Indian Institute of Technology Bhubaneswar

School of Infrastructure

Solution of Mid Semester (Autumn) Examination – 2025

Subject Name : Solid Mechanics	Subject Code: CE2L001

Instructions:

(1) Zeroth-order tensors or scalars are represented by small letters. For eg. a.

(2) First-order tensors or vectors are represented by bold small letters. For eg. $\boldsymbol{a}$.

(3) Second-order tensors are represented by bold capital letters. For eg. $\boldsymbol{A}$.

(4) Second-order identity tensor is represented by $\boldsymbol{I}$.

1. (a) Explain the following statement through theoretical derivations and examples with illustrative figures. "A tensor is independent of the coordinate system, but its components are not".

   (b) Consider a two-dimensional orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ in which a two-dimensional tensor $\boldsymbol{T}$ has the representation $\boldsymbol{T} = T_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, where $i, j = 1, 2$. The component matrix of $\boldsymbol{T}$ has values

$$[\boldsymbol{T}]_{\{\boldsymbol{e}_i\}} = \begin{bmatrix} 50 & 25 \\ 25 & 100 \end{bmatrix}.$$

   Consider a second basis $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*\}$ which is related to $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ by

$$\boldsymbol{e}_1^* = \frac{1}{2}\boldsymbol{e}_1 - \frac{\sqrt{3}}{2}\boldsymbol{e}_2, \qquad \boldsymbol{e}_2^* = \frac{\sqrt{3}}{2}\boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{e}_2.$$

   Find the values of $[\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}}$ in the $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*\}$ basis.

Solution:

(a) A tensor is invariant under changes in the coordinate system. The components of a tensor change with the change in the coordinate system such that the tensor remains independent of the coordinate system. Let us consider a second-order tensor $\boldsymbol{T}$. Let two orthonormal basis $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}_i^*\}$. Tensor is independent of the coordinate system implies that $\boxed{\boldsymbol{T} = T_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = T^*_{ij}(\boldsymbol{e}^*_i \otimes \boldsymbol{e}^*_j)}$, where T_{ij} and T^*_{ij} are the components of the tensor, $\boldsymbol{T}$ in the orthonormal basis $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}^*_i\}$, respectively.

The matrix representation of a second-order tensor is the components of the tensor in a particular coordinate system. A matrix is simply an array of numbers organized into rows and columns, and its entries change when the coordinate system (or basis) changes. Therefore, the matrix representation is not invariant, it depends on the chosen coordinate system.

The components of the tensor $\boldsymbol{T}$ in orthonormal basis $\{\boldsymbol{e}^*_i\}$ can be expressed as

$$\begin{aligned} T^*_{ij} &= \boldsymbol{e}^*_i \cdot \boldsymbol{T}\boldsymbol{e}^*_j \\ &= \boldsymbol{e}^*_i \cdot T_{kl}\left(\boldsymbol{e}_k \otimes \boldsymbol{e}_l\right)\boldsymbol{e}^*_j \\ &= T_{kl}\boldsymbol{e}^*_i \cdot \left(\boldsymbol{e}_l \cdot \boldsymbol{e}^*_j\right)\boldsymbol{e}_k \\ &= T_{kl}\left(\boldsymbol{e}^*_i \cdot \boldsymbol{e}_k\right)\left(\boldsymbol{e}_l \cdot \boldsymbol{e}^*_j\right)) \\ \implies T^*_{ij} &= Q_{ki}T_{kl}Q_{lj} \end{aligned} \tag{1}$$

where $\boxed{Q_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{Q}\boldsymbol{e}_j = \boldsymbol{e}_i \cdot \boldsymbol{e}^*_j}$. Defining the components of the tensor $\boldsymbol{T}$ in $\{\boldsymbol{e}_i\}$ basis, $[\boldsymbol{T}]_{\{\boldsymbol{e}_i\}} := (T_{ij})_{i,j\in(1,2,3)}$, the components of the tensor $\boldsymbol{T}$ in $\{\boldsymbol{e}^*_i\}$ basis, $[\boldsymbol{T}]_{\{\boldsymbol{e}^*_i\}} := \left(T^*_{ij}\right)_{i,j\in(1,2,3)}$, one can rewrite Eq. (1) as

$$[\boldsymbol{T}]_{\{\boldsymbol{e}^*_i\}} = [\boldsymbol{Q}]^T[\boldsymbol{T}]_{\{\boldsymbol{e}_i\}}[\boldsymbol{Q}]. \tag{2}$$

