Indian Institute of Technology Bhubaneswar

School of Infrastructure

Subject Name : Solid Mechanics	Subject Code: CE2L001

Tutorial No. 3

1. The three principal invariants of a second-order tensor $\boldsymbol{T} = T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ are defined as $I_1 = \text{trace}\,(\boldsymbol{T}) = T_{ii}$, $I_2 = (1/2)\left((\text{trace}\,(\boldsymbol{T}))^2 - \text{trace}\left(\boldsymbol{T}^2\right)\right) = (1/2)\,(T_{ii}T_{jj} - T_{ij}T_{ij})$ and $I_3 = \det(\boldsymbol{T}) = (1/6)\,\epsilon_{lmn}\,\epsilon_{ijk}\,T_{li}\,T_{mj}\,T_{nk}$. Prove that I_1, I_2 and I_3 are indeed invariants, i.e., they do not change with the change in the coordinate system.

Solution: The second order tensor $\boldsymbol{T}$ can be expressed in an orthonormal basis $\{\boldsymbol{e}_i\}$ as $\boldsymbol{T} = T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. The first principal invariant of this tensor is defined as $I_1 = \text{trace}\,(\boldsymbol{T}) = T_{ii}$. To prove that the given I_1 is indeed an invariant, one can determine the same in another orthonormal basis, $\{\boldsymbol{e}_i^*\}$, which should be the same. The second-order tensor $\boldsymbol{T}$ in the basis $\{\boldsymbol{e}_i^*\}$ can be expressed as $\boldsymbol{T} = T_{ij}^*\boldsymbol{e}_i^* \otimes \boldsymbol{e}_j^*$. Thus the components can be written as

$$
\begin{aligned}
T_{ij}^* &= \boldsymbol{e}_i^* \cdot \left(\boldsymbol{T}\boldsymbol{e}_j^*\right) \\
&= \boldsymbol{e}_i^* \cdot \left(T_{kl}(\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\boldsymbol{e}_j^*\right) \\
&= \boldsymbol{e}_i^* \cdot \left(T_{kl}(\boldsymbol{e}_j^* \cdot \boldsymbol{e}_l)\boldsymbol{e}_k\right) \qquad \text{since } (\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\boldsymbol{e}_j^* = (\boldsymbol{e}_j^* \cdot \boldsymbol{e}_l)\boldsymbol{e}_k \\
&= T_{kl}\left((\boldsymbol{e}_k \cdot \boldsymbol{e}_i^*)(\boldsymbol{e}_l \cdot \boldsymbol{e}_j^*)\right) \\
\implies T_{ij}^* &= Q_{ki}T_{kl}Q_{lj} \qquad \text{since } Q_{ki} = \boldsymbol{e}_k \cdot \boldsymbol{e}_i^* \quad \& \quad Q_{lj} = \boldsymbol{e}_l \cdot \boldsymbol{e}_j^*
\end{aligned}
\tag{1}
$$

With the definition $Q_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j^*$, the orthogonal transformation tensor $\boldsymbol{Q} = Q_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ can be defined as

$$\boldsymbol{Q} = (\boldsymbol{e}_i \cdot \boldsymbol{e}_j^*)(\boldsymbol{e}_i \otimes \boldsymbol{e}_j). \tag{2}$$

Thus, one can find the components of the tensor $\boldsymbol{T}$ in $\{\boldsymbol{e}_i^*\}$ basis, i.e., $[\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}} := \left(T_{ij}^*\right)_{i,j\in(1,2,3)}$, in terms of the components of the tensor $\boldsymbol{T}$ in $\{\boldsymbol{e}_i\}$ basis, i.e., $[\boldsymbol{T}]_{\{\boldsymbol{e}_i\}} := (T_{ij})_{i,j\in(1,2,3)}$, as

$$[\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}} = [\boldsymbol{Q}]^T[\boldsymbol{T}]_{\{\boldsymbol{e}_i\}}[\boldsymbol{Q}]. \tag{3}$$

Proof for first-invariant: From Eq. (1), the first invariant in the basis $\{\boldsymbol{e}_i^*\}$ can be expressed as

$$
\begin{aligned}
I_1^* &= T_{ii}^* = Q_{ki}T_{kl}Q_{li} && \text{(From Eq. 1)}\\
&= \delta_{kl}T_{kl} && \text{(since } Q_{ki}Q_{li} = \delta_{kl}\text{)}\\
&= T_{kk} = T_{ii}\\
\implies I_1^* &= I_1
\end{aligned}
$$

