Indian Institute of Technology Bhubaneswar

School of Infrastructure

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Notes on Tensors

Course Instructor: Dr. Mohammad Masiur Rahaman

<u>Notations :</u>
Zeroth-order tensors or scalars are represented by small letters. For eg. $a$
First-order tensors or vectors are represented by bold small letters. For eg. $\boldsymbol{a}$.
Second-order tensors are represented by bold capital letters. For eg. $\boldsymbol{A}$

# 1 Tensors

Tensors can be classified by their order: zeroth-order, first-order, second-order, and higher. Among these, second-order tensors hold particular significance in practical applications, especially in fields like continuum mechanics, where they play a crucial role in describing various physical quantities.

# 2 Zeroth and first-order tensors

A tensor of order zero is called a scalar, while a first-order tensor is a vector.

# 3 Second order tensors

A second-order tensor is a linear transformation that maps vectors to vectors as shown in Figure below.

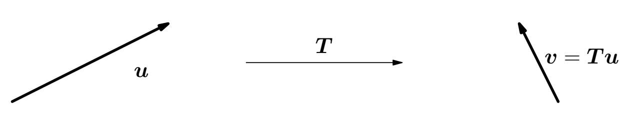


Figure 1: The schematic representation of a linear transformation of a vector $\boldsymbol{u}$ into $\boldsymbol{v}$

Remarks : Special cases of the second order tensor

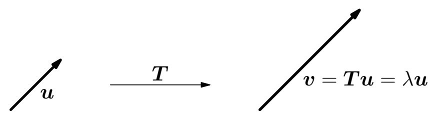


Figure 2: The schematic representation of a linear transformation of a vector $\boldsymbol{u}$ into $\boldsymbol{v}$ whose magnitutde is increased by $\lambda$, but the direction is not changed. The vectors that do not rotate upon transformation with the second-order tensor is called as eigen vectors. Here, the constant $\lambda$ is the eigen value and $\boldsymbol{u}$ is an eigen vector.

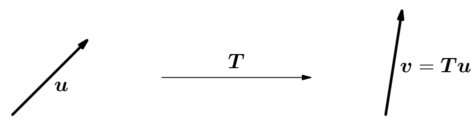


Figure 3: The schematic representation of a linear transformation of a vector $\boldsymbol{u}$ into $\boldsymbol{v}$ whose direction is changed, but the magnitude is not changed

Assume $\boldsymbol{T}$ is a second-order tensor that maps vector $\boldsymbol{u}$ to vector $\boldsymbol{v}$ i.e.

$$\boldsymbol{T}\boldsymbol{u} = \boldsymbol{v}. \tag{1}$$

Then to become a linear transformation it must satisfy the property

$$\boldsymbol{T}(a\boldsymbol{u} + b\boldsymbol{v}) = a\,\boldsymbol{T}\boldsymbol{u} + b\,\boldsymbol{T}\boldsymbol{v},\ \forall \boldsymbol{u}, \boldsymbol{v} \in V,\ \text{and } a, b \in \mathbb{R} \tag{2}$$

For example, the second-order tensor can be considered a linear operator that transforms all the vectors into its mirror image as shown in Fig. 4, where $\boldsymbol{T}$ is the second-order tensor.

