Indian Institute of Technology Bhubaneswar

School of Infrastructure

| Subject Name : Solid Mechanics | Subject Code: CE2L001 |
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Problem Sheet No. 2 Date: September 12, 2025

Instructions:

Provide neatly labelled diagrams whenever necessary.

Notations :

Zeroth-order tensors or scalars are represented by small letters. For eg. $a$

First-order tensors or vectors are represented by bold small letters. For eg. $\boldsymbol{a}$.

Second-order tensors are represented by bold capital letters. For eg. $\boldsymbol{A}$.

1. Show that the dot product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be interpreted as the magnitude of $\boldsymbol{u}$ times the component of $\boldsymbol{v}$ in the direction of $\boldsymbol{u}$.

2. Write the following in index notation: $\boldsymbol{v}$ , $\boldsymbol{v} \cdot \boldsymbol{e}_1$, $\boldsymbol{v} \cdot \boldsymbol{e}_k$.

3. The work done by a force, represented by a vector $\boldsymbol{f}$ in moving an object a given distance is the product of the component of force in the given direction times the distance moved. If the vector $\boldsymbol{u}$ represents the direction and magnitude (distance) the object is moved, show that the work done is equivalent to $\boldsymbol{f} \cdot \boldsymbol{u}$.

4. Explain the concepts and physical significance of zeroth, first, second, third, and fourth-order tensors through equations, figures, and examples.

5. Explain the concepts and physical significance of gradient and divergence of first-, second-, and third-order tensor fields using equations, figures, and examples.

6. Explain the concepts and physical significance of inner product (scalar product) and outer product (dyadic product) of first and second order tensors using equations, figures, and examples.

7. Prove that the dot product is commutative, $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$.

8. Evaluate $\boldsymbol{u} \cdot \boldsymbol{v}$ where $\boldsymbol{u} = \boldsymbol{e}_1 + 3\boldsymbol{e}_2 - 2\boldsymbol{e}_3$ and $\boldsymbol{v} = 4\boldsymbol{e}_1 - 2\boldsymbol{e}_2 + 4\boldsymbol{e}_3$.

9. Prove that for any vector $\boldsymbol{u}$, one can write $\boldsymbol{u} = (\boldsymbol{u} \cdot \boldsymbol{e}_1)\, \boldsymbol{e}_1 + (\boldsymbol{u} \cdot \boldsymbol{e}_2)\, \boldsymbol{e}_2 + (\boldsymbol{u} \cdot \boldsymbol{e}_3)\, \boldsymbol{e}_3$.

10. Find the projection of the vector $\boldsymbol{u} = \boldsymbol{e}_1 - 2\boldsymbol{e}_2 + 2\boldsymbol{e}_3$ on the vector $\boldsymbol{v} = 4\boldsymbol{e}_1 - 4\boldsymbol{e}_2 + 7\boldsymbol{e}_3$.

11. Find the angle between the vector $\boldsymbol{u} = 3\boldsymbol{e}_1 + 2\boldsymbol{e}_2 - 6\boldsymbol{e}_3$ on the vector $\boldsymbol{v} = 4\boldsymbol{e}_1 - 3\boldsymbol{e}_2 + \boldsymbol{e}_3$.

12. Show that $\delta_{ij} a_i b_j$ is equivalent to $\mathbf{a} \cdot \mathbf{b}$.

13. Show that $\det[\mathbf{A}] = \epsilon_{ijk} A_{1i} A_{2j} A_{3k} = \epsilon_{ijk} A_{i1} A_{j2} A_{k3}$.

14. Verify that $\epsilon_{kij}\epsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$ and hence show that $\epsilon_{ijk}\epsilon_{ijp} = 2\delta_{pk}$.

15. Evaluate or simplify the following expressions:
    (i) $\delta_{kk}$.
    (ii) $\delta_{ij}\delta_{ij}$.
    (iii) $\delta_{ij}\delta_{jk}$.
    (iv) $\epsilon_{1jk}\delta_{3j}v_k$.
    (v) $\delta_{ij}\delta_{jk}\delta_{kp}\delta_{pi}$.
    (vi) $\epsilon_{mjk}\epsilon_{njk}$.

