



## Indian Institute of Technology Bhubaneswar

### School of Infrastructure

Session: Autumn 2024

Solid Mechanics (CE2L001)

Solution of Assignment No. 1

#### Notations :

Zeroth-order tensors or scalars are represented by small letters. For eg.  $a$

First-order tensors or vectors are represented by bold small letters. For eg.  $\mathbf{a}$ .

Second-order tensors are represented by bold capital letters. For eg.  $\mathbf{A}$

1. Simplify the following expressions using the contraction property of  $\delta$  and the  $\epsilon - \delta$  relation.

(a)  $\delta_{ij}\delta_{jk}\delta_{kl}\delta_{lm}\delta_{mn}$  (b)  $\epsilon_{jkq}\epsilon_{jkq}$ .

**Solution:** (a) The Kronecker delta function  $\delta_{ij}$  is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (1)$$

To simplify the expression  $\delta_{ij}\delta_{jk}\delta_{kl}\delta_{lm}\delta_{mn}$ , use the contraction properties of the Kronecker delta  $\delta_{ij}$ .

$$\begin{aligned} \delta_{ij}\delta_{jk}\delta_{kl}\delta_{lm}\delta_{mn} &= \delta_{ik}\delta_{kl}\delta_{lm}\delta_{mn} && (\text{since } \delta_{ij}\delta_{jk} = \delta_{ik}) \\ &= \delta_{il}\delta_{lm}\delta_{mn} && (\text{since } \delta_{ik}\delta_{kl} = \delta_{il}) \\ &= \delta_{im}\delta_{mn} && (\text{since } \delta_{il}\delta_{lm} = \delta_{im}) \\ &= \delta_{in} = 0 && (\text{since, } i \neq n) \quad (\text{using Eq.(1)}). \end{aligned}$$

(b) The  $\epsilon - \delta$  relation is given by

$$\epsilon_{jkq}\epsilon_{jkq} = \delta_{jj}\delta_{kk} - \delta_{jk}\delta_{jk} = (3) \times (3) - \delta_{jj} \quad (2)$$

$$= 9 - 3 = 6 \quad (\text{since } \delta_{jj}, \delta_{kk} = 3) \quad (3)$$

2. Consider two vectors  $\mathbf{a}$  and  $\mathbf{b}$  whose matrix of components relative to an orthonormal basis  $\{\mathbf{e}_i\}$  are

$$[\mathbf{a}] = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \text{ and } [\mathbf{b}] = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

Compute (a)  $|\mathbf{a}|$ , magnitude of  $\mathbf{a}$  vector, (b) the angle between  $\mathbf{a}$  and  $\mathbf{b}$  vectors (c) the area of the parallelogram bounded by  $\mathbf{a}$  and  $\mathbf{b}$  vectors (d)  $\mathbf{b} \times \mathbf{a}$ .

**Solution:**

- (a) The norm or magnitude of a vector can be determined as

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14} \approx 3.741.$$

- (b) The angle ( $\theta$ ) between two vectors  $\mathbf{a}$  and  $\mathbf{b}$  can be determined as

$$\theta = \cos^{-1} \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right), \quad (4)$$

where  $\mathbf{a} \cdot \mathbf{b}$  can be found as

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= 3 \times 1 + (-2) \times 4 + 1 \times (-2) = 3 - 8 - 2 = -7. \end{aligned} \quad (5)$$

One can determine  $|\mathbf{b}|$  as

$$|\mathbf{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{1 + 16 + 4} = \sqrt{21} = 4.582. \quad (6)$$

From Eq. (4), the angle between vectors  $\mathbf{a}$  and  $\mathbf{b}$  can be determined as

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right) = \cos^{-1} \left( \frac{-7}{\sqrt{14}\sqrt{21}} \right) \\ \Rightarrow \theta &\approx 114.09^\circ. \end{aligned}$$

