldsymbol{E}] = \frac{1}{2}\begin{bmatrix} 0 & (a+b)x_3 & (a+c)x_2 \\ (b+a)x_3 & 0 & (b+c)x_1 \\ (c+a)x_2 & (c+b)x_1 & 0 \end{bmatrix}$$

c) Given that $u_1 = -a\,x_1\,x_2$, $u_2 = (bx_1^2 + cx_2^2 - cx_3^2)$ and $u_3 = c\,x_2\,x_3$. Thus $\nabla\boldsymbol{u}$ can be expressed as

$$\begin{aligned}[\nabla\boldsymbol{u}] &= \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \\ &= \begin{bmatrix} -ax_2 & -ax_1 & 0 \\ 2bx_1 & 2cx_2 & -2cx_3 \\ 0 & cx_3 & cx_2 \end{bmatrix}\end{aligned}$$

The transpose of $\nabla\boldsymbol{u}$ can be expressed as

$$[\nabla\boldsymbol{u}^T] = \begin{bmatrix} -ax_2 & 2bx_1 & 0 \\ -ax_1 & 2cx_2 & cx_3 \\ 0 & -2cx_3 & cx_2 \end{bmatrix}$$

Thus the linear strain tensor $\boldsymbol{E} = (1/2)(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ can be written as

$$[\boldsymbol{E}] = \frac{1}{2}\begin{bmatrix} -2ax_2 & (-a+2b)x_1 & 0 \\ (-a+2b)x_1 & 4cx_2 & -cx_3 \\ 0 & -cx_3 & 2cx_2 \end{bmatrix}$$

d) Given that $u_1 = a\,(3x_1^2 + x_2)$, $u_2 = a\,(2x_2^2 + x_3)$ and $u_3 = a\,(4x_3^2 + x_1)$. Thus $\nabla\boldsymbol{u}$ can be expressed as

$$\begin{aligned}[\nabla\boldsymbol{u}] &= \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \\ &= \begin{bmatrix} 6ax_1 & a & 0 \\ 0 & 4ax_2 & a \\ a & 0 & 8ax_3 \end{bmatrix}\end{aligned}$$

The transpose of $\nabla\boldsymbol{u}$ can be expressed as

$$[\nabla\boldsymbol{u}^T] = \begin{bmatrix} 6ax_1 & 0 & a \\ a & 4ax_2 & 0 \\ 0 & a & 8ax_3 \end{bmatrix}$$

Thus the linear strain tensor $\boldsymbol{E} = (1/2)(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ can be written as

$$[\boldsymbol{E}] = \frac{1}{2}\begin{bmatrix} 12ax_1 & a & a \\ a & 8ax_2 & a \\ a & a & 16ax_3 \end{bmatrix}$$

3. The displacement field in a turbine blade of a jet engine shown in Fig. 2 can be given in a Cartesian reference frame by $\boldsymbol{u} = u_1\boldsymbol{e_1} + u_2\boldsymbol{e_2} + u_3\boldsymbol{e_3}$, where

$$u_1 = c\,(x_1^2 + 10)$$
$$u_2 = 2\,c\,x_2\,x_3$$
$$u_3 = c\,(-x_1x_2 + x_3^2),$$

with $c = 10^{-4}$ mm. Determine the components of the strain tensor $\boldsymbol{E} = (1/2)(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$ and linear strain tensor $\boldsymbol{E}_s = (1/2)(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ at $(x_1,\, x_2,\, x_3) = (0,\, 2,\, 1)$ mm. Comment on the results.

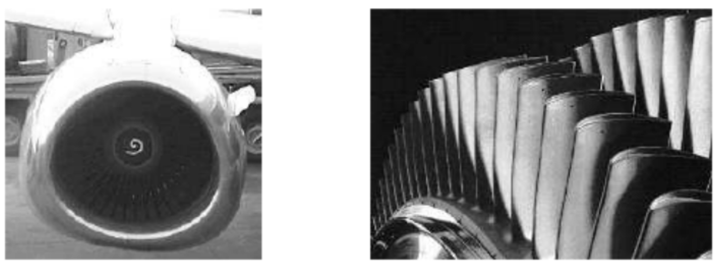

Figure 2: Turbine blades of a jet engine

[10]

Solution:

The displacement field is given by:

$$u_1 = c(x_1^2 + 10)$$

$$u_2 = 2cx_2x_3$$

$$u_3 = c(-x_1x_2 + x_3^2)$$

with $c = 10^{-4}$.

Linear Strain Tensor ($\boldsymbol{E}_s$)

The linear strain tensor is defined as $\boldsymbol{E}_s = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$, where $\nabla \boldsymbol{u}$ is the gradient of the displacement field. First, we compute the displacement gradient tensor:

$$[\nabla \boldsymbol{u}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2cx_1 & 0 & 0 \\ 0 & 2cx_3 & 2cx_2 \\ -cx_2 & -cx_1 & 2cx_3 \end{bmatrix}$$

Evaluating this tensor at the point $(0, 2, 1)$:

$$[\nabla \boldsymbol{u}]_{(0,2,1)} = c \begin{bmatrix} 2(0) & 0 & 0 \\ 0 & 2(1) & 2(2) \\ -2 & -(0) & 2(1) \end{bmatrix} = c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ -2 & 0 & 2 \end{bmatrix}$$

The transpose is:

$$[\nabla \boldsymbol{u}]^T_{(0,2,1)} = c \begin{bmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$$

Now, we calculate the linear strain tensor $\boldsymbol{E}_s$:

$$[\boldsymbol{E}_s] = \frac{1}{2}([\nabla \boldsymbol{u}] + [\nabla \boldsymbol{u}]^T) = \frac{c}{2}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ -2 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}\right) = c \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

Substituting $c = 10^{-4}$:

$$[\boldsymbol{E}_s] = 10^{-4} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

Green-Lagrange Strain Tensor ($\boldsymbol{E}$)

The Green-Lagrange strain tensor $\boldsymbol{E}$ is defined as $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$, where $\boldsymbol{F}$ is the deformation gradient tensor given by $\boldsymbol{F} = \boldsymbol{I} + \nabla \boldsymbol{u}$.