16. If $\boldsymbol{e}$ is a unit vector and $\boldsymbol{a}$ an arbitrary vector, show that $\boldsymbol{a} = (\boldsymbol{a} \cdot \boldsymbol{e})\, \boldsymbol{e} + \boldsymbol{e} \times (\boldsymbol{a} \times \boldsymbol{e})$.

17. Verify that: $\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijm}\mathbf{e}_m$. Hence, by dotting each side with $\mathbf{e}_k$, show that $\epsilon_{ijk} = (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k$..

18. Show that $\mathbf{u} \times \mathbf{v} = \epsilon_{ijk} u_i v_j \mathbf{e}_k$.

19. Show that $A_{ij} = \epsilon_{ijk} a_k$ is skew-symetric i.e., $A_{ji} = -A_{ij}$, where $\epsilon_{ijk}$ is the standard permutation symbol.

20. Show that $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{b})^2$.

21. Show that $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$.

22. Show that $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})$.

23. Show that $\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\nabla \times \boldsymbol{a}) - \boldsymbol{a} \cdot (\nabla \times \boldsymbol{b})$.

24. Show that $\nabla^2 \boldsymbol{a} = \nabla(\nabla \cdot \boldsymbol{a}) - \nabla \times (\nabla \times \boldsymbol{a})$.

25. Prove the following identities
    (i) $\nabla \times (\nabla \phi) = 0$, where $\phi$ is a scalar.
    (ii) $\nabla \cdot (\nabla \times \boldsymbol{a}) = 0$, where $\boldsymbol{a}$ is a vector.

26. Show that $(\boldsymbol{A}\boldsymbol{a}) \cdot (\boldsymbol{B}\boldsymbol{b}) = \boldsymbol{a} \cdot \left(\boldsymbol{A}^T \boldsymbol{B}\right) \boldsymbol{b}$.

27. Show that $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v})$.

28. The density distribution throughout a material is given by $\rho = 1 + \boldsymbol{x} \cdot \boldsymbol{x}$.
    (i) What sort of function is this?
    (ii) The density is given in symbolic notation - write it in index notation.
    (iii) Evaluate the gradient of $\rho$.
    (iv) Give a unit vector in the direction in which the density is increasing the most.
    (v) Give a unit vector in any direction in which the density is not increasing.
    (vi) Take any unit vector other than the base vectors and the other vectors you used above and calculate $d\rho/dx$ in the direction of this unit vector.
    (vii) Evaluate and sketch all these quantities for the point $(2, 1)$.

    In parts (iii-iv), give your answer in (a) symbolic, (b) index, and (c) full notation.

29. Consider the scalar field defined by $\phi = x^2 + 3yx + 2z$.
    (i) Find the unit normal to the surface of constant $\phi$ at the origin $(0, 0, 0)$.
    (ii) What is the maximum value of the directional derivative of $\phi$ at the origin?
    (iii) Evaluate $d\phi/dx$ at the origin if $d\boldsymbol{x} = ds\,(\boldsymbol{e}_1 + \boldsymbol{e}_2)$.

30. If $\boldsymbol{u} = x_1 x_2 x_3\, \boldsymbol{e}_1 + x_1 x_2\, \boldsymbol{e}_2 + x_1\, \boldsymbol{e}_3$, determine $div\, \boldsymbol{u} := \nabla \cdot \boldsymbol{u}$ and $curl\, \boldsymbol{u} := \nabla \times \boldsymbol{u}$.

31. Determine the constant $a$ so that the vector $\mathbf{v} = (x_1 + 3x_2)\, \mathbf{e}_1 + (x_2 - x_3)\, \mathbf{e}_2 + (x_1 + a x_3)\, \mathbf{e}_3$ is solenoidal.

32. Show that $\delta_{ij} A_{ij} = \mathrm{tr}(\mathbf{A})$.

33. Show that the dyad is a linear operator, in other words, show that
    $(\mathbf{u} \otimes \mathbf{v})(\alpha \mathbf{w} + \beta \mathbf{x}) = \alpha (\mathbf{u} \otimes \mathbf{v})\mathbf{w} + \beta (\mathbf{u} \otimes \mathbf{v})\mathbf{x}$.