- (c) The area of the parallelogram,  $A$  bounded by  $\mathbf{a}$  and  $\mathbf{b}$  can be determined

$$A = |\mathbf{a} \times \mathbf{b}|, \quad (7)$$

Let,  $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ . The components of the vector  $\mathbf{v} = v_i \mathbf{e}_i$  can be determined as  $v_i = \epsilon_{ijk} a_j b_k$ . Thus, the components along each basis vector can be determined as

$$\begin{aligned} v_1 &= \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2 = 4 - 4 = 0, \\ v_2 &= \epsilon_{231} a_3 b_1 + \epsilon_{213} a_1 b_3 = a_3 b_1 - a_1 b_3 = 1 - 6 = -5, \\ v_3 &= \epsilon_{312} a_1 b_2 + \epsilon_{321} a_2 b_1 = a_1 b_2 - a_2 b_1 = 12 - 3 = 9. \end{aligned}$$

$$\implies \mathbf{a} \times \mathbf{b} = \mathbf{v} = v_i \mathbf{e}_i = 0\mathbf{e}_1 + 7\mathbf{e}_2 + 14\mathbf{e}_3.$$

Thus the area,  $A$  can be found as

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{7^2 + 14^2} = \sqrt{49 + 196} \approx 15.65.$$

(d) The vector  $\mathbf{b} \times \mathbf{a}$  can be determined as

$$\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) = -7\mathbf{e}_2 - 14\mathbf{e}_3.$$

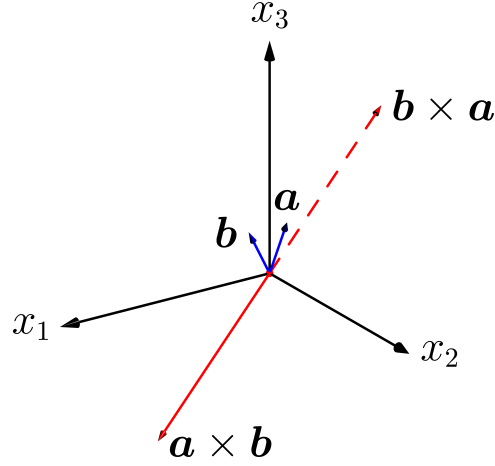


Figure 1: Schematic representation of the cross product between two vectors  $\mathbf{a}$  and  $\mathbf{b}$

3. Rewrite the expression  $\epsilon_{mni}a_ib_jc_md_ne_j$  in direct notation using the scalar and cross products of vectors.

**Solution:** To simplify the expression  $\epsilon_{mni}a_ib_jc_md_ne_j$  using direct notation as follows:

$$\begin{aligned} \epsilon_{mni}a_ib_jc_md_ne_j &= a_ib_j(\mathbf{c} \times \mathbf{d})_i \mathbf{e}_j \quad (\text{since } \epsilon_{mni}c_md_n = (\mathbf{c} \times \mathbf{d})_i) \\ &= (a_i(\mathbf{c} \times \mathbf{d})_i)(b_j \mathbf{e}_j) \quad (\text{Rearranging the terms}) \\ &= (\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d}))\mathbf{b} \quad (\text{since } a_i(\mathbf{c} \times \mathbf{d})_i = \mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})) \end{aligned}$$

4. For a two-dimensional (2D) problem, let the components of a second-order tensor  $\mathbf{A}$  be  $A_{11} = 2, A_{12} = 4 = A_{21}, A_{22} = 5$ . Let the components of a vector  $\mathbf{v}$  be  $v_1 = 3, v_2 = -1$ , all in the same orthogonal basis. Compute the components of the vector,  $\mathbf{w} = \mathbf{A} \mathbf{v}$  using the relation  $w_i = A_{ij}v_j$ .