First, we calculate $\boldsymbol{F}$ at the point $(0, 2, 1)$:

$$[\boldsymbol{F}] = [\boldsymbol{I}] + [\nabla \boldsymbol{u}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ -2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+2c & 4c \\ -2c & 0 & 1+2c \end{bmatrix}$$

Next, we compute the product $\boldsymbol{F}^T\boldsymbol{F}$:

$$[\boldsymbol{F}^T\boldsymbol{F}] = \begin{bmatrix} 1 & 0 & -2c \\ 0 & 1+2c & 0 \\ 0 & 4c & 1+2c \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+2c & 4c \\ -2c & 0 & 1+2c \end{bmatrix} = \begin{bmatrix} 1+4c^2 & 0 & -2c-4c^2 \\ 0 & 1+4c+4c^2 & 4c+8c^2 \\ -2c-4c^2 & 4c+8c^2 & 1+4c+20c^2 \end{bmatrix}$$

Finally, we find $\boldsymbol{E}$:

$$[\boldsymbol{E}] = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}\begin{bmatrix} 4c^2 & 0 & -2c-4c^2 \\ 0 & 4c+4c^2 & 4c+8c^2 \\ -2c-4c^2 & 4c+8c^2 & 4c+20c^2 \end{bmatrix}$$

$$[\boldsymbol{E}] = \begin{bmatrix} 2c^2 & 0 & -c-2c^2 \\ 0 & 2c+2c^2 & 2c+4c^2 \\ -c-2c^2 & 2c+4c^2 & 2c+10c^2 \end{bmatrix}$$

Substituting $c = 10^{-4}$ and $c^2 = 10^{-8}$:

$$[\boldsymbol{E}] = \begin{bmatrix} 2\times10^{-8} & 0 & -1\times10^{-4} - 2\times10^{-8} \\ 0 & 2\times10^{-4} + 2\times10^{-8} & 2\times10^{-4} + 4\times10^{-8} \\ -1\times10^{-4} - 2\times10^{-8} & 2\times10^{-4} + 4\times10^{-8} & 2\times10^{-4} + 10\times10^{-8} \end{bmatrix}$$

Comment on the Results

Numerical value of the finite strain tensor $[\boldsymbol{E}]$ is not the same as $[\boldsymbol{E}_s]$. However, $[\boldsymbol{E}] \approx [\boldsymbol{E}_s]$ if we assume $10^{-8} \approx 0$. It is consistent with the fact that $\boldsymbol{E}_s$ is the linear approximation (neglecting the higher order non-linear terms) of fine strain tensor $\boldsymbol{E}$.

4. Give an interpretation and determine the invariants of the strain tensors given in component form as

(a) $[\boldsymbol{E}] = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $[\boldsymbol{E}] = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $[\boldsymbol{E}] = \begin{bmatrix} -\epsilon & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $[\boldsymbol{E}] = \begin{bmatrix} -\epsilon & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & -\epsilon \end{bmatrix}$

(e) $[\boldsymbol{E}] = \begin{bmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

where ϵ and γ are small positive numbers. [20]

Solution: The three principal invariants of a strain tensor $\boldsymbol{E}$ are:

- **First Invariant (I_1):** $I_1 = \mathrm{tr}(\boldsymbol{E}) = E_{11} + E_{22} + E_{33}$
- **Second Invariant (I_2):** $I_2 = E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11} - E_{12}^2 - E_{23}^2 - E_{31}^2$
- **Third Invariant (I_3):** $I_3 = \det(\boldsymbol{E})$

(a) $[\boldsymbol{E}] = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- **Interpretation:** This tensor represents a uniaxial (one-directional) stretching along the x_1-axis. There is an elongation in the x_1 direction, but no change in length in the x_2 or x_3 directions and no change in the angles between the axes. This is a state of simple tension.
- **Invariants:**
  - $I_1 = \epsilon + 0 + 0 = \epsilon$
  - $I_2 = (\epsilon)(0) + (0)(0) + (0)(\epsilon) - 0^2 - 0^2 - 0^2 = 0$
  - $I_3 = \det(\boldsymbol{E}) = 0$

(b) $[\boldsymbol{E}] = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- **Interpretation:** This represents a uniform biaxial (two-directional) stretching in the x_1-x_2 plane. A material element is stretched equally in the x_1 and x_2 directions, with no change in the x_3 direction. This is a state of **plane strain**.
- **Invariants:**
  - $I_1 = \epsilon + \epsilon + 0 = 2\epsilon$
  - $I_2 = (\epsilon)(\epsilon) + (\epsilon)(0) + (0)(\epsilon) - 0^2 - 0^2 - 0^2 = \epsilon^2$
  - $I_3 = \det(\boldsymbol{E}) = 0$

(c) $[\boldsymbol{E}] = \begin{bmatrix} -\epsilon & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- **Interpretation:** This is the opposite of case (b). It represents a **uniform biaxial compression** in the x_1-x_2 plane. The material is compressed equally in the x_1 and x_2 directions.
- **Invariants:**
  - $I_1 = -\epsilon - \epsilon + 0 = -2\epsilon$

- $I_2 = (-\epsilon)(-\epsilon) + (-\epsilon)(0) + (0)(-\epsilon) - 0^2 - 0^2 - 0^2 = \epsilon^2$
- $I_3 = \det(\boldsymbol{E}) = 0$

