Indian Institute of Technology Bhubaneswar

School of Infrastructure

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Solid Mechanics (CE2L001): Notes on Vector and Vector-field

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Notations :

Zeroth-order tensors or scalars are represented by small letters. For eg. a

First-order tensors or vectors are represented by bold small letters. For eg. $\boldsymbol{a}$.

Second-order tensors are represented by bold capital letters. For eg. $\boldsymbol{A}$

1 Indicial Notations

1.1 Free index

A free index appears in every expression of an equation, except for expressions that contain real numbers (scalars) only. Index i in the equation $f_i = a_i b_j c_j$ is Free index. Free indices are only used once per quantity and can take the integer values 1, 2, and 3. For example,

$$a_i \Leftrightarrow a_1, a_2, a_3.$$

Similarly, we can have two (or more) free indices as follows:

$$a_{ij} \Leftrightarrow a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}.$$

1.2 Dummy index

The repeated index, also called the dummy index or summation index, appears twice within an additive term of an expression.

$$a_{ii} \Leftrightarrow \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}.$$

$$a_i b_i \Leftrightarrow \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$a_{ij}b_{ij} \Leftrightarrow \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}b_{ij} = \sum_{i=1}^{3} a_{i1}b_{i1} + a_{i2}b_{i2} + a_{i3}b_{i3}$$

$$= a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23} + a_{31}b_{31} + a_{32}b_{32} + a_{33}b_{33}.$$

As a rule, no index must appear more than twice in an expression. Index j in the equation $f_i = a_i b_j c_j$ is a dummy index.

2 Vectors in $\mathbb{R}^3$

To describe many physical quantities (such as force, displacement, velocity) both magnitude and direction must be given. These quantities can be described by vectors in a three-dimensional Euclidean space, $\mathbb{R}^3$. By introducing a set of right-handed orthonormal basis vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. Any vector $\boldsymbol{a}$ can be written using the summation convention as

$$\boldsymbol{a} = a_i\boldsymbol{e}_i = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 \tag{1}$$

as shown in Fig. 1.

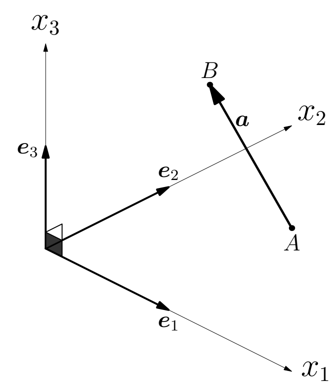


Figure 1: Schematic illustration of the vector $\boldsymbol{a}$

The coefficients a_i are the components of $\boldsymbol{a}$ for the basis vectors $\boldsymbol{e}_i$. The length or Euclidean norm of the vector $\boldsymbol{a}$ can be represented as $|\boldsymbol{a}|$.

2.1 Free vectors and Bound Vectors

Vectors that are fully characterized by the magnitude and direction only are called **free vectors** and those that are fully characterized by the magnitude, direction, and also line of support are called **line (or bound) vectors**. The displacement and velocity of a rigid body are free vectors while the force and moment of a force about a point are line vectors. In Fig. 2, the vector $\boldsymbol{a}$ (marked in red) is a bound vector while the vectors $\boldsymbol{b}$ (marked in blue) is a free vector.