Proof for second-invariant: A similar approach can be used to solve for the case of second invariant I_2 also. The second invariant in the basis $\{\boldsymbol{e}_i\}$, i.e I_2 can be defined as

$$
I_2 = \frac{1}{2}\left((\text{trace}(\boldsymbol{T}))^2 - \text{trace}(\boldsymbol{T}^2)\right) \tag{4}
$$

To prove the invariance of I_2, one need to prove the invariance of the individual terms $(\text{trace}(\boldsymbol{T}))^2$ and $\text{trace}(\boldsymbol{T}^2)$. Let's consider the first term $(\text{trace}(\boldsymbol{T}))^2$ in basis $\{\boldsymbol{e}_i\}$,

$$
(\text{trace}(\boldsymbol{T}))^2_{\{\boldsymbol{e}_i\}} = I_1^2 \quad = T_{ii}T_{jj} \tag{5}
$$

In the basis $\{\boldsymbol{e}_i^*\}$, $(\text{trace}(\boldsymbol{T}))^2$ can be written as

$$
\begin{aligned}
(\text{trace}(\boldsymbol{T}))^2_{\{\boldsymbol{e}_i^*\}} &= T_{ii}^*T_{jj}^*\\
&= (Q_{ki}T_{kl}Q_{li})(Q_{mj}T_{mn}Q_{nj}) && \text{(From Eq. 1)}\\
&= \delta_{kl}T_{kl}\delta_{mn}T_{mn} && \text{(since } Q_{ki}Q_{li} = \delta_{kl} \ \& \ Q_{mj}Q_{nj} = \delta_{mn}\text{)}\\
&= T_{kk}T_{mm} && \text{(since } \delta_{kl}T_{kl} = T_{kk} \ \& \ \delta_{mn}T_{mn} = T_{mm}\text{)}\\
\implies (\text{trace}(\boldsymbol{T}))^2_{\{\boldsymbol{e}_i^*\}} &= (\text{trace}(\boldsymbol{T}))^2_{\{\boldsymbol{e}_i\}}
\end{aligned} \tag{6}
$$

Let's consider the second term $\text{trace}(\boldsymbol{T}^2)$ in basis $\{\boldsymbol{e}_i\}$,

$$
\text{trace}(\boldsymbol{T}^2)_{\{\boldsymbol{e}_i\}} = T_{ij}T_{ji} \tag{7}
$$

$\text{trace}(\boldsymbol{T}^2)$ in basis $\{\boldsymbol{e}_i^*\}$,

$$
\begin{aligned}
\text{trace}(\boldsymbol{T}^2)_{\{\boldsymbol{e}_i^*\}} &= T_{ij}^*T_{ji}^*\\
&= (Q_{ki}T_{kl}Q_{lj})(Q_{mj}T_{mn}Q_{ni}) && \text{(From Eq. 1)}\\
&= \delta_{kn}\delta_{lm}T_{kl}T_{mn} && \text{(since } Q_{ki}Q_{ni} = \delta_{kn} \ \& \ Q_{lj}Q_{mj} = \delta_{lm}\text{)}\\
&= T_{km}T_{mk} && \text{(since } \delta_{kn}T_{mn} = T_{km} \ \& \ \delta_{lm}T_{kl} = T_{mk}\text{)}\\
\text{trace}(\boldsymbol{T}^2)_{\{\boldsymbol{e}_i^*\}} &= \text{trace}(\boldsymbol{T}^2)_{\{\boldsymbol{e}_i\}}
\end{aligned} \tag{8}
$$

Since the terms $(\text{trace}(\boldsymbol{T}))^2$ and $\text{trace}(\boldsymbol{T}^2)$ are invariant, the second invariant I_2 is invariant.

Proof for third-invariant: The third invariant in basis $\{\boldsymbol{e}_i\}$ can be derived as

$$
I_3 = \det\left([\boldsymbol{T}]_{\{\boldsymbol{e}_i\}}\right) \tag{9}
$$

The third invariant in basis $\{\boldsymbol{e}_i^*\}$ can be derived as

$$\begin{aligned} I_3^* &= \det\left([\boldsymbol{T}]_{\{\boldsymbol{e}_i^*\}}\right) \\ &= \det\left([\boldsymbol{Q}]^T [\boldsymbol{T}]_{\{\boldsymbol{e}_i\}} [\boldsymbol{Q}]\right) \\ &= \det\left([\boldsymbol{Q}]^T\right) \det\left([\boldsymbol{T}]_{\{\boldsymbol{e}_i\}}\right) \det\left([\boldsymbol{Q}]\right) \\ &= \det\left([\boldsymbol{T}]_{\{\boldsymbol{e}_i\}}\right) \qquad \text{(since } \det\left([\boldsymbol{Q}]^T\right) = \det\left([\boldsymbol{Q}]\right) = 1) \\ \implies I_3^* &= I_3 \end{aligned} \tag{10}$$