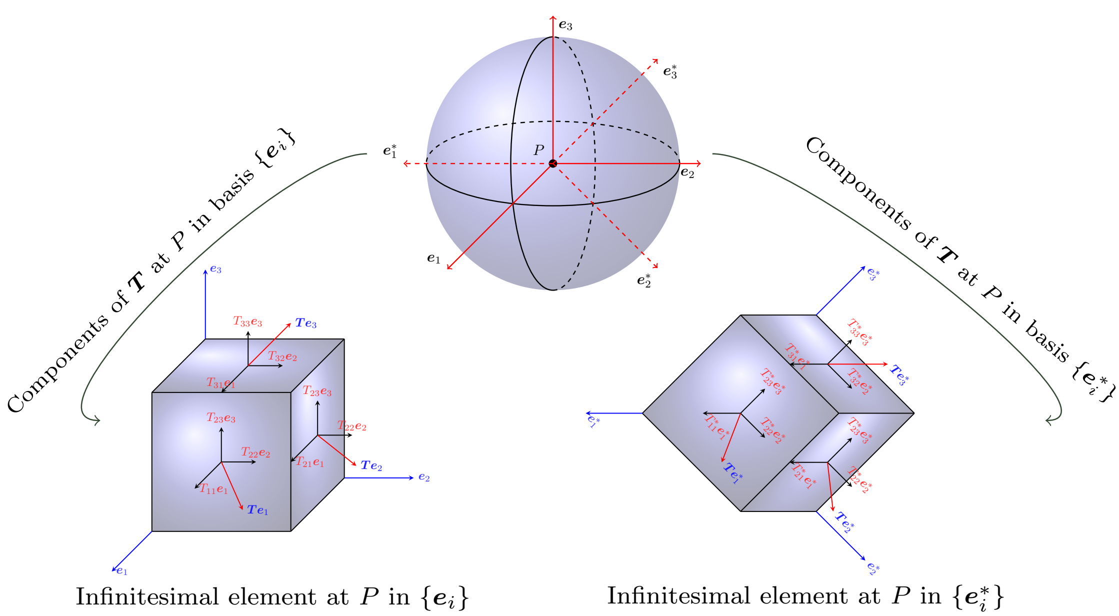


Figure 1: The representation of the tensor $\boldsymbol{T}$ in two different basis, $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}^*_i\}$.

(b) The components of $\boldsymbol{Q}$ in matrix form can be computed as

$$[\boldsymbol{Q}] = \begin{bmatrix} \boldsymbol{e}_1 \cdot \boldsymbol{e}_1^* & \boldsymbol{e}_1 \cdot \boldsymbol{e}_2^* \\ \boldsymbol{e}_2 \cdot \boldsymbol{e}_1^* & \boldsymbol{e}_2 \cdot \boldsymbol{e}_2^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1 \cdot \left(\frac{1}{2}\boldsymbol{e}_1 - \frac{\sqrt{3}}{2}\boldsymbol{e}_2\right) & \boldsymbol{e}_1 \cdot \left(\frac{\sqrt{3}}{2}\boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{e}_2\right) \\ \boldsymbol{e}_2 \cdot \left(\frac{1}{2}\boldsymbol{e}_1 - \frac{\sqrt{3}}{2}\boldsymbol{e}_2\right) & \boldsymbol{e}_2 \cdot \left(\frac{\sqrt{3}}{2}\boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{e}_2\right) \end{bmatrix}$$

$$\implies [\boldsymbol{Q}] = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad (\text{since } \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij})$$

Thus, one can find the components of the tensor $\boldsymbol{T}$ in $\{\boldsymbol{e}_i^*\}$ basis, $[\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}} := \left(T_{ij}^*\right)_{i,j \in (1,2)}$ as

$$\begin{aligned} [\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}} &= [\boldsymbol{Q}]^T [\boldsymbol{T}]_{\{\boldsymbol{e}_i\}} [\boldsymbol{Q}] \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 50 & 25 \\ 25 & 100 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{50-25\sqrt{3}}{2} & \frac{25+50\sqrt{3}}{2} \\ \frac{25-100\sqrt{3}}{2} & \frac{100+25\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{350-50\sqrt{3}}{4} & \frac{-50-50\sqrt{3}}{4} \\ \frac{-50-50\sqrt{3}}{4} & \frac{250+50\sqrt{3}}{4} \end{bmatrix} = \begin{bmatrix} 65.85 & -34.15 \\ -34.15 & 84.15 \end{bmatrix}. \end{aligned}$$

So, $\boxed{[\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}} = \begin{bmatrix} 65.85 & -34.15 \\ -34.15 & 84.15 \end{bmatrix}}$.

2. (a) Considering δ_{ij} is the Kronecker delta symbol and ϵ_{ijk} the Permutation symbol, simplify the following expressions:
(i) $\epsilon_{ijk} u_i u_j v_k$ and (ii) $\epsilon_{ijk} \delta_{jk}$.

(b) Suppose the vector field $\boxed{\boldsymbol{v}(x_1, x_2) = -x_1\, x_2 \boldsymbol{e}_1 + x_2\, \boldsymbol{e}_2}$ (where $x_1, x_2 \in \mathbb{R}$ and $x_2 > 0$) models the flow of a fluid. Is more fluid flowing into point (1,4) than flowing out? Plot the vector field and justify your answer.

(c) Vector field $\boxed{\boldsymbol{v}(x_1, x_2) = -\left(\dfrac{x_2}{x_1^2 + x_2^2}\right) \boldsymbol{e}_1 + \left(\dfrac{x_1}{x_1^2 + x_2^2}\right) \boldsymbol{e}_2}$ models the flow of a fluid. Show that if you drop a leaf into this fluid, as the leaf moves over time, the leaf does not rotate.