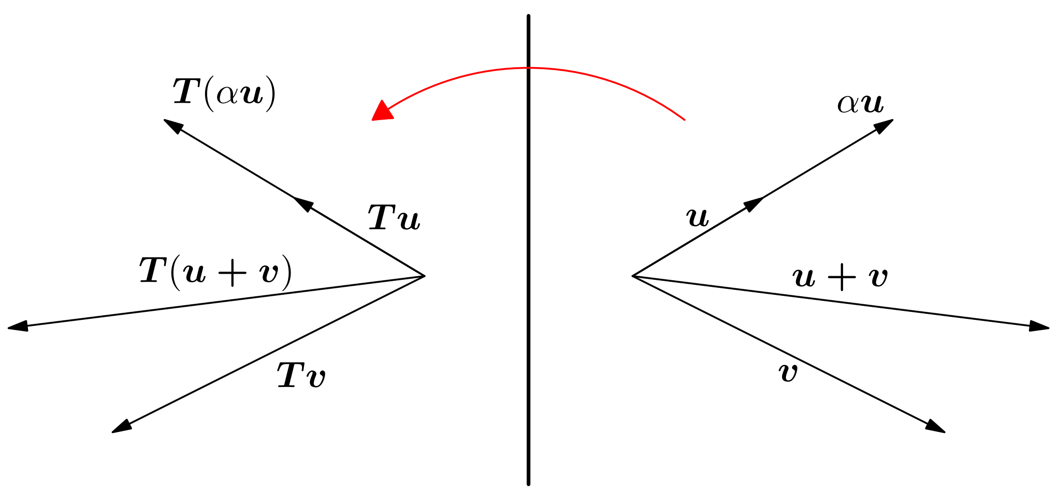


Figure 4: The schematic representation of a linear transformation of vectors into its mirror image.

**Note**: Identity tensor: A function $\boldsymbol{I} : V \to V$ defined as $\boldsymbol{I}\boldsymbol{u} = \boldsymbol{u}$, $\forall \boldsymbol{u} \in V$.

In Eq. (1), one can choose $\boldsymbol{u}$ as $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ to get three vectors that can be expressed as

$$\boldsymbol{T}\boldsymbol{e}_1 = \alpha_1\boldsymbol{e}_1 + \alpha_2\boldsymbol{e}_2 + \alpha_3\boldsymbol{e}_3$$
$$\boldsymbol{T}\boldsymbol{e}_2 = \alpha_4\boldsymbol{e}_1 + \alpha_5\boldsymbol{e}_2 + \alpha_6\boldsymbol{e}_3$$
$$\boldsymbol{T}\boldsymbol{e}_3 = \alpha_7\boldsymbol{e}_1 + \alpha_8\boldsymbol{e}_2 + \alpha_9\boldsymbol{e}_3,$$

where $\alpha_i$. $i = 1$ to 9 are scalar constants. Renaming $\alpha_i$ as $\boldsymbol{T}_{ij}$ with both $i$ and $j$ ranging from 1 to 3 we get

$$\boldsymbol{T}\boldsymbol{e}_j = T_{ij}\boldsymbol{e}_i \tag{3}$$

where, $T_{ij}$ is the component of $\boldsymbol{T}\boldsymbol{e}_j$ along the direction $\boldsymbol{e}_i$. This can be schematically represented as shown in Fig. 5 below.

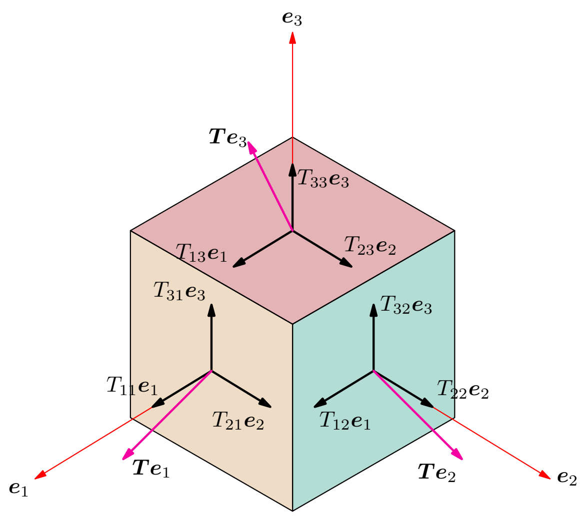


Figure 5: Schematic representation of Tensor

Now, considering the dot product on both sides of Eq. (3) with $\boldsymbol{e}_i$ for some particular $i$, we get

$$\boldsymbol{e}_i \cdot \boldsymbol{T}\boldsymbol{e}_j = \boldsymbol{e}_i \cdot T_{kj}\boldsymbol{e}_k = T_{kj}(\boldsymbol{e}_i \cdot \boldsymbol{e}_k) = T_{kj}\delta_{ik} = T_{ij} \implies \boldsymbol{e}_i \cdot \boldsymbol{T}\boldsymbol{e}_j = T_{ij} \tag{4}$$

So, we get

$$\boldsymbol{e}_i \cdot \boldsymbol{T}\boldsymbol{e}_j = T_{ij}. \tag{5}$$

Product of two second-order tensors $\boldsymbol{RS}$ is the composition of two operations $\boldsymbol{R}$, and $\boldsymbol{S}$, with $\boldsymbol{S}$ operating first and is defined as

$$(\boldsymbol{RS})\boldsymbol{u} = \boldsymbol{R}(\boldsymbol{S}\boldsymbol{u}), \forall\, \boldsymbol{u} \in V \tag{6}$$

Since $\boldsymbol{RS}$ is a linear transformation mapping vectors to vectors, $\boldsymbol{RS}$ is a second-order tensor.

