Chapter 2

Stress analysis

2.1 Forces

We are now concerned with *forces* as possible, though not unique, *agents of deformation*. About forces, we admit they are intuitively understood (we do not care here of their real, ultimate physical nature, of no importance for our context; it is sufficient for us to know that forces produce displacements and hence deformations) and that they are *represented by vectors*. There are different types of forces and it is important to understand that the *interior parts of a body* Ω *exchange forces between them*.

The general situation that we examine is that of a body Ω of which we consider a *material part* $\beta \subset \Omega$, with frontier $\partial\beta$ and outward unit normal $\mathbf{n}$, see Fig. 2.1. A material part is a subset of Ω composed by a set of material points, i.e., during deformation, the points remain exactly the same and their quantity is preserved.

Generally speaking, some forces act upon β and they can be of two types:

i. *volume* or *body* forces: these forces are directly applied to the material points in β for the simple reason that they exist. They are *remote* forces, result of the presence of one or more force fields: gravitational, electrostatic, magnetic etc. As such, these forces normally depend upon the position and they admit a *density*:

   - a *volume* density $\mathbf{b}=\mathbf{b}(p)$, or
   - a *mass* density $\mathbf{r}=\mathbf{r}(p) \rightarrow \mathbf{b}=\rho\ \mathbf{r}$,

   with ρ the volume mass (density of the matter). These forces are extensive quantities, so the total remote force acting upon β is

$$\mathbf{F}_\beta = \int_\beta \mathbf{b}\ dv = \int_\beta \rho\ \mathbf{r}\ dv; \tag{2.1}$$

ii. *surface* forces: these are the forces that Ω exchange with the environment, by contact through its boundary $\partial\Omega$, like pressure or thrusts exerted by some devices or other bodies, or the forces that β exchange with the rest of Ω still by contact through its frontier $\partial\beta$, called also *interior* forces; these last are the direct consequence of the same idea of continuum.

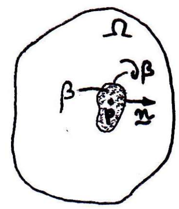

Figure 2.1: Material part.

The surface forces too admit a density, in this case of course a *surface density*, $\mathbf{t}$:

$$\mathbf{f}_\beta = \int_{\partial\beta} \mathbf{t}\ ds. \tag{2.2}$$

The density of surface forces $\mathbf{t}$ is called *traction* or *stress vector*. About $\mathbf{t}$, we admit the *Cauchy's postulate*: $\mathbf{t}$ is a function of the *actual* position and of the outward normal to $\partial\beta$:

$$\mathbf{t} = \mathbf{t}(p, \mathbf{n}). \tag{2.3}$$

The above statements deserve some remarks:

- there exist also attractive body forces that interior parts of a same body mutually exchange; such forces are neglected in the classical theory, but can be of course of an extreme importance in other fields, like astronomy and geophysics;
- the volume forces and the surface forces acting upon the boundary $\partial\Omega$ are *external* forces; they are considered to be *known*;
- the interior forces are *unknown* and to determine these last once the external forces known is the *major problem of continuum mechanics*;
- the Cauchy's postulate is a strong assumption: two different surfaces $\partial\beta_1$ and $\partial\beta_2$ sharing in p the same normal $\mathbf{n}$, share also the same traction $\mathbf{t}$; in particular, $\mathbf{t}$ does not depend upon the curvature of the surfaces in p;
- considering that through any point $p \in \partial\beta$ the matter exchanges only interior forces and not also interior couples is an implicit assumption that defines a class of materials, the so-called *classical continuum bodies à la Cauchy*; several classical materials can be well represented by this model, e.g. metallic alloys, wood, concrete etc, but not other ones, called *polar bodies*, like some polymers, for which the introduction of surface couples exchanged by interior parts of the body is necessary for a satisfactory description of its behavior; in this text, we will refer only to classical Cauchy bodies.

2.2 The Cauchy's theorem

The Cauchy's postulate does not specify in which way $\mathbf{t}$ is a function of $\mathbf{n}$. This is done by the

Theorem (Cauchy's theorem on stress)**.** *Traction* $\mathbf{t}$ *is a linear function of* $\mathbf{n}$*, i.e. it exists a second-rank tensor* $\boldsymbol{\sigma}$*, the Cauchy's stress tensor, such that*

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}. \tag{2.4}$$

Proof. Let us see the classical proof based on the use of the so-called *tetrahedron of Cauchy.* We consider at a point $p \in \Omega$ a tetrahedron like in Fig. 2.2, where p is the axes origin and the fourth face, whose normal is $\mathbf{n}$, is inclined with respect to the three faces passing by the axes. Be δ the distance of p from the inclined face. For δ sufficiently

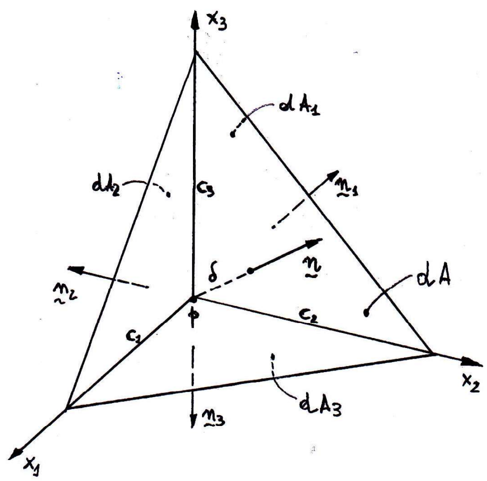

Figure 2.2: The tetrahedron of Cauchy.

small, all the tetrahedron is in Ω; be dA the area of the inclined face, with outward unit normal $\mathbf{n}$, while dA_i is the area of the face orthogonal to axis x_i, of outward unit normal $\mathbf{n}_i = -\mathbf{e}_i$. Be $\mathbf{t} = (t_1, t_2, t_3)$ the traction on the inclined surface and $\mathbf{b}$ the body force.

About the area of the surfaces of the tetrahedron, we know that[1]

$$dA_i = dA\ \mathbf{n} \cdot \mathbf{e}_i \quad \forall i = 1, 2, 3, \tag{2.5}$$

[1] The result in eq. (2.5) is known as *theorem of the cosine for the surfaces.* To prove it, we name c_i the length of the side of the tetrahedron along the axis x_i; then

$$dA_i = \frac{1}{2} c_j c_k,\ \ i, j, k = 1, 2, 3,\ \ i \neq j \neq k,$$

and

$$dA = \frac{1}{2} |(-c_1, c_2, 0) \times (-c_1, 0, c_3)| = \frac{1}{2} \sqrt{c_1^2 c_2^2 + c_2^2 c_3^2 + c_3^2 c_1^2}.$$

The normal $\mathbf{n}$ to dA is given by

$$\mathbf{n} = \frac{(-c_1, c_2, 0) \times (-c_1, 0, c_3)}{|(-c_1, c_2, 0) \times (-c_1, 0, c_3)|} = \frac{1}{\sqrt{c_1^2 c_2^2 + c_2^2 c_3^2 + c_3^2 c_1^2}} (c_2 c_3, c_1 c_3, c_1 c_2)$$

so we get

$$\mathbf{n} \cdot \mathbf{e}_i = \frac{dA_i}{dA},$$

i.e. eq. (2.5).

while the volume of the tetrahedron is

$$dV = \frac{1}{3}\delta\ dA. \tag{2.6}$$

We write now the balance of the forces acting upon the tetrahedron, using the *Euler's axiom*: when a body Ω is in equilibrium, then all of its material parts β are in equilibrium. Then, imaging the tetrahedron as a separated part of Ω, it will be in equilibrium under the action of the body forces and of the surface (contact) forces that it exchanges with the rest of Ω through its four surfaces. This gives the balance equation:

$$\mathbf{t}\ dA + \mathbf{t}_i dA_i + \mathbf{b}\ dV = \mathbf{o}, \tag{2.7}$$

and, for the above formulae for the areas and volume we get, after dividing by dA,

$$\mathbf{t} + \mathbf{t}_i\ \mathbf{n}\cdot\mathbf{e}_i + \frac{1}{3}\mathbf{b}\ \delta = \mathbf{o}. \tag{2.8}$$

Hence, when $\delta \to 0$, the point p tends to the surface dA whose normal is $\mathbf{n}$ and the body forces vanish; because $\mathbf{n}_i = -\mathbf{e}_i$, we obtain

$$\mathbf{t} = -\mathbf{t}_i\ \mathbf{n}\cdot\mathbf{e}_i = -(\mathbf{t}_i\otimes\mathbf{e}_i)\mathbf{n} = (\mathbf{t}_i\otimes\mathbf{n}_i)\mathbf{n}. \tag{2.9}$$

