Indian Institute of Technology Bhubaneswar

School of Infrastructure

Subject Name : Solid Mechanics	Subject Code: CE2L001

Solution of Tutorial No. 8

1. (a) What are the assumptions made in the double integration method for calculating beam deflections?

   (b) What are the limitations of the double integration method, and when is it not applicable?

   (c) How do you handle discontinuous loading conditions, such as point loads or sudden changes in distributed loads, when using the double integration method?

   (d) How does the double integration method account for the effects of shear deformation on beam deflections?

Solution: (a) **Assumptions Made in the Double Integration Method:**

The double integration method is based on the Euler–Bernoulli beam theory. The key assumptions are:

1. Plane sections remain plane and perpendicular to the neutral axis after bending (shear deformations are neglected).

2. The material is linearly elastic, homogeneous, and isotropic, so Hooke's law applies.

3. Deflections and slopes are small, allowing the linear curvature relation:

$$\frac{1}{\rho} \approx \frac{d^2y}{dx^2}.$$

4. The beam has a prismatic cross-section, so the flexural rigidity EI is constant (or piecewise constant).

5. The moment–curvature relationship is valid:

$$M(x) = EI\frac{d^2y}{dx^2}.$$

6. No significant axial loads are present which could influence deflections.

(b) **Limitations of the Double Integration Method:**

The double integration method has the following limitations:

1. Shear deformation is neglected; therefore, it is inaccurate for deep beams or cases where shear effects are significant.

2. Not suitable for large deflections, where geometric nonlinearities become important.

3. Difficult to apply directly to beams with variable EI, non-prismatic geometry, or complex loading unless treated piecewise.

4. Complicated for beams with many discontinuities such as several point loads, abrupt changes in distributed load, internal hinges, or multiple spans.

5. Requires determination of multiple integration constants, which becomes error-prone for many segments.

6. Not applicable when the material behavior is nonlinear (e.g., plastic bending), since $M = EI\kappa$ no longer holds.

(c) **Handling Discontinuous Loading Conditions:**

For discontinuous loading conditions such as point loads, point moments, or sudden changes in distributed load:

1. The beam is divided into multiple segments between discontinuities.

2. For each segment, a separate bending moment function $M(x)$ is written.

3. The governing equation

$$EI \frac{d^2 y}{dx^2} = M(x)$$

   is integrated twice for each segment.

4. At the boundaries between segments, continuity conditions are applied:

   - continuity of slope $\theta(x)$,
   - continuity of deflection $y(x)$.

5. Discontinuities are handled as follows:

   - A point load causes a jump in shear force, but bending moment remains continuous.
   - A point moment causes a jump in bending moment, while slope remains continuous.
   - A discontinuous distributed load $w(x)$ leads to piecewise definitions of $M(x)$.

6. Global boundary conditions (e.g., $y = 0$ at supports) are applied to solve for constants.

Alternatively, Macaulay functions or singularity functions may be used to integrate the entire beam in one expression, avoiding piecewise definitions.

(d) **Accounting for Shear Deformation** The classical double integration method **does not** account for shear deformation. It is derived from the Euler–Bernoulli beam theory, which assumes:

$$\gamma = 0 \qquad \text{(no shear strain)}.$$

Thus, total deflection is assumed to arise only from bending curvature. If shear deformation must be included, Timoshenko beam theory is used, in which:

$$v(x) = v_b(x) + v_s(x),$$

where

$$v_b(x) = \iint \frac{M(x)}{EI} \, dx \, dx, \qquad v_s(x) = \int \frac{V(x)}{kGA} \, dx.$$

Here:

- $V(x)$ = shear force,

- k = shear correction factor,

- G = shear modulus,

- A = cross-sectional area.

Thus, the double integration method contributes only the bending part of deflection and must be supplemented when shear effects are significant.