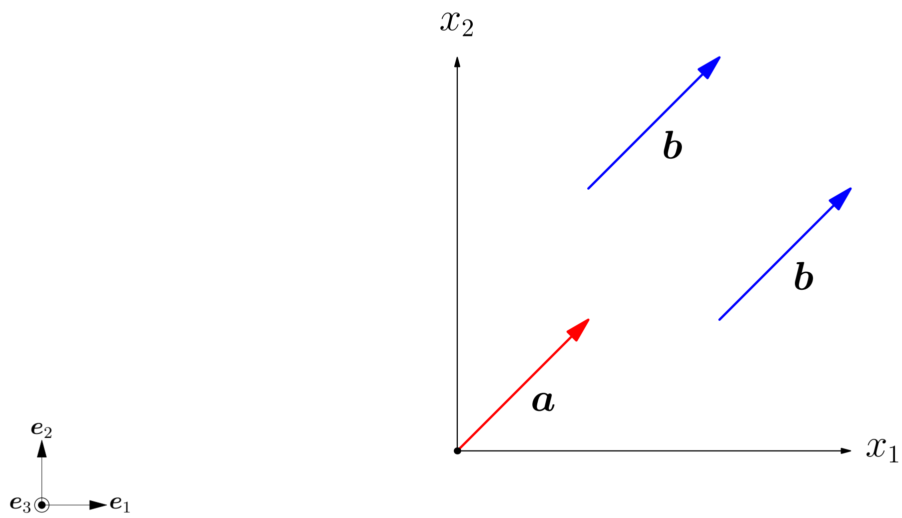


Figure 2: Schematic illustration of the free vector and bound vector

2.2 Dot product

The dot product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be expressed as

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_j \delta_{ij} \tag{2}$$

as shown in Fig. 3 below.

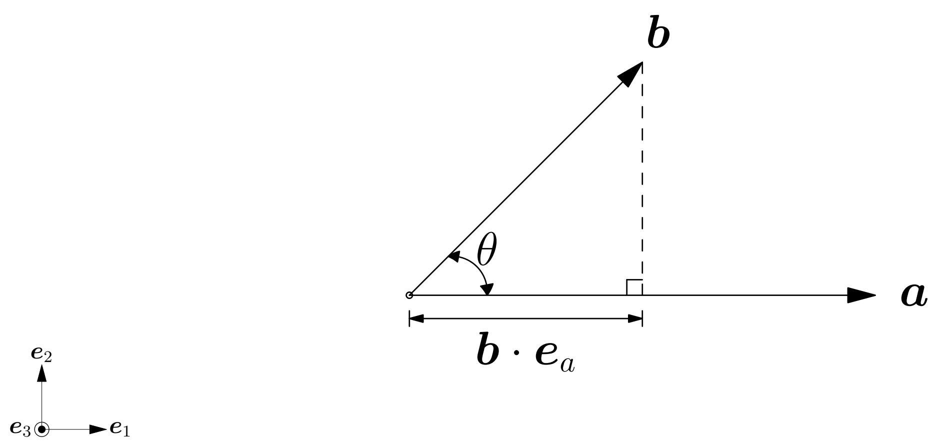


Figure 3: The schematic illustration of the dot product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The symbol $\boldsymbol{e}_a$ denotes the unit vector along the vector $\boldsymbol{a}$ and the dot product $(\boldsymbol{b} \cdot \boldsymbol{e}_a) = |\boldsymbol{b}| \cos\theta$

In Fig. 3, δ_{ij} is the Kronecker delta symbol which allows a simple representation of the dot product (or scalar product) of orthonormal vectors in a right-handed basis system. The Kronecker delta is defined as follows.

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{3}$$

Thus the components of the vector $\boldsymbol{a}$ can be written as

$$a_i = \boldsymbol{a} \cdot \boldsymbol{e}_i. \tag{4}$$

The dot product between the basis vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ can be defined as

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \tag{5}$$

The Kronecker delta δ_{ij} modifies (or contracts) the subscripts in the coefficients of an expression in which it appears as

$$a_i\delta_{ij} = a_j \tag{6}$$
$$a_i b_j \delta_{ij} = a_i b_i = a_j b_j \tag{7}$$
$$\delta_{ij}\delta_{ik} = \delta_{jk} \tag{8}$$

2.3 Vector Product

In an orthonormal basis, the vector products can be expressed in the index form as

$$\boldsymbol{a} \times \boldsymbol{b} = \epsilon_{ijk} a_i b_j \boldsymbol{e}_k, \tag{9}$$

as shown in Fig. 4.