We put

$$\boldsymbol{\sigma} = \mathbf{t}_i\otimes\mathbf{n}_i, \tag{2.10}$$

the *Cauchy's stress tensor in* p, and finally

$$\mathbf{t} = \boldsymbol{\sigma}\ \mathbf{n}. \tag{2.11}$$

□

From eq. (2.9) we have also

$$\sigma_{ij} = \mathbf{e}_i\cdot(\mathbf{t}_k\otimes\mathbf{n}_k)\mathbf{e}_j = \mathbf{t}_k\cdot\mathbf{e}_i\ \mathbf{n}_k\cdot\mathbf{e}_j = (\mathbf{t}_k)_i(\mathbf{n}_k)_j. \tag{2.12}$$

Of course, if we take $\mathbf{n} = \mathbf{e}_i$, then $\mathbf{t} = \mathbf{t}_i$, as it must be. Just as for any other second rank tensor, given a base $e = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\boldsymbol{\sigma} = \sigma_{ij}\ \mathbf{e}_i\otimes\mathbf{e}_j, \tag{2.13}$$

with

$$\sigma_{ij} = \mathbf{e}_i\cdot\boldsymbol{\sigma}\ \mathbf{e}_j. \tag{2.14}$$

It is important to remark that $\boldsymbol{\sigma}$ is a function of the place and time, not of $\mathbf{n}$:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(p,t). \tag{2.15}$$

As already done, the dependence upon time, always existing, is left tacitly understood in the equations.

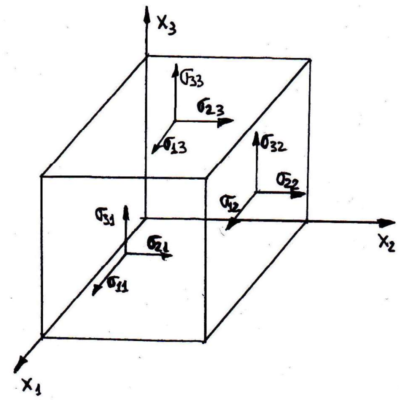

Figure 2.3: The components σ_{ij}.

2.3 Stress components

Let us apply the Cauchy's theorem to surface elements whose normal is parallel to one of the axes, $\mathbf{n} = \mathbf{e}_k$:

$$\mathbf{t}^{(k)} = \sigma_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_k = \sigma_{ij}\ \delta_{jk}\ \mathbf{e}_i = \sigma_{ik}\ \mathbf{e}_i, \tag{2.16}$$

so

$$\mathbf{t}^{(k)} = \sigma_{ik}\ \mathbf{e}_i = (\sigma_{1k}, \sigma_{2k}, \sigma_{3k}); \tag{2.17}$$

this result shows that the k-th column of the matrix representing $\boldsymbol{\sigma}$ in the base $e = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is composed by the Cartesian components of the traction acting upon the surface whose normal is $\mathbf{e}_k$. Graphically, the situation is depicted in Fig. 2.3. We remark the position of the indexes: the first one gives the direction of the component of the traction acting upon a surface whose normal is the axis indicated by the second index (e.g. σ_{13} is the component along x_1 of the traction acting upon a surface whose normal is $\mathbf{e}_3$).

To remark that the above nomenclature comes directly from the mere application of the equations; some authors chose to swap the indexes: in σ_{ij}, i is the direction of the normal to the surface upon which the traction acts, while j is the direction of the component σ_{ij} of the traction. This is not so important, because $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$, as we will see below.

Looking at Fig. 2.3, it is clear why:

- the components with equal indexes σ_{ii} are called *normal stresses*: they give the component of the traction upon a surface that is normal, i.e. perpendicular, to the same surface; because in eq. (2.4) $\mathbf{n}$ is the *outward* unit normal, a normal stress σ_{ii} is positive if it is a tension, negative if a compression; normal stresses form the diagonal of the matrix representing $\boldsymbol{\sigma}$;
- the components with different indexes σ_{ij}, $i \neq j$ are called *shear stresses*: they give a component of the traction upon a surface orthogonal to an axis that is tangential to the same surface; they are the out-of-diagonal components of the matrix representing $\boldsymbol{\sigma}$.

More generally, for each element of surface of unit normal $\mathbf{n}$, the traction $\mathbf{t} = \boldsymbol{\sigma}\ \mathbf{n}$ can be decomposed into two mutually orthogonal vectors, see Fig. 2.4:

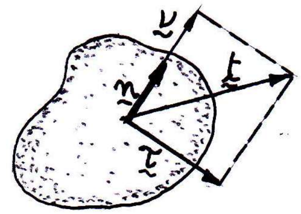

Figure 2.4: Normal, $\boldsymbol{\nu}$, and tangential, $\boldsymbol{\tau}$, stresses.

- the *normal stress* $\boldsymbol{\nu}$:

$$\boldsymbol{\nu} = (\mathbf{t} \cdot \mathbf{n})\mathbf{n} = (\mathbf{n} \otimes \mathbf{n})\mathbf{t} = (\mathbf{n} \otimes \mathbf{n})\boldsymbol{\sigma}\ \mathbf{n}; \tag{2.18}$$

- the *tangential stress* $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \mathbf{t} - \boldsymbol{\nu} = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{t} = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\boldsymbol{\sigma}\ \mathbf{n}. \tag{2.19}$$

2.4 Balance equations

We can now write the balance equations for any part β of Ω. The Euler's axioms stipulate that $\forall\beta \subset \Omega$, the force resultant and the moment resultant are null. Let us start examining first the force resultant:

$$\int_\beta \mathbf{b}\ dv + \int_{\partial\beta} \mathbf{t}\ ds = \int_\beta \rho\ \ddot{p}\ dv \quad \forall\beta \subset \Omega. \tag{2.20}$$

Applying the Cauchy's theorem we get

$$\int_\beta \mathbf{b} - \rho\ \ddot{p}\ dv + \int_{\partial\beta} \boldsymbol{\sigma}\ \mathbf{n}\ ds = \mathbf{o} \quad \forall\beta \subset \Omega, \tag{2.21}$$

and for the tensor form of the Gauss theorem

$$\int_\beta (\mathbf{b} - \rho\ \ddot{p} + \mathrm{div}\boldsymbol{\sigma})\ dv = \mathbf{o} \quad \forall\beta \subset \Omega. \tag{2.22}$$

The only possibility for this integral to be null $\forall\beta \subset \Omega$ is the integrand to be identically null:

$$\mathbf{b} + \mathrm{div}\boldsymbol{\sigma} = \rho\ \ddot{p} \quad \forall p \in \Omega. \tag{2.23}$$

These are the *Cauchy-Poisson equations of motion for classical continuum bodies*. They generalize to each point of a deformable body the second principle of dynamics of Newton. In case of equilibrium, $\ddot{p} = \mathbf{o}$ and we obtain the *equilibrium equations*

$$\mathbf{b} + \mathrm{div}\boldsymbol{\sigma} = \mathbf{o} \quad \forall p \in \Omega. \tag{2.24}$$

In terms of components, the above equations read like

$$b_i + \sigma_{ij,j} = \rho\, \ddot{p}_i, \quad i,j = 1,2,3. \tag{2.25}$$

Let us now turn the attention on the moment resultant on β:

$$\int_\beta (p-o)\times \mathbf{b}\ dv + \int_{\partial\beta} (p-o)\times \mathbf{t}\ ds = \int_\beta \rho(p-o)\times \ddot{p}\ dv \ \ \forall\beta\subset\Omega. \tag{2.26}$$

Still using the Cauchy's theorem we get

$$\int_\beta (p-o)\times (\mathbf{b}-\rho\,\ddot{p})\ dv + \int_{\partial\beta} (p-o)\times (\boldsymbol{\sigma}\mathbf{n})\ ds = \mathbf{o} \ \ \forall\beta\subset\Omega \tag{2.27}$$

and introducing, first, the axial tensor $\mathbf{W}$ of $(p-o)$

$$\int_\beta \mathbf{W}(\mathbf{b}-\rho\,\ddot{p})\ dv + \int_{\partial\beta} \mathbf{W}(\boldsymbol{\sigma}\mathbf{n})\ ds = \mathbf{o} \ \ \forall\beta\subset\Omega, \tag{2.28}$$

then the motion equation and the Gauss theorem, we obtain

$$\int_\beta \mathrm{div}(\mathbf{W}\boldsymbol{\sigma}) - \mathbf{W}\mathrm{div}\boldsymbol{\sigma}\ dv = \mathbf{o} \ \ \forall\beta\subset\Omega, \tag{2.29}$$

that, for being true $\forall\beta\subset\Omega$, gives the condition

$$\mathrm{div}(\mathbf{W}\boldsymbol{\sigma}) = \mathbf{W}\mathrm{div}\boldsymbol{\sigma} \ \ \forall p\in\Omega. \tag{2.30}$$