# 4 Tensor product

Tensor product or dyadic product of two vectors $\boldsymbol{a}$, and $\boldsymbol{b}$ are defined as

$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}, \, \forall \, \boldsymbol{c} \in V. \tag{7}$$

Note that $\boldsymbol{a} \otimes \boldsymbol{b}$, maps a third vector $\boldsymbol{c}$ to another vector $(\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$. Hence, if we could prove that the transformation is linear, $\boldsymbol{a} \otimes \boldsymbol{b}$ can be considered as a second-order tensor. For any arbitrary scalars $c$, and $d \in \mathbb{R}$, and $\boldsymbol{x}$, and $\boldsymbol{y} \in V$ we have

$$\begin{aligned}(\boldsymbol{a} \otimes \boldsymbol{b})(c\boldsymbol{x} + d\boldsymbol{y}) &= [\boldsymbol{b} \cdot (c\boldsymbol{x} + d\boldsymbol{y})]\boldsymbol{a} \\ &= (c\boldsymbol{b} \cdot \boldsymbol{x} + d\boldsymbol{b} \cdot \boldsymbol{y})\boldsymbol{a} \\ &= c(\boldsymbol{b} \cdot \boldsymbol{x})\boldsymbol{a} + d(\boldsymbol{b} \cdot \boldsymbol{y})\boldsymbol{a}) \\ &= c[(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{x}] + d[(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{y}] \end{aligned} \tag{8}$$

which proves that $\boldsymbol{a} \otimes \boldsymbol{b}$ is a linear transformation and hence $\boldsymbol{a} \otimes \boldsymbol{b}$ is a second-order tensor. Action of $\boldsymbol{T}$, a second-order tensor on any arbitrary vector $\boldsymbol{u}$ can be written as

$$\boldsymbol{T}\boldsymbol{u} = v_i \, \boldsymbol{e}_i \quad (\text{let } \boldsymbol{v} = \boldsymbol{T}\boldsymbol{u} = v_i \, \boldsymbol{e}_i) \tag{9}$$

$$\begin{aligned} &= [\boldsymbol{e}_i \cdot (\boldsymbol{T}\boldsymbol{u})]\boldsymbol{e}_i \\ &= [\boldsymbol{e}_i \cdot (\boldsymbol{T} u_j \boldsymbol{e}_j)]\boldsymbol{e}_i \\ &= [u_j \, \boldsymbol{e}_i \cdot (\boldsymbol{T}\boldsymbol{e}_j)]\boldsymbol{e}_i \\ &= \{(\boldsymbol{u} \cdot \boldsymbol{e}_j) \, [\boldsymbol{e}_i \cdot (\boldsymbol{T}\boldsymbol{e}_j)]\}\boldsymbol{e}_i \\ &= [\boldsymbol{e}_i \cdot (\boldsymbol{T}\boldsymbol{e}_j)](\boldsymbol{u} \cdot \boldsymbol{e}_j) \, \boldsymbol{e}_i \\ &= [T_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)]\boldsymbol{u} \end{aligned}$$

$$\implies \boldsymbol{T} = T_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \tag{10}$$

The second order tensor $\boldsymbol{T}$ can be written in terms of components and base vectors as follows

$$\begin{aligned}\boldsymbol{T} = T_{11}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1) + T_{12}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2) + T_{13}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_3) + T_{21}(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + T_{22}(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) \\ + T_{23}(\boldsymbol{e}_2 \otimes \boldsymbol{e}_3) + T_{31}(\boldsymbol{e}_3 \otimes \boldsymbol{e}_1) + T_{32}(\boldsymbol{e}_3 \otimes \boldsymbol{e}_2) + T_{33}(\boldsymbol{e}_3 \otimes \boldsymbol{e}_3) \end{aligned} \tag{11}$$