34. Show that $\frac{1}{2} \nabla (\boldsymbol{u} \cdot \boldsymbol{u}) = (\nabla \boldsymbol{u})^T \boldsymbol{u}$.

35. When is $\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{b} \otimes \boldsymbol{a}$?

36. Consider the dyadic (tensor) $(\boldsymbol{a} \otimes \boldsymbol{a} + \boldsymbol{b} \otimes \boldsymbol{b})$. Show that this tensor orthogonally projects every vector $\boldsymbol{v}$ onto the plane formed by $\boldsymbol{a}$ and $\boldsymbol{b}$ (sketch a diagram).

37. Draw a sketch to show the meaning of $\mathbf{u} \cdot (\mathbf{P}\mathbf{v})$, where $\mathbf{P}$ is the projection tensor. What is the order of the resulting tensor?

38. Show that
(i) $(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \otimes \boldsymbol{d})$.
(ii) $(\boldsymbol{e}_i \otimes \boldsymbol{e}_i) = \boldsymbol{I}$.
(iii) $\boldsymbol{T}(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{T}\boldsymbol{a}) \otimes \boldsymbol{b}$.
(iv) $\boldsymbol{S}\,(\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{S}\boldsymbol{u}) \otimes \boldsymbol{v}$.

39. Consider the vector field $\mathbf{v} = x_1^2\,\mathbf{e}_1 + x_3^2\,\mathbf{e}_2 + x_2^2\,\mathbf{e}_3$.
(a) find the matrix representation of the gradient of $\mathbf{v}$,
(b) find the vector $(\text{grad}\mathbf{v})\mathbf{v}$.

40. Show that the component $T_{11}$ of a tensor $\boldsymbol{T}$ can be evaluated from $\boldsymbol{e}_1 \cdot \boldsymbol{T}\boldsymbol{e}_1$, and that $T_{12} = \boldsymbol{e}_1 \cdot \boldsymbol{T}\boldsymbol{e}_2$ (and so on, so that $T_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{T}\boldsymbol{e}_j$).

41. Consider two second-order tensors $\boldsymbol{D}$ and $\boldsymbol{F}$ given by

$$\begin{aligned}
\boldsymbol{D} &= 3\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1) + 2\,(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) - (\boldsymbol{e}_2 \otimes \boldsymbol{e}_3) + 5\,(\boldsymbol{e}_3 \otimes \boldsymbol{e}_3)\,, \\
\boldsymbol{F} &= 4\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_3) + 6\,(\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) - 3\,(\boldsymbol{e}_3 \otimes \boldsymbol{e}_2) + (\boldsymbol{e}_3 \otimes \boldsymbol{e}_3)\,,
\end{aligned}$$

where $\{\boldsymbol{e}_i\}$ are the standard orthonormal Cartesian basis vectors. Compute the following: (i) $\boldsymbol{D}\boldsymbol{F}$ and (ii) $\boldsymbol{D} : \boldsymbol{F}$.

42. Consider the second-order tensor $\boldsymbol{T}$ given by

$$\boldsymbol{T} = 3\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1) - 4\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2) + 2\,(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1) + (\boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + (\boldsymbol{e}_3 \otimes \boldsymbol{e}_3)\,.$$

Determine the image of the vector $\boldsymbol{r} = 4\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + 5\boldsymbol{e}_3$ when $\boldsymbol{T}$ operates on it.

43. Expand the following terms
(a) $B_{ii}$
(b) $C_{kkj}$
(c) $B_{mn}$
(d) $a_i b_j A_{ij}$.
Are these the components of scalars, vectors or second order tensors?

44. Write $(\boldsymbol{a} \otimes \boldsymbol{b}) : (\boldsymbol{c} \otimes \boldsymbol{d})$ in terms of components of four vectors. What is the order of the resulting tensor?

