Indian Institute of Technology Bhubaneswar

School of Infrastructure

Session: Autumn 2025

Solid Mechanics (CE2L001) Compensatory Class Test 1

<u>Notations :</u>
Zeroth-order tensors or scalars are represented by small letters. For eg. a
First-order tensors or vectors are represented by bold small letters. For eg. $\boldsymbol{a}$.
Second-order tensors are represented by bold capital letters. For eg. $\boldsymbol{A}$

1. (a) Simplify the following expressions:
   (i) $\delta_{ij}\,\delta_{ik}\,\delta_{jk}$, (ii) $\epsilon_{ijk}\,\epsilon_{ijk}$ and (iii) $\epsilon_{ijk}u_i u_j v_k$.
   (b) Show that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

.

Solution: (i) The Kronecker delta function δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1}$$

To simplify the expression $\delta_{ij}\,\delta_{ik}\,\delta_{jk}$, one can use the contraction properties of the Kronecker delta symbol, δ_{ij}.

$$\begin{aligned} \delta_{ij}\,\delta_{ik}\,\delta_{jk} &= \delta_{ij}\delta_{ij} \qquad (\text{since } \delta_{ik}\delta_{jk} = \delta_{ij}) \\ &= \delta_{ii} \qquad (\text{since } \delta_{ij}\delta_{ij} = \delta_{ii}) \\ &= (\delta_{11} + \delta_{22} + \delta_{33}) \qquad (\text{since dummy index implies sum}) \\ &= (1 + 1 + 1) \quad (\text{using Eq.(1)}) \\ &= 3. \end{aligned}$$

(ii) The $\epsilon - \delta$ relation is given by

$$\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}. \tag{2}$$

Using the $\epsilon - \delta$ relation given in Eq. (2),

$$\begin{aligned}
\epsilon_{ijk}\epsilon_{ijk} &= \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} \\
&= 3 \times 3 - \delta_{ii} \quad (\text{since } \delta_{ii} = \delta_{jj} = 3 \;\; \& \;\; \delta_{ij}\delta_{ji} = \delta_{ii}) \\
\implies \epsilon_{ijk}\epsilon_{ijk} &= 9 - 3 = 6.
\end{aligned}$$

(iii) $\epsilon_{ijk}u_i u_j v_k$ can be expressed as follows:

$$\begin{aligned}
\epsilon_{ijk}u_i u_j v_k &= \epsilon_{ijk}u_i u_j \,(\boldsymbol{e}_k \cdot \boldsymbol{v}) && (\text{since } v_k = \boldsymbol{v} \cdot \boldsymbol{e}_k = \boldsymbol{e}_k \cdot \boldsymbol{v}) \\
&= (\epsilon_{ijk}u_i u_j \boldsymbol{e}_k) \cdot \boldsymbol{v} \\
&= (\boldsymbol{u} \times \boldsymbol{u}) \cdot \boldsymbol{v} && (\text{since } \epsilon_{ijk}u_i u_j \boldsymbol{e}_k = \boldsymbol{u} \times \boldsymbol{u}) \\
&= 0 && (\text{since } \boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0} \text{ and } \boldsymbol{0} \cdot \boldsymbol{v} = 0)
\end{aligned}$$

(b) Using indicial notations, $\boldsymbol{a} = a_m e_m$ and $\boldsymbol{b} \times \boldsymbol{c} = \epsilon_{ijk} b_j c_k \boldsymbol{e}_i$

$$\begin{aligned}
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (a_m e_m) \times (\epsilon_{ijk} b_j c_k \boldsymbol{e}_i) \\
&= (a_m b_j c_k \epsilon_{ijk}\epsilon_{nmi})\boldsymbol{e}_n = (a_m b_j c_k \epsilon_{ijk}\epsilon_{inm})\boldsymbol{e}_n \\
&= a_m b_j c_k (\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn})\boldsymbol{e}_n \\
&= a_m b_j c_k (\delta_{jn}\delta_{km})\boldsymbol{e}_n - a_m b_j c_k (\delta_{jm}\delta_{kn})\boldsymbol{e}_n
\end{aligned}$$