Also the transpose of the tensor $\boldsymbol{T}$ can be written as $\boldsymbol{T}^T = T_{ji}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$.
Using Eq. (5), we can find the components of $\boldsymbol{a} \otimes \boldsymbol{b}$ also. Let $\boldsymbol{T} = \boldsymbol{a} \otimes \boldsymbol{b}$

$$\begin{aligned} T_{ij} &= \boldsymbol{e}_i \cdot (\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{e}_j \\ &= \boldsymbol{e}_i \cdot (\boldsymbol{b} \cdot \boldsymbol{e}_j)\boldsymbol{a} \\ &= (\boldsymbol{a} \cdot \boldsymbol{e}_i)(\boldsymbol{b} \cdot \boldsymbol{e}_j) \\ &= a_i b_j \end{aligned} \tag{12}$$

The tensor product can be defined for other higher-order tensors as follows:

$$\begin{aligned}(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c} &= (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}\\ (\boldsymbol{a}\otimes\boldsymbol{b})(\boldsymbol{c}\otimes\boldsymbol{d}) &= (\boldsymbol{b}\cdot\boldsymbol{c})(\boldsymbol{a}\otimes\boldsymbol{d})\\ (\boldsymbol{a}\otimes\boldsymbol{b})(\boldsymbol{c}\otimes\boldsymbol{d}\otimes\boldsymbol{e}) &= (\boldsymbol{b}\cdot\boldsymbol{c})(\boldsymbol{a}\otimes\boldsymbol{d}\otimes\boldsymbol{e})\\ (\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{c})(\boldsymbol{d}\otimes\boldsymbol{e}\otimes\boldsymbol{f}) &= (\boldsymbol{c}\cdot\boldsymbol{d})(\boldsymbol{a}\otimes\boldsymbol{b}\otimes\boldsymbol{e}\otimes\boldsymbol{f})\end{aligned} \tag{13}$$

**Note:** To work with tensors of any order, it is crucial to understand the fundamental operation of tensor contraction, where adjacent vectors in the tensor expressions are contracted or "dotted" together.

## 4.1 Simple contraction

This operation is known as simple contraction, as it reduces the order of the tensors involved. For example, when a second-order tensor contracts with a first-order tensor, the result is a tensor of order one. Let us take an example as given below

$$\begin{aligned}\boldsymbol{Ta} &= T_{ij}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)a_k\boldsymbol{e}_k\\ &= T_{ij}a_k(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\boldsymbol{e}_k\\ &= T_{ij}a_k\delta_{jk}\boldsymbol{e}_i\\ \implies \boldsymbol{Ta} &= T_{ij}a_j\boldsymbol{e}_i\end{aligned} \tag{14}$$

The simple contraction of two second-order tensors can be expressed as

$$\begin{aligned}\boldsymbol{TS} &= T_{ij}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)S_{kl}(\boldsymbol{e}_k\otimes\boldsymbol{e}_l)\\ &= T_{ij}S_{kl}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)(\boldsymbol{e}_k\otimes\boldsymbol{e}_l)\\ &= T_{ij}S_{kl}(\boldsymbol{e}_j\cdot\boldsymbol{e}_k)(\boldsymbol{e}_i\otimes\boldsymbol{e}_l)\\ &= T_{ij}S_{kl}\delta_{jk}(\boldsymbol{e}_i\otimes\boldsymbol{e}_l)\\ \implies \boldsymbol{TS} &= T_{ij}S_{jl}(\boldsymbol{e}_i\otimes\boldsymbol{e}_l)\end{aligned} \tag{15}$$

From the above, the simple contraction of two second-order tensors results in another second-order tensor. If we write $\boldsymbol{A}=\boldsymbol{TS}$, then we can write

$$A_{ij} = T_{ik}S_{kj}, \text{ where } \boldsymbol{A} = A_{ij}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j) \tag{16}$$

**Note:**

$$\begin{aligned}\boldsymbol{AB} &\neq \boldsymbol{BA}\\ (\boldsymbol{AB})\boldsymbol{C} &= \boldsymbol{A}(\boldsymbol{BC})\\ \boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) &= \boldsymbol{AB}+\boldsymbol{AC}\end{aligned}$$

