CHAPTER 12

Orthogonal Functions and Fourier Series

CHAPTER CONTENTS

Our goal in Part 4 of this text is solve certain kinds of linear **partial differential equations** in an applied context. Although we do not solve any PDEs in this chapter, the material covered sets the stage for the procedures discussed later.

In calculus you saw that a sufficiently differentiable function f could often be expanded in a Taylor series, which essentially is an infinite series consisting of powers of x. The principal concept examined in this chapter also involves expanding a function in an infinite series. In the early 1800s, the French mathematician **Joseph Fourier** (1768–1830) advanced the idea of expanding a function f in a series of trigonometric functions. It turns out that **Fourier series** are just special cases of a more general type of series representation of a function using an infinite set of **orthogonal functions**. The notion of orthogonal functions leads us back to eigenvalues and the corresponding set of eigenfunctions. Since eigenvalues and eigenfunctions are the linchpins of the procedures in the two chapters that follow, you are encouraged to review Example 2 in Section 3.9.

12.1 Orthogonal Functions

Introduction In certain areas of advanced mathematics, a function is considered to be a generalization of a vector. In this section we shall see how the two vector concepts of inner, or dot, product and orthogonality of vectors can be extended to functions. The remainder of the chapter is a practical application of this discussion.

Inner Product Recall, if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ are two vectors in R^3 or 3-space, then the inner product or dot product of $\mathbf{u}$ and $\mathbf{v}$ is a real number, called a **scalar**, defined as the sum of the products of their corresponding components:

$$(\mathbf{u}, \mathbf{v}) = u_1v_1 + u_2v_2 + u_3v_3 = \sum_{k=1}^{3} u_k v_k.$$

In Chapter 7, the inner product was denoted by $\mathbf{u} \cdot \mathbf{v}$.

The inner product $(\mathbf{u}, \mathbf{v})$ possesses the following properties:

(*i*) $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$
(*ii*) $(k\mathbf{u}, \mathbf{v}) = k(\mathbf{u}, \mathbf{v})$, k a scalar
(*iii*) $(\mathbf{u}, \mathbf{u}) = 0$ if $\mathbf{u} = \mathbf{0}$ and $(\mathbf{u}, \mathbf{u}) > 0$ if $\mathbf{u} \neq \mathbf{0}$
(*iv*) $(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$.

We expect any generalization of the inner product to possess these same properties.

Suppose that f_1 and f_2 are piecewise-continuous functions defined on an interval $[a, b]$.* Since a definite integral on the interval of the product $f_1(x)f_2(x)$ possesses properties (*i*)–(*iv*) of the inner product of vectors, whenever the integral exists we are prompted to make the following definition.

Definition 12.1.1 Inner Product of Functions

The **inner product** of two functions f_1 and f_2 on an interval $[a, b]$ is the number

$$(f_1, f_2) = \int_a^b f_1(x)f_2(x)\,dx.$$

Orthogonal Functions Motivated by the fact that two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal whenever their inner product is zero, we define **orthogonal functions** in a similar manner.

Definition 12.1.2 Orthogonal Functions

Two functions f_1 and f_2 are said to be **orthogonal** on an interval $[a, b]$ if

$$(f_1, f_2) = \int_a^b f_1(x)f_2(x)\,dx = 0. \tag{1}$$

EXAMPLE 1 Orthogonal Functions

The functions $f_1(x) = x^2$ and $f_2(x) = x^3$ are orthogonal on the interval $[-1, 1]$. This fact follows from (1):

$$(f_1, f_2) = \int_{-1}^{1} x^2 \cdot x^3\,dx = \int_{-1}^{1} x^5\,dx = \frac{1}{6}x^6\Big]_{-1}^{1} = 0.$$

Unlike vector analysis, where the word *orthogonal* is a synonym for *perpendicular*, in this present context the term *orthogonal* and condition (1) have no geometric significance.

Orthogonal Sets We are primarily interested in infinite sets of orthogonal functions.

*The interval could also be $(-\infty, \infty)$, $[0, \infty)$, and so on.

