Indian Institute of Technology Bhubaneswar

School of Infrastructure

Subject Name : Solid Mechanics	Subject Code: CE2L001

Tutorial No. 6

1. A 45-degree strain rosette is used to measure strains on a steel surface. The readings are $\epsilon_a = 100\,\mu\text{m/m}$, $\epsilon_b = 400\,\mu\text{m/m}$, and $\epsilon_c = 900\,\mu\text{m/m}$. Determine the principal strains and stresses.

Solution: Calculate strain components ϵ_{11}, ϵ_{22}, and γ_{12}:

$$\begin{aligned}
\epsilon_{11} &= \epsilon_a = 100\,\mu\text{m/m} \\
\epsilon_{22} &= \epsilon_c = 900\,\mu\text{m/m} \\
\gamma_{12} &= 2\epsilon_b - (\epsilon_a + \epsilon_c) = 2 \times 400 - (100 + 900) = -200\,\mu\text{m/m}
\end{aligned}$$

Calculate principal strains:

$$\begin{aligned}
\epsilon_{I,II} &= \frac{\epsilon_{11} + \epsilon_{22}}{2} \pm \sqrt{\left(\frac{\epsilon_{11} - \epsilon_{22}}{2}\right)^2 + \left(\frac{\gamma_{12}}{2}\right)^2} \\
\epsilon_{I,II} &= \frac{100 + 900}{2} \pm \sqrt{\left(\frac{100 - 900}{2}\right)^2 + \left(\frac{-200}{2}\right)^2} \\
\epsilon_I &= 912.31\,\mu\text{m/m}, \quad \epsilon_{II} = 87.69\,\mu\text{m/m}
\end{aligned}$$

Calculate principal stresses:

$$\begin{aligned}
\sigma_I &= \frac{E}{1 - \nu^2}(\epsilon_I + \nu\epsilon_{II}) \\
\sigma_{II} &= \frac{E}{1 - \nu^2}(\epsilon_{II} + \nu\epsilon_I)
\end{aligned}$$

For a given steel with $E = 205\,\text{GPa}$ and $\nu = 0.29$, we can calculate the principal stresses as $\sigma_I = 209.85$ MPa and $\sigma_{II} = 78.83$ MPa.

2. Consider a 60° strain gauge rosette to be mounted on the surface of a specimen as shown in Fig. 1.

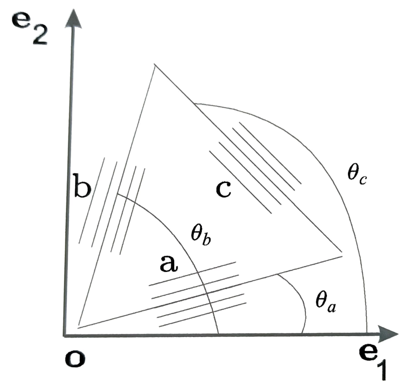

Figure 1: Schematic representation of a 60° strain-gauge rosette.

(a) Let $\{\boldsymbol{e}_a, \boldsymbol{e}_b, \boldsymbol{e}_c\}$ denote three non-collinear unit vectors which represent the directions in which the three strain gauges in a rosette are arranged, and let

$$E_{aa} = \boldsymbol{e}_a \cdot \boldsymbol{E}\boldsymbol{e}_a, \qquad E_{bb} = \boldsymbol{e}_b \cdot \boldsymbol{E}\boldsymbol{e}_b, \qquad E_{cc} = \boldsymbol{e}_c \cdot \boldsymbol{E}\boldsymbol{e}_c,$$

denote the components of the strain measured in the directions $\{\boldsymbol{e}_a, \boldsymbol{e}_b, \boldsymbol{e}_c\}$. Determine a general expression for the components of strain, $\boldsymbol{E}$, (i.e E_{11}, E_{22}, $E_{12} = E_{21}$) with respect to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, as functions of ($E_{aa}$, E_{bb}, E_{cc}) and the orientations $(\theta_a, \theta_b, \theta_c)$.