## 4.2 Double contraction

Double contraction, as the name suggests, reduces the tensor order by twice as much as simple contraction. In simple contraction, the sum of the orders of two tensors is reduced by two; in double contraction, this sum is reduced by four. The double contraction operation is typically denoted by a colon (:), for example,

$\boldsymbol{T} : \boldsymbol{S}$. The double contraction of simple tensors can be defined as

$$(\boldsymbol{a} \otimes \boldsymbol{b}) : (\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d})$$

Thus the double contraction of $\boldsymbol{T}$ and $\boldsymbol{S}$ can be described as

$$\begin{aligned} \boldsymbol{T} : \boldsymbol{S} &= T_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : S_{kl}(\boldsymbol{e}_k \otimes \boldsymbol{e}_l) \\ &= T_{ij}S_{kl}(\boldsymbol{e}_i \cdot \boldsymbol{e}_k)(\boldsymbol{e}_j \cdot \boldsymbol{e}_l) \\ &= T_{ij}S_{kl}\delta_{ik}\delta_{jl} \\ \implies \boldsymbol{T} : \boldsymbol{S} &= T_{ij}S_{ij}, \end{aligned} \tag{17}$$

which is a scalar quantity (zeroth order tensor).

# 5 Properties of a second order tensor

## 5.1 Trace of a tensor

The trace of a second-order tensor $\boldsymbol{T}$, denoted by $\mathrm{tr}\boldsymbol{T}$, is a scalar equal to the sum of the diagonal elements of its matrix representation. Thus, it can be expressed as

$$\mathrm{tr}(\boldsymbol{T}) = \boldsymbol{I} : \boldsymbol{T} = T_{ii} \tag{18}$$

where $\boldsymbol{I}$ is a second order identity tensor.

## 5.2 Norm of a tensor

The norm of a second order tensor $\boldsymbol{T}$, denoted by $|\boldsymbol{T}|$ or $||\boldsymbol{T}||$ is defined as

$$|\boldsymbol{T}| = \sqrt{\boldsymbol{T} : \boldsymbol{T}} \tag{19}$$

## 5.3 Determinant of a Tensor

The determinant of a second-order tensor $\boldsymbol{T}$ is defined as

$$\mathrm{det}\boldsymbol{T} = \epsilon_{ijk}T_{1i}T_{2j}T_{3k} \tag{20}$$

Following are some of the standard properties of the determinant of a second-order tensor.

$$\begin{aligned} \mathrm{det}(\boldsymbol{AB}) &= \mathrm{det}(\boldsymbol{A})\mathrm{det}(\boldsymbol{B}) \\ \mathrm{det}(\boldsymbol{A}^T) &= \mathrm{det}(\boldsymbol{A}) \\ \mathrm{det}(\alpha\, \boldsymbol{A}) &= \alpha^3\mathrm{det}(\boldsymbol{A}) \end{aligned} \tag{21}$$

### 5.4 Orthogonal Tensor

An orthogonal tensor $\boldsymbol{T}$ is a second-order tensor which follows the following conditions

$$\boldsymbol{T}\boldsymbol{u} \cdot \boldsymbol{T}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V \tag{22}$$

It can be schematically shown in Fig. below

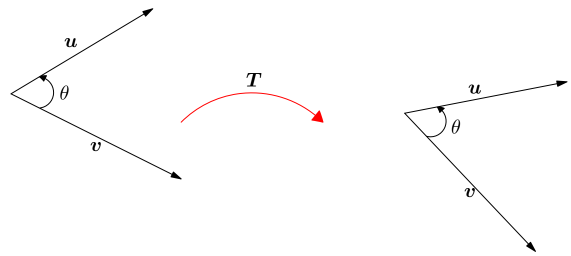


Figure 6: Schematic representation of Orthogonal tensor

Thus the magnitude of the vectors and the angle between the vectors is preserved.