Definition 12.1.3 Orthogonal Set

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\phi_n(x)\,dx = 0, \quad m \neq n. \tag{2}$$

Orthonormal Sets The norm, or length $\|\mathbf{u}\|$, of a vector $\mathbf{u}$ can be expressed in terms of the inner product. The expression $(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|^2$ is called the square norm, and so the norm is $\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})}$. Similarly, the **square norm** of a function ϕ_n is $\|\phi_n(x)\|^2 = (\phi_n, \phi_n)$, and so the **norm**, or its generalized length, is $\|\phi_n(x)\| = \sqrt{(\phi_n, \phi_n)}$. In other words, the square norm and norm of a function ϕ_n in an orthogonal set $\{\phi_n(x)\}$ are, respectively,

$$\|\phi_n(x)\|^2 = \int_a^b \phi_n^2(x)\,dx \quad \text{and} \quad \|\phi_n(x)\| = \sqrt{\int_a^b \phi_n^2(x)\,dx}. \tag{3}$$

If $\{\phi_n(x)\}$ is an orthogonal set of functions on the interval $[a, b]$ with the property that $\|\phi_n(x)\| = 1$ for $n = 0, 1, 2, \dots$, then $\{\phi_n(x)\}$ is said to be an **orthonormal set** on the interval.

EXAMPLE 2 Orthogonal Set of Functions

Show that the set $\{1, \cos x, \cos 2x, \dots\}$ is orthogonal on the interval $[-\pi, \pi]$.

SOLUTION If we make the identification $\phi_0(x) = 1$ and $\phi_n(x) = \cos nx$, we must then show that $\int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)\,dx = 0$, $n \neq 0$, and $\int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)\,dx = 0$, $m \neq n$. We have, in the first case, for $n \neq 0$,

$$\begin{aligned}(\phi_0, \phi_n) &= \int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)\,dx = \int_{-\pi}^{\pi} \cos nx\,dx \\ &= \frac{1}{n}\sin nx\Big]_{-\pi}^{\pi} = \frac{1}{n}[\sin n\pi - \sin(-n\pi)] = 0,\end{aligned}$$

and in the second, for $m \neq n$,

$$\begin{aligned}(\phi_m, \phi_n) &= \int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)\,dx = \int_{-\pi}^{\pi} \cos mx \cos nx\,dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x]\,dx \quad \leftarrow \text{trigonometric identity} \\ &= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0.\end{aligned}$$

≡

EXAMPLE 3 Norms

Find the norms of each function in the orthogonal set given in Example 2.

SOLUTION For $\phi_0(x) = 1$ we have from (3)

$$\|\phi_0(x)\|^2 = \int_{-\pi}^{\pi} dx = 2\pi$$

so that $\|\phi_0(x)\| = \sqrt{2\pi}$. For $\phi_n(x) = \cos nx$, $n > 0$, it follows that

$$\|\phi_n(x)\|^2 = \int_{-\pi}^{\pi} \cos^2 nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [1 + \cos 2nx]\,dx = \pi.$$

Thus for $n > 0$, $\|\phi_n(x)\| = \sqrt{\pi}$.

≡

Any orthogonal set of nonzero functions $\{\phi_n(x)\}$, $n = 0, 1, 2, \ldots$, can be *normalized*—that is, made into an orthonormal set—by dividing each function by its norm. It follows from Examples 2 and 3 that the set

An orthogonal set can be made into an orthonormal set.

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots\right\}$$

is orthonormal on the interval $[-\pi, \pi]$.

Vector Analogy We shall make one more analogy between vectors and functions. Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are three mutually orthogonal nonzero vectors in 3-space. Such an orthogonal set can be used as a basis for 3-space; that is, any three-dimensional vector can be written as a linear combination

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3, \tag{4}$$

where the c_i, $i = 1, 2, 3$, are scalars called the components of the vector. Each component c_i can be expressed in terms of $\mathbf{u}$ and the corresponding vector $\mathbf{v}_i$. To see this we take the inner product of (4) with $\mathbf{v}_1$:

$$(\mathbf{u}, \mathbf{v}_1) = c_1(\mathbf{v}_1, \mathbf{v}_1) + c_2(\mathbf{v}_2, \mathbf{v}_1) + c_3(\mathbf{v}_3, \mathbf{v}_1) = c_1\|\mathbf{v}_1\|^2 + c_2 \cdot 0 + c_3 \cdot 0.$$