We now develop:

$$\begin{aligned}\mathrm{div}(\mathbf{W}\boldsymbol{\sigma}) &= (\mathbf{W}\boldsymbol{\sigma})_{ij,j}\ \mathbf{e}_i = (W_{ik}\ \sigma_{kj})_{,j}\ \mathbf{e}_i\\ &= W_{ik,j}\ \sigma_{kj}\ \mathbf{e}_i + W_{ik}\ \sigma_{kj,j}\ \mathbf{e}_i = W_{ik,j}\ \sigma_{kj}\ \mathbf{e}_i + \mathbf{W}\ \mathrm{div}\boldsymbol{\sigma},\end{aligned} \tag{2.31}$$

and injecting this result into eq. (2.30) gives

$$W_{ik,j}\sigma_{kj} = 0\ \forall i = 1,2,3. \tag{2.32}$$

For a generic point $p = (p_1, p_2, p_3)\in\Omega$,

$$\mathbf{W} = \left[\begin{array}{ccc} 0 & -p_3 & p_2\\ p_3 & 0 & -p_1\\ -p_2 & p_1 & 0\end{array}\right], \tag{2.33}$$

so that $W_{12,3} = -1, W_{13,2} = 1$ etc. Injecting these results into eq. (2.32) for $i = 1,2,3$ gives

$$\begin{array}{l} i=1\ \rightarrow\ \sigma_{23}=\sigma_{32},\\ i=2\ \rightarrow\ \sigma_{13}=\sigma_{31},\\ i=3\ \rightarrow\ \sigma_{12}=\sigma_{21},\end{array} \quad \Rightarrow \quad \boldsymbol{\sigma}=\boldsymbol{\sigma}^\top. \tag{2.34}$$

So, for classical continuum bodies, the balance of the couples corresponds to the symmetry of $\boldsymbol{\sigma}$.

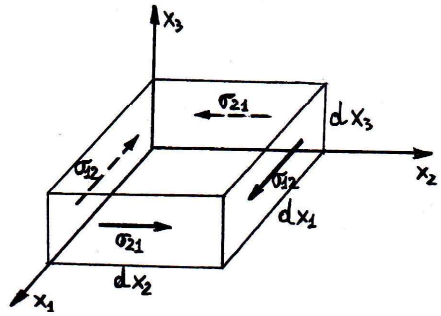

Figure 2.5: Reciprocity of the shear stresses.

There are at least two other ways to prove the *reciprocity of the shear stresses*, i.e. the symmetry of $\boldsymbol{\sigma}$, both of them more *mechanical* then the previous one. In the first one, we consider a parallelepiped with the faces parallel to the axes, like in Fig. 2.5. If, e.g., we focus on the balance of the torque around axis x_3, body forces and tractions on the horizontal faces give higher order contributions and can be discarded, so we have

$$(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_1} dx_1) dx_1 dx_2 dx_3 = (\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_2} dx_2) dx_1 dx_2 dx_3, \tag{2.35}$$

and neglecting higher order terms we get $\sigma_{12} = \sigma_{21}$; in a similar way we obtain also $\sigma_{13} = \sigma_{31}$ and $\sigma_{23} = \sigma_{32}$.

The other method to prove the symmetry of $\boldsymbol{\sigma}$ is based upon the use of the classical *Principle of Virtual Displacements*[2]: for each possible infinitesimal rigid displacement field $\mathbf{w}$, the balance equations are satisfied if and only if

$$\int_{\partial\beta} \mathbf{t} \cdot \mathbf{w}\; ds + \int_{\beta} (\mathbf{b} - \rho\, \ddot{p}) \cdot \mathbf{w}\; dv = 0. \tag{2.36}$$

In fact, using the Cauchy's and Gauss's theorems we have

$$\begin{aligned} \int_{\partial\beta} \mathbf{t} \cdot \mathbf{w}\; ds &= \int_{\partial\beta} \boldsymbol{\sigma}\, \mathbf{n} \cdot \mathbf{w}\; ds = \int_{\partial\beta} \boldsymbol{\sigma}^\top \mathbf{w} \cdot \mathbf{n}\; ds \\ &= \int_{\beta} \operatorname{div}(\boldsymbol{\sigma}^\top \mathbf{w}) dv = \int_{\beta} (\mathbf{w} \cdot \operatorname{div}\boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \nabla \mathbf{w}) dv. \end{aligned} \tag{2.37}$$

Using the equation of mouvement (2.23) for expressing div$\boldsymbol{\sigma}$, we have

$$\int_{\partial\beta} \mathbf{t} \cdot \mathbf{w}\; ds + \int_{\beta} (\mathbf{b} - \rho\, \ddot{p}) \cdot \mathbf{w}\; dv = \int_{\beta} \boldsymbol{\sigma} \cdot \nabla \mathbf{w}\; dv \quad \forall \beta \subset \Omega. \tag{2.38}$$

The left-hand member is null for a body at equilibrium, for the Principle of Virtual Displacements; so, because the above equation must be satisfied $\forall \beta \subset \Omega$, we obtain the condition

$$\boldsymbol{\sigma} \cdot \nabla \mathbf{w} = 0 \quad \forall p \in \Omega, \tag{2.39}$$

[2] The Principle of Virtual Displacements as used here is just the principle as known, usually, for rigid bodies mechanics; the key point for the principle in this form is the virtual displacement field to be infinitesimal and rigid; in such a circumstance, as used here, the principle is exactly the same used in classical rigid mechanics. A more general form of the Principle of Virtual Displacements exists for deformable bodies, it is presented in Section 2.7.

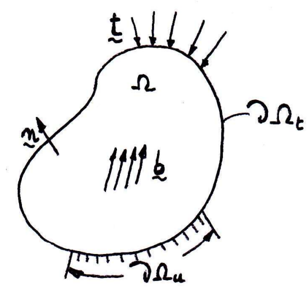

Figure 2.6: Scheme of the boundary conditions.

to be satisfied $\forall \mathbf{w}$ rigid and infinitesimal $\Rightarrow \quad \nabla\mathbf{w} = -\nabla^\top\mathbf{w}$ [3], so that $\boldsymbol{\sigma}$ is necessarily symmetric[4].

2.5 Boundary conditions

The balance equations (2.23) must be completed by adequate boundary conditions. To this purpose, we consider the general situation depicted in Fig. (2.6):

- the whole boundary $\partial\Omega$ is composed of two complementary parts, $\partial\Omega_u$ and $\partial\Omega_t$, such that
$$\partial\Omega = \partial\Omega_u \cup \partial\Omega_t, \quad \partial\Omega_u \cap \partial\Omega_t = \emptyset; \tag{2.40}$$
- on $\partial\Omega_u$ the displacement vector is known:
$$\mathbf{u} = \mathbf{u}_0, \tag{2.41}$$
typically $\mathbf{u}_0 = \mathbf{o}$; these are the *kinematical boundary conditions*;
- on $\partial\Omega_t$ the traction vector is known:
$$\mathbf{t} = \boldsymbol{\sigma}\ \mathbf{n} = \mathbf{t}_0; \tag{2.42}$$

[3]For any rigid displacement, $\boldsymbol{\varepsilon} = \mathbf{O}$, which implies $\nabla\mathbf{w} = -\nabla^\top\mathbf{w}$.

[4]This is a consequence of the following

Theorem. *A tensor* $\mathbf{L}$ *is orthogonal to any skew tensor* $\mathbf{W} \iff \mathbf{L} = \mathbf{L}^\top$.