2. Find the slope at point A, the deflection at point C and E of the beam shown in Fig. 1. Comment on the results.
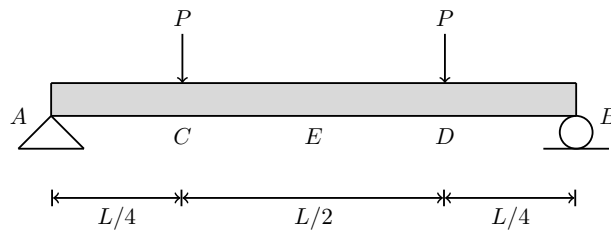

Figure 1: Simply supported beam with four-point loads

Solution: Due to the symmetric loading in the simply supported beam, we are going to analyse half of the beam using double integration method.

At section x from left, $(0 \leq x \leq \frac{L}{2})$

$$EI\frac{d^2y}{dx^2} = M = Px - P\left\langle x - \frac{L}{4} \right\rangle, \tag{1}$$

that implies

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{P}{EI}x - \frac{P}{EI}\left\langle x - \frac{L}{4} \right\rangle. \tag{2}$$

Integrating Eq. (2) with respect to x one can get that

$$\frac{dy}{dx} = \int \left(\frac{P}{EI}x - \frac{P}{EI}\left\langle x - \frac{L}{4} \right\rangle \right) dx + C_1 = \frac{P}{EI}\frac{x^2}{2} - \frac{P}{2EI}\left\langle x - \frac{L}{4} \right\rangle^2 + C_1. \tag{3}$$

Integrating Eq. (3) with respect to x leads to

$$y = \int \left(\frac{P}{EI}\frac{x^2}{2} - \frac{P}{2EI}\left\langle x - \frac{L}{4} \right\rangle^2 + C_1 \right) dx + C_2 = \frac{P}{EI}\frac{x^3}{6} - \frac{P}{6EI}\left\langle x - \frac{L}{4} \right\rangle^3 + C_1 x + C_2. \tag{4}$$

By applying the boundary conditions at $x = 0$, deflection, $y = 0$ due to hinge support, one can determine $C_2 = 0$ from Eq. (4). From the symmetric loading condition, one can see that slope at mid span is zero, i.e., $\frac{dy}{dx} = 0$ at $x = \frac{L}{2}$. Use of slope boundary condition in Eq. (3) leads to

$$0 = \frac{P}{EI}\frac{L^2}{8} - \frac{P}{2EI}\left\langle \frac{L}{2} - \frac{L}{4} \right\rangle^2 + C_1 \implies C_1 = -\frac{3PL^2}{32EI}.$$

Therefore,

$$\frac{dy}{dx} = \frac{P}{EI}\frac{x^2}{2} - \frac{P}{2EI}\left\langle x - \frac{L}{4} \right\rangle^2 - \frac{3PL^2}{32EI} \tag{5}$$

$$y = \frac{P}{EI}\frac{x^3}{6} - \frac{P}{6EI}\left\langle x - \frac{L}{4} \right\rangle^3 - \frac{3PL^2}{32EI}x \tag{6}$$

Slope at A can be determined by evaluating Eq. (5) at $x = 0$ as

$$\theta_A = \left[\frac{dy}{dx} \right]_{x=0} = -\frac{3PL^2}{32EI}. \quad \text{(-ve means } \circlearrowleft\text{)}$$

One can determine the deflection at C by using Eq. (6) at $x = L/4$ as

$$y_C = \frac{P}{EI}\frac{L^3}{64 \times 6} - \frac{3PL^2}{32EI}\frac{L}{4} = \frac{-PL^3}{48EI}. \quad \text{(-ve means } \downarrow\text{)}$$

Deflection at E can be found by using Eq. (6) at $x = L/2$ as

$$y_E = \frac{P}{EI}\frac{L^3}{8 \times 6} - \frac{P}{6EI}\frac{L^3}{64} - \frac{3PL^2}{32EI}\frac{L}{2} = \frac{-11PL^3}{384EI}. \quad \text{(-ve means } \downarrow\text{)}$$