(d) $[\boldsymbol{E}] = \begin{bmatrix} -\epsilon & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & -\epsilon \end{bmatrix}$

- **Interpretation:** This tensor represents a state of pure volumetric (hydrostatic) compression. The material is compressed equally in all three directions. This causes a decrease in volume without any change in shape (distortion).
- **Invariants:**
  - $I_1 = -\epsilon - \epsilon - \epsilon = -3\epsilon$
  - $I_2 = (-\epsilon)(-\epsilon) + (-\epsilon)(-\epsilon) + (-\epsilon)(-\epsilon) - 0^2 - 0^2 - 0^2 = 3\epsilon^2$
  - $I_3 = \det(\boldsymbol{E}) = -\epsilon^3$

(e) $[\boldsymbol{E}] = \begin{bmatrix} 0 & \frac{\gamma}{2} & 0 \\ \frac{\gamma}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- **Interpretation:** This tensor represents a state of **pure shear** in the x_1-x_2 plane. There is no stretching or compression along the coordinate axes. Instead, the off-diagonal elements indicate a distortion where the initially right angle between the x_1 and x_2 axes decreases by an amount γ.
- **Invariants:**
  - $I_1 = 0 + 0 + 0 = 0$
  - $I_2 = (0)(0) + (0)(0) + (0)(0) - (\frac{\gamma}{2})^2 - 0^2 - 0^2 = -\frac{\gamma^2}{4}$
  - $I_3 = \det(\boldsymbol{E}) = 0$

5. In the direct extrusion process, a round billet is placed in a chamber and forced through a die opening by a hydraulic-driven ram in Fig. 3. The extrusion pressure is affected by the die angle, the reduction in cross-section, extrusion speed, billet temperature, and lubrication. In the vicinity of the corner of the die a rectangular block of material is considered, with its axes oriented along an ortho-normal basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. Its dimension along each of the three axes is l. The block is deformed as shown in Fig. 4. The thickness remains unchanged. Determine the deformation gradient tensor $\boldsymbol{F}$. [10]

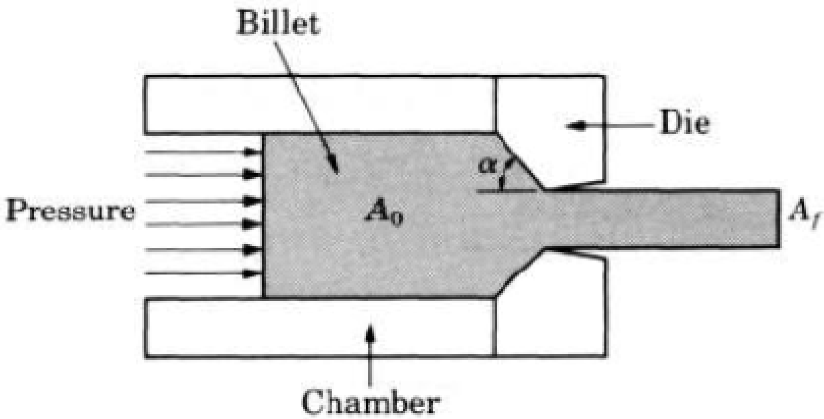

Figure 3: The direct extrusion process

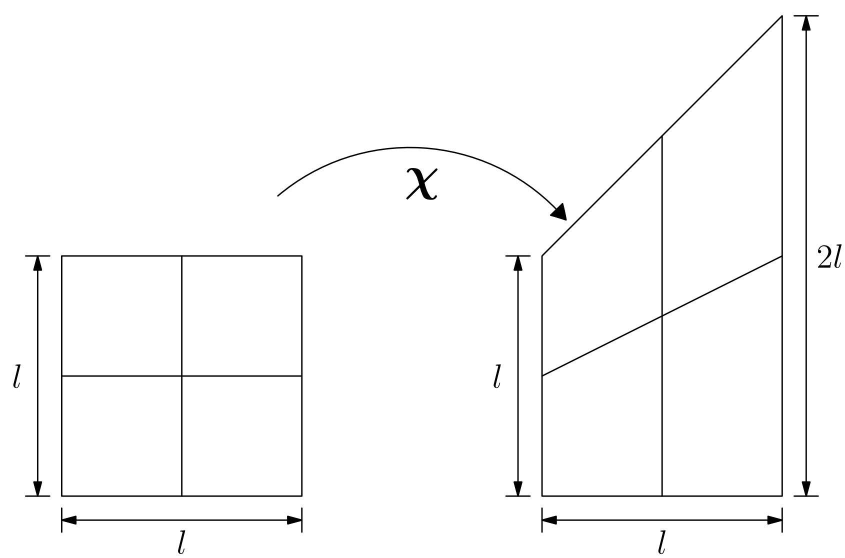

Figure 4: Deformation of body

Solution: Consider the deformed coordinates to be a function of the coordinates x_1 and x_2. Then, a general expression for the displacement components can be written as

$$x'_1(x_1, x_2) = a + b\,x_1 + c\,x_2 + d\,x_1\,x_2$$
$$x'_2(x_1, x_2) = p + q\,x_1 + r\,x_2 + s\,x_1\,x_2$$
$$x'_3(x_1, x_2) = x_3$$

From the deformed shape of the solid block, x_1 at each four corners of the solid block can be written as

$$
\begin{aligned}
x_1'(0,0) &= a = 0 \implies a = 0 \\
x_1'(l,0) &= a + b\,l = l \implies b = 1 \\
x_1'(0,l) &= a + c\,l = 0 \implies c = 0 \\
x_1'(l,l) &= a + b\,l + c\,l + d\,l^2 = l \implies d = 0
\end{aligned}
$$