2. In two dimensions, let us consider two orthogonal basis vectors $\boldsymbol{e_i}$ and $\boldsymbol{e_i^*}$ such that $\boldsymbol{e_1^*}$ is oriented at an angle θ with respect to $\boldsymbol{e_1}$. σ_{ij} and σ_{ij}^* are, respectively, the components of a stress tensor $\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e_i}\otimes\boldsymbol{e_j} = \sigma_{ij}^*\boldsymbol{e_i^*} \otimes \boldsymbol{e_j^*}$ expressed in the $\boldsymbol{e_i}$ and $\boldsymbol{e_i^*}$ bases. Using the expression: $\sigma_{ij}^* = \sigma_{lk}\left(\boldsymbol{e_l} \cdot \boldsymbol{e_i}^*\right)\left(\boldsymbol{e_j}^* \cdot \boldsymbol{e_k}\right)$,

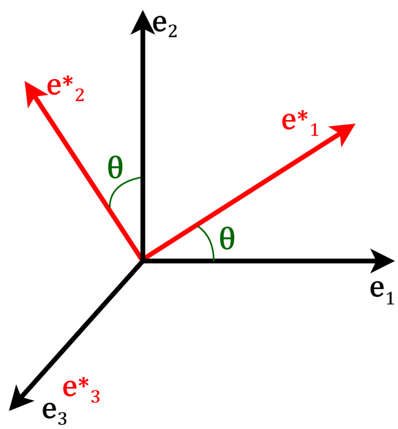


Figure 1: Transformation of coordinate system.

(a) Derive the following relations:

$$\begin{aligned} \sigma_{11}^* &= \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + \sigma_{12}\sin 2\theta \\ \sigma_{22}^* &= \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - \sigma_{12}\sin 2\theta \\ \sigma_{12}^* &= -\frac{\sigma_{11} - \sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta \end{aligned}$$

(b) Show that these equations are related to Mohr's circle for plane stress problems.

Solution: We first note the trigonometric relations between the vectors $\boldsymbol{e_i}$ and $\boldsymbol{e_j^*}$:

$$\boldsymbol{e_1} \cdot \boldsymbol{e_1}^* = \cos\theta, \quad \boldsymbol{e_2} \cdot \boldsymbol{e_2}^* = \cos\theta, \quad \boldsymbol{e_1} \cdot \boldsymbol{e_2}^* = -\sin\theta, \quad \boldsymbol{e_2} \cdot \boldsymbol{e_1}^* = \sin\theta$$

Using the transformation expression, the stresses in the rotated system are:

$$\begin{aligned}\sigma_{11}^{*} &= \sigma_{11}(\boldsymbol{e_1}\cdot\boldsymbol{e_1}^{*})^2 + \sigma_{22}(\boldsymbol{e_2}\cdot\boldsymbol{e_2}^{*})^2 + 2\sigma_{12}(\boldsymbol{e_1}\cdot\boldsymbol{e_1}^{*})(\boldsymbol{e_2}\cdot\boldsymbol{e_2}^{*}) \\ &= \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + \sigma_{12}\sin 2\theta \end{aligned} \tag{11}$$

$$\begin{aligned}\sigma_{22}^{*} &= \sigma_{11}(\boldsymbol{e_1}\cdot\boldsymbol{e_2}^{*})^2 + \sigma_{22}(\boldsymbol{e_2}\cdot\boldsymbol{e_1}^{*})^2 + 2\sigma_{12}(\boldsymbol{e_1}\cdot\boldsymbol{e_2}^{*})(\boldsymbol{e_2}\cdot\boldsymbol{e_1}^{*}) \\ &= \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - \sigma_{12}\sin 2\theta \end{aligned} \tag{12}$$

$$\begin{aligned}\sigma_{12}^{*} &= -\tfrac{\sigma_{11}}{2}\sin 2\theta + \tfrac{\sigma_{22}}{2}\sin 2\theta + \sigma_{12}(\cos^2\theta - \sin^2\theta) \\ &= -\tfrac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta \end{aligned} \tag{13}$$

Thus, the stress transformation relations are

$$\boxed{\begin{aligned}&\sigma_{11}^{*} = \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + \sigma_{12}\sin 2\theta \\ &\sigma_{22}^{*} = \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - \sigma_{12}\sin 2\theta \\ &\sigma_{12}^{*} = -\tfrac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta\end{aligned}}$$