3. Find the slope at point A and deflection at point C of the beam shown in Fig. 2. Comment on the results.
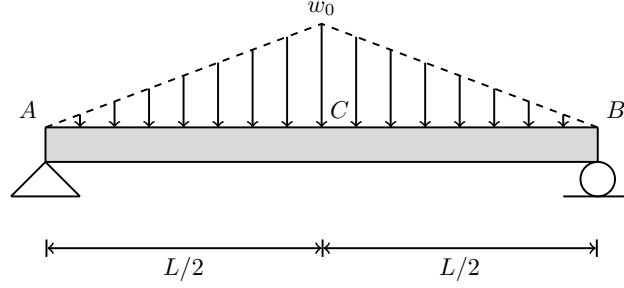

Figure 2

Solution: Due to the symmetric loading in the simply supported beam, we are going to analyze half of the beam using the double integration method.

At section x from left, $(0 \leq x < \frac{L}{2})$,

$$EI\frac{d^2y}{dx^2} = M = \frac{w_0L}{4}x - \frac{w_0}{3L}\langle x-0 \rangle^3 \tag{7}$$

$$\implies \frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{w_0L}{4EI}x - \frac{w_0}{3LEI}\langle x-0 \rangle^3 \tag{8}$$

Integrating Eq. (8) with respect to x, one can get,

$$\frac{dy}{dx} = \int \left(\frac{w_0L}{4EI}x - \frac{w_0}{3LEI}\langle x-0 \rangle^3 \right) dx + C_1 = \frac{w_0L}{4EI}\frac{x^2}{2} - \frac{w_0}{3LEI \times 4}\langle x-0 \rangle^4 + C_1 \tag{9}$$

Integrating Eq. (9) with respect to x, one can get,

$$y = \int \left(\frac{w_0L}{4EI}\frac{x^2}{2} - \frac{w_0}{3LEI \times 4}\langle x-0 \rangle^4 + C_1 \right) dx + C_2 = \frac{w_0L}{4EI}\frac{x^3}{6} - \frac{w_0}{3LEI \times 4 \times 5}\langle x-0 \rangle^5 + C_1 x + C_2 \tag{10}$$

By applying the boundary conditions at $x = 0, y = 0$ due to hinge support, one can find from Eq. (10) that $C_2 = 0$. Due to symmetric loading at mid-span, the slope becomes zero .ie at $x = L/2, \frac{dy}{dx} = 0$. By using this boundary condition in Eq. (9), one can get

$$0 = \frac{w_0L^3}{32EI} - \frac{w_0L^3}{12EI \times 16} + C_1 \implies C_1 = -\frac{5w_0L^3}{192EI}$$

Therefore,

$$\frac{dy}{dx} = \frac{w_0L}{4EI}\frac{x^2}{2} - \frac{w_0}{3LEI \times 4}\langle x-0 \rangle^4 - \frac{5w_0L^3}{192EI} \tag{11}$$

$$y = \frac{w_0L}{4EI}\frac{x^3}{6} - \frac{w_0}{3LEI \times 4 \times 5}\langle x-0 \rangle^5 - \frac{5w_0L^3}{192EI}x \tag{12}$$

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Slope at A can be determined by using Eq. (11) at $x = 0$,

$$\theta_A = \left[\frac{dy}{dx}\right]_{x=0} = -\frac{5w_0 L^3}{192EI} \quad \text{(-ve means $\circlearrowleft$)}$$

One can determine the deflection at C using Eq. (12) at $x = L/2$ as

$$y_C = \frac{w_0 L}{4EI} \frac{L^3}{6 \times 8} - \frac{w_0}{3LEI \times 4 \times 5} \frac{L^5}{2^5} - \frac{5w_0 L^3}{192EI} \frac{L}{2} = \frac{-w_0 L^4}{120EI}. \quad \text{(-ve means $\downarrow$)}$$

4. A cantilever beam of length L is subjected to a uniformly distributed load w_0 per unit length. Using the double integration method, determine the deflection at the free end of the beam.