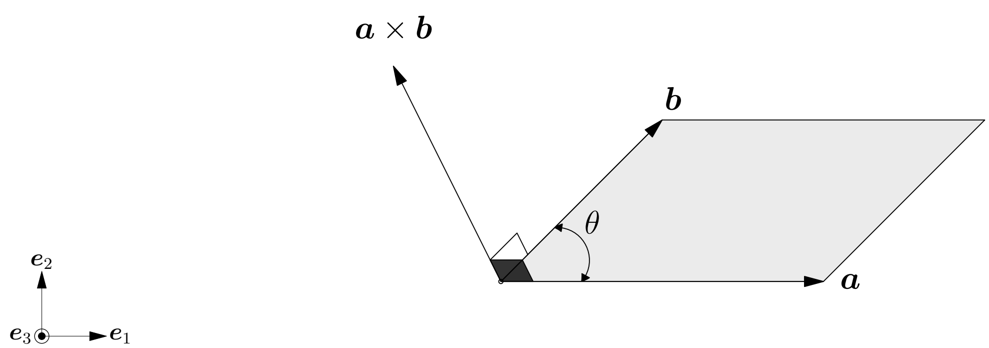


Figure 4: The schematic illustration of the vector product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

The permutation symbol, ϵ_{ijk} is a rank-3 antisymmetric tensor defined as

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } i,\, j,\, k \text{ are in cyclic order and not repeated} \\ 0, & \text{any two indices are the same} \\ -1, & \text{if } i,\, j,\, k \text{ are not in cyclic order and not repeated} \end{cases} \tag{10}$$

By definition, the subscripts of the permutation symbol can be permuted without changing its value; an interchange of any two subscripts will change the sign (hence, interchange of two subscripts twice keeps the value unchanged). We define the cross product of two base vectors $\boldsymbol{e}_j$ and $\boldsymbol{e}_k$ as

$$\boldsymbol{e}_j \times \boldsymbol{e}_k := \epsilon_{ijk} \boldsymbol{e}_i \tag{11}$$

Considering dot product with $\boldsymbol{e}_m$ on both sides of Eq. (11), we get

$$\begin{aligned}
\boldsymbol{e}_m \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) &= \boldsymbol{e}_m \cdot (\epsilon_{ijk} \boldsymbol{e}_i) \\
&= \epsilon_{ijk} (\boldsymbol{e}_m \cdot \boldsymbol{e}_i) \\
&= \epsilon_{ijk} (\delta_{mi}) \\
&= \epsilon_{mjk} \\
\implies \boldsymbol{e}_m \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) &= \epsilon_{mjk}.
\end{aligned} \tag{12}$$

Replacing index m, with i, we get

$$\epsilon_{ijk} = \boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) \tag{13}$$

2.4 Scalar triple product

Scalar triple product of three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ is denoted as $[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$, and is defined as

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] := \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}). \tag{14}$$

which is illustrated in Fig. 5.