45. Show that the components of the second-order identity tensor are given by $I_{ij} = \delta_{ij}$.

46. Show that
(a) $(\boldsymbol{u} \otimes \boldsymbol{v})\, \boldsymbol{A} = \left(\boldsymbol{u} \otimes \boldsymbol{A}^T \boldsymbol{v}\right)$
(b) $\boldsymbol{A} : (\boldsymbol{B}\boldsymbol{C}) = \left(\boldsymbol{B}^T \boldsymbol{A}\right) : \boldsymbol{C} = \left(\boldsymbol{A}\,\boldsymbol{C}^T\right) : \boldsymbol{B}$.

47. For the second-order identity tensor, show that $\boldsymbol{I}^T = \boldsymbol{I}$.

48. Show that $\mathrm{tr}(\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$.

49. Formally derive the index notation for the functions
$\mathrm{tr}\boldsymbol{A}^2$ , $\mathrm{tr}\boldsymbol{A}^3$ , $(\mathrm{tr}\boldsymbol{A})^2$ , $(\mathrm{tr}\boldsymbol{A})^3$.

50. Show that $\boldsymbol{A} : \boldsymbol{B} = \mathrm{tr}\left(\boldsymbol{A}^T \boldsymbol{B}\right)$.

51. Prove that $(\boldsymbol{T}\boldsymbol{a} \times \boldsymbol{T}\boldsymbol{b}) \cdot \boldsymbol{T}\boldsymbol{c} = (\det \boldsymbol{T})\,[(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}]$.

52. Show that $\left(\boldsymbol{A}^{-1}\right)^T : \boldsymbol{A} = 3$.

53. Show that
(i) $\boldsymbol{S} = \frac{1}{2}(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a})$ is a symmetric and $\boldsymbol{W} = \frac{1}{2}(\boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a})$ a skew-symmetric tensor.
(ii) Show that the axial vector of tensor $\boldsymbol{W}$ can be given by $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{b} \times \boldsymbol{a})$.
(iii) $\mathrm{tr}\,(\boldsymbol{S}\boldsymbol{W}) = 0$.

54. Find the spherical (volumetric) and deviatoric parts of the tensor $\boldsymbol{A}$ for which $A_{ij} = 1$.

55. Explain the physical meaning of the matrices given below using figures and examples.

$$[\boldsymbol{Q}] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad [\boldsymbol{Q}] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad [\boldsymbol{Q}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

56. (i) Consider the tensor $\boldsymbol{S} = 2(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_2)$. Compute the principal variants of $\boldsymbol{S}$.
(ii) Consider the tensor $\boldsymbol{S} = (\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + 2(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_1 \otimes \boldsymbol{e}_2)$ . Express $\boldsymbol{S}$ in spectral form.

57. Find the eigen-values, (normalized) eigen-vectors and principal invariants of
$\boldsymbol{T} = \boldsymbol{I} + \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1$.

58. Show that
(i) the trace of a second-order tensor $\boldsymbol{T}$, $\mathrm{tr}\,(\boldsymbol{T}) = T_{ii}$ is an invariant.
(ii) $\boldsymbol{a} \cdot (\boldsymbol{T}\boldsymbol{a}) = T_{ij} a_i a_j$ is an invariant.

59. Consider a two-dimensional orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ in which a two-dimensional tensor $\boldsymbol{T}$ has the representation

$$\boldsymbol{T} = T_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad i, j = 1, 2,$$

and the component matrix of $\boldsymbol{T}$ has values

$$[\boldsymbol{T}] = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}.$$

Consider a second basis $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*\}$ which is related to $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ by

$$\boldsymbol{e}_1^* = \frac{1}{2}\boldsymbol{e}_1 + \frac{\sqrt{3}}{2}\boldsymbol{e}_2, \qquad \boldsymbol{e}_2^* = -\frac{\sqrt{3}}{2}\boldsymbol{e}_1 + \frac{1}{2}\boldsymbol{e}_2,$$

Find the value of $T_{11}^*$ in the $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*\}$ basis.