(b) Evaluate E_{11}, E_{22} and E_{12} for $\theta_a = 0°$, $\theta_b = 60°$, and $\theta_c = 120°$.

Solution: (a) Given that the components of the strain measured in the directions $\{\boldsymbol{e}_a, \boldsymbol{e}_b, \boldsymbol{e}_c\}$ is

$$E_{aa} = \boldsymbol{e}_a \cdot \boldsymbol{E}\boldsymbol{e}_a, \qquad E_{bb} = \boldsymbol{e}_b \cdot \boldsymbol{E}\boldsymbol{e}_b, \qquad E_{cc} = \boldsymbol{e}_c \cdot \boldsymbol{E}\boldsymbol{e}_c \tag{1}$$

where the strain $\boldsymbol{E}$ with respect to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ can be defined as $\boldsymbol{E} = E_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Thus Eq. (1) can be expressed as

$$\begin{aligned}
E_{aa} &= \boldsymbol{e}_a \cdot (E_{ij}\,(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{e}_a) \\
&= \boldsymbol{e}_a \cdot (E_{ij}(\boldsymbol{e}_j \cdot \boldsymbol{e}_a)\boldsymbol{e}_i) \qquad \text{since} \quad (\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{e}_a = (\boldsymbol{e}_j \cdot \boldsymbol{e}_a)\boldsymbol{e}_i \\
&= E_{ij}(\boldsymbol{e}_i \cdot \boldsymbol{e}_a)(\boldsymbol{e}_j \cdot \boldsymbol{e}_a) \\
\implies E_{aa} &= E_{11}(\boldsymbol{e}_1 \cdot \boldsymbol{e}_a)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_a) + E_{12}(\boldsymbol{e}_1 \cdot \boldsymbol{e}_a)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_a) + E_{21}(\boldsymbol{e}_2 \cdot \boldsymbol{e}_a)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_a) + E_{22}(\boldsymbol{e}_2 \cdot \boldsymbol{e}_a)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_a)
\end{aligned}$$

Similarly E_{bb} and E_{cc} can be determined as

$$\begin{aligned}
E_{bb} &= \boldsymbol{e}_b \cdot (E_{ij}\,(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{e}_b) \\
&= \boldsymbol{e}_b \cdot (E_{ij}(\boldsymbol{e}_j \cdot \boldsymbol{e}_b)\boldsymbol{e}_i) \qquad \text{since} \quad (\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{e}_b = (\boldsymbol{e}_j \cdot \boldsymbol{e}_b)\boldsymbol{e}_i \\
&= E_{ij}(\boldsymbol{e}_i \cdot \boldsymbol{e}_b)(\boldsymbol{e}_j \cdot \boldsymbol{e}_b)
\end{aligned}$$

$$\implies E_{bb} = E_{11}(\boldsymbol{e}_1 \cdot \boldsymbol{e}_b)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_b) + E_{12}(\boldsymbol{e}_1 \cdot \boldsymbol{e}_b)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_b) + E_{21}(\boldsymbol{e}_2 \cdot \boldsymbol{e}_b)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_b) + E_{22}(\boldsymbol{e}_2 \cdot \boldsymbol{e}_b)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_b),$$

and

$$\begin{aligned}
E_{cc} &= \boldsymbol{e}_c \cdot (E_{ij}\,(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{e}_c) \\
&= \boldsymbol{e}_c \cdot (E_{ij}(\boldsymbol{e}_j \cdot \boldsymbol{e}_c)\boldsymbol{e}_i) \qquad \text{since} \quad (\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{e}_c = (\boldsymbol{e}_j \cdot \boldsymbol{e}_c)\boldsymbol{e}_i \\
&= E_{ij}(\boldsymbol{e}_i \cdot \boldsymbol{e}_c)(\boldsymbol{e}_j \cdot \boldsymbol{e}_c) \\
\implies E_{cc} &= E_{11}(\boldsymbol{e}_1 \cdot \boldsymbol{e}_c)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_c) + E_{12}(\boldsymbol{e}_1 \cdot \boldsymbol{e}_c)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_c) + E_{21}(\boldsymbol{e}_2 \cdot \boldsymbol{e}_c)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_c) + E_{22}(\boldsymbol{e}_2 \cdot \boldsymbol{e}_c)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_c)
\end{aligned}$$