Using the contraction property of δ,

$$\begin{aligned}
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (a_m b_n c_m - a_m b_m c_n)\boldsymbol{e}_n \\
&= (a_m c_m) b_n \boldsymbol{e}_n - (a_m b_m) c_n \boldsymbol{e}_n
\end{aligned}$$

Since $a_m c_m = \boldsymbol{a} \cdot \boldsymbol{c}$, $a_m b_m = \boldsymbol{a} \cdot \boldsymbol{b}$, $b_n \boldsymbol{e}_n = \boldsymbol{b}$, and $c_n \boldsymbol{e}_n = \boldsymbol{c}$

$$\begin{aligned}
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \\
\implies \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}
\end{aligned}$$

2. (a) Which of the following vector fields acts as source, sink, or solenoidal @ co-ordinate (1, 2):
(a) $2x\boldsymbol{e}_1 + 5y\boldsymbol{e}_2$,
(b) $-6x\boldsymbol{e}_1 - 3y\boldsymbol{e}_2$,
(c) $(4x + y^2)\,\boldsymbol{e}_1 + (x^4 + 2y)\,\boldsymbol{e}_2$.
Justify your answer using plots of the vector fields. Consider the following Cartesian coordinates (x, y) for plotting the vector field.

$[x, y]$	[-1,0]	[-1,-1]	[-1,1]	[0,0]	[0,-1]	[0,1]	[1,-1]	[1,0]	[1,1]

Solution: (a) The divergence of vector field $\boldsymbol{f} = 2x\,\boldsymbol{e}_1 + 5y\,\boldsymbol{e}_2$ is:

$$\nabla \cdot \boldsymbol{f} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(5y) = 7. \tag{3}$$

We know that if $\nabla \cdot \boldsymbol{f} > 0$

Then we can say that the given vector field acts as a Source. The divergence of a given vector field is $\nabla \cdot \boldsymbol{f} < 0$. Therefore it acts as a Source. Also, the plot of the given vector field is shown in Fig. 1

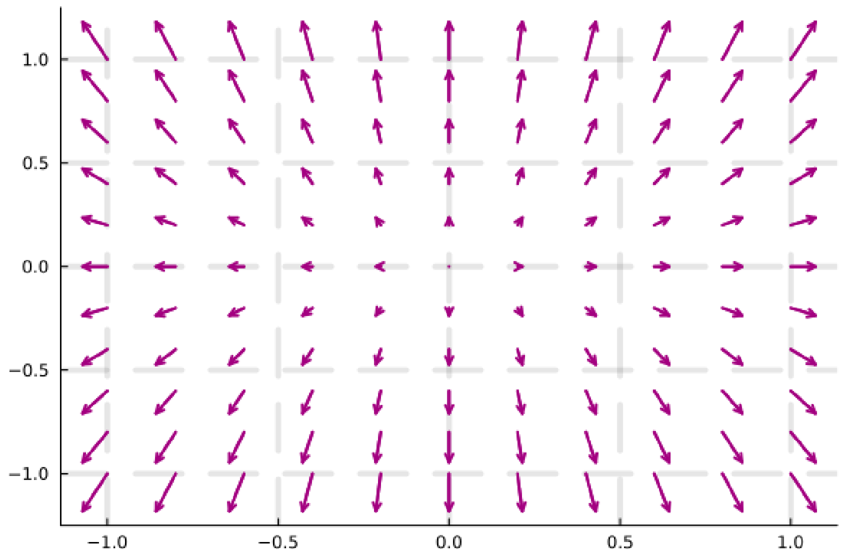

Figure 1: Vector field $\boldsymbol{f} = 2x\,\boldsymbol{e}_1 + 5y\,\boldsymbol{e}_2$

(b) The divergence of vector field $\boldsymbol{f} = -6x\boldsymbol{e}_1 + -3y\boldsymbol{e}_2$ is:

$$\nabla \cdot \boldsymbol{f} = \frac{\partial}{\partial x}(-6x) + \frac{\partial}{\partial y}(-3y) = -9 \tag{4}$$