Similarly, the displacement along x_2 at each four corners of the solid block can be written as

$$
\begin{aligned}
x_2'(0,0) &= p = 0 \implies p = 0 \\
x_2'(l,0) &= p + q\,l = 0 \implies q = 0 \\
x_2'(0,l) &= p + r\,l = l \implies r = 1 \\
x_2'(l,l) &= p + q\,l + r\,l + s\,l^2 = 2l \implies s = \frac{1}{l}
\end{aligned}
$$

Thus the expression for the deformed coordinates can be written as

$$
\begin{aligned}
x_1'(x_1, x_2) &= x_1 \\
x_2'(x_1, x_2) &= x_2 + \frac{1}{l}\, x_1\, x_2 \\
x_3'(x_1, x_2) &= x_3
\end{aligned}
$$

The components of the deformation gradient tensor $\boldsymbol{F}$ can be written as

$$
F_{ij} = \frac{\partial x_i'}{\partial x_j}
$$

In the component form, it can written as

$$
[\boldsymbol{F}] = \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} & \frac{\partial x_1'}{\partial x_3} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} & \frac{\partial x_2'}{\partial x_3} \\ \frac{\partial x_3'}{\partial x_1} & \frac{\partial x_3'}{\partial x_2} & \frac{\partial x_3'}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{l} x_2 & \frac{1}{l} x_1 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

6. Consider the following transformation

$$
\begin{aligned}
x_1^{'} &= x_1 \\
x_2^{'} &= x_3, \\
x_3^{'} &= -x_2,
\end{aligned}
$$

(a) Is the transformation linear?

(b) Calculate the components of the deformation gradient $\boldsymbol{F}$. Compute det $(\boldsymbol{F})$, and $\boldsymbol{F}^{-1}$, where det $(\cdot)$ and $(\cdot)^{-1}$ denote the determinant and the inverse of a second order tensor, respectively.

(c) Study the transformation over a unit cube defined by the coordinates of the corner points (0,0,0) , (1,0,0) , (1,1,0) , (0,1,0), (0,0,1) , (1,0,1) , (1,1,1) , (0,1,1).

Solution: a) Yes, the given transformation is linear.

b) The deformation tensor field $\boldsymbol{F} = \frac{\partial x_i'}{\partial x_j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ can be computed as

$$[\boldsymbol{F}] = \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} \end{bmatrix}$$

Since $x_1^{'} = x_1$, $x_2^1 = x_3$ and $x_3^{'} = -x_2$, $[\boldsymbol{F}]$ can be written as

$$\begin{aligned} [\boldsymbol{F}] &= \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} & \frac{\partial x_1'}{\partial x_3} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} & \frac{\partial x_2'}{\partial x_3} \\ \frac{\partial x_3'}{\partial x_1} & \frac{\partial x_3'}{\partial x_2} & \frac{\partial x_3'}{\partial x_3} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \end{aligned}$$

The determinant of $\boldsymbol{F}$ can be determined as

$$\begin{aligned} \det(\boldsymbol{F}) &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} \\ &= 1 \end{aligned}$$

The inverse of the $\boldsymbol{F}$ can be written as

$$[\boldsymbol{F}^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

d) The mapping between the reference and deformed configurations is given in Table 2,

(x_1, x_2, x_3)	(x'_1, x'_2, x'_3)
(0,0,0)	(0,0,0)
(1,0,0)	(1,0,0)
(1,1,0)	(1,0,-1)
(0,1,0)	(0,0,-1)
(0,0,1)	(0,1,0)
(1,0,1)	(1,1,0)
(1,1,1)	(1,1,-1)
(0,1,1)	(0,1,-1)

Table 2: Reference and deformed coordinates

The reference and deformed configuration for the given mapping function is depicted in Fig. 5 below (Reverse the x_3 and x'_3 axes and change it to the right-handed coordinate system).

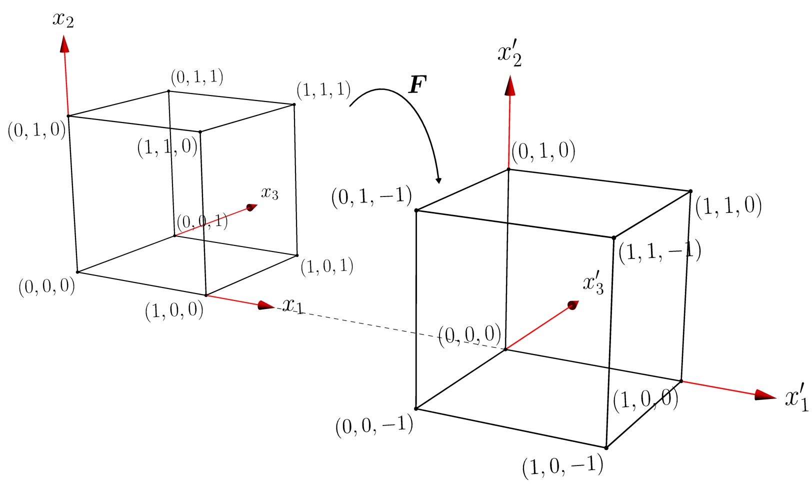

Figure 5: Deformation mapping

7. The following transformation is assigned

$$x_1^{'} = x_1 + \alpha x_2$$
$$x_2^{'} = x_2,$$
$$x_3^{'} = x_3,$$

where α is a generic constant.

(a) Study the deformation of a unit cube defined by the coordinates of the corner points (0,0,0) , (1,0,0) , (1,1,0) , (0,1,0), (0,0,1) , (1,0,1) , (1,1,1) , (0,1,1).

(b) Calculate the components of tensor $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$.