Proof. We prove first that if $\mathbf{L}$ is symmetric and $\mathbf{W}$ skew, then they are necessarily orthogonal:
$$\mathbf{W}\cdot\mathbf{L} = \mathrm{tr}(\mathbf{W}^\top\mathbf{L}) = -\mathrm{tr}(\mathbf{W}\mathbf{L}) = -\mathrm{tr}(\mathbf{L}\mathbf{W}) = -\mathrm{tr}(\mathbf{L}^\top\mathbf{W}) = -\mathbf{L}\cdot\mathbf{W} = -\mathbf{W}\cdot\mathbf{L} \iff \mathbf{W}\cdot\mathbf{L} = 0.$$
To complete the proof, we must prove that if $\mathbf{L}\cdot\mathbf{W} = 0\ \forall\mathbf{W}:\ \mathbf{W} = -\mathbf{W}^\top$, then $\mathbf{L} = \mathbf{L}^\top$; to this end, let us suppose that $\mathbf{L} \neq \mathbf{L}^\top$ and decompose $\mathbf{L}$ in its symmetric and skew parts:
$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2,\ \ \mathbf{L}_1 = \frac{\mathbf{L}+\mathbf{L}^\top}{2},\ \mathbf{L}_1 = \mathbf{L}_1^\top,\ \ \mathbf{L}_2 = \frac{\mathbf{L}-\mathbf{L}^\top}{2},\ \mathbf{L}_2 = -\mathbf{L}_2^\top.$$
So,
$$\mathbf{L}\cdot\mathbf{W} = \mathbf{L}_1\cdot\mathbf{W} + \mathbf{L}_2\cdot\mathbf{W} = 0;$$
the first term on the right-hand side is null, as we have just proved, because $\mathbf{L}_1$ is symmetric and $\mathbf{W}$ is skew; so, it must be $\mathbf{L}_2\cdot\mathbf{W} = 0\ \forall\mathbf{W} = -\mathbf{W}^\top$. Because $\mathbf{L}_2$ is skew, we can chose $\mathbf{W} = \mathbf{L}_2$; then, for the same definition of scalar product, we get $\mathbf{L}_2\cdot\mathbf{L}_2 = 0 \iff \mathbf{L}_2 = \mathbf{O}$, which proves the theorem. □

these are the *natural boundary conditions*.

2.6 Principal stresses

The symmetry of $\boldsymbol{\sigma}$, just as for $\boldsymbol{\varepsilon}$, brings, through the spectral theorem, the existence of three real eigenvalues, say $\sigma_1 \geq \sigma_2 \geq \sigma_3$: the *principal stresses*. The eigenvectors of $\boldsymbol{\sigma}$ form a base, say $v = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, the base of the *principal directions of stress*; in the base $v, \boldsymbol{\sigma}$ is diagonal:

$$\boldsymbol{\sigma} = \sigma_i\ \mathbf{v}_i \otimes \mathbf{v}_i,\ \ i = 1, 2, 3, \quad \to\ \boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}. \tag{2.43}$$

It is then clear, using the Cauchy's theorem, that the traction on surfaces orthogonal to the principal directions of stress v_i is composed uniquely by a normal stress: the principal directions are normal to surfaces where the shear stress is null.

The envelop, throughout Ω, of the principal directions of the stress form a family of lines called the *isostatic lines*, that have the following property: along an isostatic line, the matter is simply subjected to tension or compression, not to shear too. The isostatic lines are hence the *lines of best use of the matter*: an effective structure is a structure where the matter follows as much as possible the isostatic lines. In Nature, the selection has produced a great amount of exemples where the matter tends to be distributed along the isostatic lines, e.g. in the bones, trees etc.

For the property of maximality of the eigenvalues, see Exercise 6, Chapter 1, σ_1 is the highest value of the normal stress, σ_3 the minimal value and σ_2 an intermediate value (a local extremal).

2.7 The Principle of Virtual Displacements

We give here a form of the Principle of Virtual Displacements more general than that used for rigid bodies mechanics: the only requirement of the virtual displacement is to be *compatible*, not necessarily rigid, which is just a particular case.

A virtual displacement field $\delta\mathbf{u}$ on Ω is said to be *compatible* if:

i. $\delta\mathbf{u}$ is an infinitesimal, regular, time independent field of displacement;

ii. it satisfies to the boundary conditions on $\partial\Omega_u : \delta\mathbf{u}|_{\partial\Omega_u} = \mathbf{o}$;

iii. it satisfies to the geometric relations with $\boldsymbol{\varepsilon}$: $\delta\mathbf{u}$ defines a virtual strain field $\delta\boldsymbol{\varepsilon}$ as

$$\delta\boldsymbol{\varepsilon} = \frac{\nabla\delta\mathbf{u} + \nabla^\top\delta\mathbf{u}}{2}. \tag{2.44}$$

We further assume that the body is in equilibrium, which implies that the equilibrium equation is satisfied everywhere in Ω :

$$\mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \mathbf{o}. \tag{2.45}$$

Then, the work done by the surface tractions $\mathbf{t}$ applied to Ω on $\partial\Omega_t$ for the virtual displacement $\delta\mathbf{u}$ is, for the theorem of Cauchy,

$$\int_{\partial\Omega} \mathbf{t} \cdot \delta\mathbf{u}\ ds = \int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n} \cdot \delta\mathbf{u}\ ds; \tag{2.46}$$

so, using successively the theorem of Gauss, the identity

$$\mathrm{div}(\mathbf{S}^\top \mathbf{v}) = \mathbf{S} \cdot \nabla\mathbf{v} + \mathbf{v} \cdot \mathrm{div}\mathbf{S} \quad \forall\mathbf{v} \in \mathcal{V} \text{ and } \forall\mathbf{S} \in Lin(\mathcal{V}), \tag{2.47}$$

the equilibrium equation (2.45) and the fact that[5]

$$\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot \nabla\mathbf{u}, \tag{2.48}$$

we get

$$\begin{aligned}
\int_{\partial\Omega} \mathbf{t} \cdot \delta\mathbf{u}\ ds &= \int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n} \cdot \delta\mathbf{u}\ ds = \int_{\partial\Omega} \boldsymbol{\sigma}^\top \delta\mathbf{u} \cdot \mathbf{n}\ ds \\
&= \int_{\Omega} \mathrm{div}(\boldsymbol{\sigma}^\top \delta\mathbf{u}) d\omega = \int_{\Omega} (\boldsymbol{\sigma} \cdot \nabla\delta\mathbf{u} + \delta\mathbf{u} \cdot \mathrm{div}\boldsymbol{\sigma}) d\omega \\
&= \int_{\Omega} \boldsymbol{\sigma} \cdot \nabla\delta\mathbf{u}\ d\omega - \int_{\Omega} \mathbf{b} \cdot \delta\mathbf{u}\ d\omega \\
&= \int_{\Omega} \boldsymbol{\sigma} \cdot \delta\boldsymbol{\varepsilon}\ d\omega - \int_{\Omega} \mathbf{b} \cdot \delta\mathbf{u}\ d\omega,
\end{aligned} \tag{2.49}$$

and finally

$$\int_{\Omega} \boldsymbol{\sigma} \cdot \delta\boldsymbol{\varepsilon}\ d\omega = \int_{\Omega} \mathbf{b} \cdot \delta\mathbf{u}\ d\omega + \int_{\partial\Omega} \mathbf{t} \cdot \delta\mathbf{u}\ ds \quad \forall \text{ compatible } \delta\mathbf{u}. \tag{2.50}$$

The theorem so proved is the *Principle of Virtual Displacements* (PVD), valid for any kind of deformable body[6]; as the same proof of the theorem shows, it is completely equivalent to the equilibrium equations. For its importance, and for a matter of tradition, this theorem is often called a principle, like we do.

The PVD states that, at the equilibrium, the virtual work of the internal forces, the left-hand side term, equals the virtual work of the external forces, the right-hand side term,

---

[5]

$$\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot \frac{\nabla\mathbf{u} + \nabla^\top\mathbf{u}}{2} = \frac{1}{2}(\boldsymbol{\sigma} \cdot \nabla\mathbf{u} + \boldsymbol{\sigma} \cdot \nabla^\top\mathbf{u}) = \frac{1}{2}(\boldsymbol{\sigma} \cdot \nabla\mathbf{u} + \boldsymbol{\sigma}^\top \cdot \nabla^\top\mathbf{u})$$

because $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$; but, generally speaking, for any two tensors $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \cdot \mathbf{B} = \mathbf{A}^\top \cdot \mathbf{B}^\top$ so that

$$\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{\sigma} \cdot \nabla\mathbf{u} + \boldsymbol{\sigma} \cdot \nabla\mathbf{u}) = \boldsymbol{\sigma} \cdot \nabla\mathbf{u}.$$

[6]In fact, the PVD is completely general because no constitutive law has been used for proving it; in particular, its use is not exclusively reserved to elastic bodies, see Chapter 3.

not only for the real displacement field, but more generally *for any compatible virtual displacement field*, i.e. for any infinitesimal displacement field that satisfies the kinematical boundary conditions and that is linked to the virtual strain field by eq. (2.44).

The PVD is hence the principle of equilibrium and it has several and remarkable applications, like in the resolution of hyperstatic structures, see Chapter 5, or in the proof of the elasticity theorems, Section 3.10.