We know that if $\nabla \cdot \boldsymbol{f} < 0$

Then we can say that the given vector field acts as a Sink. The divergence of a given vector field is $\nabla \cdot \boldsymbol{f} < 0$. Therefore it acts as a Sink. Also, the plot of the given vector field is shown in Fig. 2

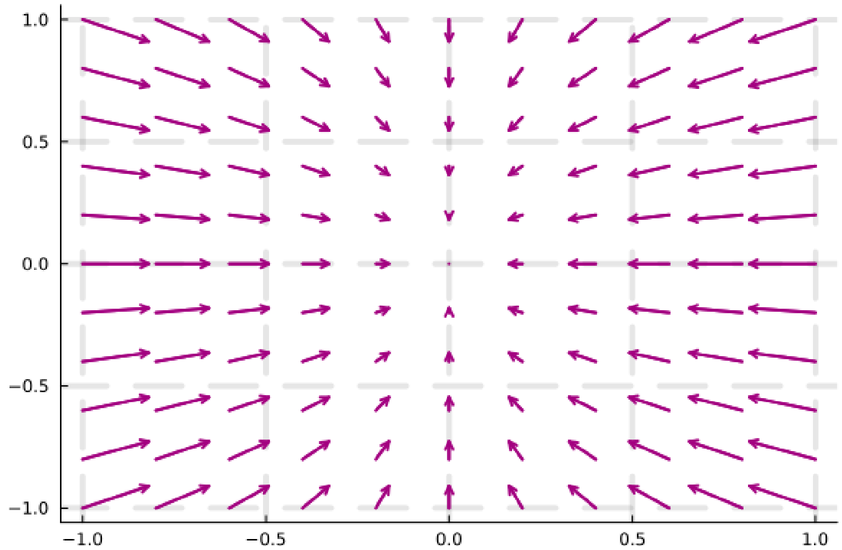

Figure 2: vector field $\boldsymbol{f} = -6x\boldsymbol{e}_1 + -3y\boldsymbol{e}_2$

(c) The divergence of vector field $\boldsymbol{f} = (4x + y^2)\,\boldsymbol{e}_1 + (x^4 + 2y)\,\boldsymbol{e}_2$ is:

$$\nabla \cdot \boldsymbol{f} = \frac{\partial}{\partial x}(4x + y^2) + \frac{\partial}{\partial y}(x^4 + 2y) = 6 \tag{5}$$

We know that if $\nabla \cdot \boldsymbol{f} > 0$

Then we can say that the given vector field acts as a Source. The divergence of a given vector field is $\nabla \cdot \boldsymbol{f} < 0$. Therefore it acts as a Source. Also, the plot of the given vector field is shown in Fig. 3

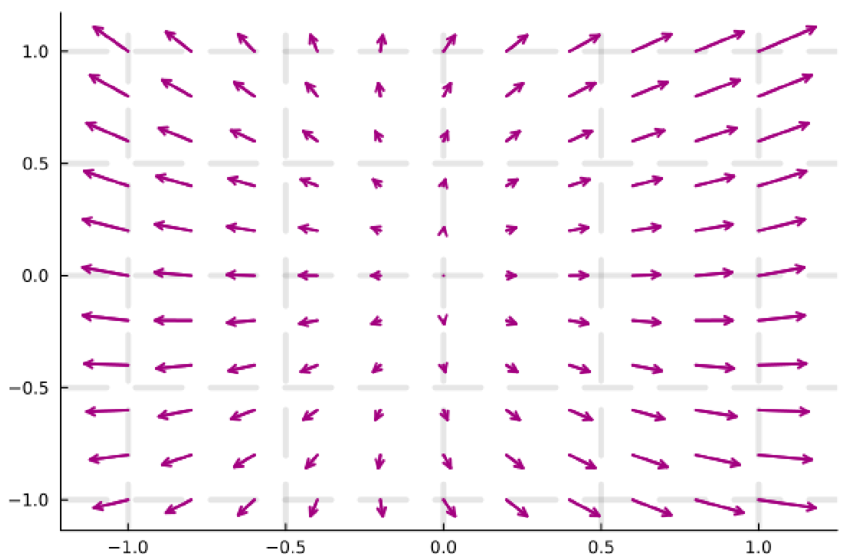

Figure 3: vector field $\boldsymbol{f} = (4x + y^2)\,\boldsymbol{e}_1 + (x^4 + 2y)\,\boldsymbol{e}_2$