(b) Using the relations $\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta}$, one can rewrite the above equations as:

$$\begin{aligned}\sigma_{11}^{*} &= \frac{\sigma_{11}}{2}(1+\cos 2\theta) + \frac{\sigma_{22}}{2}(1-\cos 2\theta) + \sigma_{12}\sin 2\theta \\ &= \frac{(\sigma_{11}+\sigma_{22})}{2} + \frac{(\sigma_{11}-\sigma_{22})}{2}\cos 2\theta + \sigma_{12}\sin 2\theta \\ \sigma_{22}^{*} &= \frac{\sigma_{11}}{2}(1-\cos 2\theta) + \frac{\sigma_{22}}{2}(1+\cos 2\theta) - \sigma_{12}\sin 2\theta \\ &= \frac{(\sigma_{11}+\sigma_{22})}{2} - \frac{(\sigma_{11}-\sigma_{22})}{2}\cos 2\theta - \sigma_{12}\sin 2\theta \\ \sigma_{12}^{*} &= -\frac{(\sigma_{11}-\sigma_{22})}{2}\sin 2\theta + \sigma_{12}\cos 2\theta\end{aligned}$$

Denoting $\sigma_{11}^{*} := \sigma$ and $\sigma_{12}^{*} := \tau$, one can write that

$$\left(\sigma - \frac{\sigma_{11}+\sigma_{22}}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2$$

One can identify the above equation as an equation of a circle (comparing with the standard one, $(x-x_0)^2 + (y-y_0)^2 = R^2$) with center $\left(\frac{\sigma_{11}+\sigma_{22}}{2}, 0\right)$ and radius $R = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$. This was first identified by the German engineer Christian Otto Mohr, and it is called Mohr's circle, a two-dimensional graphical representation of the transformation law for the components of Cauchy stress tensor.

3. Suppose we have a material under plane stress conditions with the following stress state: $\sigma_{11} = 80$ MPa, $\sigma_{22} = 20$ MPa, and $\sigma_{12} = 40$ MPa. Draw the Mohr's circle for this state of stress and determine the principal stresses and principal planes.

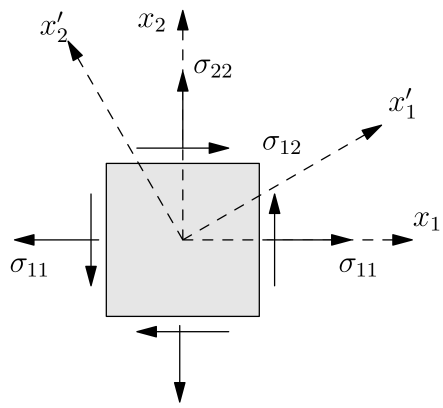
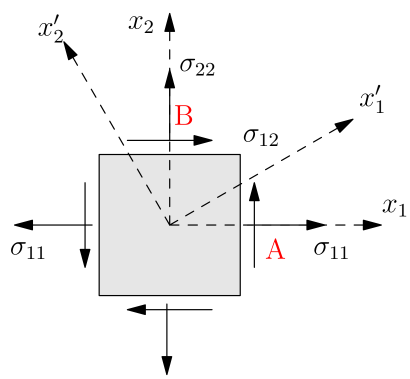
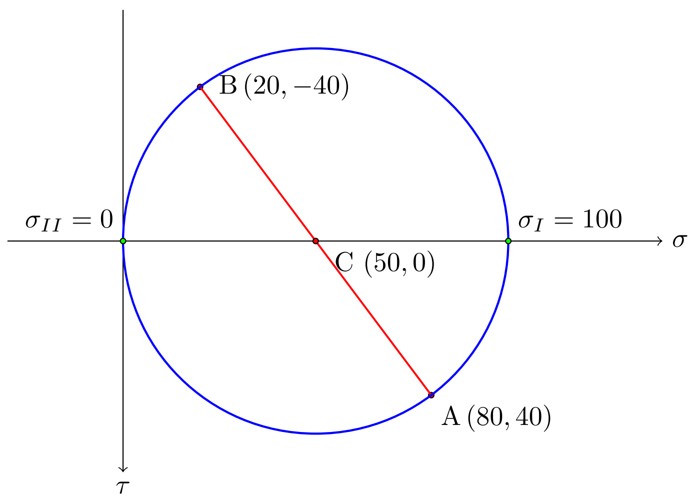


Figure 2: 2D stress element corresponding to the given stress state.