Hence
$$c_1 = \frac{(\mathbf{u}, \mathbf{v}_1)}{\|\mathbf{v}_1\|^2}.$$

In like manner we find that the components c_2 and c_3 are given by

$$c_2 = \frac{(\mathbf{u}, \mathbf{v}_2)}{\|\mathbf{v}_2\|^2} \quad \text{and} \quad c_3 = \frac{(\mathbf{u}, \mathbf{v}_3)}{\|\mathbf{v}_3\|^2}.$$

Hence (4) can be expressed as

$$\mathbf{u} = \frac{(\mathbf{u}, \mathbf{v}_1)}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{(\mathbf{u}, \mathbf{v}_2)}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \frac{(\mathbf{u}, \mathbf{v}_3)}{\|\mathbf{v}_3\|^2}\mathbf{v}_3 = \sum_{n=1}^{3}\frac{(\mathbf{u}, \mathbf{v}_n)}{\|\mathbf{v}_n\|^2}\mathbf{v}_n. \tag{5}$$

Orthogonal Series Expansion Suppose $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a, b]$. We ask: If $y = f(x)$ is a function defined on the interval $[a, b]$, is it possible to determine a set of coefficients c_n, $n = 0, 1, 2, \ldots$, for which

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots? \tag{6}$$

As in the foregoing discussion on finding components of a vector, we can find the coefficients c_n by utilizing the inner product. Multiplying (6) by $\phi_m(x)$ and integrating over the interval $[a, b]$ gives

$$\begin{aligned}\int_a^b f(x)\phi_m(x)\,dx &= c_0\int_a^b \phi_0(x)\phi_m(x)\,dx + c_1\int_a^b \phi_1(x)\phi_m(x)\,dx + \cdots + c_n\int_a^b \phi_n(x)\phi_m(x)\,dx + \cdots \\ &= c_0(\phi_0, \phi_m) + c_1(\phi_1, \phi_m) + \cdots + c_n(\phi_n, \phi_m) + \cdots.\end{aligned}$$

By orthogonality, each term on the right-hand side of the last equation is zero *except* when $m = n$. In this case we have

$$\int_a^b f(x)\phi_n(x)\,dx = c_n\int_a^b \phi_n^2(x)\,dx.$$

It follows that the required coefficients are

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}, \quad n = 0, 1, 2, \ldots.$$

In other words,
$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x), \tag{7}$$

where
$$c_n = \frac{\int_a^b f(x)\phi_n(x)\, dx}{\|\phi_n(x)\|^2}. \tag{8}$$

With inner product notation, (7) becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n(x)\|^2} \phi_n(x). \tag{9}$$

Thus (9) is seen to be the function analogue of the vector result given in (5).

Definition 12.1.4 Orthogonal Set/Weight Function

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$ is said to be **orthogonal with respect to a weight function** $w(x)$ on an interval $[a, b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)\, dx = 0, \quad m \neq n.$$

The usual assumption is that $w(x) > 0$ on the interval of orthogonality $[a, b]$. The set $\{1, \cos x, \cos 2x, \dots\}$ in Example 2 is orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-\pi, \pi]$.

If $\{\phi_n(x)\}$ is orthogonal with respect to a weight function $w(x)$ on the interval $[a, b]$, then multiplying (6) by $w(x)\phi_n(x)$ and integrating yields

$$c_n = \frac{\int_a^b f(x)\, w(x)\, \phi_n(x)\, dx}{\|\phi_n(x)\|^2}, \tag{10}$$

where
$$\|\phi_n(x)\|^2 = \int_a^b w(x)\phi_n^2(x)\, dx. \tag{11}$$

The series (7) with coefficients given by either (8) or (10) is said to be an **orthogonal series expansion** of f or a **generalized Fourier series**.