Solution: (a) (i)

$$\begin{aligned} \epsilon_{ijk} u_i u_j v_k &= (\boldsymbol{e}_i \times \boldsymbol{e}_j) \cdot \boldsymbol{e}_k\, u_i u_j v_k && (\text{since } \epsilon_{ijk} := (\boldsymbol{e}_i \times \boldsymbol{e}_j) \cdot \boldsymbol{e}_k) \\ &= (u_i \boldsymbol{e}_i \times u_j \boldsymbol{e}_j) \cdot v_k \boldsymbol{e}_k && (\text{since } u_i, u_j \text{ and } v_k \text{ are scalar quantity}) \\ &= (\boldsymbol{u} \times \boldsymbol{u}) \cdot \boldsymbol{v} = 0 && (\text{since } \boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0} \text{ and } \boldsymbol{0} \cdot \boldsymbol{v} = 0) \end{aligned}$$

Hence, $\boxed{\epsilon_{ijk} u_i u_j v_k = 0}$.

(ii) The permutation symbol, ϵ_{ijk} is coefficients of a rank-3 antisymmetric tensor and can be defined as

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } i,\ j,\ k \text{ are in cyclic order and not repeated} \\ 0, & \text{any two indices are the same} \\ -1, & \text{if } i,\ j,\ k \text{ are not in cyclic order and not repeated} \end{cases} \tag{3}$$

$$\begin{aligned} \epsilon_{ijk}\delta_{jk} &= \epsilon_{ijj} && \text{(using the contraction property of } \delta_{ij}) \\ &= 0_i && \text{(where } 0_i = 0 \text{ for } i = 1,2,3) \end{aligned}$$

Hence, $\boxed{\epsilon_{ijk}\delta_{jk} = 0_i}$.

(b) The divergence of the vector field is given by $\nabla \cdot \boldsymbol{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$. Let us calculate the partial derivatives:

$$\frac{\partial v_1}{\partial x_1} = \frac{\partial(-x_1 x_2)}{\partial x_1} = -x_2, \qquad \frac{\partial v_2}{\partial x_2} = \frac{\partial(x_2)}{\partial x_2} = 1.$$

Thus, the divergence is $\nabla \cdot \boldsymbol{v} = -x_2 + 1$. At point $(1,4)$, $x_2 = 4$. Therefore, $\boxed{\nabla \cdot \boldsymbol{v} = -4 + 1 = -3}$. A negative divergence indicates that more fluid is flowing into the point than flowing out. Therefore, based on the divergence calculation, we conclude that at point $(1,4)$, the fluid flow is characterized by more fluid flowing into the point than flowing out.

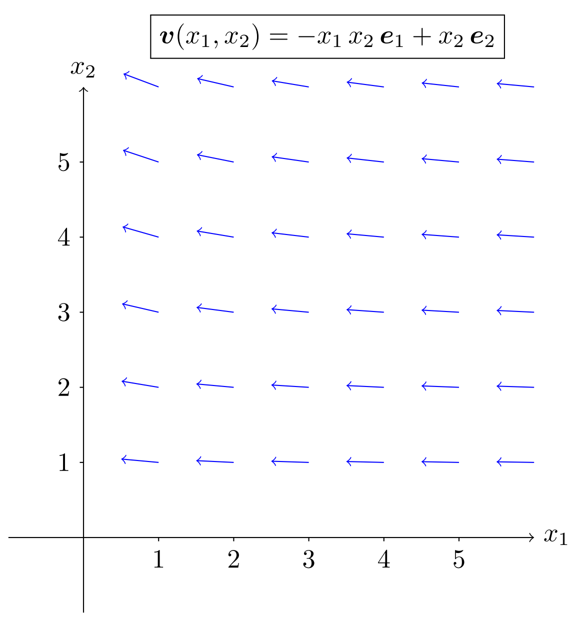


Figure 2: Vector field plot.

(c) Given the vector field $\boldsymbol{v}(x_1, x_2) = -\left(\frac{x_2}{x_1^2+x_2^2}\right)\boldsymbol{e}_1 + \left(\frac{x_1}{x_1^2+x_2^2}\right)\boldsymbol{e}_2$, we want to show that if we drop a leaf into this fluid, as the leaf moves over time, the leaf does not rotate.

For a leaf to not rotate, the curl of the velocity field $\boldsymbol{v}$ must be zero. The curl of the vector field $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2$ is given by $\nabla \times \boldsymbol{v} = \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\boldsymbol{e}_3$.

Let's calculate the partial derivatives:

$$\frac{\partial v_1}{\partial x_2} = \frac{\partial}{\partial x_2}\left(-\frac{x_2}{x_1^2 + x_2^2}\right) = -\frac{(x_1^2 + x_2^2) - x_2(2x_2)}{(x_1^2 + x_2^2)^2} = -\frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}$$

$$\frac{\partial v_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{x_1}{x_1^2 + x_2^2}\right) = \frac{(x_1^2 + x_2^2) - x_1(2x_1)}{(x_1^2 + x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}$$

The curl is $\nabla \times \boldsymbol{v} = \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\boldsymbol{e}_3 = \left(\frac{x_2^2 - x_1^2}{(x_1^2+x_2^2)^2} - \left(-\frac{x_1^2 - x_2^2}{(x_1^2+x_2^2)^2}\right)\right)\boldsymbol{e}_3$.