Solution: **Solution**:

(a) The mapping between the reference and deformed configurations is given in Table 3,

(x_1, x_2, x_3)	(x_1', x_2', x_3')
(0,0,0)	(0,0,0)
(1,0,0)	(1,0,0)
(1,1,0)	(1+α,0,-1)
(0,1,0)	(α,1,0)
(0,0,1)	(0,0,1)
(1,0,1)	(1,0,1)
(1,1,1)	(1+α,1,1)
(0,1,1)	(α,1,1)

Table 3: Reference and deformed coordinates

The reference and deformed configuration for the given mapping function is depicted in Fig. 6 below (Reverse the x_3 and x_3' axes and change it to the right-handed coordinate system).

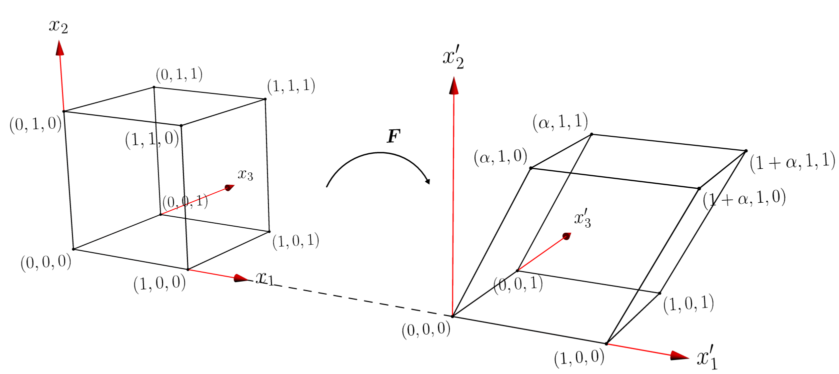

Figure 6: Deformation mapping

b) The deformation tensor field $\boldsymbol{F} = \frac{\partial x_i'}{\partial x_j}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ can be computed as

$$[\boldsymbol{F}] = \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} & \frac{\partial x_1'}{\partial x_3} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} & \frac{\partial x_1'}{\partial x_3} \\ \frac{\partial x_3'}{\partial x_1} & \frac{\partial x_3'}{\partial x_3} & \frac{\partial x_3'}{\partial x_3} \end{bmatrix}$$

Since $x_1^{'} = x_1$, $x_2^{1} = x_3$ and $x_3^{'} = -x_2$, $[\boldsymbol{F}]$ can be written as

$$
\begin{aligned}
[\boldsymbol{F}] &= \begin{bmatrix} \frac{\partial x_1'}{\partial x_1} & \frac{\partial x_1'}{\partial x_2} & \frac{\partial x_1'}{\partial x_3} \\ \frac{\partial x_2'}{\partial x_1} & \frac{\partial x_2'}{\partial x_2} & \frac{\partial x_2'}{\partial x_3} \\ \frac{\partial x_3'}{\partial x_1} & \frac{\partial x_3'}{\partial x_2} & \frac{\partial x_3'}{\partial x_3} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

c) The components of the tensor $\boldsymbol{C} = F^T F$ can be written as

$$
\begin{aligned}
[\boldsymbol{C}] &= [\boldsymbol{F}^T \boldsymbol{F}] \\
&= \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & \alpha & 0 \\ \alpha & \alpha^2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\implies [\boldsymbol{C}] &= \begin{bmatrix} 1 & \alpha & 0 \\ \alpha & \alpha^2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

8. The motion of the body is described as

$$
\begin{aligned}
x_1^{'} &= \frac{1}{\sqrt{2}}(x_1 - x_2 + 5) \\
x_2^{'} &= \frac{1}{\sqrt{2}}(x_1 + x_2 + 3), \\
x_3^{'} &= x_3 + 6,
\end{aligned}
$$

(a) Find the deformation gradient, $\boldsymbol{F}$, for the motion.

(b) Calculate
(i) $\boldsymbol{B} = \boldsymbol{F}\,\boldsymbol{F}^T$,
(ii) $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$,
(iii) $\boldsymbol{E} = (1/2)(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$ and
(iv) $\boldsymbol{E}_s = (1/2)(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$.
Comment on the results.

[10]

Solution: The given motion is described by the equations:

$$x'_1 = \frac{1}{\sqrt{2}}(x_1 - x_2 + 5)$$

$$x'_2 = \frac{1}{\sqrt{2}}(x_1 + x_2 + 3)$$

$$x'_3 = x_3 + 6$$

This represents a transformation from the initial coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ to the final coordinates $\boldsymbol{x'} = (x'_1, x'_2, x'_3)$.

(a) Deformation Gradient, $\boldsymbol{F}$

The **deformation gradient tensor** $\boldsymbol{F}$ is defined by its components $F_{ij} = \frac{\partial x'_i}{\partial x_j}$. We compute the partial derivatives for each component:

$$\begin{aligned}
F_{11} &= \frac{\partial x'_1}{\partial x_1} = \frac{1}{\sqrt{2}} & F_{12} &= \frac{\partial x'_1}{\partial x_2} = -\frac{1}{\sqrt{2}} & F_{13} &= \frac{\partial x'_1}{\partial x_3} = 0 \\
F_{21} &= \frac{\partial x'_2}{\partial x_1} = \frac{1}{\sqrt{2}} & F_{22} &= \frac{\partial x'_2}{\partial x_2} = \frac{1}{\sqrt{2}} & F_{23} &= \frac{\partial x'_2}{\partial x_3} = 0 \\
F_{31} &= \frac{\partial x'_3}{\partial x_1} = 0 & F_{32} &= \frac{\partial x'_3}{\partial x_2} = 0 & F_{33} &= \frac{\partial x'_3}{\partial x_3} = 1
\end{aligned}$$

Assembling these components into matrix form gives:

$$[\boldsymbol{F}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix represents a pure rotation of 45° about the x_3-axis.