We can write the PVD as

$$\delta U_\Omega = \delta W_b + \delta W_t, \tag{2.51}$$

with

$$\begin{aligned}\delta U_\Omega &= \int_\beta \boldsymbol{\sigma} \cdot \delta\boldsymbol{\varepsilon}\; dv,\\ \delta W_b &= \int_\beta \mathbf{b} \cdot \delta\mathbf{u}\; dv,\\ \delta W_t &= \int_{\partial\beta} \mathbf{t} \cdot \delta\mathbf{u}\; ds,\end{aligned} \tag{2.52}$$

respectively the virtual work done by the internal actions for the deformation $\delta\boldsymbol{\varepsilon}$, the virtual work of the body forces on Ω and that of the surface tractions on $\partial\Omega_t$. If in the PVD we take, as virtual displacement field, the real one, which is obviously compatible, then δW_b and δW_t are real works and δU_Ω represents the true work done by the internal actions. This last can be interpreted also as the variation of the energy stored in Ω, as consequence of the deformation of the body, for the variation of deformation $\delta\boldsymbol{\varepsilon}$. Its volume density is

$$\delta U = \boldsymbol{\sigma} \cdot \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot \nabla\delta\mathbf{u}. \tag{2.53}$$

2.8 Exercises

1. Consider the *plane stress state*

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

   i. find the normal, $\boldsymbol{\nu}$, and tangential, $\boldsymbol{\tau}$, stress on a surface of normal $\mathbf{n} = (\cos\theta, \sin\theta, 0)$;

   ii. show that, in the plane $\nu - \tau$, the points representing the stress state belong to a circle (the *Mohr's circle*);

   iii. which is the physical meaning of the centre, radius and intersection with the axes of the Mohr's circle?

   iv. generalize the Mohr's circle to 3D stress states.

2. Show that $\boldsymbol{\sigma}$ is uniquely determined by the system of applied forces.

3. Assume that $\mathbf{t} = \mathbf{o}$ on $\partial\Omega$; show that $\forall p \in \partial\Omega$ the traction $\mathbf{t}$ on each plane orthogonal to $\partial\Omega$ is tangent to $\partial\Omega$.

4. Study the following cases of elementary stress states:

   a) *hydrostatic stress*: it is that of a fluid at rest, that can exert only a compressive normal stress;

   b) *pure extension*: $\boldsymbol{\sigma} = \sigma\ \mathbf{e} \otimes \mathbf{e}, \quad |\mathbf{e}| = 1, \quad \sigma \in \mathbb{R}$;

   c) *pure shear*: $\boldsymbol{\sigma} = \tau(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}), \quad |\mathbf{m}| = |\mathbf{n}| = 1,\ \mathbf{m} \cdot \mathbf{n} = 0,\ \tau \in \mathbb{R}$.

   For each one of these cases:

   i. describe the stress state;

   ii. find analytically the principal stresses and the principal directions of the stress;

   iii. trace and study the Mohr's circle.

5. Be $\sigma_1 > \sigma_2 > \sigma_3$ the eigenvalues of $\boldsymbol{\sigma}$ (principal stresses);

   i. show that $\boldsymbol{\sigma}_1 > \mathbf{n} \cdot \boldsymbol{\sigma}\ \mathbf{n} > \sigma_3 \quad \forall \mathbf{n}, |\mathbf{n}| = 1$;

   ii. be $\mathbf{m} \cdot \mathbf{n} = 0,\ |\mathbf{m}| = 1$; then show that

   $$\max(\mathbf{m} \cdot \boldsymbol{\sigma}\ \mathbf{n}) = \frac{1}{2}(\sigma_1 - \sigma_3),$$

   and that it is attained for

   $$\mathbf{m} = \frac{1}{\sqrt{2}}(\mathbf{e}_3 - \mathbf{e}_1), \quad \mathbf{n} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_3),$$

   with $\mathbf{e}_i$ the principal directions of the stress;

   iii. interpret all this with the Mohr's circle.

6. Consider a vector $\mathbf{x} = p - o = \ell\mathbf{n},\ |\mathbf{n}| = 1$ and the value ν of the normal stress on a surface orthogonal to $\mathbf{n}$ in o:

   $$\nu = \mathbf{n} \cdot \boldsymbol{\sigma}\ \mathbf{n} \ \rightarrow \ \nu\ell^2 = \mathbf{x} \cdot \boldsymbol{\sigma}\ \mathbf{x};$$

   the quadric

   $$\mathbf{x} \cdot \boldsymbol{\sigma}\ \mathbf{x} = \pm\frac{k^2}{\ell^2}$$

   is called the *stress quadric*.

   i. write the stress quadric in the principal base of the stress;

   ii. which is the utility of the stress quadric?

   iii. examine the cases:

   a) $\sigma_1 > \sigma_2 > \sigma_3 > 0$,

   b) $\sigma_1 > \sigma_2 > 0 > \sigma_3$,

   c) $\sigma_1 > 0 > \sigma_2 > \sigma_3$, considering for the three cases the two possible situations $\pm k^2/\ell^2$;

iv. find the stress quadric for the three elementary stress states of Ex. 4.

7. Find the *principal shearing stress*, i.e. the stationary values, with respect to the direction $\mathbf{n}$, of the tangential stress τ on an element of normal $\mathbf{n}$. Express then the same result with $\boldsymbol{\sigma}$ given in the principal base of the stress and represent the results with the circles of Mohr.

8. Find τ_{oct}, the *octahedral shearing stress*, i.e. the value of the shearing stress on a surface element orthogonal to the trisectrix of the first octant of the principal stress directions frame.

9. The decomposition of $\boldsymbol{\varepsilon}$ introduced in Sect. 1.8 in spherical and deviatoric parts is, of course, possible for $\boldsymbol{\sigma}$ too. Write this decomposition and give a physical interpretation of the scalar that appears in the expression of the spherical part. Find then this decomposition for the elementary cases of stress state of Ex. 4.

10. Define the *principal invariants* of $\boldsymbol{\sigma}$, as well as of any other 2^{nd}–rank tensor, like

$$\mathcal{I}_1 = \mathrm{tr}\boldsymbol{\sigma}, \quad \mathcal{I}_2 = \frac{1}{2}(\mathrm{tr}\boldsymbol{\sigma}^2 - \mathrm{tr}^2\boldsymbol{\sigma}), \quad \mathcal{I}_3 = \det\boldsymbol{\sigma};$$

then, show that:

i. $\mathcal{I}_2^d = \dfrac{2}{3}(\tau_1^2 + \tau_2^2 + \tau_3^2)$,

ii. $\mathcal{I}_2^d = \dfrac{3}{2}\tau_{oct}^2$,

where $\mathcal{I}_2^d$ is the second principal invariant of $\boldsymbol{\sigma}_d$, the deviatoric part of $\boldsymbol{\sigma}$, while the τ_i are the principal shearing stresses.

11. Show that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^d$ share the same principal directions but not necessarily the same principal values.

12. A stress state is defined by

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & 0 & \sigma_{13} \\ 0 & \sigma_{22} & 0 \\ \sigma_{13} & 0 & \sigma_{33} \end{bmatrix}, \quad \text{with } \sigma_{33} = (1 + \frac{t}{\tau})\frac{x_1^2 + x_3^2}{\alpha^2}, \ \alpha, \tau \in \mathbb{R},$$

t being the time. Find the principal stresses and the principal directions of stress everywhere and $\forall t$. Give the Mohr's representation of the stress state for $t = 0, x_1 = x_3 = 1$.

13. Show that the vector $(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\boldsymbol{\sigma}\ \mathbf{n},\ |\mathbf{n}| = 1$ takes its minimum norm, zero, if and only if $\mathbf{n}$ is a principal direction for $\boldsymbol{\sigma}$.

14. Be $\boldsymbol{\sigma} = \sigma_1 \mathbf{e} \otimes \mathbf{e} + \sigma_2(\mathbf{I} - \mathbf{e} \otimes \mathbf{e}),\ |\mathbf{e}| = 1$ and $\sigma \in [\sigma_1, \sigma_2]$. Show that $\forall \mathbf{n}, |\mathbf{n}| = 1$, such that $\mathbf{n} \cdot \boldsymbol{\sigma}\ \mathbf{n} = \sigma$, the norm of the vector of Ex. 13 has constant value

$$\tau = \sqrt{(\sigma - \sigma_1)(\sigma_2 - \sigma)}.$$

Chapter 3

Classical elasticity

3.1 Constitutive equations

Let us consider the most general problem of the equilibrium of deformable bodies (refer to Fig. 2.6): a deformable body Ω is

- acted upon by body forces $\mathbf{b}$ on Ω;
- subjected to tractions $\mathbf{t}_0$ on $\partial\Omega_t$;
- constraint to the displacement $\mathbf{u}_0$ on $\partial\Omega_u$.

The problem is to find:

- the deformed configuration Ω_t, i.e. the vector field of the displacement $\mathbf{u} = \mathbf{u}(p)$;
- the tensor field of infinitesimal strain $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(p)$;
- the tensor field of stress $\boldsymbol{\sigma} = \boldsymbol{\sigma}(p)$.