Solution: Sign convention and governing equation Take the x–axis measured from the fixed end ($x = 0$) toward the free end ($x = L$). Positive y is upward. For Euler–Bernoulli beams

$$EI \frac{d^2 y}{dx^2} = M(x),$$

where E is Young's modulus, I the area moment of inertia, and $M(x)$ the internal bending moment (positive as per chosen sign convention).

Bending moment Consider a section at coordinate x. The portion of beam to the right (from x to L) carries the resultant of the UDL:

$$\text{resultant} = w_0(L - x),$$

acting at a distance $(L - x)/2$ from the section. Taking bending moment due to that load (negative because it causes clockwise curvature with our sign convention),

$$M(x) = -\frac{w_0(L - x)^2}{2}.$$

The differential equation becomes

$$EI \frac{d^2 y}{dx^2} = -\frac{w_0}{2}(L - x)^2.$$

Integrate once to get slope $\theta(x) = \frac{dy}{dx}(x)$:

$$EI \frac{dy}{dx}(x) = -\frac{w_0}{2} \int (L - x)^2 \, dx = -\frac{w_0}{2}\left[-\frac{(L - x)^3}{3}\right] + C_1 = \frac{w_0}{6}(L - x)^3 + C_1.$$

Thus

$$\theta(x) = \frac{w_0}{6EI}(L - x)^3 + \frac{C_1}{EI}.$$

Integrate again to get deflection $y(x)$:

$$EI\, y(x) = \frac{w_0}{6} \int (L-x)^3 \, dx + C_1 x + C_2 = \frac{w_0}{6} \left[-\frac{(L-x)^4}{4} \right] + C_1 x + C_2 = -\frac{w_0}{24}(L-x)^4 + C_1 x + C_2,$$

so

$$y(x) = -\frac{w_0}{24EI}(L-x)^4 + \frac{C_1}{EI}\, x + \frac{C_2}{EI}.$$

Boundary conditions: For a cantilever fixed at $x = 0$:

$$y(0) = 0, \qquad \theta(0) = 0.$$

Apply $\theta(0) = 0$:

$$0 = \frac{w_0}{6EI}L^3 + \frac{C_1}{EI} \quad \Longrightarrow \quad C_1 = -\frac{w_0 L^3}{6}.$$

Apply $y(0) = 0$:

$$0 = -\frac{w_0}{24EI}L^4 + \frac{C_2}{EI} \quad \Longrightarrow \quad C_2 = \frac{w_0 L^4}{24}.$$

Deflection at the free end: Evaluate $y(L)$:

$$y(L) = -\frac{w_0}{24EI}(L-L)^4 + \frac{C_1}{EI}L + \frac{C_2}{EI} = \frac{C_1 L + C_2}{EI}.$$

Substitute C_1, C_2:

$$y(L) = \frac{1}{EI}\left(-\frac{w_0 L^3}{6}L + \frac{w_0 L^4}{24} \right) = \frac{w_0 L^4}{EI}\left(-\frac{1}{6} + \frac{1}{24} \right) = -\frac{w_0 L^4}{8EI}.$$

Thus, the vertical deflection at the free end (downward) is

$$\boxed{\delta_{\text{tip}} = y(L) = -\frac{w_0 L^4}{8EI}}$$

The negative sign indicates deflection in the downward direction (with the upward-positive sign convention used above). The magnitude of tip deflection is

$$\boxed{|\delta_{\text{tip}}| = \frac{w_0 L^4}{8EI}.}$$

5. A beam of length L is fixed at both ends and carries a point load P at its center. Using the double integration method, determine the deflection at the center of the beam.