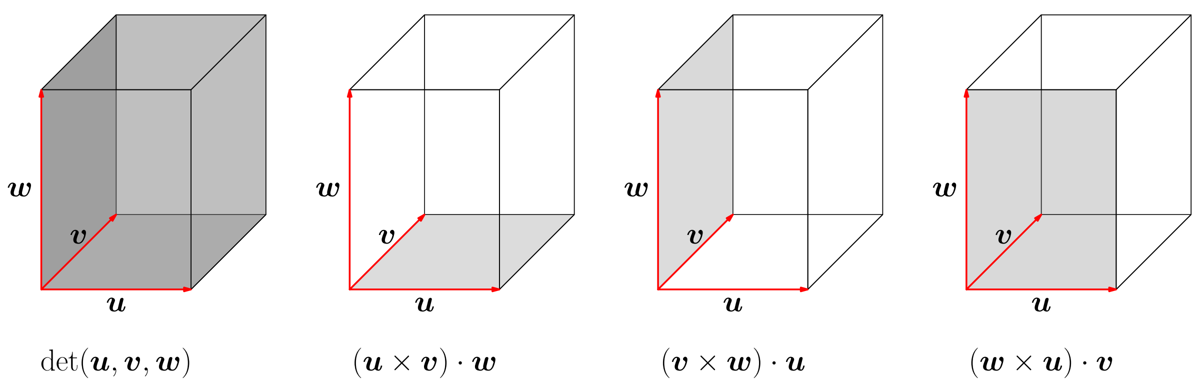


Figure 5: Schematic illustration of the scalar triple product where $\det(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = [\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = (\boldsymbol{v} \times \boldsymbol{w}) \cdot \boldsymbol{u} = (\boldsymbol{w} \times \boldsymbol{u}) \cdot \boldsymbol{v}$

In indicial notation, the scalar triple product can be expressed as

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = \epsilon_{ijk} u_i v_j w_k \tag{15}$$

3 Scalar and Vector fields

3.1 Scalar fields

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be a Cartesian basis in three dimensional space. Let $\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$ denotes the position vector of a point in space. A scalar field is a scalar-valued function of position in space. A scalar field is a function of the components of the position vector, and so may be expressed as $\phi(x_1, x_2, x_3)$. The value of ϕ at a particular point in space must be independent of the choice of basis vectors. A scalar field may be a function of time (and possibly other parameters) as well as position in space.

3.2 Vector fields

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be a Cartesian basis in three dimensional space. Let $\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$ denotes the position vector of a point in space. A vector field is a vector-valued function of position in space. A vector field is a function of the components of the position vector, and so may be expressed as $\boldsymbol{v}(x_1, x_2, x_3)$. The vector may also be expressed as components in the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$

$$\boldsymbol{v}(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)\boldsymbol{e}_1 + v_2(x_1, x_2, x_3)\boldsymbol{e}_2 + v_3(x_1, x_2, x_3)\boldsymbol{e}_3 \tag{16}$$

The magnitude and direction of $\boldsymbol{v}$ at a particular point in space is independent of the choice of basis vectors. A vector field may be a function of time (and possibly other parameters) as well as position in space.

3.3 Del operator

The del operator is defined by

$$\nabla = \boldsymbol{e}_1\frac{\partial()}{\partial x_1} + \boldsymbol{e}_2\frac{\partial()}{\partial x_2} + \boldsymbol{e}_3\frac{\partial()}{\partial x_3}. \tag{17}$$

The del operator is a vector differential operator. In the summation convention, it can be written as

$$\nabla = \boldsymbol{e}_i\frac{\partial()}{\partial x_i}. \tag{18}$$

The del operator can be used for different operations in mathematics like gradient, divergence and curl.

3.4 Gradient of a scalar field

The gradient of a scalar field ϕ will result in a vector field and can be represented as

$$\nabla\phi = \boldsymbol{e}_1\frac{\partial\phi}{\partial x_1} + \boldsymbol{e}_2\frac{\partial\phi}{\partial x_2} + \boldsymbol{e}_3\frac{\partial\phi}{\partial x_3} \tag{19}$$

Examples on gradient of a scalar field:

A hill is described by the scalar function $h(x,\ y)$ = 200 - x^2 - $2y^2$, where h is the height of the hill at any point (x, y) on the $x - y$ plane. If you are standing at the point $(1, 2)$, in which direction should you walk to climb the hill most rapidly, and what is the rate of increase of height in that direction?