Solution:





The center of Mohr's circle is defined as

$$C = \left(\frac{\sigma_{11} + \sigma_{22}}{2}, 0\right) = \left(\frac{80 + 20}{2}, 0\right) = (50, 0).$$

The radius of Mohr's circle is given by

$$R = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2} = \sqrt{\left(\frac{80 - 20}{2}\right)^2 + 40^2} = \sqrt{30^2 + 40^2} = 50.$$

Thus, Mohr's circle is centered at $(50, 0)$ with radius 50. The principal stresses are obtained as

$$\sigma_{I,II} = C \pm R$$

Substituting values,

$$\sigma_I = 100 \text{ MPa}, \qquad \sigma_{II} = 0 \text{ MPa}.$$

The orientation of the principal planes is determined from

$$\tan 2\theta_p = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}} == \frac{80}{60} = \frac{4}{3}$$

So,

$$2\theta_p = 53.13°, \quad \Longrightarrow \theta_p = 26.57°.$$

$$\boxed{\sigma_I = 100 \text{ MPa}, \quad \sigma_{II} = 0 \text{ MPa}, \quad \theta_p = 26.57°}.$$

4. Give practical examples corresponding to each of the stress states shown in Fig. 3 and sketch the Mohr's circles. For each case, explain the usefulness of the Mohr's circle. Assume suitable numerical values for $\sigma_0 > 0$ and $\tau_o > 0$.

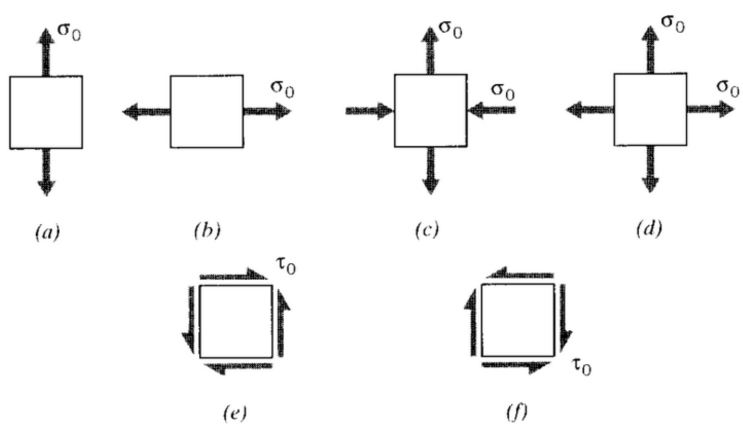


Figure 3: Different state of stress

Solution: The Mohr's circle for each state of stress can be plotted as shown in Fig. 4.

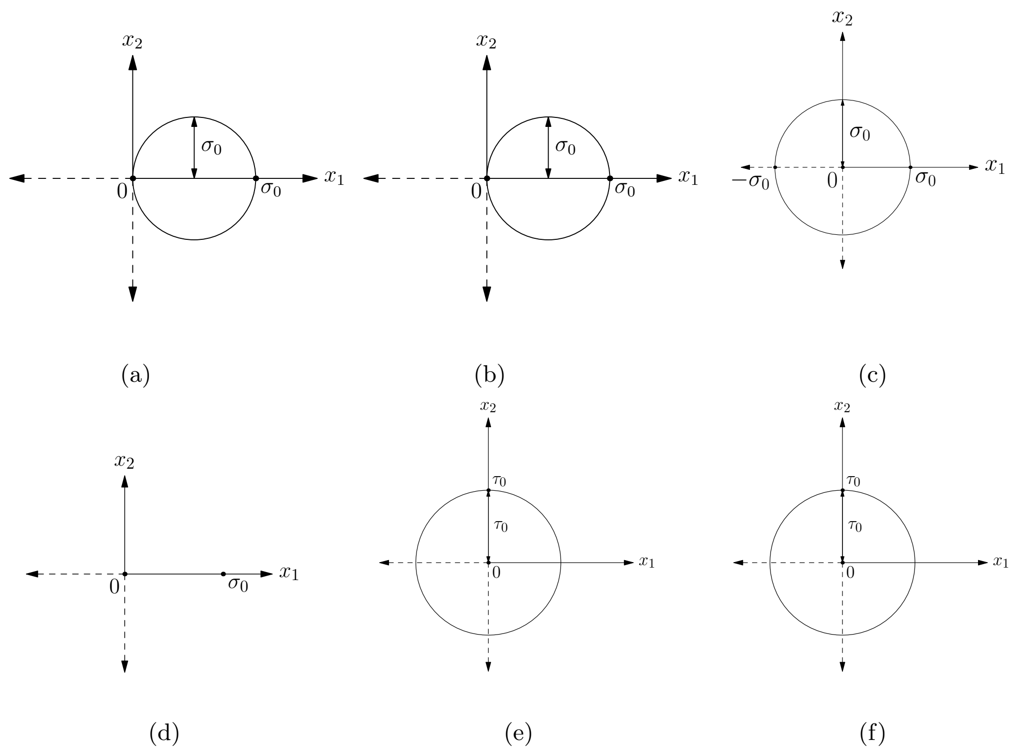


Figure 4

The practical application and usefulness of Mohr's circle for each case:

(a) The practical application is the stress state in a bar or cable under vertical tensile load. There is only one normal stress component σ_0 and all other components including shear stress components are zero. This results in a Mohr's circle with a center at coordinate $(\sigma_0, 0)$ with a radius of σ_0. From the obtained one, it can be understood that the principal stress value is σ_0 and the maximum shear stress is σ_0 at a plane 45°.