3. (a) Let $\boldsymbol{r} = x\ \boldsymbol{e}_1 + y\ \boldsymbol{e}_2 + z\ \boldsymbol{e}_3$ be the position vector field in $\mathbb{R}^3$. Find
   (a) The gradient of $f(\boldsymbol{r}) = \boldsymbol{r} \cdot \boldsymbol{r}$,
   (b) The divergence of $\boldsymbol{r}$,
   (c) The curl of $\boldsymbol{r}$.

Solution: (a) Given $\boldsymbol{r}$ the position vector,

$$f(\boldsymbol{r}) = \boldsymbol{r} \cdot \boldsymbol{r} = x^2 + y^2 + z^2 \tag{6}$$

It has to be noted that $\boldsymbol{r} \cdot \boldsymbol{r}$ is a scalar and the gradient of any scalar field can be determined using the del operator as given in Eq. (7) below

$$\nabla = \ \boldsymbol{e}_1 \frac{\partial()}{\partial x} + \ \boldsymbol{e}_2 \frac{\partial()}{\partial y} + \ \boldsymbol{e}_3 \frac{\partial()}{\partial z} \tag{7}$$

Thus the gradient of the scalar field $f(\boldsymbol{r})$ can be written as

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x}\ \boldsymbol{e}_1 + \frac{\partial f}{\partial y}\ \boldsymbol{e}_2 + \frac{\partial f}{\partial z}\ \boldsymbol{e}_3 \\ &= 2x\ \boldsymbol{e}_1 + 2y\ \boldsymbol{e}_2 + 2z\ \boldsymbol{e}_3 = 2\ \boldsymbol{r} \end{aligned} \tag{8}$$

(b) The divergence of $\boldsymbol{r}$ can be calculated as,

$$\nabla \cdot \boldsymbol{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3 \tag{9}$$

(c) The curl of $\boldsymbol{r}$ can be calculated as,

$$\begin{aligned} \nabla \times \boldsymbol{r} &= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\ &= (0-0)\ \boldsymbol{e_1} - (0-0)\ \boldsymbol{e_2} + (0-0)\, \boldsymbol{e_3} \qquad (10) \\ &= \boldsymbol{0} \qquad (11) \end{aligned}$$

4. (a) Consider the second-order tensor $\boldsymbol{A}$ given by

$$\boldsymbol{A} = 3\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1) - 4\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2) + 2\,(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + (\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + (\boldsymbol{e}_3 \otimes \boldsymbol{e}_3)\,.$$

Determine the image of the vector $\boldsymbol{v} = 4\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + 5\boldsymbol{e}_3$ when $\boldsymbol{A}$ operates on it.

(b) Consider a two-dimensional orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ in which a two-dimensional tensor $\boldsymbol{T}$ has the representation

$$\boldsymbol{T} = T_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad i, j = 1, 2,$$

and the component matrix of $\boldsymbol{T}$ has values

$$[\boldsymbol{T}] = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}.$$

Consider a second basis $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*\}$ which is related to $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ by

$$\boldsymbol{e}_1^* = \frac{1}{2}\boldsymbol{e}_1 + \frac{\sqrt{3}}{2}\boldsymbol{e}_2, \qquad \boldsymbol{e}_2^* = -\frac{\sqrt{3}}{2}\boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{e}_2,$$

Find the value of T_{11}^* in the $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*\}$ basis.