60. (i) Consider an orthonormal basis $\{\boldsymbol{e}_i\}$, and define $\boldsymbol{E}_{ij} := \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ with the orthonormality property $\boldsymbol{E}_{ij} : \boldsymbol{E}_{kl} = \delta_{ik}\delta_{jl}$ where $\delta_{ij}$ is the Kronecker delta. Using this notation, the components $C_{ijkl}$ of the fourth order tensor $\mathbb{C}$ are defined as $C_{ijkl} = \boldsymbol{E}_{ij} : \mathbb{C}\boldsymbol{E}_{kl}$. Consider a fourth-order tensor with components

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right),$$

where $\lambda$ and $\mu$ are given constants. Determine $C_{1111}$, $C_{2222}$, $C_{3333}$, $C_{1212}$, $C_{2323}$, $C_{1313}$, $C_{1112}$, $C_{1222}$ and $C_{1333}$.

(ii) Given any second order skew-symmetric tensor $\boldsymbol{\Omega}$, there is a unique vector $\boldsymbol{\omega}$, called the axial vector of $\boldsymbol{\Omega}$ such that

$$\boldsymbol{\Omega}\,\boldsymbol{u} = \boldsymbol{\omega} \times \boldsymbol{u} \quad \text{for all vectors } \boldsymbol{u}.$$

Show that the indicial notation components of $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ are related by

$$\Omega_{ij} = -\epsilon_{ijk}\omega_k \quad \text{and} \quad \omega_i = -\frac{1}{2}\epsilon_{ijk}\Omega_{jk},$$

where $\epsilon_{ijk}$ is the permutation symbol.

61. In two dimensions, any orthogonal tensor can be expressed as

$$\boldsymbol{R} = \cos\theta\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \sin\theta\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \sin\theta\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \cos\theta\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2$$

(i) Show that $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{I}$ (where $\boldsymbol{I}$ is the second order identity tensor), i.e. prove that the inverse of an orthogonal tensor is its transpose.
(ii) Show that $|\boldsymbol{v}| = |\boldsymbol{R}\boldsymbol{v}|$ for all $\boldsymbol{v}$; i.e. an orthogonal tensor does not change the length of a vector.

62. Establish the following identities:
(i) $(\boldsymbol{S} + \boldsymbol{F})^{\mathrm{T}} = \boldsymbol{S}^{\mathrm{T}} + \boldsymbol{F}^{\mathrm{T}}$.
(ii) $(\boldsymbol{S}\boldsymbol{F})^{\mathrm{T}} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}}$.
(iii) $\det(\alpha\boldsymbol{S}) = \alpha^3 \det\boldsymbol{S}$.
(iv) $|\boldsymbol{T}|^2 = |\mathrm{sym}\ \boldsymbol{T}|^2 + |\mathrm{skew}\ \boldsymbol{T}|^2$.

63. (i) Consider a ball, $B$, of radius 4 centered at the origin. Using the divergence theorem, compute $\int\limits_{\partial B} \boldsymbol{x} \cdot \boldsymbol{n}\; da$, where $\boldsymbol{x}$ is the position vector and $\boldsymbol{n}$ is the unit outward normal to the surface $\partial B$.
(ii) Use the divergence theorem to show that

$$\int\limits_{R} 5x_i x_j dv = \int\limits_{\partial R} x_i x_j x_k n_k \; da.$$

64. Establish the following identities using the divergence theorem:

(i) $\int\limits_{\partial R} \boldsymbol{n} \cdot \mathrm{curl}\, \boldsymbol{v}\; da = 0.$

(ii) $\int\limits_{\partial R} \boldsymbol{n} \times \boldsymbol{v}\; da = \int\limits_{R} \mathrm{curl}\, \boldsymbol{v}\; dv.$

(iii) $\int\limits_{\partial R} \boldsymbol{v} \otimes \boldsymbol{n}\; da = \int\limits_{R} \mathrm{grad}\, \boldsymbol{v}\; dv.$

(iv) $\int\limits_{\partial R} \boldsymbol{T}\boldsymbol{n} \otimes \boldsymbol{v}\; da = \int\limits_{R} \left( (\mathrm{div}\boldsymbol{T}) \otimes \boldsymbol{v} + \boldsymbol{T}(\mathrm{grad}\, \boldsymbol{v})^{T} \right)\; dv.$

(v) $\int\limits_{\partial R} \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{n})\; da = \int\limits_{R} (\boldsymbol{u}\, \mathrm{div}\, \boldsymbol{v} + (\mathrm{grad}\, \boldsymbol{u})\boldsymbol{v})\; dv.$