Simplifying, we get $\boxed{\nabla \times \boldsymbol{v} = \dfrac{0}{(x_1^2 + x_2^2)^2}\boldsymbol{e}_3 = 0}$. Since the curl of the velocity field is zero, the leaf will not rotate as it moves with the fluid.

3. (a) Explain the physical meaning of the components of the stress tensor using figures and examples.

   (b) The stress distribution in a body $\mathcal{B}$ is given as

$$\begin{aligned}\boldsymbol{\sigma} &= (3x_1^2 + Ax_1x_2 - 8x_2^2)\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + (-Bx_1^2 - 6x_1x_2 - 2x_2^2)\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)\\ &\quad + (2x_1^2 + x_1x_2 + Cx_2^2)\,\boldsymbol{e}_2 \otimes \boldsymbol{e}_2,\end{aligned}$$

   where all scalar multipliers and the constants A, B, and C have units of force per area-squared. For what values of A, B, and C does this stress distribution represent an equilibrium stress distribution? Assume zero body forces and no accelerations.

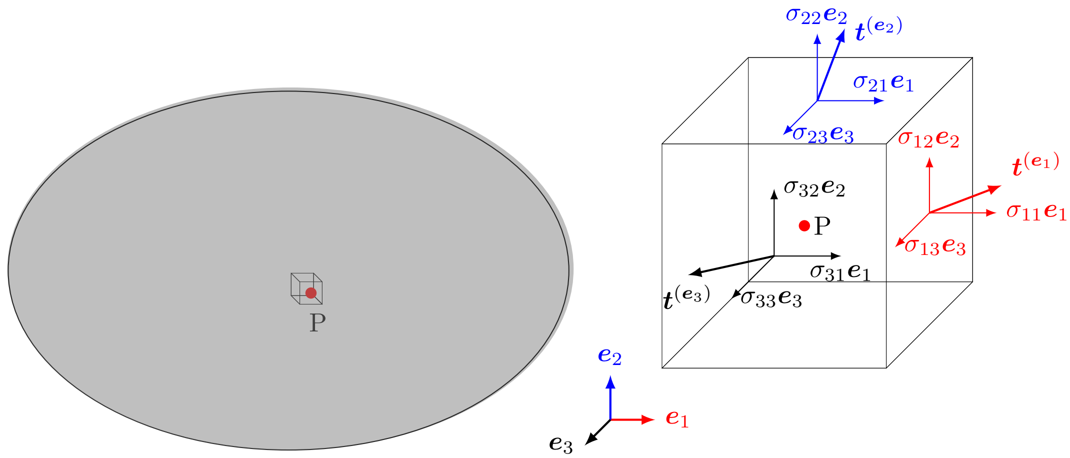
   (c) Prove the following:
   (i) $(\boldsymbol{e}_i \otimes \boldsymbol{e}_i) = \boldsymbol{I}$ and (ii) $(\boldsymbol{S}\boldsymbol{F})^T = \boldsymbol{F}^T\boldsymbol{S}^T$.

Solution:

(a) The stress tensor is a second-order tensor that describes the normal and shear stresses at a material point (see Fig. 3) of a continuum body. The components of the stress tensor, $\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, can be given by:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

where σ_{ij} represents the stress components.



(a) To define stress state at a point P of a solid body, an infinitesimal volume with three independent planes is considered.

(b) Stress state at point P of a solid body is defined by the stress tensor $\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$.

Figure 3

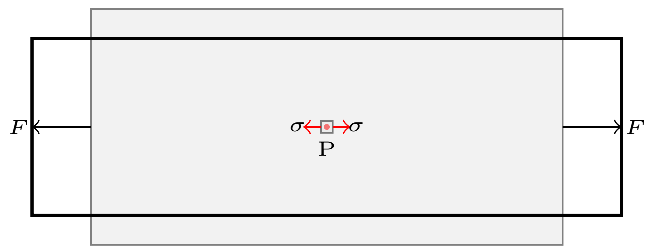
The diagonal components (σ_{11}, σ_{22}, σ_{33}) represent normal stresses. For example, σ_{11} represents the normal stress in the x_1 direction on a plane perpendicular to the x_1 axis. The off-diagonal components ($\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, $\sigma_{23} = \sigma_{32}$) represent shear stresses. For example, σ_{12} represents the shear stress in the x_2 direction on a plane perpendicular to the x_1 axis.

Examples

(i) Consider a bar under uniaxial tension or compression. If the bar is aligned with the x_1 axis, the stress tensor might simplify to:

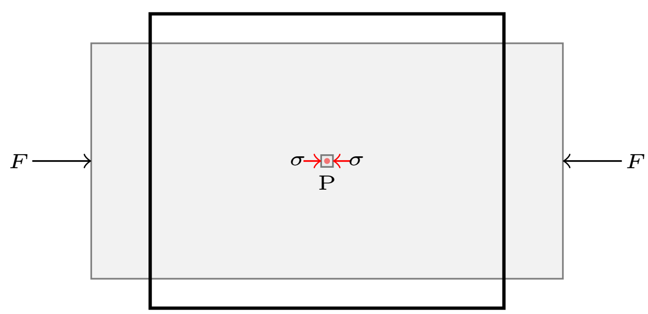
$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, $\sigma_{11} = \sigma$ represents the normal stress in the x_1 direction due to the tension applied in the x_1 direction. Here, $\sigma > 0$ and $\sigma < 0$ represent that the bar is under tension and compression, respectively.