**Solution:** The components of  $\mathbf{w} = \mathbf{A} \mathbf{v}$  are related by  $w_i = A_{ij}v_j$ , where  $\mathbf{w} = w_i \mathbf{e}_i$ ,  $\mathbf{v} = v_j \mathbf{e}_j$  and  $\mathbf{A} = A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ . Thus the individual components of  $\mathbf{w}$  along each basis vector can be determined as

$$\begin{aligned} w_1 &= A_{11}v_1 + A_{12}v_2 = 2 \times 3 + 4 \times (-1) = 2 \\ w_2 &= A_{21}v_1 + A_{22}v_2 = 4 \times 3 + 5 \times (-1) = 7. \end{aligned}$$

5. Show that  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ .

**Solution:**

$$\begin{aligned} (\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_i &= \epsilon_{ijk} a_j (\mathbf{b} \times \mathbf{c})_k \\ &= \epsilon_{ijk} a_j \epsilon_{klm} b_l c_m \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_j b_l c_m \\ &= a_j b_i c_j - a_j b_j c_i \\ &= (\mathbf{a} \cdot \mathbf{c}) b_i - (\mathbf{a} \cdot \mathbf{b}) c_i \\ \Rightarrow \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \end{aligned}$$

6. Show that  $\mathbf{a} \times (\nabla \times \mathbf{a}) = \frac{1}{2} \nabla (\mathbf{a} \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla) \mathbf{a}$

**Solution:**

$$\begin{aligned}
\mathbf{a} \times (\nabla \times \mathbf{a}) &= a_i \mathbf{e}_i \times \left( \frac{\partial a_k}{\partial x_j} \mathbf{e}_j \times \mathbf{e}_k \right) & (\mathbf{a} = a_i \mathbf{e}_i, \nabla \times \mathbf{a} = \frac{\partial a_k}{\partial x_j} \mathbf{e}_j \times \mathbf{e}_k) \\
&= \epsilon_{jkm} a_i \frac{\partial a_k}{\partial x_j} (\mathbf{e}_i \times \mathbf{e}_m) & (\mathbf{e}_j \times \mathbf{e}_k = \epsilon_{jkm} \mathbf{e}_m) \\
&= \epsilon_{jkm} \epsilon_{imn} a_i \frac{\partial a_k}{\partial x_j} \mathbf{e}_n & (\mathbf{e}_i \times \mathbf{e}_m = \epsilon_{imn} \mathbf{e}_n) \\
&= a_i \frac{\partial a_k}{\partial x_j} (\delta_{jn} \delta_{ki} - \delta_{ji} \delta_{kn}) \mathbf{e}_n & (\epsilon_{jkm} \epsilon_{imn} = \delta_{jn} \delta_{ki} - \delta_{ji} \delta_{kn}) \\
&= a_i \frac{\partial a_i}{\partial x_j} \mathbf{e}_j - a_i \frac{\partial a_k}{\partial x_i} \mathbf{e}_k & (\delta_{jn} \mathbf{e}_n = \mathbf{e}_j, \delta_{ji} \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_i}) \\
&= \frac{1}{2} \frac{\partial (a_i a_i)}{\partial x_j} \mathbf{e}_j - a_i \frac{\partial a_k}{\partial x_i} \mathbf{e}_k & \left( a_i \frac{\partial a_i}{\partial x_j} = \frac{1}{2} \frac{\partial (a_i a_i)}{\partial x_j} \right) \\
&= \frac{1}{2} \frac{\partial (a_i a_i)}{\partial x_j} \mathbf{e}_j - a_i \frac{\partial \mathbf{a}}{\partial x_i} & (a_k \mathbf{e}_k = \mathbf{a}) \\
&= \frac{1}{2} \nabla (\mathbf{a} \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla) \mathbf{a} & (\nabla f = \mathbf{e}_j \frac{\partial f}{\partial x_j}, a_i \frac{\partial \mathbf{a}}{\partial x_i} = \mathbf{a} \cdot \nabla \mathbf{a})
\end{aligned}$$

7. Prove the following identities:

a)  $\nabla \cdot (\nabla \phi) = \nabla^2 \phi$ , where  $\phi$  is a scalar field.