### 5.5 Symmetric and Skew Tensors

A tensor $\boldsymbol{T}$ is said to be symmetric if it is identical to the transposed tensor, $\boldsymbol{T} = \boldsymbol{T}^T$, and skew (antisymmetric) if $\boldsymbol{T} = -\boldsymbol{T}^T$. Any tensor can be uniquely divided into its symmetric ($\boldsymbol{S}$) and skew symmetric part ($\boldsymbol{W}$) as

$$\begin{aligned} \boldsymbol{S} &= \frac{1}{2}\left(\boldsymbol{T} + \boldsymbol{T}^T\right) = \boldsymbol{S}^T \\ \boldsymbol{W} &= \frac{1}{2}\left(\boldsymbol{T} - \boldsymbol{T}^T\right) = -\boldsymbol{W}^T \end{aligned} \tag{23}$$

An antisymmetric tensor $\boldsymbol{W}$ when broken down into its components, can be written as

$$[\boldsymbol{W}] = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix} \tag{24}$$

Therefore, an antisymmetric second-order tensor has 3 independent components, namely $W_{12}$, $W_{13}$, and $W_{23}$. Indeed, a skew tensor can always be written in the form

$$\boldsymbol{W}\boldsymbol{u} = \boldsymbol{w} \times \boldsymbol{u}, \quad \forall \boldsymbol{u} \in V \tag{25}$$

where $\boldsymbol{w}$ is an axial vector of the skew-symmetric tensor $\boldsymbol{W}$. Thus, the components of $\boldsymbol{W}$ can be expressed as

$$\begin{aligned} W_{ij} &= \boldsymbol{e}_i \cdot (\boldsymbol{W}\boldsymbol{e}_j) \\ &= \boldsymbol{e}_i \cdot (\boldsymbol{w} \times \boldsymbol{e}_j) = \boldsymbol{e}_i \cdot (w_k \boldsymbol{e}_k \times \boldsymbol{e}_j) \\ &= w_k \boldsymbol{e}_i \cdot (\epsilon_{kjp} \boldsymbol{e}_p) = w_k \epsilon_{kjp} \delta_{ip} \\ \implies W_{ij} &= -w_k \epsilon_{ijk} \end{aligned} \tag{26}$$

Thus the components of the axial vector $w$ can be defined in terms of the skew-symmetric tensor $\boldsymbol{W}$ as

$$w_1 = -W_{23}, \;\; w_2 = -W_{13}, \;\;\; w_3 = -W_{12} \tag{27}$$

This is schematically shown in Fig. 7. The axial movement of the screw under a rotation can be considered as a practical example of this (see Fig. 8).