Complete Sets The procedure outlined for determining the coefficients c_n was *formal*; that is, basic questions on whether an orthogonal series expansion such as (7) is actually possible were ignored. Also, to expand f in a series of orthogonal functions, it is certainly necessary that f not be orthogonal to each ϕ_n of the orthogonal set $\{\phi_n(x)\}$. (If f were orthogonal to every ϕ_n, then $c_n = 0$, $n = 0, 1, 2, \dots$.) To avoid the latter problem we shall assume, for the remainder of the discussion, that an orthogonal set is **complete**. This means that the only continuous function orthogonal to each member of the set is the zero function.

Remarks

Suppose that $\{f_0(x), f_1(x), f_2(x), \dots\}$ is an infinite set of real-valued functions that are continuous on an interval $[a, b]$. If this set is *linearly independent* on $[a, b]$ (see page 346 for the definition of an infinite linearly independent set), then it can always be made into an orthogonal set and, as described earlier in this section, can be made into an orthonormal set. See Problem 22 in Exercises 12.1.

12.1 Exercises

Answers to selected odd-numbered problems begin on page ANS-29.

In Problems 1–6, show that the given functions are orthogonal on the indicated interval.

1. $f_1(x) = x, f_2(x) = x^2$; $[-2, 2]$
2. $f_1(x) = x^3, f_2(x) = x^2 + 1$; $[-1, 1]$
3. $f_1(x) = e^x, f_2(x) = xe^{-x} - e^{-x}$; $[0, 2]$
4. $f_1(x) = \cos x$, $f_2(x) = \sin^2 x$; $[0, \pi]$
5. $f_1(x) = x, f_2(x) = \cos 2x$; $[-\pi/2, \pi/2]$
6. $f_1(x) = e^x, f_2(x) = \sin x$; $[\pi/4, 5\pi/4]$

In Problems 7–12, show that the given set of functions is orthogonal on the indicated interval. Find the norm of each function in the set.

7. $\{\sin x,\ \sin 3x,\ \sin 5x, \ldots\}$; $[0, \pi/2]$
8. $\{\cos x,\ \cos 3x,\ \cos 5x, \ldots\}$; $[0, \pi/2]$
9. $\{\sin nx\}$, $n = 1, 2, 3, \ldots$; $[0, \pi]$
10. $\left\{\sin \frac{n\pi}{p}x\right\}$, $n = 1, 2, 3, \ldots$; $[0, p]$
11. $\left\{1, \cos \frac{n\pi}{p}x\right\}$, $n = 1, 2, 3, \ldots$; $[0, p]$
12. $\left\{1, \cos \frac{n\pi}{p}x, \sin \frac{m\pi}{p}x\right\}$, $n = 1, 2, 3, \ldots$, $m = 1, 2, 3, \ldots$; $[-p, p]$

In Problems 13 and 14, verify by direct integration that the functions are orthogonal with respect to the indicated weight function on the given interval.