(b) The practical application is the stress state in a bar or cable under horizontal tensile load. The Mohr;s circle is drawn in the same way as the case (a).

(c) The practical application is the stress state at a point above the neutral axis of a beam under transverse loads. In this case, since one of the normal stress component is tensile and other is compressive, the center of the Mohr's circle lies at the origin (0,0) with a radius σ_0. From the Mohr's circle, it can be seen that the principal stresses are σ_0 and $-\sigma_0$ with a maximum shear stress of σ_0 at a plane 45°.

(d) The practical application is the stress state at a point in a pressurized vessel or thin-walled cylinder. In this case, since both the normal stress components are tensile, the Mohr's circle is a point itself at $(\sigma_0, 0)$. Thus it can be inferred from the Mohr's circle that the shear stress is zero at any plane along this point and the principal stress component is σ_0.

(e) The practical application is the stress state at a point in a shaft under torsion where a pure shear stress develops. In this case, since there is no normal stress components, the center of the Mohr's circle remains at origin $(0, 0)$ and the radius of the circle is τ_0. From the Mohr's circle, it can be understood that both the maximum shear stress component and the principal stress component is τ_0.

(f) The practical applications are similar to the case of (e). The Mohr's circle is also drawn in the same way as the case (e).

5. To test a glue, two plates are glued together as shown in Fig. 5. The bar formed by the joined plates is then subjected to tensile axial loads of 200 N. Using stress states and related Mohr's circle, determine the normal and shear stresses act on the plane where the plates are glued together (In other words, what stresses must the glue support?). [7]

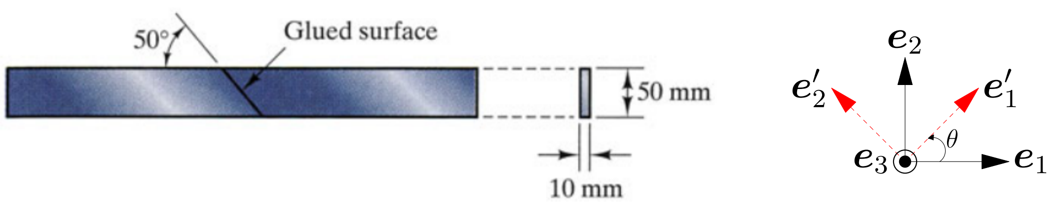


Figure 5

Solution: To determine the stresses acting on any plane that is at an angle θ from the earlier coordinate axis, the following expression can be used,

$$[\boldsymbol{\sigma}]_{\{\boldsymbol{e}'_i\}} = [\boldsymbol{Q}]^T[\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i\}}[\boldsymbol{Q}] \tag{14}$$

where $\boldsymbol{Q}$ is the orthogonal transformation tensor which is defined as

$$[\boldsymbol{Q}] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{15}$$

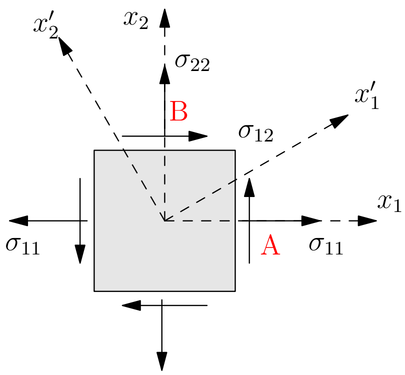
The components of the stress in the initial plane can be determined as

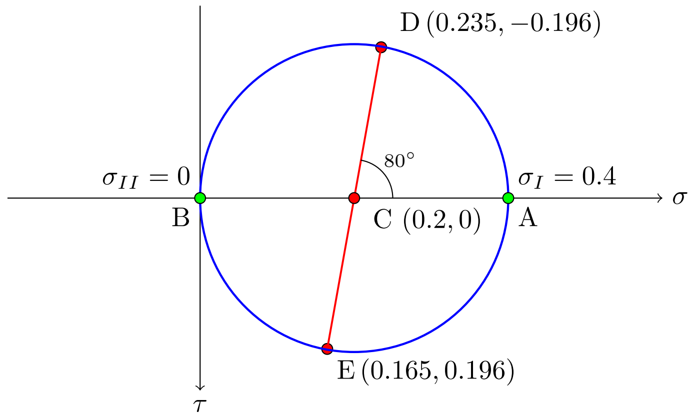
$$\begin{aligned} \sigma_{11} &= \frac{200}{50 \times 10} = 0.4 \text{ N/mm}^2 \\ \sigma_{22} &= 0 \\ \sigma_{12} &= 0 \end{aligned} \tag{16}$$