Tensile stress developed at a point P of a solid body due to applied tensile force.

Figure 4



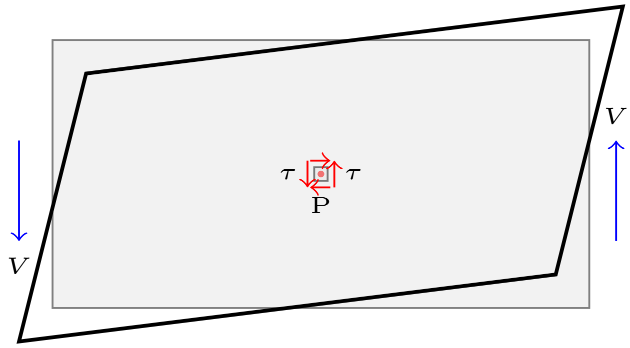
Compressive stress developed at a point P of a solid body due to applied compressive force.

Figure 5

(ii) Consider a material under pure shear stress, where the stress tensor is:

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, $\sigma_{21} = \sigma_{12} = \tau$ represents the shear stress in the x_1 direction on a plane perpendicular to the x_2 axis, and vice versa.



Shear stress developed at a point P of a solid body due to applied shear force.

Figure 6

(b) The equilibrium condition for a stress distribution in the absence of body forces and accelerations is given by the equation $\boxed{\nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \implies \sigma_{ij,j} = 0}$. For the given plane stress distribution (i.e., $i, j = 1, 2$), this translates to:

$$\sigma_{1j,j} = 0 \implies \sigma_{11,1} + \sigma_{12,2} = 0$$

$$\sigma_{2j,j} = 0 \implies \sigma_{21,1} + \sigma_{22,2} = 0$$

Given $\boldsymbol{\sigma}$, the components are:

$$\sigma_{11} = 3x_1^2 + Ax_1x_2 - 8x_2^2, \qquad \sigma_{22} = 2x_1^2 + x_1x_2 + Cx_2^2,$$
$$\sigma_{12} = \sigma_{21} = -Bx_1^2 - 6x_1x_2 - 2x_2^2.$$

So,

$$\sigma_{11,1} = \frac{\partial \sigma_{11}}{\partial x_1} = 6x_1 + Ax_2, \qquad \sigma_{12,2} = \frac{\partial \sigma_{12}}{\partial x_2} = -6x_1 - 4x_2$$
$$\sigma_{21,1} = \frac{\partial \sigma_{21}}{\partial x_1} = -2Bx_1 - 6x_2, \qquad \sigma_{22,2} = \frac{\partial \sigma_{22}}{\partial x_2} = x_1 + 2Cx_2$$

For equilibrium, we must have:

$$\sigma_{11,1} + \sigma_{12,2} = 0 \Rightarrow 6x_1 + Ax_2 - 6x_1 - 4x_2 = 0 \Rightarrow A = 4$$

$$\sigma_{21,1} + \sigma_{22,2} = 0 \Rightarrow -2Bx_1 - 6x_2 + x_1 + 2Cx_2 = 0 \Rightarrow (-2B + 1)x_1 + (2C - 6)x_2 = 0$$

For this to hold for all x_1 and x_2, $\boxed{-2B + 1 = 0}$ and $\boxed{2C - 6 = 0}$. Solving these gives $B = \frac{1}{2}$ and $C = 3$. Thus, the values of A, B, and C for which the stress distribution represents an equilibrium stress distribution are:

$$\boxed{A = 4, \quad B = \frac{1}{2}, \quad C = 3}$$

(c) Consider any arbitrary vector, $\boldsymbol{u} = u_i \boldsymbol{e}_i$.

(i)

$$\begin{aligned}(\boldsymbol{e}_i \otimes \boldsymbol{e}_i)\, \boldsymbol{u} &= (\boldsymbol{e}_i \cdot \boldsymbol{u})\, \boldsymbol{e}_i \qquad (\text{since} \;\; (\boldsymbol{e}_i \otimes \boldsymbol{e}_i)\, \boldsymbol{u} = (\boldsymbol{e}_i \cdot \boldsymbol{u})\, \boldsymbol{e}_i) \\ &= u_i\, \boldsymbol{e}_i \qquad (\text{since} \;\; u_i = \boldsymbol{u} \cdot \boldsymbol{e}_i = \boldsymbol{e}_i \cdot \boldsymbol{u}) \\ &= \boldsymbol{u} \qquad (\text{From the definition,} \;\; \boldsymbol{u} = u_i\, \boldsymbol{e}_i) \\ &= \boldsymbol{I}\, \boldsymbol{u} \qquad (\text{since} \;\; \boldsymbol{I}\, \boldsymbol{u} = \boldsymbol{u} \text{ from the definition of } \;\boldsymbol{I}) \,.\end{aligned}$$

Hence, $\boxed{(\boldsymbol{e}_i \otimes \boldsymbol{e}_i) = \boldsymbol{I}}$ (proved), as the above relation is valid for all $\boldsymbol{u}$.