Solution: (a) Given the tensor $\boldsymbol{A}$ and vector $\boldsymbol{v}$:

$$\boldsymbol{A} = 3\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1) - 4\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2) + 2\,(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + (\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + (\boldsymbol{e}_3 \otimes \boldsymbol{e}_3)$$

$$\boldsymbol{v} = 4\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + 5\boldsymbol{e}_3$$

To find the image of $\boldsymbol{v}$ under $\boldsymbol{A}$, we compute $\boldsymbol{A}\boldsymbol{v}$. Using the definition of the tensor product, $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$, we can calculate the action of $\boldsymbol{A}$ on $\boldsymbol{v}$. The action of each term of $\boldsymbol{A}$ on $\boldsymbol{v}$ is:

$$\begin{aligned}
3\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1):&\quad 3(\boldsymbol{e}_1 \cdot \boldsymbol{v})\boldsymbol{e}_1 = 3(4)\boldsymbol{e}_1 = 12\boldsymbol{e}_1 \\
-4\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2):&\quad -4(\boldsymbol{e}_2 \cdot \boldsymbol{v})\boldsymbol{e}_1 = -4(2)\boldsymbol{e}_1 = -8\boldsymbol{e}_1 \\
2\,(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1):&\quad 2(\boldsymbol{e}_1 \cdot \boldsymbol{v})\boldsymbol{e}_2 = 2(4)\boldsymbol{e}_2 = 8\boldsymbol{e}_2 \\
(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2):&\quad (\boldsymbol{e}_2 \cdot \boldsymbol{v})\boldsymbol{e}_2 = (2)\boldsymbol{e}_2 = 2\boldsymbol{e}_2 \\
(\boldsymbol{e}_3 \otimes \boldsymbol{e}_3):&\quad (\boldsymbol{e}_3 \cdot \boldsymbol{v})\boldsymbol{e}_3 = (5)\boldsymbol{e}_3 = 5\boldsymbol{e}_3
\end{aligned}$$

Combining the results:

$$\boldsymbol{A}\boldsymbol{v} = 12\boldsymbol{e}_1 - 8\boldsymbol{e}_1 + 8\boldsymbol{e}_2 + 2\boldsymbol{e}_2 + 5\boldsymbol{e}_3 = (12-8)\boldsymbol{e}_1 + (8+2)\boldsymbol{e}_2 + 5\boldsymbol{e}_3 = 4\boldsymbol{e}_1 + 10\boldsymbol{e}_2 + 5\boldsymbol{e}_3$$

Therefore, the final answer is:

$$\boxed{\boldsymbol{A}\boldsymbol{v} = 4\boldsymbol{e}_1 + 10\boldsymbol{e}_2 + 5\boldsymbol{e}_3}$$

(b) From the definition $\boldsymbol{e}_i^* = \boldsymbol{Q}\boldsymbol{e}_i$, the components Q_{ij} of orthogonal transformation tensor $\boldsymbol{Q} = Q_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ can be defined as

$$Q_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{Q}\boldsymbol{e}_j = \boldsymbol{e}_i \cdot \boldsymbol{e}_j^*. \tag{12}$$

Since, tensor is independent of coordinates system, $\boxed{\boldsymbol{T} = T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = T_{ij}^*\boldsymbol{e}_i^* \otimes \boldsymbol{e}_j^*}$. Thus the components T^*_{ij} can be expressed as

$$\begin{aligned}
T^*_{ij} &= \boldsymbol{e}^*_i \cdot \left(\boldsymbol{T}\boldsymbol{e}^*_j\right) \\
&= \boldsymbol{e}^*_i \cdot \left(T_{kl}(\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\boldsymbol{e}^*_j\right) \\
&= \boldsymbol{e}^*_i \cdot \left(T_{kl}(\boldsymbol{e}^*_j \cdot \boldsymbol{e}_l)\boldsymbol{e}_k\right) \qquad \text{(since } (\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\boldsymbol{e}^*_j = (\boldsymbol{e}^*_j \cdot \boldsymbol{e}_l)\boldsymbol{e}_k\text{)} \\
&= T_{kl}\left((\boldsymbol{e}_k \cdot \boldsymbol{e}^*_i)(\boldsymbol{e}_l \cdot \boldsymbol{e}^*_j)\right) \\
\implies T^*_{ij} &= Q_{ki}T_{kl}Q_{lj} \qquad \text{(using Eq. (12))}
\end{aligned} \tag{13}$$