Thus the stresses acting on an inclined plane at an angle $\theta = 90^\circ - 50^\circ = 40^\circ$ can be determined as

$$
\begin{aligned}
[\boldsymbol{\sigma}]_{\{\boldsymbol{e}'_i\}} &= [\boldsymbol{Q}]^T[\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i\}}[\boldsymbol{Q}] \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos(40) & \sin(40) \\ -\sin(40) & \cos(40) \end{bmatrix} \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(40) & -\sin(40) \\ \sin(40) & \cos(40) \end{bmatrix} \\
\implies [\boldsymbol{\sigma}]_{\{\boldsymbol{e}'_i\}} &= \begin{bmatrix} 0.235 & -0.196 \\ -0.196 & 0.165 \end{bmatrix}
\end{aligned}
$$

The normal stress components on the inclined plane are 0.235 MPa and 0.165 MPa, while the shear stress is 0.196 MPa.





6. A point p of the car's frame is subjected to the components of plane stress $\sigma_{x'x'} = 32$ MPa, $\sigma_{y'y'} = -16$ MPa, and $\tau_{x'y'} = -24$ MPa: If $\theta = 35^\circ$, what are the stresses σ_{xx}, σ_{yy}, and τ_{xy} at p?

Figure 6

Solution: The component of the transformation tensor $\boldsymbol{Q}$ that transforms the stress tensor in the $x-y$ coordinate axis to $x'-y'$ coordinate axis is

$$\begin{aligned}[\boldsymbol{Q}] &= \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} \\ &= \begin{bmatrix}\cos 35 & -\sin 35 \\ \sin 35 & \cos 35\end{bmatrix}\end{aligned} \tag{17}$$

and the stress tensor in the $x'-y'$ coordinate axis can be expressed as

$$[\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i'\}} = [\boldsymbol{Q}]^T[\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i\}}[\boldsymbol{Q}] \tag{18}$$

In the given problem, the components of the stress tensor in $x'-y'$ coordinate system, $\boldsymbol{\sigma}'$ is given. One can determine the stress tensor in $x-y$ coordinate axis , $\boldsymbol{\sigma}$ as

$$[\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i\}} = [\boldsymbol{Q}]^{-T}[\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i'\}}[\boldsymbol{Q}]^{-1} \tag{19}$$

Thus the components of the stress tensor $\boldsymbol{\sigma}'$ can be determined as

$$\begin{aligned}[\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i\}} &= [\boldsymbol{Q}^{-T}][\boldsymbol{\sigma}]_{\{\boldsymbol{e}_i'\}}[\boldsymbol{Q}^{-1}] \\ &= \begin{bmatrix}\cos 35 & -\sin 35 \\ \sin 35 & \cos 35\end{bmatrix}^{-T}\begin{bmatrix}32 & -24 \\ -24 & -16\end{bmatrix}\begin{bmatrix}\cos 35 & -\sin 35 \\ \sin 35 & \cos 35\end{bmatrix}^{-1} \\ \implies [\boldsymbol{\sigma}] &= \begin{bmatrix}38.76 & 14.34 \\ 14.34 & -22.76\end{bmatrix}\end{aligned}$$

The stress tensor $\boldsymbol{\sigma}$ in the tensorial form as

$$\boldsymbol{\sigma} = 38.76(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1) + 14.34(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1) - 22.76(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) \tag{20}$$

So, the stress components σ_{xx}, σ_{yy}, and τ_{xy} at p are

$$\sigma_{xx} = 38.76\,\text{MPa}, \quad \sigma_{yy} = -22.76\,\text{MPa}, \quad \tau_{xy} = 14.34\,\text{MPa}$$

7. For the stress tensor given below, determine the principal stresses and their corresponding principal directions.

$$[\boldsymbol{\sigma}] = \begin{pmatrix}10 & 2 & 0 \\ 2 & 20 & 0 \\ 0 & 0 & 30\end{pmatrix}\text{MPa}$$

Solution: The given stress tensor is

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 10 & 2 & 0 \\ 2 & 20 & 0 \\ 0 & 0 & 30 \end{bmatrix} \text{ MPa.}$$