**Solution:**

$$\begin{aligned}
\nabla \cdot (\nabla \phi) &= \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot \mathbf{e}_j \frac{\partial \phi}{\partial x_j} & (\text{since } \nabla = \mathbf{e}_i \frac{\partial}{\partial x_i}, \nabla \phi = \mathbf{e}_i \frac{\partial \phi}{\partial x_i}) \\
&= \frac{\partial}{\partial x_i} \left( \frac{\partial \phi}{\partial x_j} \right) (\mathbf{e}_i \cdot \mathbf{e}_j) & (\text{note } \frac{\partial}{\partial x_i} \left( \frac{\partial \phi}{\partial x_j} \right) \text{ is a scalar quantity}) \\
&= \frac{\partial^2 \phi}{\partial x_i \partial x_j} \delta_{ij} & (\text{since } \frac{\partial}{\partial x_i} \left( \frac{\partial \phi}{\partial x_j} \right) = \frac{\partial^2 \phi}{\partial x_i \partial x_j} \text{ and } \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}) \\
&= \frac{\partial^2 \phi}{\partial x_i^2} & (\text{using contraction property of } \delta_{ij}) \\
&= \nabla^2 \phi & (\text{since } \nabla^2 \phi := \frac{\partial^2 \phi}{\partial x_i^2})
\end{aligned}$$

b)  $\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$ , where  $\mathbf{a}$  is a vector field.

**Solution:**

$$\begin{aligned}
[\nabla \times (\nabla \times \mathbf{a})]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times \mathbf{a})_k & ([\nabla \times \mathbf{v}]_i &= \epsilon_{ijk} \frac{\partial v_k}{\partial x_j}) \\
&= \epsilon_{ijk} \frac{\partial}{\partial x_j} \left( \epsilon_{klm} \frac{\partial a_m}{\partial x_l} \right) & ((\nabla \times \mathbf{a})_k &= \epsilon_{klm} \frac{\partial a_m}{\partial x_l}) \\
&= \epsilon_{ijk} \epsilon_{klm} \frac{\partial^2 a_m}{\partial x_j \partial x_l} & \left( \frac{\partial}{\partial x_j} (\epsilon_{klm} \frac{\partial a_m}{\partial x_l}) &= \epsilon_{klm} \frac{\partial^2 a_m}{\partial x_j \partial x_l} \right) \\
&= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial^2 a_m}{\partial x_j \partial x_l} & (\epsilon_{ijk} \epsilon_{klm} &= \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \\
&= \frac{\partial^2 a_j}{\partial x_i \partial x_j} - \frac{\partial^2 a_i}{\partial x_j^2} & (\delta\text{-contractions, } \frac{\partial^2 a_j}{\partial x_j \partial x_i} &= \frac{\partial^2 a_j}{\partial x_i \partial x_j}) \\
&= \frac{\partial}{\partial x_i} \left( \frac{\partial a_j}{\partial x_j} \right) - \nabla^2 a_i & ([\nabla(\nabla \cdot \mathbf{a})]_i &= \frac{\partial}{\partial x_i} (\frac{\partial a_j}{\partial x_j}), [\nabla^2 \mathbf{a}]_i = \frac{\partial^2 a_i}{\partial x_j^2}) \\
&= [\nabla(\nabla \cdot \mathbf{a})]_i - [\nabla^2 \mathbf{a}]_i & (\nabla f = \mathbf{e}_i \frac{\partial f}{\partial x_i}, \nabla \cdot \mathbf{a} &= \frac{\partial a_i}{\partial x_i}) \\
\Rightarrow \nabla \times (\nabla \times \mathbf{a}) &= \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}
\end{aligned}$$

8. Consider a cyclone in the northern hemisphere described by the velocity vector field of the wind:

$$\mathbf{v}(x, y) = x \mathbf{e}_1 - y^2 \mathbf{e}_2$$

(a) Calculate the divergence and curl of the vector field  $\mathbf{v}(x, y)$ .