(b) Tensor Calculations

(i) $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$

First, we find the transpose of $\boldsymbol{F}$:

$$[\boldsymbol{F}^T] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, we compute the product $\boldsymbol{F}\boldsymbol{F}^T$:

$$[\boldsymbol{B}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (\frac{1}{2} + \frac{1}{2}) & (\frac{1}{2} - \frac{1}{2}) & 0 \\ (\frac{1}{2} - \frac{1}{2}) & (\frac{1}{2} + \frac{1}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

(ii) $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$

Next, we compute the product $\boldsymbol{F}^T\boldsymbol{F}$:

$$[\boldsymbol{C}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (\frac{1}{2}+\frac{1}{2}) & (-\frac{1}{2}+\frac{1}{2}) & 0 \\ (-\frac{1}{2}+\frac{1}{2}) & (\frac{1}{2}+\frac{1}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

(iii) Green-Lagrange Strain Tensor, $\boldsymbol{E} = (1/2)(\boldsymbol{C} - \boldsymbol{I})$

Using the result for $\boldsymbol{C}$ from the previous step:

$$[\boldsymbol{E}] = \frac{1}{2}([\boldsymbol{C}] - [\boldsymbol{I}]) = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boldsymbol{0}$$

(iv) Linear Strain Tensor, $\boldsymbol{E}_s = (1/2)(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$

First, we find the displacement gradient tensor $\nabla\boldsymbol{u} = \boldsymbol{F} - \boldsymbol{I}$:

$$[\nabla\boldsymbol{u}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} - 1 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we compute $\boldsymbol{E}_s = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$:

$$[\boldsymbol{E}_s] = \frac{1}{2}\left(\begin{bmatrix} \frac{1}{\sqrt{2}} - 1 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{2}} - 1 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$[\boldsymbol{E}_s] = \frac{1}{2}\begin{bmatrix} \frac{2}{\sqrt{2}} - 2 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} - 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} - 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \approx \begin{bmatrix} -0.293 & 0 & 0 \\ 0 & -0.293 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Comment on the Results

The overall motion described is a **rigid body motion**, consisting of a rotation and a translation.

9. Consider a linearized strain field $\boldsymbol{E}_s(\boldsymbol{x})$ whose components are given by

$$[\boldsymbol{E}_s(\boldsymbol{x})] = \begin{bmatrix} 3x_1 & 5x_2 + 6x_3 & (x_3)^3 \\ 5x_2 + 6x_3 & 0 & (x_1)^2 + (x_2)^2 \\ (x_3)^3 & (x_1)^2 + (x_2)^2 & \exp(x_1) \end{bmatrix} \times 10^{-6}$$

(a) Find the principal strains and directions at $x_i = (1, 2, 3)$.

(b) What is the normal strain in the direction $n_i = (1, 1, 1)$ at the point $x_i = (2, 2, 0)$?
(c) What is the change in angle between $v_i^{(1)} = (1, 1, 1)$ and $v_i^{(2)} = (2, 1, 3)$ at the point $x_i = (1, 1, 1)$?
(d) What is the volumetric strain at $x_i = (0, 0, 0)$? [10]

Solution:

(a) Given the strain tensor $\boldsymbol{E}_s$ at $x_i = (1, 2, 3)$:

$$[\boldsymbol{E}_s] = \begin{bmatrix} 3 & 28 & 27 \\ 28 & 0 & 5 \\ 27 & 5 & e \end{bmatrix} \times 10^{-6}$$

Let us denote,

$$[\boldsymbol{A}] = \begin{bmatrix} 3 & 28 & 27 \\ 28 & 0 & 5 \\ 27 & 5 & e \end{bmatrix}.$$

Invariants and characteristic polynomial of matrix $[\boldsymbol{A}]$

For a 3×3 matrix $[\boldsymbol{A}]$ the characteristic polynomial is

$$p(\lambda) = \det([\boldsymbol{A}] - \lambda[\boldsymbol{I}]) = \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3,$$

where the principal invariants are

$$\begin{aligned} I_1 &= \operatorname{tr}(\boldsymbol{A}), \\ I_2 &= \tfrac{1}{2}\big((\operatorname{tr}\boldsymbol{A})^2 - \operatorname{tr}(\boldsymbol{A}^2)\big), \\ I_3 &= \det(\boldsymbol{A}). \end{aligned}$$

We compute these for $[\boldsymbol{A}]$:

$$\begin{aligned} I_1 &= \operatorname{tr}([\boldsymbol{A}]) = 3 + 0 + e = 3 + e \approx 5.7183, \\ I_2 &= \tfrac{1}{2}\big(I_1^2 - \operatorname{tr}([\boldsymbol{A}]^2)\big) \approx -1529.8452, \\ I_3 &= \det([\boldsymbol{A}]) \approx 5353.8670. \end{aligned}$$

Thus the cubic for $[\boldsymbol{A}]$ is

$$\lambda^3 - (5.7183)\lambda^2 + (-1529.8452)\lambda - (5353.8670) = 0.$$

Since $[\boldsymbol{E}_s] = 10^{-6} \times [\boldsymbol{A}]$, the eigenvalues of $[\boldsymbol{E}_s]$ are the eigenvalues of $[\boldsymbol{A}]$ scaled by 10^{-6}.