We use the stress invariants I_1, I_2, I_3. For a 3×3 stress tensor

$$I_1 = \mathrm{tr}(\boldsymbol{\sigma}), \qquad I_2 = \tfrac{1}{2}\left[(\mathrm{tr}\,\boldsymbol{\sigma})^2 - \mathrm{tr}(\boldsymbol{\sigma}^2)\right], \qquad I_3 = \det(\boldsymbol{\sigma}).$$

Compute them for the given $\boldsymbol{\sigma}$:

$$I_1 = 10 + 20 + 30 = 60,$$

$$I_2 = (10)(20) + (20)(30) + (30)(10) - (2)^2 - 0 - 0 = 200 + 600 + 300 - 4 = 1096,$$

$$I_3 = \det[\boldsymbol{\sigma}] = 10 \cdot 20 \cdot 30 - 30 \cdot (2)^2 = 6000 - 120 = 5880.$$

The characteristic equation for the principal stresses λ is

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0,$$

so here

$$\lambda^3 - 60\lambda^2 + 1096\lambda - 5880 = 0.$$

Its roots (principal stresses) are (show all the steps as discussed in the class)

$$\lambda_1 = 30, \qquad \lambda_2 = 15 + \sqrt{29}, \qquad \lambda_3 = 15 - \sqrt{29}.$$

Therefore the principal stresses are

$$\sigma_I = 30.00 \text{ MPa}, \qquad \sigma_{II} = 15 + \sqrt{29} \text{ MPa} \approx 20.385 \text{ MPa}, \qquad \sigma_{III} = 15 - \sqrt{29} \text{ MPa} \approx 9.615 \text{ MPa}.$$

Now find the corresponding principal directions (eigenvectors).

For $\sigma_I = 30$ MPa :

$$(\boldsymbol{\sigma} - 30\mathbf{I})\mathbf{n}^{(1)} = \mathbf{0} \quad \Rightarrow \quad [\mathbf{n}^{(1)}] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ (do the detail calculations as shown in the class).

For $\sigma_{II} = 20.385$ MPa :

$$[\mathbf{n}^{(2)}] = \begin{bmatrix} 0.1891 \\ 0.9820 \\ 0 \end{bmatrix}. \quad \text{(do the detail calculations as shown in the class)}$$

For $\sigma_{III} = 9.615$ MPa :

$$[\mathbf{n}^{(3)}] = \begin{bmatrix} -0.9820 \\ 0.1891 \\ 0 \end{bmatrix}. \quad \text{(do the detail calculations as shown in the class)}$$

8. A point (say, P) of a solid body is subjected to the state of stress (in MPa):

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Determine the principal stresses and the absolute maximum shear stress at P.

Solution: A point P is subjected to the stress

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix} \text{ MPa.}$$

We use the stress invariants I_1, I_2, I_3:

$$I_1 = \mathrm{tr}(\boldsymbol{\sigma}), \qquad I_2 = \tfrac{1}{2}\big[(\mathrm{tr}\,\boldsymbol{\sigma})^2 - \mathrm{tr}(\boldsymbol{\sigma}^2)\big], \qquad I_3 = \det(\boldsymbol{\sigma}).$$

Compute them for the given tensor:

$$\begin{aligned} I_1 &= 4 + 2 + 3 = 9, \\ \mathrm{tr}(\boldsymbol{\sigma}^2) &= 21 + 9 + 11 = 41, \\ I_2 &= \tfrac{1}{2}\big(9^2 - 41\big) = \tfrac{1}{2}(81 - 41) = 20, \\ I_3 &= \det[\boldsymbol{\sigma}] = 10. \end{aligned}$$

The characteristic equation for the principal stresses λ is

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0,$$

so here

$$\lambda^3 - 9\lambda^2 + 20\lambda - 10 = 0.$$

Its roots (principal stresses) are (show all the steps as discussed in the class)

$$\lambda_1 \approx 5.8951 \text{ MPa}, \qquad \lambda_2 \approx 2.3973 \text{ MPa}, \qquad \lambda_3 \approx 0.7076 \text{ MPa}.$$

The absolute maximum shear stress at P is

$$\tau_{\max} = \frac{1}{2}\big(\sigma_{\max} - \sigma_{\min}\big) = \frac{1}{2}\big(5.89510652 - 0.70759841\big) \approx 2.5938 \text{ MPa}.$$

Final Answer:

$$\sigma_I \approx 5.8951 \text{ MPa}, \qquad \sigma_{II} \approx 2.3973 \text{ MPa}, \qquad \sigma_{III} \approx 0.7076 \text{ MPa},$$

$$\tau_{\max} \approx 2.5938 \text{ MPa}.$$