**Solution:** (a) **Divergence:**

$$\begin{aligned}
\mathbf{v}(x, y) &= x \mathbf{e}_1 - y^2 \mathbf{e}_2 & (\text{given}) \\
\nabla \cdot \mathbf{v} &= \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} & (\nabla \cdot \mathbf{v} = \partial v_i / \partial x_i, \mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2) \\
&= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y^2) & (v_1 = x, v_2 = -y^2 \text{ substituted}) \\
&= 1 - 2y & \left( \frac{\partial}{\partial x} x = 1, \frac{\partial}{\partial y} (-y^2) = -2y \right)
\end{aligned}$$

$$\boxed{\nabla \cdot \mathbf{v} = 1 - 2y}$$

(b) **Curl:**

$$\mathbf{v}(x, y) = x \mathbf{e}_1 - y^2 \mathbf{e}_2 \quad (\text{given})$$

$$(\nabla \times \mathbf{v})_3 = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \quad (\text{note other components will be zero})$$

$$= \frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(x) \quad (v_1 = x, v_2 = -y^2)$$

$$= 0 - 0 \quad \left( \frac{\partial(-y^2)}{\partial x} = 0, \frac{\partial x}{\partial y} = 0 \right)$$

$$= 0 \quad (\text{so the scalar } z\text{-component is } 0)$$

$$\nabla \times \mathbf{v} = 0 \mathbf{e}_1 + 0 \mathbf{e}_2 + 0 \mathbf{e}_3 \quad (\text{extend to 3D vector})$$

$$= \mathbf{0} \quad (\text{zero vector})$$

$$\boxed{\nabla \times \mathbf{v} = \mathbf{0}}$$

(b) Explain the physical significance of the divergence and curl in the context of a cyclone.

**Solution:** (b) **Physical significance:**

Divergence of vector field tells us how the air is spreading or compressing. - Positive when  $y < 0.5$ : indicates source (air spreading). - Negative when  $y > 0.5$ : indicates sink (air converging).

Curl of the vector field is zero: the field is irrotational, meaning there's no local spinning or swirl in the wind.

(c) Based on the curl, determine the direction of rotation of the cyclone.

**Solution:** (c) Since the curl is zero, the cyclone exhibits no rotational behavior — the wind field is irrotational.

9. In Geo technical engineering, understanding the flow of water in a dam's vicinity is crucial. The potential function  $\phi(x, y)$  of a water flow around a dam is given by:

$$\phi(x, y) = xy$$

- (a) Calculate the velocity vector field  $\mathbf{v}(x, y)$  from the potential function  $\phi(x, y)$ .

**Solution:** (a) Velocity field is the gradient of potential:

$$\begin{aligned}\mathbf{v} &:= \nabla \phi = \mathbf{e}_i \left( \frac{\partial \phi}{\partial x_i} \right) \\ &= \mathbf{e}_1 \left( \frac{\partial \phi}{\partial x_1} \right) + \mathbf{e}_2 \left( \frac{\partial \phi}{\partial x_2} \right) && \text{(for 2D case, } i = 1, 2) \\ &= \left( \frac{\partial \phi}{\partial x} \right) \mathbf{e}_1 + \left( \frac{\partial \phi}{\partial y} \right) \mathbf{e}_2 && \text{(here, } x_1 = x \text{ and } x_2 = y) \\ &= y \mathbf{e}_1 + x \mathbf{e}_2. && \text{(since } \frac{\partial \phi}{\partial x} = y \text{ and } \frac{\partial \phi}{\partial y} = x)\end{aligned}$$

- (b) Determine the divergence and curl of the velocity vector field.

**Solution:** (b) **Divergence:**

$$\nabla \cdot \mathbf{v} = \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} = 0 + 0 = 0$$

**Curl:**

$$(\nabla \times \mathbf{v})_3 = \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} = 1 - 1 = 0$$

- (c) Draw the vector field and discuss the water flow behavior around the dam.

**Solution:** (c) **Flow interpretation:** - The flow is both **incompressible** (divergence = 0) and **irrotational** (curl = 0). - The streamlines are symmetric diagonal lines. - Water flows smoothly without swirling or compressing, ideal for modelling laminar groundwater flow near dams.



10. Given a vector  $\mathbf{a} = a_i \mathbf{e}_i = a_i^* \mathbf{e}_i^*$  defined with respect to the basis  $\mathbf{e}_i$  by

$$\mathbf{a} = 3\mathbf{e}_1 - 2\mathbf{e}_2 + 4\mathbf{e}_3$$

Find the components  $a_i^*$  of  $\mathbf{a}$  with respect to the basis  $\mathbf{e}_i^*$  defined in Fig. 2.