The fundamental assumption is that strain and displacement are infinitesimal, so that $\Omega_t \simeq \Omega$, so that the equilibrium equations can be written on the reference configuration Ω. The unknown of the problem are 15 scalar fields:

- the 3 components of $\mathbf{u}: \ u_i = u_i(p), \ i = 1, 2, 3$;
- the 6 distinct components of $\boldsymbol{\varepsilon}: \ \varepsilon_{ij} = \varepsilon_{ij}(p), \ i, j = 1, 2, 3, \ \varepsilon_{ij} = \varepsilon_{ji}$;
- the 6 distinct components of $\boldsymbol{\sigma}: \ \sigma_{ij} = \sigma_{ij}(p), \ i, j = 1, 2, 3, \ \sigma_{ij} = \sigma_{ji}$.

The equations at our disposal are 9:

- the 6 relations displacement-strain:

$$\boldsymbol{\varepsilon} = \frac{\nabla\mathbf{u} + \nabla^\top\mathbf{u}}{2} \ \rightarrow \ \varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}, \ i, j = 1, 2, 3, \ \varepsilon_{ij} = \varepsilon_{ji}; \tag{3.1}$$

- the 3 equilibrium equations:

$$\mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \mathbf{o} \ \rightarrow \ \sigma_{ij,j} + b_i = 0, \ i, j = 1, 2, 3. \tag{3.2}$$

These are the only general, i.e. valid for any solid, that can be written. There is hence a lack of 6 equations. This fact shows that the description of the equilibrium problem by uniquely geometry and mechanical balance is not sufficient: 6 other equations are needed for the problem closure.

These 6 equations must introduce what is still absent in the general equations: the behavior of the material. Such equations are called *constitutive equations*, and they give the link between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$. Generally speaking,

$$\sigma_{ij} = \sigma_{ij}(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\omega}, \boldsymbol{\varepsilon}_{,t}, \boldsymbol{\varepsilon}_{,p}, \boldsymbol{\omega}_{,t}, \boldsymbol{\omega}_{,p}). \tag{3.3}$$

Nonetheless, some requirements must be fulfilled by the constitutive equations:

- the mechanical behavior of a body must be independent from the place and orientation; as a consequence, any dependence from rigid translations and rotations must vanish $\Rightarrow \boldsymbol{\sigma}$ cannot be a function of $\mathbf{u}$ nor of $\boldsymbol{\omega}$;
- as a consequence, we are left with

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{,t}, \boldsymbol{\varepsilon}_{,p}); \tag{3.4}$$

- materials whose constitutive equations depend only upon $\boldsymbol{\varepsilon}_{,t}$:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}_{,t}) \tag{3.5}$$

  are *viscous fluids*, like the *Newtonian fluids*:

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\dot{\boldsymbol{\varepsilon}} + \lambda \mathrm{tr}\dot{\boldsymbol{\varepsilon}}\mathbf{I}, \tag{3.6}$$

  with p the pressure, μ and λ the coefficients of viscosity;
- materials whose constitutive equations depend on both $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}_{,p}$ are *polar materials*, like some polymers; for them, non-local effects are possible;
- materials whose constitutive equations are of the type

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) \tag{3.7}$$

  are *classical solids*, like metals, wood, concrete etc.; in this case, internal stresses σ_{ij} are only functions of the changes in length and in angle of fibers, described by the ε_{ij}.

3.2 Classical elasticity

A *natural state* for a solid is a state for which in the body $\boldsymbol{\varepsilon} = \mathbf{O}$ when applied forces and imposed displacements are null.

Then, *classical elasticity* is a theory concerned with

i. bodies with a natural state;

ii. infinitesimal strain;

iii. bodies for which $\boldsymbol{\sigma}$ is a linear function of $\boldsymbol{\varepsilon}$.

These assumptions give the following type of constitutive law:

$$\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}; \tag{3.8}$$

this is the *generalized Hooke's law* that, actually, generalizes to 3D elastic bodies the celebrated *Hooke's law*, 1660: *ut tensio sic vis*[1]. The Hooke's law concerned, at the origin, the behavior of springs (Hooke tested clock's springs), or, as he said, of *any springy body*, i.e. of any body whose behavior is similar to that of a spring: elastic bodies. The generalization of the Hooke's law to 3D elastic bodies is due to Cauchy, 1821.

$\mathbb{C}$ is the *elastic (stiffness) tensor*; it describes, by the value of its components, the behavior of the material; relating two second-rank tensors, it is a fourth-rank tensor[2]:

$$\mathbb{C} = \mathbb{C}_{ijkl}\ \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \quad i, j, k, l = 1, 2, 3, \tag{3.9}$$

which gives, for the components of $\boldsymbol{\sigma}$,

$$\sigma_{ij} = \mathbb{C}_{ijkl}\ \varepsilon_{kl} \quad \forall\ i, j, k, l = 1, 2, 3. \tag{3.10}$$

A material whose constitutive equation is of this type is called a *material of Cauchy*. This law implies that for $\boldsymbol{\varepsilon} = \mathbf{O}$, i.e. in the absence of applied forces, $\boldsymbol{\sigma} = \mathbf{O}$ and, of course, the converse: for any null stress state, $\boldsymbol{\varepsilon} = \mathbf{O}$: the body takes its original undeformed configuration when it is not stressed, i.e. when it is not acted upon. This is the most peculiar characteristic of elastic bodies.

The *elastic moduli* $\mathbb{C}_{ijkl}$ are 81; their value must be determined experimentally. This is of course very cumbersome, because 81 independent experimental measures should be done. Nevertheless, we will see that in the end, for the cases interesting for us and very common in practice, only two elastic coefficients are to be determined by laboratory tests.

To this purpose, we introduce two concepts:

i. *homogeneous elastic bodies*; in this case, $\mathbb{C}$ is independent from the position: the $\mathbb{C}_{ijkl}$ are constant all over Ω;

ii. *isotropic elastic bodies*; in this case, $\mathbb{C}$ is insensitive to any rotation: the $\mathbb{C}_{ijkl}$ do not depend upon the direction.

A *homogeneous, isotropic, elastic body* is hence a body whose response is elastic, independent from the position and from the direction. Many important materials, like metal alloys, are of this type. The study of this type of materials is the domain of *classical elasticity*. The following of this text is concerned with problems of classical elasticity.

[1] Hooke discovered this law, empirically, in 1660, but he revealed it, under the form of an anagram, *ceiiinosssttuu*, only in 1676 and finally under the final form only in 1678 in his book *De Potentia Restitutiva*.

[2] $\forall\ \mathbf{A}, \mathbf{B}$ and $\mathbf{L} \in Lin(\mathcal{V})$, $\mathbf{A}\otimes\mathbf{B}$ is the fourth-rank tensor defined by the operation $(\mathbf{A}\otimes\mathbf{B})\mathbf{L} := (\mathbf{B}\cdot\mathbf{L})\mathbf{A}$. Applying this rule to the dyads of a basis, we get a fundamental result: $[(\mathbf{e}_i \otimes \mathbf{e}_j) \otimes (\mathbf{e}_k \otimes \mathbf{e}_l)]\,(\mathbf{e}_p \otimes \mathbf{e}_q) = (\mathbf{e}_k \otimes \mathbf{e}_l) \cdot (\mathbf{e}_p \otimes \mathbf{e}_q)(\mathbf{e}_i \otimes \mathbf{e}_j) = \delta_{kp}\delta_{lq}(\mathbf{e}_i \otimes \mathbf{e}_j)$.

3.3 Reduction of the number of elastic moduli

Let us see now how from 81 moduli we arrive to only 2. The first reduction is due to the symmetry of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$:

$$\begin{aligned}\sigma_{ij} = \sigma_{ji} \;&\rightarrow\; \mathbb{C}_{ijkl}\ \varepsilon_{kl} = \mathbb{C}_{jikl}\ \varepsilon_{kl} \;\Rightarrow\; \mathbb{C}_{ijkl} = \mathbb{C}_{jikl},\\ \varepsilon_{kl} = \varepsilon_{lk} \;&\rightarrow\; \mathbb{C}_{ijkl}\ \varepsilon_{kl} = \mathbb{C}_{jilk}\ \varepsilon_{lk} \;\Rightarrow\; \mathbb{C}_{ijkl} = \mathbb{C}_{jilk}.\end{aligned} \tag{3.11}$$

Hence, we have the following 45 conditions, called *minor symmetries*[3],

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk} = \mathbb{C}_{jilk}, \tag{3.12}$$

that reduce the number of independent elastic moduli from 81 to 36.