Thus, one can find the components of the tensor $\boldsymbol{T}$ in $\{\boldsymbol{e}^*_i\}$ basis, i.e., $[\boldsymbol{T}]_{\{\boldsymbol{e}^*_i\}} := \left(T^*_{ij}\right)_{i,j\in(1,2,3)}$, in terms of the components of the tensor $\boldsymbol{T}$ in $\{\boldsymbol{e}_i\}$ basis, i.e., $[\boldsymbol{T}]_{\{\boldsymbol{e}_i\}} := (T_{ij})_{i,j\in(1,2,3)}$, as

$$[\boldsymbol{T}]_{\{\boldsymbol{e}^*_i\}} = [\boldsymbol{Q}]^T[\boldsymbol{T}]_{\{\boldsymbol{e}_i\}}[\boldsymbol{Q}]. \tag{14}$$

The components of $\boldsymbol{Q}$ in matrix form can be computed as

$$\begin{aligned}
[\boldsymbol{Q}] &= \begin{bmatrix} \boldsymbol{e}_1 \cdot \boldsymbol{e}^*_1 & \boldsymbol{e}_1 \cdot \boldsymbol{e}^*_2 \\ \boldsymbol{e}_2 \cdot \boldsymbol{e}^*_1 & \boldsymbol{e}_2 \cdot \boldsymbol{e}^*_2 \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{e}_1 \cdot \left(\frac{1}{2}\boldsymbol{e}_1 + \frac{\sqrt{3}}{2}\boldsymbol{e}_2\right) & \boldsymbol{e}_1 \cdot \left(-\frac{\sqrt{3}}{2}\boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{e}_2\right) \\ \boldsymbol{e}_2 \cdot \left(\frac{1}{2}\boldsymbol{e}_1 + \frac{\sqrt{3}}{2}\boldsymbol{e}_2\right) & \boldsymbol{e}_2 \cdot \left(-\frac{\sqrt{3}}{2}\boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{e}_2\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad \text{(using } \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}\text{)}
\end{aligned}$$

Thus, one can find the components of the tensor $\boldsymbol{T}$ in $\{\boldsymbol{e}^*_i\}$ basis, $[\boldsymbol{T}]_{\{\boldsymbol{e}^*_i\}} := \left(T^*_{ij}\right)_{i,j\in(1,2)}$ as

$$\begin{aligned}
[\boldsymbol{T}]_{\{\boldsymbol{e}^*_i\}} &= [\boldsymbol{Q}]^T[\boldsymbol{T}]_{\{\boldsymbol{e}_i\}}[\boldsymbol{Q}] \\
&= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{5+10\sqrt{3}}{2} & \frac{10-5\sqrt{3}}{2} \\ \frac{15+20\sqrt{3}}{2} & \frac{20-15\sqrt{3}}{2} \end{bmatrix}.
\end{aligned}$$

Hence, $\boxed{T^*_{11} = \dfrac{5+10\sqrt{3}}{4} + \dfrac{15\sqrt{3}+60}{4} = \dfrac{25\sqrt{3}+65}{4}}$.

5. (a) Prove that Mohr's circle of stress is a graphical representation corresponding to the coordinate transformation of stress components.

   (b) The components of plane stress on an element of an industrial robot are shown in Fig. 4. Determine the stresses σ and τ by using Mohr's circle.