(ii) Consider two arbitrary vectors, $\boldsymbol{u} = u_i \boldsymbol{e}_i$ and $\boldsymbol{v} = v_i \boldsymbol{e}_i$.

$$\begin{aligned}\boldsymbol{u} \cdot (\boldsymbol{SF})^T\, \boldsymbol{v} &= (\boldsymbol{SF})\, \boldsymbol{u} \cdot \boldsymbol{v} \qquad \left(\text{since} \;\; \boldsymbol{u} \cdot \boldsymbol{A v} \; = \boldsymbol{A}^T \boldsymbol{u} \cdot \boldsymbol{v} \text{ from the definition of dot product}\right) \\ &= (\boldsymbol{F u}) \cdot (\boldsymbol{S}^T \boldsymbol{v}) \qquad (\text{since} \;\; (\boldsymbol{SF})\, \boldsymbol{u} = \boldsymbol{S}\, (\boldsymbol{F u})) \\ &= \boldsymbol{u} \cdot \left(\boldsymbol{F}^T \boldsymbol{S}^T\right) \boldsymbol{v}\end{aligned}$$

Hence, $\boxed{(\boldsymbol{SF})^T = \boldsymbol{F}^T \boldsymbol{S}^T}$ (proved), as the above relation is valid for all $\boldsymbol{u}$ and $\boldsymbol{v}$.

4. (a) To test a glue, two plates are glued together as shown in Fig. 7. The bar formed by the joined plates is then subjected to tensile axial loads of 200 N. Using stress states and related **Mohr's circle**, determine the normal and shear stresses act on the plane where the plates are glued together (In other words, what stresses must the glue support?).

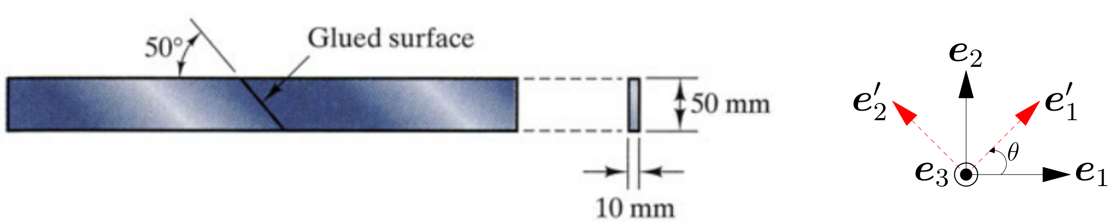


Figure 7

(b) Consider the unit ball $\mathcal{B} = \{\boldsymbol{x} |\, |\boldsymbol{x}| < 1\}$ with boundary $\partial \mathcal{B} = \{\boldsymbol{x} |\, |\boldsymbol{x}| = 1\}$. Using the divergence theorem, compute

$$\int_{\mathcal{B}} \operatorname{div} \left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|} \right) dV.$$

(c) Express the following divergence theorem in terms of components using index notation.

$$\int_{\partial\mathcal{B}} \boldsymbol{\sigma}\, \boldsymbol{n}\, dS = \int_{\mathcal{B}} \mathrm{div}\,(\boldsymbol{\sigma})\ dV.$$

Solution:

(a) To determine the stresses acting on any plane that is at an angle θ from the earlier coordinate axis, the following expression can be used,

$$[\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i'\}} = [\boldsymbol{Q}]^T[\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i\}}[\boldsymbol{Q}] \tag{4}$$

where $\boldsymbol{Q}$ is the orthogonal transformation tensor which is defined as

$$[\boldsymbol{Q}] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{5}$$

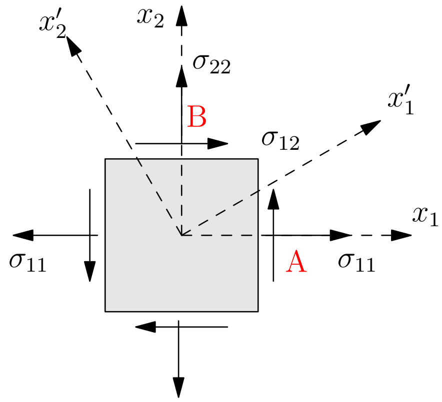
The components of the stress in the initial plane can be determined as

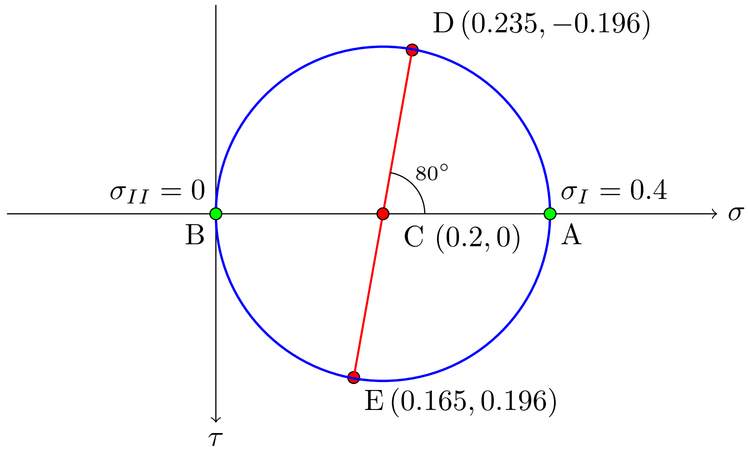
$$\begin{aligned} \sigma_{11} &= \frac{200}{50 \times 10} = 0.4\ \mathrm{N/mm^2} \\ \sigma_{22} &= 0 \\ \sigma_{12} &= 0 \end{aligned} \tag{6}$$