13. $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$; $w(x) = e^{-x^2}$, $(-\infty, \infty)$
14. $L_0(x) = 1, L_1(x) = -x + 1, L_2(x) = \frac{1}{2}x^2 - 2x + 1$; $w(x) = e^{-x}$, $[0, \infty)$
15. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a, b]$ such that $\phi_0(x) = 1$. Show that $\int_a^b \phi_n(x)\,dx = 0$ for $n = 1, 2, \ldots$.
16. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a, b]$ such that $\phi_0(x) = 1$ and $\phi_1(x) = x$. Show that $\int_a^b (\alpha x + \beta)\phi_n(x)\,dx = 0$ for $n = 2, 3, \ldots$ and any constants α and β.
17. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a, b]$. Show that $\|\phi_m(x) + \phi_n(x)\|^2 = \|\phi_m(x)\|^2 + \|\phi_n(x)\|^2$, $m \neq n$.
18. From Problem 1 we know that $f_1(x) = x$ and $f_2(x) = x^2$ are orthogonal on $[-2, 2]$. Find constants c_1 and c_2 such that $f_3(x) = x + c_1x^2 + c_2x^3$ is orthogonal to both f_1 and f_2 on the same interval.
19. The set of functions $\{\sin nx\}$, $n = 1, 2, 3, \ldots$, is orthogonal on the interval $[-\pi, \pi]$. Show that the set is not complete.
20. Suppose f_1, f_2, and f_3 are functions continuous on the interval $[a, b]$. Show that $(f_1 + f_2, f_3) = (f_1, f_3) + (f_2, f_3)$.
21. A real-valued function f is said to be **periodic** with period T if $f(x + T) = f(x)$. For example, 4π is a period of $\sin x$ since $\sin(x + 4\pi) = \sin x$. The smallest value of T for which $f(x + T) = f(x)$ holds is called the **fundamental period** of f. For example, the fundamental period of $f(x) = \sin x$ is $T = 2\pi$. What is the fundamental period of each of the following functions?
    (a) $f(x) = \cos 2\pi x$
    (b) $f(x) = \sin \frac{4}{L}x$
    (c) $f(x) = \sin x + \sin 2x$
    (d) $f(x) = \sin 2x + \cos 4x$
    (e) $f(x) = \sin 3x + \cos 2x$
    (f) $f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos \frac{n\pi}{p}x + B_n \sin \frac{n\pi}{p}x\right)$, A_n and B_n depend only on n
22. The **Gram–Schmidt process** for constructing an orthogonal set that was discussed in Section 7.7 carries over to a linearly independent set $\{f_0(x), f_1(x), f_2(x), \ldots\}$ of real-valued functions continuous on an interval $[a, b]$. With the inner product $(f_n, \phi_n) = \int_a^b f_n(x)\phi_n(x)\,dx$, define the functions in the set $B' = \{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ to be

$$\phi_0(x) = f_0(x)$$

$$\phi_1(x) = f_1(x) - \frac{(f_1, \phi_0)}{(\phi_0, \phi_0)}\phi_0(x)$$

$$\phi_2(x) = f_2(x) - \frac{(f_2, \phi_0)}{(\phi_0, \phi_0)}\phi_0(x) - \frac{(f_2, \phi_1)}{(\phi_1, \phi_1)}\phi_1(x)$$

$$\vdots \qquad\qquad \vdots$$

and so on.
    (a) Write out $\phi_3(x)$ in the set.
    (b) By construction, the set $B' = \{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is orthogonal on $[a, b]$. Demonstrate that $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$ are mutually orthogonal.

Discussion Problems

23. (a) Consider the set of functions $\{1, x, x^2, x^3, \ldots\}$ defined on the interval $[-1, 1]$. Apply the Gram–Schmidt process given in Problem 22 to this set and find $\phi_0(x), \phi_1(x), \phi_2(x)$, and $\phi_3(x)$ of the orthogonal set B'.
    (b) Discuss: Do you recognize the orthogonal set?
24. Verify that the inner product (f_1, f_2) in Definition 12.1.1 satisfies properties (*i*)–(*iv*) given on page 655.
25. In R^3, give an example of a set of orthogonal vectors that is not complete. Give a set of orthogonal vectors that is complete.

12.2 Fourier Series

Introduction We have just seen in the preceding section that if $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is a set of real-valued functions that is orthogonal on an interval $[a, b]$ and if f is a function defined on the same interval, then we can formally expand f in an orthogonal series $c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots$. In this section we shall expand functions in terms of a special orthogonal set of trigonometric functions.

Trigonometric Series In Problem 12 in Exercises 12.1, you were asked to show that the set of trigonometric functions

$$\left\{1, \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cos\frac{3\pi}{p}x, \ldots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x, \sin\frac{3\pi}{p}x, \ldots\right\} \tag{1}$$

is orthogonal on the interval $[-p, p]$. This set will be of special importance later on in the solution of certain kinds of boundary-value problems involving linear partial differential equations. In those applications we will need to expand a function f defined on $[-p, p]$ in an orthogonal series consisting of the trigonometric functions in (1); that is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x + b_n\sin\frac{n\pi}{p}x\right). \tag{2}$$

The coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$, can be determined in exactly the same manner as in the general discussion of orthogonal series expansions on pages 657 and 658. Before proceeding, note that we have chosen to write the coefficient of 1 in the set (1) as $\frac{1}{2}a_0$ rather than a_0; this is for convenience only because the formula of a_n will then reduce to a_0 for $n = 0$.

This is why $\frac{1}{2}a_0$ is used instead of a_0.