From Fig. 1, the dot product between the basis vectors can be determined as follows:

$$\begin{aligned}
\boldsymbol{e}_1 \cdot \boldsymbol{e}_a &= \cos\theta_a, \quad \boldsymbol{e}_1 \cdot \boldsymbol{e}_b = \cos\theta_b, \quad \boldsymbol{e}_1 \cdot \boldsymbol{e}_c = \cos\theta_c \\
\boldsymbol{e}_2 \cdot \boldsymbol{e}_a &= \sin\theta_a, \quad \boldsymbol{e}_2 \cdot \boldsymbol{e}_b = \sin\theta_b, \quad \boldsymbol{e}_2 \cdot \boldsymbol{e}_c = \sin\theta_c
\end{aligned}$$

Thus the components of the strain measured in the directions $\{\boldsymbol{e}_a, \boldsymbol{e}_b, \boldsymbol{e}_c\}$ can be expressed as (by considering a symmetric tensor i.e, $E_{12} = E_{21}$) as

$$\begin{aligned}
E_{aa} &= E_{11}\cos^2\theta_a + 2E_{12}\cos\theta_a\sin\theta_a + E_{22}\sin^2\theta_a \\
E_{bb} &= E_{11}\cos^2\theta_b + 2E_{12}\cos\theta_b\sin\theta_b + E_{22}\sin^2\theta_b \\
E_{cc} &= E_{11}\cos^2\theta_c + 2E_{12}\cos\theta_c\sin\theta_c + E_{22}\sin^2\theta_c,
\end{aligned} \tag{2}$$

which can be expressed in the matrix form as

$$\begin{bmatrix} E_{aa} \\ E_{bb} \\ E_{cc} \end{bmatrix} = \begin{bmatrix} \cos^2\theta_a & 2\cos\theta_a\sin\theta_a & \sin^2\theta_a \\ \cos^2\theta_b & 2\cos\theta_b\sin\theta_b & \sin^2\theta_b \\ \cos^2\theta_c & 2\cos\theta_c\sin\theta_c & \sin^2\theta_c \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{12} \\ E_{22} \end{bmatrix}$$

Thus, the general expression for the components of strain, $\boldsymbol{E}$, (i.e E_{11}, E_{22}, $E_{12} = E_{21}$) with respect to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ can be found as

$$\begin{bmatrix} E_{11} \\ E_{12} \\ E_{22} \end{bmatrix} = \begin{bmatrix} \cos^2\theta_a & 2\cos\theta_a\sin\theta_a & \sin^2\theta_a \\ \cos^2\theta_b & 2\cos\theta_b\sin\theta_b & \sin^2\theta_b \\ \cos^2\theta_c & 2\cos\theta_c\sin\theta_c & \sin^2\theta_c \end{bmatrix}^{-1} \begin{bmatrix} E_{aa} \\ E_{bb} \\ E_{cc} \end{bmatrix} \tag{3}$$

(b) Given that $\theta_a = 0°$, $\theta_b = 60°$, and $\theta_c = 120°$, the components of strain, E_{11}, E_{22}, E_{12} can be determined by substituting the values of θ_a, θ_b, θ_c in Eq. (3) as

$$\begin{bmatrix} E_{11} \\ E_{12} \\ E_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{\sqrt{3}}{2} & \frac{3}{4} \\ \frac{1}{4} & -\frac{\sqrt{3}}{2} & \frac{3}{4} \end{bmatrix}^{-1} \begin{bmatrix} E_{aa} \\ E_{bb} \\ E_{cc} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} E_{aa} \\ E_{bb} \\ E_{cc} \end{bmatrix}. \tag{4}$$