Thus the stresses acting on an inclined plane at an angle $\theta = 90° - 50° = 40°$ can be determined as

$$\begin{aligned} [\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i'\}} &= [\boldsymbol{Q}]^T[\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i\}}[\boldsymbol{Q}] \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(40) & \sin(40) \\ -\sin(40) & \cos(40) \end{bmatrix} \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(40) & -\sin(40) \\ \sin(40) & \cos(40) \end{bmatrix} \\ \implies [\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i'\}} &= \begin{bmatrix} 0.235 & -0.196 \\ -0.196 & 0.165 \end{bmatrix} \end{aligned}$$

The normal stress components on the inclined plane are 0.235 MPa and 0.165 MPa, while the shear stress is 0.196 MPa.





(b) The divergence theorem for a vector $\boldsymbol{v} = v_i\, \boldsymbol{e}_i$ can be written in index notation as

$$\int_{\mathcal{B}} v_{i,i}\, dV = \int_{\partial\mathcal{B}} v_i\, n_i\, dS.$$

For the given problem, identifying $\boldsymbol{v} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$, we can find

$$\begin{aligned}
\int_{\mathcal{B}} \operatorname{div}\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) dV &= \int_{\mathcal{B}} \left(\frac{x_i}{|\boldsymbol{x}|}\right)_{,i} dV \\
&= \int_{\partial\mathcal{B}} \left(\frac{x_i}{|\boldsymbol{x}|}\right) n_i\, dS \\
&= \int_{\partial\mathcal{B}} n_i\, n_i\, dS \\
&= \int_{\partial\mathcal{B}} 1\, dS \\
\implies \int_{\mathcal{B}} \operatorname{div}\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) dV &= 4\,\pi.
\end{aligned}$$

(c) Let $\{\boldsymbol{e}_i\}$ be the basis vectors in a Cartesian coordinate system. The stress tensor $\boldsymbol{\sigma}$ can be represented as $\boxed{\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j}$, where σ_{ij} are the components of the stress tensor. The unit normal vector $\boldsymbol{n}$ can be represented as $\boxed{\boldsymbol{n} = n_j\boldsymbol{e}_j}$ where n_j are the components of the normal vector. The surface integral can be written as:

$$\begin{aligned}
\int_{\partial\mathcal{B}} \boldsymbol{\sigma}\, \boldsymbol{n}\, dS &= \int_{\partial\mathcal{B}} (\sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\,(n_k\boldsymbol{e}_k)\; dS \\
&= \int_{\partial\mathcal{B}} \sigma_{ij} n_k \delta_{jk}\, \boldsymbol{e}_i\, dS, \quad (\text{since } (\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\, \boldsymbol{e}_k = (\boldsymbol{e}_j \cdot \boldsymbol{e}_k)\, \boldsymbol{e}_i = \delta_{jk}\, \boldsymbol{e}_i) \\
&= \int_{\partial\mathcal{B}} \sigma_{ij} n_j\, \boldsymbol{e}_i\, dS \quad (\text{using } n_k\delta_{jk} = n_j)\,.
\end{aligned}$$

The divergence of the stress tensor $\boldsymbol{\sigma}$ is: $\boxed{\operatorname{div}(\boldsymbol{\sigma}) = \frac{\partial \sigma_{ij}}{\partial x_j}\boldsymbol{e}_i = \sigma_{ij,j}\boldsymbol{e}_i}$, where $\sigma_{ij,j}$ denotes the partial derivative of σ_{ij} with respect to x_j. The volume integral can be written as:

$$\int_{\mathcal{B}} \operatorname{div}(\boldsymbol{\sigma})\, dV = \int_{\mathcal{B}} \sigma_{ij,j}\boldsymbol{e}_i\, dV$$

Thus, the divergence theorem in index notation is:

$$\boxed{\int_{\partial\mathcal{B}} \sigma_{ij} n_j\, dS = \int_{\mathcal{B}} \sigma_{ij,j}\, dV}.$$

5. The components of stress at a critical location in a structural member are given by:

$$[\boldsymbol{\sigma}] = 150 \times \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{MPa}$$

with respect to a rectangular Cartesian coordinate system with base vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$.
(a) Evaluate the mean normal pressure, $p = -(1/3)\operatorname{tr}(\boldsymbol{\sigma})$.
(b) Evaluate the components of the stress deviator, $\boldsymbol{\sigma}^{\text{dev}} = \boldsymbol{\sigma} + p\boldsymbol{I}$.
(c) Evaluate the traction vector $\boldsymbol{t}$ associated with the unit normal $\boxed{\boldsymbol{n} = (1/\sqrt{2})\,\boldsymbol{e}_1 + (1/\sqrt{2})\,\boldsymbol{e}_2}$. Show that $\boldsymbol{n}$ is a principal direction. What is the corresponding principal value of stress?
(d) Determine all the principal values and principal directions of stress.
(e) Evaluate the Von-Mises equivalent stress,

$$\sigma^{\text{eq}} := \sqrt{\frac{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}{2}}$$

where σ_I, σ_{II} and σ_{III} are the principal stresses.