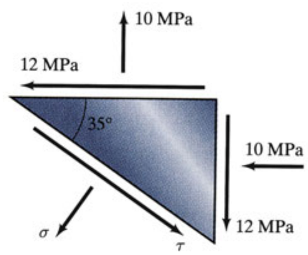

Figure 4

Solution: (a) Stress tensor is independent of coordinate system, i.e., $\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \sigma^*_{ij}\boldsymbol{e}^*_i \otimes \boldsymbol{e}^*_j$. So, the stress componets in the rotated basis $\{\boldsymbol{e}^*_i\}$ can be expressed as

$$\sigma^*_{ij} = \boldsymbol{e}^*_i \cdot \boldsymbol{\sigma}\boldsymbol{e}^*_j = \boldsymbol{e}^*_i \cdot (\sigma_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\,\boldsymbol{e}^*_j = \sigma_{kl}\,(\boldsymbol{e}^*_i \cdot \boldsymbol{e}_k)\,\big(\boldsymbol{e}^*_j \cdot \boldsymbol{e}_l\big)$$

Note the trigonometric relations between the vectors $\boldsymbol{e_i}$ and $\boldsymbol{e^*_j}$:

$$\boldsymbol{e_1} \cdot \boldsymbol{e_1}^* = \cos\theta, \quad \boldsymbol{e_2} \cdot \boldsymbol{e_2}^* = \cos\theta, \quad \boldsymbol{e_1} \cdot \boldsymbol{e_2}^* = -\sin\theta, \quad \boldsymbol{e_2} \cdot \boldsymbol{e_1}^* = \sin\theta$$

Thus, the stress transformation relations are

$$\begin{aligned}\sigma^*_{11} &= \sigma_{11}(\boldsymbol{e_1} \cdot \boldsymbol{e_1}^*)^2 + \sigma_{22}(\boldsymbol{e_2} \cdot \boldsymbol{e_2}^*)^2 + 2\sigma_{12}(\boldsymbol{e_1} \cdot \boldsymbol{e_1}^*)(\boldsymbol{e_2} \cdot \boldsymbol{e_2}^*)\\ &= \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + \sigma_{12}\sin 2\theta\end{aligned} \tag{15}$$

$$\begin{aligned}\sigma^*_{22} &= \sigma_{11}(\boldsymbol{e_1} \cdot \boldsymbol{e_2}^*)^2 + \sigma_{22}(\boldsymbol{e_2} \cdot \boldsymbol{e_1}^*)^2 + 2\sigma_{12}(\boldsymbol{e_1} \cdot \boldsymbol{e_2}^*)(\boldsymbol{e_2} \cdot \boldsymbol{e_1}^*)\\ &= \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - \sigma_{12}\sin 2\theta\end{aligned} \tag{16}$$

$$\begin{aligned}\sigma^*_{12} &= -\tfrac{\sigma_{11}}{2}\sin 2\theta + \tfrac{\sigma_{22}}{2}\sin 2\theta + \sigma_{12}(\cos^2\theta - \sin^2\theta)\\ &= -\tfrac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta\end{aligned} \tag{17}$$

$$\boxed{\begin{aligned}\sigma_{11}^* &= \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + \sigma_{12}\sin 2\theta\\ \sigma_{22}^* &= \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - \sigma_{12}\sin 2\theta\\ \sigma_{12}^* &= -\tfrac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta\end{aligned}}$$

Using the relations $\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta}$, one can rewrite the above equations as:

$$\begin{aligned}\sigma_{11}^* &= \frac{\sigma_{11}}{2}(1+\cos 2\theta) + \frac{\sigma_{22}}{2}(1-\cos 2\theta) + \sigma_{12}\sin 2\theta\\ &= \frac{(\sigma_{11}+\sigma_{22})}{2} + \frac{(\sigma_{11}-\sigma_{22})}{2}\cos 2\theta + \sigma_{12}\sin 2\theta\\ \sigma_{22}^* &= \frac{\sigma_{11}}{2}(1-\cos 2\theta) + \frac{\sigma_{22}}{2}(1+\cos 2\theta) - \sigma_{12}\sin 2\theta\\ &= \frac{(\sigma_{11}+\sigma_{22})}{2} - \frac{(\sigma_{11}-\sigma_{22})}{2}\cos 2\theta - \sigma_{12}\sin 2\theta\\ \sigma_{12}^* &= -\frac{(\sigma_{11}-\sigma_{22})}{2}\sin 2\theta + \sigma_{12}\cos 2\theta\end{aligned}$$