The values of E_{11}, E_{22}, E_{12} can be found from Eq. (4) as

$$E_{11} = E_{aa}, \qquad E_{12} = \frac{E_{bb} - E_{cc}}{\sqrt{3}}, \qquad E_{22} = \frac{-E_{aa} + 2E_{bb} + 2E_{cc}}{3}.$$

3. Stress is not a directly measurable quantity for most materials and is usually computed from the strain measurements in a complex engineering system. A common method for measuring the state of strain is to use *strain gauges* which are simple electrical devices that can measure only the normal strain along its length.

   A strain rosette having three strain gauges a, b and c is installed on a block as shown in Fig.2. During a static test of the block in plane strain ($\epsilon_{zz} = 0, \gamma_{xz} = 0$ and $\gamma_{yz} = 0$), the strain rosettes read $\epsilon_a = 0.003$, $\epsilon_b = 0.001$ and $\epsilon_c = 0.001$.

   1) Calculate the shear strain γ_{xy} for an element oriented along the xy plane (Round your answer to 4 decimal points). Note that for a strain gauge oriented at an angle of θ to the x-axis , the gauge reading ϵ_θ can be expressed as:

   $$\epsilon_\theta = \epsilon_{xx} \cos^2(\theta) + \epsilon_{yy} \sin^2(\theta) + \gamma_{xy} \sin(\theta) \cos(\theta)$$

   2) If the block is made of a material with elastic modulus $E = 100$ GPa and Poisson's ratio $\nu = 0.3$, use Hooke's law to find the stress components in the $x - y$ plane.

   3) Determine the principal stresses in the block.

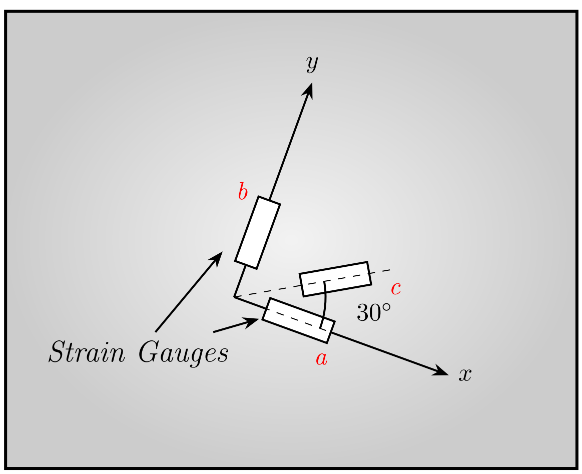

Figure 2: Strain Gauges

Solution: We use the given strain transformation equation for each gauge:

- Gauge a ($\theta_a = 0°$):

$$\begin{aligned}
\epsilon_a &= \epsilon_{xx} \cos^2(0°) + \epsilon_{yy} \sin^2(0°) + \gamma_{xy} \sin(0°) \cos(0°) \\
0.003 &= \epsilon_{xx}(1) + \epsilon_{xy}(0) + \gamma_{xy}(0) \\
\implies \epsilon_{xx} &= 0.003
\end{aligned}$$

- Gauge b ($\theta_b = 90°$):

$$\begin{aligned}
\epsilon_b &= \epsilon_{xx} \cos^2(90°) + \epsilon_{yy} \sin^2(90°) + \gamma_{xy} \sin(90°) \cos(90°) \\
0.001 &= \epsilon_{xx}(0) + \epsilon_{yy}(1) + \gamma_{xy}(0) \\
\implies \epsilon_{yy} &= 0.001
\end{aligned}$$