Solution: By symmetry the reactions at the two supports are equal:

$$R_A = R_B = \frac{P}{2}.$$

Also the deflection curve is symmetric about the midspan, so the slope at the center is zero:

$$\theta\left(\frac{L}{2}\right) = \frac{dy}{dx}\left(\frac{L}{2}\right) = 0.$$

Internal bending moment (left half): Consider the left half $0 \leq x \leq \frac{L}{2}$. For a section at coordinate x measured from the left end, the internal bending moment (taking positive for sagging) is obtained from the left reaction and the fixed-end moment contribution. Using the simple equilibrium for the left portion (no point load acts for $x < L/2$), we may write

$$M(x) = R_A x + M_A.$$

With $R_A = \frac{P}{2}$ and noting that M_A (the fixed-end moment at the left support) will be determined by the boundary/compatibility conditions (we avoid computing it separately here), we can instead write the explicit moment function for the left span by using the standard expression valid for this symmetric case:

$$M(x) = \frac{P}{2}x - \frac{PL}{8}, \qquad 0 \leq x \leq \frac{L}{2}.$$

(One can check $M(0) = -PL/8 = M_A$ and $M(L/2) = PL/8$ which is the midspan sagging moment.)

Governing equation and double integration: From Euler–Bernoulli beam theory,

$$EI\frac{d^2y}{dx^2} = M(x) = \frac{P}{2}x - \frac{PL}{8}, \qquad 0 \leq x \leq \frac{L}{2}.$$

Integrate once to obtain slope $\theta(x) = \frac{dy}{dx}(x)$:

$$EI\frac{dy}{dx}(x) = \int\left(\frac{P}{2}x - \frac{PL}{8}\right)dx = \frac{P}{4}x^2 - \frac{PL}{8}x + C_1.$$

Therefore

$$\theta(x) = \frac{P}{4EI}x^2 - \frac{PL}{8EI}x + \frac{C_1}{EI}.$$

Integrate again to obtain deflection:

$$EI\,y(x) = \int\left(\frac{P}{4}x^2 - \frac{PL}{8}x + C_1\right)dx = \frac{P}{12}x^3 - \frac{PL}{16}x^2 + C_1 x + C_2,$$

so

$$y(x) = \frac{P}{12EI}x^3 - \frac{PL}{16EI}x^2 + \frac{C_1}{EI}x + \frac{C_2}{EI}.$$

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Solution: Boundary and symmetry conditions: For the left end (clamped at $x = 0$):

$$y(0) = 0, \qquad \theta(0) = 0.$$

From these,

$$\theta(0) = 0 \implies 0 = \frac{C_1}{EI} \implies C_1 = 0,$$

$$y(0) = 0 \implies 0 = \frac{C_2}{EI} \implies C_2 = 0.$$

Hence the slope and deflection expressions simplify to

$$\theta(x) = \frac{P}{4EI}x^2 - \frac{PL}{8EI}x, \qquad y(x) = \frac{P}{12EI}x^3 - \frac{PL}{16EI}x^2.$$

Use the symmetry condition at midspan $(x = L/2)$:

$$\theta\left(\frac{L}{2}\right) = 0,$$

which is already satisfied by the expression above (this is consistent with the assumed reactions and moments).

Deflection at the center: Evaluate $y(x)$ at $x = \frac{L}{2}$:

$$y\left(\frac{L}{2}\right) = \frac{P}{12EI}\left(\frac{L}{2}\right)^3 - \frac{PL}{16EI}\left(\frac{L}{2}\right)^2 = \frac{PL^3}{96EI} - \frac{PL^3}{64EI}.$$

Combine terms:

$$y\left(\frac{L}{2}\right) = \frac{PL^3}{EI}\left(\frac{1}{96} - \frac{1}{64}\right) = -\frac{PL^3}{192EI}.$$

The deflection at the center (downward) is

$$\boxed{y\left(\frac{L}{2}\right) = -\frac{PL^3}{192\,EI}},$$

so the magnitude of the central deflection is

$$\boxed{\left|y(L/2)\right| = \frac{PL^3}{192\,EI}}.$$