Eigenvalues (principal strains)

Numerically computing the eigenvalues of $[\boldsymbol{A}]$ (ascending order) gives:

$$\lambda_1([\boldsymbol{A}]) \approx -34.32,$$

$$\lambda_2([\boldsymbol{A}]) \approx -3.57,$$

$$\lambda_3([\boldsymbol{A}]) \approx 43.61.$$

Therefore the principal strains (eigenvalues of $[\mathbf{E}]$) are

$$\varepsilon_i = 10^{-6}\,\lambda_i([\boldsymbol{A}]),$$

So, the principal strains are:

$$\begin{aligned} \varepsilon_1 &= -0.3432 \times 10^{-4}, \\ \varepsilon_2 &= -0.0357 \times 10^{-4}, \\ \varepsilon_3 &= 0.4361 \times 10^{-4}. \end{aligned}$$

Eigenvectors (principal directions):

For each eigenvalue $\lambda_i([\boldsymbol{A}])$, we solve

$$([\boldsymbol{A}] - \lambda_i[\boldsymbol{I}])[\boldsymbol{n}] = \mathbf{0}.$$

(Show the detail for solving of the above equation for each eigen value)

For $\lambda_1([\boldsymbol{A}]) = -34.31$, solve $([\boldsymbol{A}] - \lambda_1[\boldsymbol{A}])[\boldsymbol{n}] = \mathbf{0}$. A nontrivial null vector (normalized) is:

$$[\boldsymbol{n}^{(1)}] \approx \begin{bmatrix} -0.7212 \\ 0.5221 \\ 0.4553 \end{bmatrix}.$$

For $\lambda_2([\boldsymbol{A}]) = -3.57$:

$$[\boldsymbol{n}^{(2)}] \approx \begin{bmatrix} -0.0413 \\ -0.6885 \\ 0.7241 \end{bmatrix}.$$

For $\lambda_3([\boldsymbol{A}]) = 43.61$:

$$[\boldsymbol{n}^{(3)}] \approx \begin{bmatrix} 0.6915 \\ 0.5034 \\ 0.5181 \end{bmatrix}.$$

So, the corresponding unit eigenvectors for $[\boldsymbol{E}_s]$ are:

$$[\boldsymbol{n}^{(1)}] = \begin{bmatrix} -0.7212 \\ 0.5221 \\ 0.4553 \end{bmatrix}, \qquad [\boldsymbol{n}^{(2)}] = \begin{bmatrix} -0.0413 \\ -0.6885 \\ 0.7241 \end{bmatrix}, \qquad [\boldsymbol{n}^{(3)}] = \begin{bmatrix} 0.6915 \\ 0.5034 \\ 0.5181 \end{bmatrix}.$$

(b) Normal strain in the direction $[\boldsymbol{n}] = (1, 1, 1)$ at $x_i = (2, 2, 0)$

At this point,

$$[\boldsymbol{E}_s] = 10^{-6} \begin{bmatrix} 6 & 10 & 0 \\ 10 & 0 & 8 \\ 0 & 8 & 7.389 \end{bmatrix}.$$

Normalize the direction vector:

$$[\boldsymbol{n}] = \frac{1}{\sqrt{3}}(1, 1, 1).$$

The normal strain is

$$\varepsilon_n = n_i E_{ij} n_j = [\boldsymbol{n}^T][\boldsymbol{E}_s][\boldsymbol{n}].$$

Compute:

$$[\boldsymbol{E}_s][1, 1, 1]^T = [16, 18, 15.389]^T,$$

$$\varepsilon_n = 10^{-6} \times \frac{1}{3}(16 + 18 + 15.389) = 1.65 \times 10^{-5}.$$

$$\boxed{\varepsilon_n = 1.65 \times 10^{-5}}$$

(c) Change in angle between $[\boldsymbol{v}^{(1)}] = (1, 1, 1)$ and $[\boldsymbol{v}^{(2)}] = (2, 1, 3)$ at $x_i = (1, 1, 1)$

At this point,

$$[\boldsymbol{E}_s] = \begin{bmatrix} 3 & 11 & 1 \\ 11 & 0 & 2 \\ 1 & 2 & 2.718 \end{bmatrix} \times 10^{-6}.$$

Unit vectors:

$$[\boldsymbol{n}^{(1)}] = \frac{1}{\sqrt{3}}(1, 1, 1), \quad [\boldsymbol{n}^{(2)}] = \frac{1}{\sqrt{14}}(2, 1, 3).$$

The change in angle between them is

$$\delta\theta = -2E_{ij}n_i^{(1)}n_j^{(2)}.$$

Compute:

$$\boldsymbol{E}_s\boldsymbol{n}^{(2)} = 10^{-6} \times \frac{1}{\sqrt{14}}\begin{bmatrix} 20 \\ 28 \\ 10.154 \end{bmatrix}, \qquad \boldsymbol{n}^{(1)} \cdot (\boldsymbol{E}_s\boldsymbol{n}^{(2)}) = 10^{-6} \times \frac{58.154}{\sqrt{42}} = 8.97 \times 10^{-6}.$$

Hence

$$\boxed{\delta\theta = -2(8.97 \times 10^{-6}) = -1.79 \times 10^{-5} \text{ rad}}.$$

(d) Volumetric strain at $x_i = (0, 0, 0)$

$$[\boldsymbol{E}_s] = 10^{-6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \varepsilon_v = \mathrm{tr}(\boldsymbol{E}_s) = 10^{-6}(1) = 10^{-6}.$$

$$\boxed{\varepsilon_v = 1.0 \times 10^{-6}}$$

10. The linearized strain at a particular point in a body is given by

$$[\boldsymbol{E}_s] = \begin{bmatrix} 7 & 8 & 0 \\ 8 & 9 & 3 \\ 0 & 3 & 55 \end{bmatrix} \times 10^{-5}$$

in the $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ basis where $\boldsymbol{a} = \frac{1}{\sqrt{3}}(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)$, $\boldsymbol{b} = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 - \boldsymbol{e}_2)$, and $\boldsymbol{c} = \frac{1}{\sqrt{6}}(\boldsymbol{e}_1 + \boldsymbol{e}_2 - 2\boldsymbol{e}_3)$.
(a) Find the max normal and shear strains at this point.
(b) Find the normal strain in the $\boldsymbol{e}_1$ direction at this point.
(c) Find the angle change between the $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ directions at this point. [10]

Solution: The strain tensor is given in the basis $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ as:

$$[\boldsymbol{E}_s]_{\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}} = \begin{bmatrix} 7 & 8 & 0 \\ 8 & 9 & 3 \\ 0 & 3 & 55 \end{bmatrix} \times 10^{-5}.$$

The standard Cartesian basis is $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$.