A further reduction is obtained postulating that the material is a *material of Green* (1839). To introduce this concept, let us consider again the volume density of the work of internal actions, see Sect. 2.7:

$$\delta U = \boldsymbol{\sigma} \cdot \delta\boldsymbol{\varepsilon} = \sigma_{ij}\delta\varepsilon_{ij}. \tag{3.13}$$

This work can be interpreted as the variation of the deformation energy, stored in a unit volume of the body, produced by a small variation of the strain state.

Let us consider a transformation of an elastic body from a state A to a state B. Then, we say that the body is made of a material of Green if the variation δU in passing from A to B is independent from the transformation itself, but it depends uniquely upon the initial and final states:

$$\delta U_{A\rightarrow B} = \int_A^B \delta U = U_B - U_A. \tag{3.14}$$

The consequence of this assumption is that δU must be the exact differential of the function U, i.e.

$$\delta U = dU = \boldsymbol{\sigma} \cdot d\boldsymbol{\varepsilon} = \sigma_{ij}\ d\varepsilon_{ij}, \tag{3.15}$$

which gives the *Green's formula*:

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}. \tag{3.16}$$

The function U is called the (*density of*) *strain energy* or *elastic potential*. So, in the end, a material is an *elastic material of Green if it admits an elastic potential* U, i.e., if it is possible to define a scalar function U that relates the components of stress to those of strain through the Green's formula.

In this case, through the Hooke's law, the Green's formula and the Schwarz theorem, we get

$$\left.\begin{aligned}\mathbb{C}_{ijkl} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial^2 U}{\partial \varepsilon_{kl}\partial \varepsilon_{ij}},\\ \mathbb{C}_{klij} = \frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}} = \frac{\partial^2 U}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}},\end{aligned}\right. \;\Rightarrow\; \mathbb{C}_{ijkl} = \mathbb{C}_{klij}\ \forall\ i,j,k,l = 1,2,3. \tag{3.17}$$

[3]The word *symmetry* is used here to signify the invariance of an elastic modulus with respect to a permutation of the indexes. The same word, symmetry, is used in elasticity for indicating a transformation that preserves the elastic behavior. The reader should be aware of this somewhat ambiguous double meaning of the same word in the same context, that of elasticity.

These 15 relations are called the *major symmetries*; they reduce the number of distinct elastic moduli from 36 to 21. This reduction is hence given by the existence of an elastic potential.

No further reduction can be obtained in the most general case, i.e. without introducing special properties (namely, elastic symmetries) of a given elastic material.

To remark that a material of Cauchy is not necessarily a material of Green, and vice-versa. In fact, a material of Cauchy is also of Green if it admits an elastic potential, U; this fact has always been verified experimentally for all the elastic materials. A material of Green is also of Cauchy if $\boldsymbol{\sigma}$ is a linear function of $\boldsymbol{\varepsilon}$; this is not always the case.

The most important class of elastic materials is that of *hyperelastic materials*, i.e. of materials that are at the same time of Cauchy and of Green. In such a case, $\boldsymbol{\sigma}$ is a linear function of $\boldsymbol{\varepsilon}$, the material admits an elastic potential U and the Green's formula is valid (the above proof of the existence of the major symmetries, eq. (3.17), has been done with the implicit assumption of hyperelastic behavior).

An important consequence for hyperelastic materials is that

$$U = \frac{1}{2}\mathbb{C}_{ijkl}\ \varepsilon_{ij}\ \varepsilon_{kl}, \tag{3.18}$$

i.e. U is necessarily a quadratic function of the ε_{ij}. In fact, only in this way we get, through the Green's formula,

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} = \frac{\partial}{\partial \varepsilon_{ij}}\left(\frac{1}{2}\mathbb{C}_{mnpq}\ \varepsilon_{mn}\ \varepsilon_{pq}\right) = \mathbb{C}_{ijpq}\ \varepsilon_{pq}\ \rightarrow\ \boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}, \tag{3.19}$$

i.e. we satisfy at the same time to the fundamental relations of Green and Cauchy materials. In this case, it is also

$$U = \frac{1}{2}\mathbb{C}_{ijkl}\ \varepsilon_{ij}\ \varepsilon_{kl} = \frac{1}{2}\boldsymbol{\varepsilon}\cdot\mathbb{C}\boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}. \tag{3.20}$$

We will see further that $\mathbb{C}$ is a positive definite tensor, which implies that it is inversible, i.e.

$$\exists\mathbb{S}:\ \ \boldsymbol{\varepsilon} = \mathbb{S}\boldsymbol{\sigma}\ \Rightarrow\ \mathbb{S} = \mathbb{C}^{-1}. \tag{3.21}$$

So,

$$\varepsilon_{ij} = \mathbb{S}_{ijkl}\sigma_{kl}, \tag{3.22}$$

which injected in the general expression (3.18) of U gives

$$U = \frac{1}{2}\mathbb{S}_{ijkl}\sigma_{ij}\sigma_{kl}, \tag{3.23}$$

so that, deriving with respect to σ_{ij}, we get

$$\varepsilon_{ij} = \mathbb{S}_{ijkl}\sigma_{kl} = \frac{\partial U}{\partial \sigma_{ij}}, \tag{3.24}$$

which is the dual, for the strains, of the Green's formula.

In the most general case, the behavior of hyperelastic materials depends upon 21 distinct moduli: this is the case of *completely anisotropic* or *triclinic* materials. The behavior of an anisotropic material depends upon the direction, hence the moduli $\mathbb{C}_{ijkl}$ are frame-dependent quantities.

This cannot be the case of isotropic materials, whose elastic response is insensitive to a change of frame: the elastic moduli of an isotropic material cannot be frame-dependent. This means that for an isotropic material, U cannot depend upon the ε_{ij}, that are frame-dependent quantities, but rather on the *invariants* of $\boldsymbol{\varepsilon}$[4]. As a consequence, being U a quadratic fonction of the ε_{ij}, the general expression of U must be of the type

$$U = \frac{1}{2}c_1 I_1^2 + c_2 I_2, \tag{3.25}$$

with[5]

$$I_1 = \mathrm{tr}\boldsymbol{\varepsilon} = \varepsilon_{ii}, \quad I_2 = \frac{\mathrm{tr}^2\boldsymbol{\varepsilon} - \mathrm{tr}\boldsymbol{\varepsilon}^2}{2} = \frac{\varepsilon_{ii}\ \varepsilon_{jj} - \varepsilon_{ij}\ \varepsilon_{ji}}{2}. \tag{3.26}$$

The third order invariant of $\boldsymbol{\varepsilon}$, i.e. $\det\boldsymbol{\varepsilon}$, cannot enter in the expression of U, because it is a cubic function of the ε_{ij}, while U must be a quadratic function of the ε_{ij}. Then,

$$U = \frac{1}{2}\left[(c_1 + c_2)\varepsilon_{ii}\ \varepsilon_{jj} - c_2\ \varepsilon_{ij}\ \varepsilon_{ji}\right], \tag{3.27}$$

so that[6]

$$\begin{aligned}
\sigma_{\underline{ii}} &= \frac{\partial U}{\partial \varepsilon_{\underline{ii}}} = (c_1 + c_2)\varepsilon_{ii} - c_2\ \varepsilon_{\underline{ii}},\\
\sigma_{ij} &= \frac{\partial U}{\partial \varepsilon_{ij}} = -c_2\ \varepsilon_{ji} = -c_2\ \varepsilon_{ij}.
\end{aligned} \tag{3.28}$$

For instance:

$$\begin{aligned}
\sigma_{11} &= \frac{\partial U}{\partial \varepsilon_{11}} = (c_1 + c_2)(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) - c_2\ \varepsilon_{11},\\
\sigma_{12} &= \frac{\partial U}{\partial \varepsilon_{12}} = -c_2\ \varepsilon_{12} \quad \text{etc.}
\end{aligned} \tag{3.29}$$

We see hence that in the case of isotropic materials, only two constants are sufficient to characterize the elastic behavior. This fundamental result comes directly from the assumption that the material admits an elastic potential, i.e. from the definition of elastic material as material of Green. Hence, such an approach, basically an energetic approach, eventually implies that two independent parameters are needed to describe the elastic behavior of an isotropic material. The Green's approach allowed, during the XIXth

[4]The elastic energy U is, as any other quantity derived by a scalar product, an invariant, i.e. it is not frame-dependent. Hence, because $\mathbb{C}$ for an isotropic material is frame independent, the expression of U cannot depend upon frame-dependent quantities, the ε_{ij}, but only upon frame-independent functions of the ε_{ij}: the invariants of $\boldsymbol{\varepsilon}$.