- Gauge c ($\theta_c = 30°$):

$$\begin{aligned}
\epsilon_c &= \epsilon_{xx} \cos^2(30°) + \epsilon_{yy} \sin^2(30°) + \gamma_{xy} \sin(30°) \cos(30°) \\
0.001 &= (0.003)(\cos 30°)^2 + (0.001)(\sin 30°)^2 + \gamma_{xy}(\sin 30°)(\cos 30°) \\
0.001 &= (0.003)(0.75) + (0.001)(0.25) + \gamma_{xy}(0.5)(0.866) \\
0.001 &= 0.00225 + 0.00025 + 0.433\gamma_{xy} \\
0.001 &= 0.0025 + 0.433\gamma_{xy} \\
-0.0015 &= 0.433\gamma_{xy} \\
\implies \gamma_{xy} &\approx -0.0035
\end{aligned}$$

First, calculate the shear modulus G:

$$G = \frac{E}{2(1+\nu)} = \frac{100 \text{ GPa}}{2(1+0.3)} = 38.46 \text{ GPa}$$

The shear stress τ_{xy} is:

$$\tau_{xy} = G\gamma_{xy} = (38.46 \text{ GPa})(-0.0035) = -0.1346 \text{ GPa}$$

For plane strain ($\epsilon_{zz} = 0$), we use the 3D Hooke's Law equations:

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \\
\epsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})] \\
\epsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] = 0
\end{aligned}$$

Substituting known values (with E=100 GPa):

$$
\begin{aligned}
0.003 &= \frac{1}{100}[\sigma_{xx} - 0.3(\sigma_{yy} + \sigma_{zz})] \implies 300 = \sigma_{xx} - 0.3\sigma_{yy} - 0.3\sigma_{zz} \\
0.001 &= \frac{1}{100}[\sigma_{yy} - 0.3(\sigma_{xx} + \sigma_{zz})] \implies 100 = \sigma_{yy} - 0.3\sigma_{xx} - 0.3\sigma_{zz} \\
0 &= \frac{1}{100}[\sigma_{zz} - 0.3(\sigma_{xx} + \sigma_{yy})] \implies \sigma_{zz} = 0.3(\sigma_{xx} + \sigma_{yy})
\end{aligned}
$$

Solving this system of 3 equations (all stresses in GPa):

- $\sigma_{xx} = 0.4615$ GPa
- $\sigma_{yy} = 0.3077$ GPa
- $\sigma_{zz} = 0.2308$ GPa

Since the problem is in plane strain ($\gamma_{xz} = 0, \gamma_{yz} = 0$), there are no shear stresses on the z-face. This means the z-axis is a principal direction.

Therefore, one of the principal stresses is σ_{zz}:

$$\sigma_{p2} = \sigma_{zz} = 0.2308 \text{ GPa}$$

The other two principal stresses, σ_{p1} and σ_{p3}, lie in the xy-plane. We find them using the center and radius of the Mohr's circle for the xy-plane:

$$\sigma_{ave} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{0.4615 + 0.3077}{2} = 0.3846 \text{ GPa}$$

$$
\begin{aligned}
R &= \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \\
R &= \sqrt{\left(\frac{0.4615 - 0.3077}{2}\right)^2 + (-0.1346)^2} \\
R &= \sqrt{(0.0769)^2 + (-0.1346)^2} = 0.1550 \text{ GPa}
\end{aligned}
$$

The principal stresses are:

$$
\begin{aligned}
\sigma_{p1} &= \sigma_{ave} + R = 0.3846 + 0.1550 = 0.5396 \text{ GPa} \quad \text{(Max)} \\
\sigma_{p3} &= \sigma_{ave} - R = 0.3846 - 0.1550 = 0.2296 \text{ GPa} \quad \text{(Min)}
\end{aligned}
$$

So $\sigma_{p1} = 0.5396$ GPa, $\sigma_{p2} = 0.2308$ GPa, $\sigma_{p3} = 0.2296$ GPa.