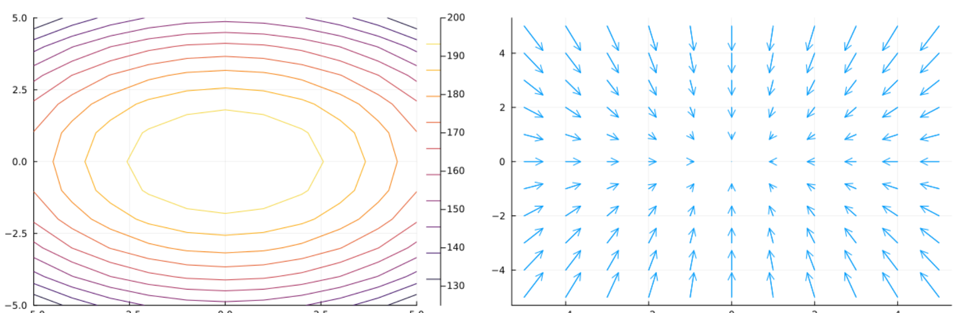
Solution:

To find the direction in which you should walk to climb the hill most rapidly and the rate of increase of height in that direction, we need to calculate the gradient of the scalar function $h(x, y)$ at the point $(1, 2)$.
The gradient of $h(x, y)$ is given by:

$\nabla h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right)$
where $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are the partial derivatives of h with respect to x and y, respectively.
First, let's find the gradient at the point $(1, 2)$:

$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}(200 - x^2 - 2y^2) = -2x$
$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}(200 - x^2 - 2y^2) = -4y$
Now, evaluate the gradient at the point $(1, 2)$:

$\nabla h(1, 2) = (-2(1), -4(2)) = (-2, -8)$
The direction in which you should walk to climb the hill most rapidly is the direction of the gradient vector $(-2, -8)$. This direction points towards the steepest increase in height.
The rate of increase of height in that direction is equal to the magnitude of the gradient vector. The magnitude of $(-2, -8)$ is calculated as follows:
$\|\nabla h(1, 2)\| = \sqrt{(-2)^2 + (-8)^2} = \sqrt{4 + 64} = \sqrt{68}$ Thus, the rate of increase of height in the direction of $(-2, -8)$ is $\sqrt{68}$ (approximately 8.246).



Plot of scalar field h (on the left), and its gradient (on the right)

3.5 Divergence of a vector field

The dot product of a del operator with a vector field is called the divergence of a vector field and is denoted by

$$\nabla \cdot \boldsymbol{a} = \frac{\partial a_i}{\partial x_i} = a_{i,i}. \tag{20}$$

Examples on divergence of vector:

a) Plot the following vector fields and determine which of them acts as source, sink, or solenoidal.

1. $\boldsymbol{f} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$
2. $\boldsymbol{f} = -x\boldsymbol{e}_1 - y\boldsymbol{e}_2$
3. $\boldsymbol{f} = \boldsymbol{e}_1 + \boldsymbol{e}_2$

Solution:

- Divergence of a vector is positive: Source (diverging).
- Divergence of a vector is zero: Solenoidal (divergence-free).
- Divergence of a vector is negative: Source (converging).