(a) Find the max normal and shear strains at this point.

The principal strains (eigenvalues) and therefore the maximum normal and shear strains are invariant under a change of basis. We can find them directly from $[\boldsymbol{E}_s'] = 10^5[\boldsymbol{E}_s]$. The principal strains ϵ' are the roots of the characteristic equation $\det([\boldsymbol{E}_s'] - \epsilon'[\boldsymbol{I}]) = 0$, where $\epsilon' = \epsilon \times 10^5$.

$$\det \begin{bmatrix} 7-\epsilon' & 8 & 0 \\ 8 & 9-\epsilon' & 3 \\ 0 & 3 & 55-\epsilon' \end{bmatrix} = 0$$

$$(7-\epsilon')[(9-\epsilon')(55-\epsilon') - 9] - 8[8(55-\epsilon')] = 0$$

This simplifies to the cubic equation:

$$\epsilon'^3 - 71\epsilon'^2 + 870\epsilon' + 118 = 0$$

Solving this equation numerically gives the principal values:

$$\epsilon_1' \approx 55.25, \quad \epsilon_2' \approx 15.88, \quad \epsilon_3' \approx -0.13$$

Hence, the principal strains are $\epsilon_1 = 55.25 \times 10^{-5}$, $\epsilon_2 = 15.88 \times 10^{-5}$, and $\epsilon_3 = -0.13 \times 10^{-5}$.

The maximum normal strain is the largest principal strain:

$$\epsilon_{\text{max}} = \epsilon_1 = 55.25 \times 10^{-5}$$

The maximum shear strain is half the difference between the maximum and minimum principal strains:

$$\tau_{\text{max}} = \frac{\epsilon_{\text{max}} - \epsilon_{\text{min}}}{2} = \frac{\epsilon_1 - \epsilon_3}{2} = \frac{55.25 - (-0.13)}{2} \times 10^{-5} = 27.69 \times 10^{-5}$$

(b) Find the normal strain in the $\boldsymbol{e}_1$ direction at this point.

The normal strain in a direction $\boldsymbol{n}$ is given by $[\epsilon_n] = [\boldsymbol{n}^T][\boldsymbol{E}_s][\boldsymbol{n}]$. To find the strain in the $\boldsymbol{e}_1$ direction, we can express $\boldsymbol{e}_1$ in the $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ basis and use the given strain tensor $[\boldsymbol{E}_s']$.

$$[\boldsymbol{e}_1]_{\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{2} \\ 1/\sqrt{6} \end{bmatrix}$$

The normal strain is $\epsilon_{e_1} = [\boldsymbol{e}_1]^T_{\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}}[\boldsymbol{E}_s'][\boldsymbol{e}_1]_{\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}}$:

$$\epsilon_{e_1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 7 & 8 & 0 \\ 8 & 9 & 3 \\ 0 & 3 & 55 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \times 10^{-5}$$

$$\epsilon_{e_1} = \left(\frac{7}{3} + \frac{9}{2} + \frac{55}{6} + \frac{2 \times 8}{\sqrt{6}} + \frac{2 \times 3}{\sqrt{12}}\right) \times 10^{-5}$$

$$\epsilon_{e_1} = \left(\frac{14 + 27 + 55}{6} + \frac{16}{\sqrt{6}} + \frac{6}{2\sqrt{3}}\right) \times 10^{-5} = \left(16 + \frac{16\sqrt{6}}{6} + \sqrt{3}\right) \times 10^{-5}$$

$$\epsilon_{e_1} = \left(16 + \frac{8\sqrt{6}}{3} + \sqrt{3}\right) \times 10^{-5} \approx 24.26 \times 10^{-5}$$

(c) Find the angle change between the e_1 and e_2 directions at this point.

The change in the angle (engineering shear strain) is $\gamma_{12} = 2[\boldsymbol{e}_1^T][\boldsymbol{E}_s][\boldsymbol{e}_2]$. We express both $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in the $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ basis. We already have $[\boldsymbol{e}_1]_{\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}}$.

$$[\boldsymbol{e}_2]_{\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{2} \\ 1/\sqrt{6} \end{bmatrix}$$

The shear strain component is $E_{12} = [\boldsymbol{e}_1]^T_{\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}}[\boldsymbol{E}_s']([\boldsymbol{e}_2]_{\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}})$:

$$E_{12} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 7 & 8 & 0 \\ 8 & 9 & 3 \\ 0 & 3 & 55 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \times 10^{-5}$$

$$E_{12} = \left(\frac{7}{3} - \frac{9}{2} + \frac{55}{6} + \frac{8}{\sqrt{6}} - \frac{8}{\sqrt{6}} + \frac{3}{\sqrt{12}} - \frac{3}{\sqrt{12}}\right) \times 10^{-5}$$

$$E_{12} = \left(\frac{14 - 27 + 55}{6}\right) \times 10^{-5} = \frac{42}{6} \times 10^{-5} = 7 \times 10^{-5}$$

The angle change is $\gamma_{12} = 2E_{12}$:

$$\gamma_{12} = 2 \times (7 \times 10^{-5}) = 14 \times 10^{-5} \text{ radians}$$