[5]$\boldsymbol{\varepsilon}^2 = \boldsymbol{\varepsilon}\boldsymbol{\varepsilon} = \varepsilon_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\ \varepsilon_{hk}\mathbf{e}_h \otimes \mathbf{e}_k = \varepsilon_{ij}\ \varepsilon_{hk}\ \mathbf{e}_j \cdot \mathbf{e}_h(\mathbf{e}_i \otimes \mathbf{e}_k) = \varepsilon_{ij}\ \varepsilon_{hk}\ \delta_{jh}(\mathbf{e}_i \otimes \mathbf{e}_k) \rightarrow \mathrm{tr}\boldsymbol{\varepsilon}^2 = \varepsilon_{ij}\ \varepsilon_{hk}\ \delta_{jh}\mathrm{tr}(\mathbf{e}_i \otimes \mathbf{e}_k) = \varepsilon_{ij}\ \varepsilon_{hk}\ \delta_{jh}\delta_{ik} = \varepsilon_{ij}\ \varepsilon_{ji}$.

[6]Following a common practice, when an index is underlined, it is not a dummy index: no summation over it.

century, to establish this important result on the basis of simple, general arguments and to solve the long lasting diatribe between the so-called *rari-constant* theory, affirming that just one parameter was sufficient to describe the elastic behavior of isotropic materials, and the *multi-constant* one, stating the necessity of two elastic parameters: experimental evidence has always confirmed the validity of the multi-constant theory.

3.4 Equations of Lamé

Classically, we pose

$$c_1 + c_2 = \lambda, \quad -\frac{c_2}{2} = \mu \ \ \Rightarrow \ c_1 = \lambda + 2\mu, \tag{3.30}$$

and we get, in compact form,

$$\sigma_{ij} = 2\mu \ \varepsilon_{ij} + \lambda \ \varepsilon_{kk} \ \delta_{ij}, \tag{3.31}$$

or, in tensor form,

$$\boldsymbol{\sigma} = 2\mu \ \boldsymbol{\varepsilon} + \lambda \ \mathrm{tr}\boldsymbol{\varepsilon} \ \mathbf{I}. \tag{3.32}$$

These are the *equations of Lamé* (1852), the constitutive equations for isotropic hyperelastic materials. They provide the 6 scalar equations (there are 6 distinct components for $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$) for the closure of the elastic problem. λ and μ are the *coefficients of Lamé*: they are the two moduli to be specified for determining the elastic behavior of a material.

The inverse of the equations of Lamé can be easily obtained:

$$\mathrm{tr}\boldsymbol{\sigma} = (2\mu + 3\lambda)\mathrm{tr}\boldsymbol{\varepsilon} \ \rightarrow \ \mathrm{tr}\boldsymbol{\varepsilon} = \frac{\mathrm{tr}\boldsymbol{\sigma}}{2\mu + 3\lambda}, \tag{3.33}$$

that replaced in eq. (3.32) gives, after simple passages,

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{2\mu + 3\lambda}\mathrm{tr}\boldsymbol{\sigma} \ \mathbf{I}\right). \tag{3.34}$$

Coefficients c_1 and c_2 are never used in the calculations, λ and μ are preferred. The components of $\mathbb{C}$ can be expressed as functions of the Lamé's coefficients (no summation over i and j):

$$\begin{aligned} &\mathbb{C}_{iiii} = c_1 = \lambda + 2\mu, \\ &\mathbb{C}_{iijj} = c_1 + c_2 = \lambda, \\ &\mathbb{C}_{ijij} = -\frac{c_2}{2} = \mu = \frac{\mathbb{C}_{iiii} - \mathbb{C}_{iijj}}{2}, \end{aligned} \qquad i, j = 1, 2, 3, \tag{3.35}$$

the other components are null.

It is often preferred to express the Lamé's equations as functions of two other parameters, the so-called *technical or engineering constants*, having a direct physical meaning and easy to be determined experimentally by a unique traction test. We consider a bar, with a cross section of area A, whose axis coincides with the x_1-axis of a reference frame, submitted to a tensile force f at its ends. We assume that (see the next Chapter on the Saint Venant problem)

$$\sigma_{11} = \frac{f}{A} \tag{3.36}$$

and it is easy to check that the stress tensor

$$\boldsymbol{\sigma} = \sigma_{11}\ \mathbf{e}_1 \otimes \mathbf{e}_1 \tag{3.37}$$

satisfies to the equilibrium equations. So, by the Lamé's equations we get

$$\begin{aligned}\boldsymbol{\varepsilon} &= \frac{1}{2\mu}\left[\sigma_{11}\ \mathbf{e}_1 \otimes \mathbf{e}_1 - \frac{\lambda}{2\mu+3\lambda}\sigma_{11}\ \mathbf{e}_i \otimes \mathbf{e}_i\right] \\ &= \frac{\sigma_{11}}{2\mu(2\mu+3\lambda)}\left[2(\lambda+\mu)\mathbf{e}_1 \otimes \mathbf{e}_1 - \lambda(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3)\right].\end{aligned} \tag{3.38}$$

Now, we introduce

- the *Young's modulus* E

$$E := \frac{\sigma_{11}}{\varepsilon_{11}}; \tag{3.39}$$

- the *Poisson's coefficient* ν

$$\nu := -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}}. \tag{3.40}$$

Of course, thanks to isotropy, nothing changes if we change the labels of the axes. It is self-evident that E measures the stiffness to extension, i.e. it gives a direct measure of the stiffness of the material. ν, on its side, gives a measure of the so-called *Poisson's effect*: a tension in a direction normally produces a contraction in the transversal directions (an expansion if tension is turned into compression).

We remark that, according to the multi-constant theory, the existence of two independent elastic parameters means that there are two distinct mechanical phenomena for stressed isotropic materials: they are the extension stiffness and the Poisson's effect.

The above formulae give us the expression of E and ν as functions of the Lamé's coefficients:

$$E = \mu\frac{2\mu+3\lambda}{\mu+\lambda}, \quad \nu = \frac{\lambda}{2(\mu+\lambda)}; \tag{3.41}$$

the converse relations are easy to be found:

$$\lambda = \frac{\nu\ E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}, \tag{3.42}$$

while the relations of the technical constants with the Cartesian components are:

$$\begin{aligned}\mathbb{C}_{1111} &= E\frac{1-\nu}{(1+\nu)(1-2\nu)}, & & E = \frac{(\mathbb{C}_{1111}-\mathbb{C}_{1122})(\mathbb{C}_{1111}+2\mathbb{C}_{1122})}{\mathbb{C}_{1111}+\mathbb{C}_{1122}}, \\ & & \rightarrow & \\ \mathbb{C}_{1122} &= E\frac{\nu}{(1+\nu)(1-2\nu)}, & & \nu = \frac{\mathbb{C}_{1122}}{\mathbb{C}_{1111}+\mathbb{C}_{1122}}.\end{aligned} \tag{3.43}$$

Technical constants can be used in place of λ and μ for writing the Lamé's equations; using the above equations, it is easy to find that the equations of Lamé can be written also in the following form:

$$\boldsymbol{\sigma} = \frac{E}{1+\nu}\left[\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}\mathrm{tr}\boldsymbol{\varepsilon}\ \mathbf{I}\right], \tag{3.44}$$

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{tr}\boldsymbol{\sigma}\ \mathbf{I}. \tag{3.45}$$

Two other technical moduli are sometimes introduced, related to two other possible mechanical situations. For a pure shear stress state, e.g.

$$\boldsymbol{\sigma} = \sigma_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \tag{3.46}$$

then

$$\boldsymbol{\varepsilon} = \frac{\sigma_{12}}{2\mu}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \tag{3.47}$$

We define *shear modulus* G the quantity

$$G := \frac{\sigma_{12}}{2\varepsilon_{12}}, \tag{3.48}$$

so that

$$G = \mu = \frac{E}{2(1+\nu)}. \tag{3.49}$$

Of course, nothing changes if the axes labels are changed. G, like E, measures a stiffness, in this case that to shearing actions.

Now, we consider a spherical stress state:

$$\boldsymbol{\sigma} = p\,\mathbf{I}, \quad p \in \mathbb{R}, \tag{3.50}$$

so that

$$\boldsymbol{\varepsilon} = \frac{p}{2\mu + 3\lambda}\mathbf{I}. \tag{3.51}$$

The change in volume is

$$\delta v = \mathrm{tr}\boldsymbol{\varepsilon} = \frac{3p}{2\mu + 3\lambda}; \tag{3.52}$$

then, we introduce the *bulk modulus* κ as

$$\kappa := \frac{p}{\delta v} = \frac{2\mu + 3\lambda}{3}. \tag{3.53}$$

κ measures the volume stiffness, i.e. the stiffness to volume changes; it is immediate to find that

$$\kappa = \frac{E}{3(1-2\nu)}. \tag{3.54}$$

To end this part, we remark that the relation (no summation over i)

$$\sigma_{ii} = E\ \varepsilon_{ii} \tag{3.55}$$