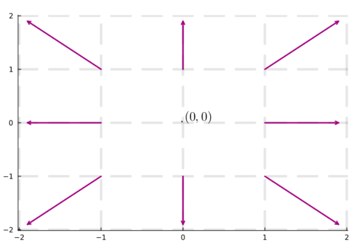


Figure 6: $\boldsymbol{f} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$

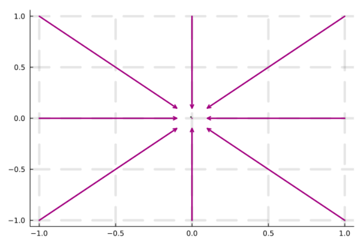


Figure 7: $\boldsymbol{f} = -x\boldsymbol{e}_1 - y\boldsymbol{e}_2$

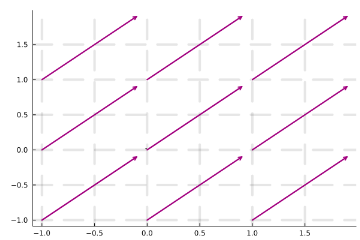


Figure 8: $\boldsymbol{f} = 1\boldsymbol{e}_1 + 1\boldsymbol{e}_2$

- The divergence of the vector field $\boldsymbol{f} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$ is
  $\nabla \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 1.$
  The vector field is a source since the divergence is greater than zero.
- The divergence of the vector field $\boldsymbol{f} = -x\boldsymbol{e}_1 - y\boldsymbol{e}_2$ is
  $\nabla \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = -1.$
  Since the divergence is less than zero, the vector field is a sink.
- The divergence of the vector field $\boldsymbol{f} = 1\boldsymbol{e}_1 + 1\boldsymbol{e}_2$ is
  $\nabla \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 0.$
  Since the divergence is equal to zero, the vector field is solenoidal.

3.6 Curl of a vector field

The curl of a vector field is defined as the del operator operating on a vector field, $\boldsymbol{a}$ by means of the cross product as

$$\begin{aligned}\nabla \times \boldsymbol{a} &= \boldsymbol{e}_i \frac{\partial ()}{\partial x_i} \times a_j\, \boldsymbol{e}_j \\ &= \boldsymbol{e}_i \frac{\partial a_j}{\partial x_i} \times \boldsymbol{e}_j \\ &= a_{j,i}\,(\boldsymbol{e}_i \times \boldsymbol{e}_j) \\ &= \epsilon_{kij} a_{j,i}\, \boldsymbol{e}_k. \end{aligned} \tag{21}$$

Examples on curl of vector:

Plot the following vector fields and determine the curl of the vector field.

a) $\boldsymbol{f} = (y+1)\boldsymbol{e}_1$

b) $\boldsymbol{f} = (x+1)\boldsymbol{e}_2$

c) $\boldsymbol{f} = \boldsymbol{e}_1 + \boldsymbol{e}_2$

Consider the following coordinates for x and y for plotting vector field

Solution:

- Curl of a vector is positive: Anticlockwise.
- Curl of a vector is negative: Clockwise.
- Curl of a vector is zero: No rotation.

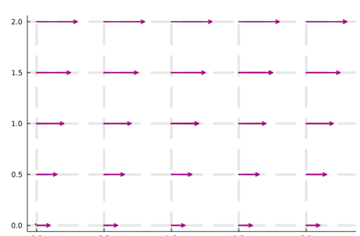


Figure 9: $\boldsymbol{f} = (y+1)\boldsymbol{e}_1$

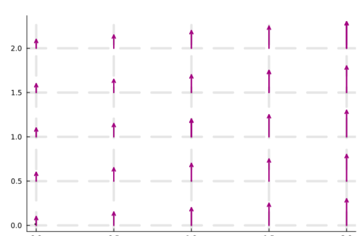


Figure 10: $\boldsymbol{f} = (x+1)\boldsymbol{e}_2$

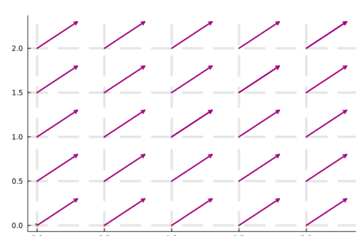


Figure 11: $\boldsymbol{f} = \boldsymbol{e}_1 + \boldsymbol{e}_2$

- The curl of the vector field $\boldsymbol{f} = (y+1)\boldsymbol{e}_1$ is $= \nabla \times \boldsymbol{f} = -2\boldsymbol{e}_3$.
  Since the curl is lesser than zero, the rotation will be in clockwise direction.
- The curl of the vector field $\boldsymbol{f} = (x+1)\boldsymbol{e}_2$ is $= \nabla \times \boldsymbol{f} = 2\boldsymbol{e}_3$.
  Since the curl is greater than zero, the rotation will be in anti-clockwise direction.
- The curl of the vector field $\boldsymbol{f} = 1\boldsymbol{e}_1 + 1\boldsymbol{e}_2$ is $= \nabla \times \boldsymbol{f} = 0$. Since the curl is equal to zero, there will be no rotation.