Denoting $\sigma_{11}^* := \sigma$ and $\sigma_{12}^* := \tau$, one can write that

$$\left(\sigma - \frac{\sigma_{11}+\sigma_{22}}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2$$

One can identify the above equation as an equation of a circle (comparing with the standard one, $(x-x_0)^2 + (y-y_0)^2 = R^2$) with center $\left(\frac{\sigma_{11}+\sigma_{22}}{2}, 0\right)$ and radius $R = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$. This was first identified by the German engineer Christian Otto Mohr, and it is called Mohr's circle, a two-dimensional graphical representation of the transformation law for the components of Cauchy stress tensor.

(b) Given the components of plane stress on an element of an industrial robot, we can determine the stresses σ and τ using Mohr's circle. The given stresses are:

$$\sigma_{xx} = -10\,\text{MPa}, \quad \sigma_{yy} = 10\,\text{MPa}, \quad \sigma_{xy} = 12\,\text{MPa}$$

The center of Mohr's circle $(\sigma_c, 0)$ is given by:

$$\sigma_c = \frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{-10+10}{2} = 0\,\text{MPa}$$

The radius (R) is given by:

$$R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} = \sqrt{\left(\frac{-10-(10)}{2}\right)^2 + (-12)^2} = \sqrt{10^2+12^2} = \sqrt{244} \approx 15.6\,\text{MPa}$$

The principal stresses (σ_1, σ_2) are:

$$\sigma_I = \sigma_c + R = 0 + 15.6 \approx 15.6\,\mathrm{MPa}$$

$$\sigma_{II} = \sigma_c - R = 0 - 15.6 \approx -15.6\,\mathrm{MPa}$$

From Mohr's circle, Rotate the line "AB" anti-clockwise by $2\theta = 110°$ to obtain the stress state A*B*. The values A* in the Mohr's circle indicates the desired value $\sigma = -7.85$ MPa and $\tau = 13.5$ MPa.

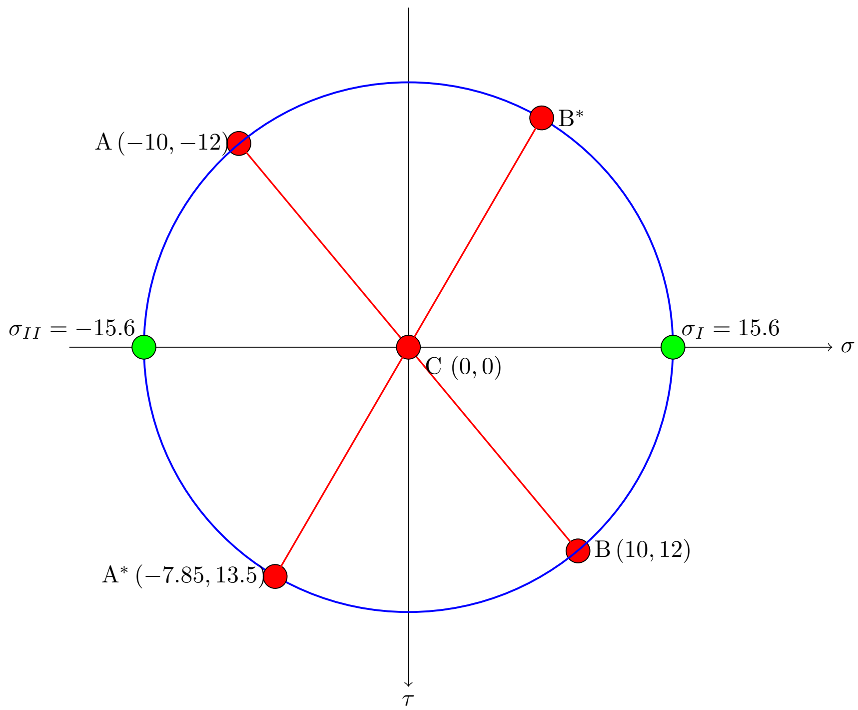

Figure 5