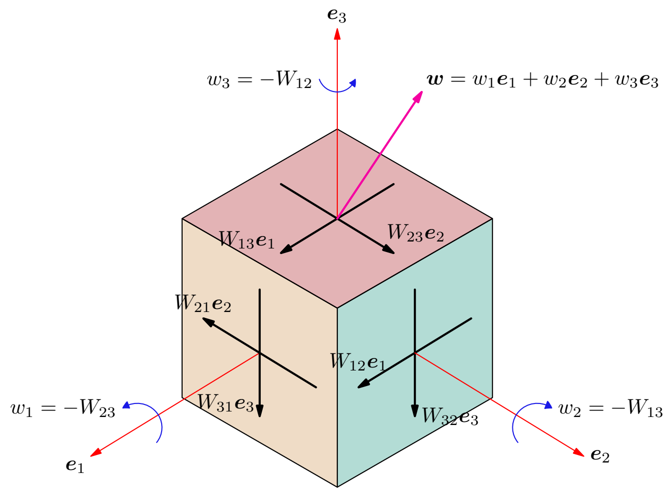


Figure 7: Schematic representation of anti-symmetric (skew) tensor

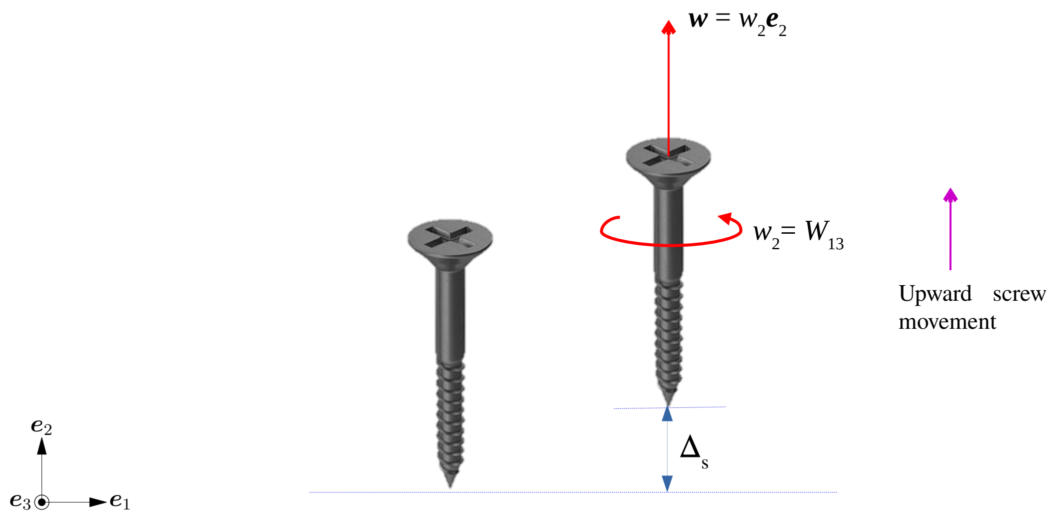


Figure 8: Schematic representation of the axial movement of the screw under a rotation. $\Delta_\mathrm{s}$ denotes the upward displacement of the screw.

### 5.6 Volumetric and Deviatoric Tensors

Every tensor $\boldsymbol{T}$ can be decomposed into its so-called volumetric (spherical) part and its deviatoric part, i.e

$$\begin{aligned}\boldsymbol{T}_{\text{vol}} &= \frac{1}{3}\text{tr}(\boldsymbol{T})\boldsymbol{I} \\ \boldsymbol{T}_{\text{dev}} &= \boldsymbol{T} - \boldsymbol{T}_{\text{vol}}\end{aligned} \tag{28}$$

Some important properties of the spherical and deviatoric tensors are

$$\begin{aligned}\text{tr}(\boldsymbol{T}_{\text{dev}}) &= 0 \\ (\boldsymbol{T}_{\text{dev}})_{\text{vol}} &= 0 \\ \boldsymbol{T}_{\text{vol}} : \boldsymbol{T}_{\text{dev}} &= 0\end{aligned}$$

## 6 Del operator

The del operator is defined by

$$\nabla = \boldsymbol{e}_1\frac{\partial()}{\partial x_1} + \boldsymbol{e}_2\frac{\partial()}{\partial x_2} + \boldsymbol{e}_3\frac{\partial()}{\partial x_3} \tag{29}$$

The del operator is a vector differential operator. In the summation convention, it can be written as

$$\nabla = \boldsymbol{e}_i\frac{\partial()}{\partial x_i} \tag{30}$$

The del operator can be used for different operation in mathematics like gradient, divergence and curl.

### 6.1 Gradient of a tensor

It has to be noted that the gradient operation will increase the order of the tensor. The gradient of a first-order tensor (vector) is a second-order tensor as given below

$$\nabla\boldsymbol{a} = (\nabla \otimes \boldsymbol{a})^T \tag{31}$$

The tensor product of the del operator with the vector $\boldsymbol{a}$ can be expressed as

$$\begin{aligned}(\nabla \otimes \boldsymbol{a}) &= \boldsymbol{e}_i\frac{\partial}{\partial x_i} \otimes (a_j\boldsymbol{e}_j) \\ &= \frac{\partial a_j}{\partial x_i}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\end{aligned} \tag{32}$$

The transpose of the tensor $(\nabla \otimes \boldsymbol{a})$ can be written as

$$(\nabla \otimes \boldsymbol{a})^T = \frac{\partial a_i}{\partial x_j}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \tag{33}$$

Thus from Eq. (31) and Eq. (33), the gradient of the vector $\boldsymbol{a}$ can be written as

$$
\begin{aligned}
\nabla \boldsymbol{a} &= \frac{\partial a_i}{\partial x_j}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \\
&= a_{i,j}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \\
\implies \nabla \boldsymbol{a} &= a_{i,j}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \qquad (34)
\end{aligned}
$$