Solution: (a) For the given stress tensor, $\operatorname{tr}(\boldsymbol{\sigma}) = 150(3 + 3 - 1) = 150 \times 5 = 750$ MPa.
So, the mean normal pressure: $\boxed{p = -(1/3)\operatorname{tr}(\boldsymbol{\sigma}) = -\frac{1}{3} \times 750 = -250 \text{ MPa}}$.

(b) The components of the stress deviator $\boldsymbol{\sigma}^{\text{dev}}$ is given by: $[\boldsymbol{\sigma}^{\text{dev}}] = [\boldsymbol{\sigma}] + p[\boldsymbol{I}]$.

$$\implies [\boldsymbol{\sigma}^{\text{dev}}] = 150 \times \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} + (-250) \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $\boxed{[\boldsymbol{\sigma}^{\text{dev}}] = \begin{bmatrix} 200 & 300 & 0 \\ 300 & 200 & 0 \\ 0 & 0 & -400 \end{bmatrix} \text{ MPa}}$

(c) The traction vector $\boldsymbol{t}$ associated with $\boldsymbol{n} = \frac{1}{\sqrt{2}}\boldsymbol{e}_1 + \frac{1}{\sqrt{2}}\boldsymbol{e}_2$ is given by $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$. The components of $\boldsymbol{t}$ can be computed as

$$[\boldsymbol{t}] = [\boldsymbol{\sigma}][\boldsymbol{n}] = 150 \times \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = 750 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \text{ MPa}$$

So, $\boldsymbol{t} = 750\,\boldsymbol{n}$. Hence, $\boldsymbol{n} = \frac{1}{\sqrt{2}}\boldsymbol{e}_1 + \frac{1}{\sqrt{2}}\boldsymbol{e}_2$ is the principal direction (proved) with the corresponding principal stress value of $\sigma_I = 750$ MPa.

(d) The previous part resulted in one principal stress value of $\boxed{\sigma_I = 750}$ MPa and a principal direction, $\boxed{\boldsymbol{n}^{(1)} = \frac{1}{\sqrt{2}}\boldsymbol{e}_1 + \frac{1}{\sqrt{2}}\boldsymbol{e}_2}$. For the given stress tensor, the shear stress components $\boxed{\sigma_{13} = \sigma_{31} = 0}$ and $\boxed{\sigma_{23} = \sigma_{32} = 0}$. Hence, $\boxed{\sigma_{III} = \sigma_{33} = -150}$ MPa is a principal stress with a principal direction, $\boxed{\boldsymbol{n}^{(3)} = \boldsymbol{e}_3}$.

From the first invariant of the stress tensor, $\boxed{\text{tr}(\boldsymbol{\sigma}) = \sigma_I + \sigma_{II} + \sigma_{III} = \sigma_{11} + \sigma_{22} + \sigma_{33}}$. Hence,

$$\sigma_I + \sigma_{II} + \sigma_{III} = 750 \implies \sigma_{II} = 150 \text{ MPa} \qquad (\text{since } \sigma_I = 750 \text{ MPa and } \sigma_{III} = -150 \text{ MPa}).$$

Hence, the second principal stress value is $\boxed{\sigma_{II} = 150 \text{ MPa}}$ with the principal direction.

$$\boldsymbol{n}^{(2)} = \boldsymbol{n}^{(3)} \times \boldsymbol{n}^{(1)} = \boldsymbol{e}_3 \times \left(\frac{1}{\sqrt{2}}\boldsymbol{e}_1 + \frac{1}{\sqrt{2}}\boldsymbol{e}_2\right) \implies \boldsymbol{n}^{(2)} = \frac{1}{\sqrt{2}}\boldsymbol{e}_2 - \frac{1}{\sqrt{2}}\boldsymbol{e}_1$$

So, the principal values and principal directions of stress are:

$$\sigma_I = 750 \text{ MPa}, \qquad \sigma_{II} = 150 \text{ MPa} \qquad \text{and } \sigma_{III} = -150 \text{ MPa}$$
$$\boldsymbol{n}^{(1)} = \frac{1}{\sqrt{2}}\boldsymbol{e}_1 + \frac{1}{\sqrt{2}}\boldsymbol{e}_2, \qquad \boldsymbol{n}^{(2)} = -\frac{1}{\sqrt{2}}\boldsymbol{e}_1 + \frac{1}{\sqrt{2}}\boldsymbol{e}_2, \text{ and } \qquad \boldsymbol{n}^{(3)} = \boldsymbol{e}_3.$$

(e) Von-Mises equivalent stress can be calculated as

$$\begin{aligned} \sigma^{\text{eq}} &= \sqrt{\frac{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}{2}} \\ &= \sqrt{\frac{(750 - 150)^2 + (150 + 150)^2 + (-150 - 750)^2}{2}} = \sqrt{\frac{(600)^2 + (300)^2 + (-900)^2}{2}} \end{aligned}$$
$$\implies \sigma^{\text{eq}} = 793.725 \text{ MPa}.$$