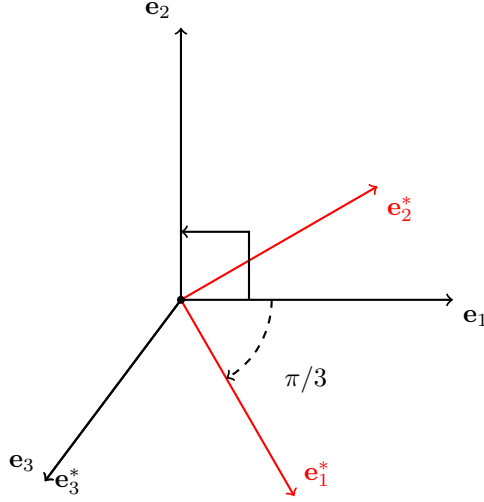


Figure 2: New ortho-normal basis  $\mathbf{e}_i^*$  is obtained by a clockwise rotation of the ortho-normal basis  $\mathbf{e}_i$  about the  $\mathbf{e}_3$ -axis.

**Solution:** The vector is independent of the coordinate system, that is,  $\mathbf{a} = a_i \mathbf{e}_i = a_i^* \mathbf{e}_i^*$ . One can show that the components of the vector in the new basis  $\mathbf{e}_i^* = \mathbf{Q} \mathbf{e}_i$  (where  $\mathbf{Q}$  is a orthogonal tensor) can be found

$$\begin{aligned} a_i^* &:= \mathbf{a} \cdot \mathbf{e}_i^* = \mathbf{a} \cdot \mathbf{Q} \mathbf{e}_i && (\text{since } \mathbf{e}_i^* = \mathbf{Q} \mathbf{e}_i) \\ &= (\mathbf{Q}^T \mathbf{a}) \cdot \mathbf{e}_i && (\text{since } \mathbf{u} \cdot \mathbf{A} \mathbf{v} = (\mathbf{A}^T \mathbf{u}) \cdot \mathbf{v}) \end{aligned}$$

So, the components in the rotated basis are

$$[\mathbf{a}]_{\{\mathbf{e}_i^*\}} = [\mathbf{Q}^T \mathbf{a}] = [\mathbf{Q}^T] [\mathbf{a}]_{\{\mathbf{e}_i\}},$$

where,  $[\mathbf{a}]_{\{\mathbf{e}_i^*\}} := (a_i^*)_{i \in (1,2,3)}$ . For the given problem, rotation about the  $\mathbf{e}_3$ -axis equals  $\theta = -\frac{\pi}{3}$  (since anticlockwise rotation is positive and clockwise rotation is negative).

The matrix representation of  $\mathbf{Q}$  can be determined as,

$$\begin{aligned} [\mathbf{Q}] &= \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \left( \text{using } Q_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j^*, \text{ for instance, } Q_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1^* = \cos \theta \right) \\ &= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \left( \text{since, } \cos \left( -\frac{\pi}{3} \right) = \frac{1}{2}, \sin \left( -\frac{\pi}{3} \right) = -\sin \left( \frac{\pi}{3} \right) = -\frac{\sqrt{3}}{2} \right). \end{aligned}$$

Hence,

$$[\mathbf{Q}]^T = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For given  $\mathbf{a} = 3\mathbf{e}_1 - 2\mathbf{e}_2 + 4\mathbf{e}_3$ , the matrix representation of  $\mathbf{a}$  can be given by

$$[\mathbf{a}]_{\{\mathbf{e}_i\}} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}.$$

Thus,

$$[\mathbf{a}]_{\{\mathbf{e}_i^*\}} = [\mathbf{Q}]^T [\mathbf{a}]_{\{\mathbf{e}_i\}} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$[\mathbf{a}]_{\{\mathbf{e}_i^*\}} = \begin{bmatrix} \frac{3+2\sqrt{3}}{2} \\ \frac{3\sqrt{3}-2}{2} \\ 4 \end{bmatrix}$$