The gradient of a second-order tensor field $\boldsymbol{T}$ is defined in a manner analogous to that of the gradient of a vector. The gradient of a second order tensor is a third-order tensor as

$$
\nabla \boldsymbol{T} = \frac{\partial \boldsymbol{T}}{\partial x_k} \otimes \boldsymbol{e}_k = \frac{\partial T_{ij}}{\partial x_k} (\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k) \qquad (35)
$$

### 6.2 Divergence of second-order tensor

It must be noted that the divergence operation will always reduce the tensor order. Thus, the divergence of a second-order tensor is a vector. It is not possible to take the divergence of a scalar. The divergence of a second order tensor, $\boldsymbol{T}$ can be written as

$$
\begin{aligned}
\nabla \cdot \boldsymbol{T} &= \boldsymbol{e}_i \frac{\partial}{\partial x_i} \cdot \big(T_{mn}(\boldsymbol{e}_m \otimes \boldsymbol{e}_n)\big) \\
&= \frac{\partial T_{mn}}{\partial x_i} \boldsymbol{e}_i \big(\boldsymbol{e}_m \otimes \boldsymbol{e}_n\big) \\
&= \frac{\partial T_{mn}}{\partial x_i} (\boldsymbol{e}_i \cdot \boldsymbol{e}_m) \boldsymbol{e}_n \\
&= \frac{\partial T_{mn}}{\partial x_i} \delta_{im} \boldsymbol{e}_n \\
&= \frac{\partial T_{in}}{\partial x_i} \boldsymbol{e}_n \\
&= T_{in,i} \boldsymbol{e}_n \qquad (36)
\end{aligned}
$$

## Relation between Kronecker delta and permutation symbol

Kronecker delta and the permutation symbol are related by the identity, known as the $\epsilon - \delta$ identity

$$
\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \qquad (37)
$$

The operations in vector notation do not have commutative or associative properties. However, all terms in index notation are scalars (although the term may represent multiple scalars in multiple equations), and only multiplication/division and addition/subtraction operations are defined. Therefore, commutative and associative properties hold.

$$\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{b} \times \boldsymbol{a}$$

$$a_i b_j = b_j a_i$$

$$a_i (b_j c_k) = (a_i b_j) c_k$$

## Example of Second order Stress Tensor

Define the traction vector $\boldsymbol{t}$ acting on a surface element within a material to be the force acting on that element divided by the area of the element,shown in figure below. Let $\boldsymbol{n}$ be a vector normal to the surface. The **stress** $\boldsymbol{\sigma}$ is defined as that second-order tensor which map $\boldsymbol{n}$ onto $\mathbf{t}$, according to

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} \qquad \textbf{The Stress Tensor} \tag{38}$$

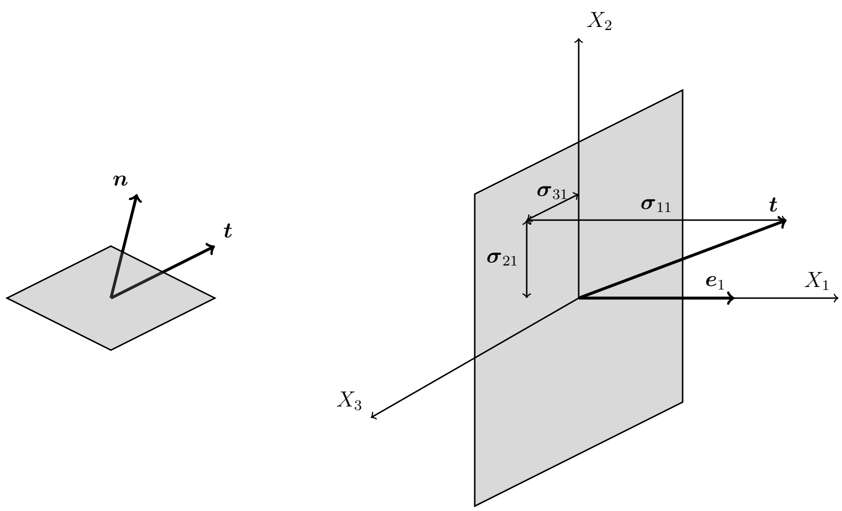


Figure 9: Stress acting on a plane