Now integrating both sides of (2) from $-p$ to p gives

$$\int_{-p}^{p} f(x)\,dx = \frac{a_0}{2}\int_{-p}^{p} dx + \sum_{n=1}^{\infty}\left(a_n\int_{-p}^{p}\cos\frac{n\pi}{p}x\,dx + b_n\int_{-p}^{p}\sin\frac{n\pi}{p}x\,dx\right). \tag{3}$$

Since $\cos(n\pi x/p)$ and $\sin(n\pi x/p)$, $n \geq 1$, are orthogonal to 1 on the interval, the right side of (3) reduces to a single term:

$$\int_{-p}^{p} f(x)\,dx = \frac{a_0}{2}\int_{-p}^{p} dx = \frac{a_0}{2}x\Big]_{-p}^{p} = pa_0.$$

Solving for a_0 yields

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx. \tag{4}$$

Now we multiply (2) by $\cos(m\pi x/p)$ and integrate:

$$\int_{-p}^{p} f(x)\cos\frac{m\pi}{p}x\,dx = \frac{a_0}{2}\int_{-p}^{p}\cos\frac{m\pi}{p}x\,dx$$
$$+ \sum_{n=1}^{\infty}\left(a_n\int_{-p}^{p}\cos\frac{m\pi}{p}x\cos\frac{n\pi}{p}x\,dx + b_n\int_{-p}^{p}\cos\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\,dx\right). \tag{5}$$

By orthogonality we have

$$\int_{-p}^{p}\cos\frac{m\pi}{p}x\,dx = 0, \quad m > 0, \qquad \int_{-p}^{p}\cos\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\,dx = 0$$

and $$\int_{-p}^{p} \cos\frac{m\pi}{p}x\cos\frac{n\pi}{p}x\,dx = \begin{cases} 0, & m \neq n \\ p, & m = n. \end{cases}$$

Thus (5) reduces to $$\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx = a_n p,$$

and so $$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx. \tag{6}$$

Finally, if we multiply (2) by $\sin(m\pi x/p)$, integrate, and make use of the results

$$\int_{-p}^{p} \sin\frac{m\pi}{p}x\,dx = 0, \quad m > 0, \qquad \int_{-p}^{p} \sin\frac{m\pi}{p}x\cos\frac{n\pi}{p}x\,dx = 0$$

and $$\int_{-p}^{p} \sin\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\,dx = \begin{cases} 0, & m \neq n \\ p, & m = n, \end{cases}$$

we find that $$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx. \tag{7}$$

The trigonometric series (2) with coefficients a_0, a_n, and b_n defined by (4), (6), and (7), respectively, is said to be the **Fourier series** of the function f. The coefficients obtained from (4), (6), and (7) are referred to as **Fourier coefficients** of f.

In finding the coefficients a_0, a_n, and b_n, we assumed that f was integrable on the interval and that (2), as well as the series obtained by multiplying (2) by $\cos(m\pi x/p)$, converged in such a manner as to permit term-by-term integration. Until (2) is shown to be convergent for a given function f, the equality sign is not to be taken in a strict or literal sense. Some texts use the symbol $\sim$ in place of $=$. In view of the fact that most functions in applications are of a type that guarantees convergence of the series, we shall use the equality symbol. We summarize the results:

Definition 12.2.1 Fourier Series

The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x + b_n\sin\frac{n\pi}{p}x\right), \tag{8}$$

where $$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \tag{9}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx \tag{10}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx. \tag{11}$$

EXAMPLE 1 Expansion in a Fourier Series

Expand $$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases} \tag{12}$$

in a Fourier series.

SOLUTION The graph of f is given in **FIGURE 12.2.1**. With $p = \pi$ we have from (9) and (10) that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}(\pi - x)\,dx\right] = \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

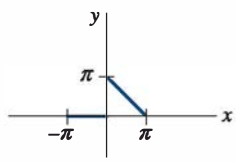


FIGURE 12.2.1 Function f in Example 1

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\Bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right]$$

$$= -\frac{1}{n\pi}\frac{\cos nx}{n}\Bigg]_0^{\pi}$$

$$= \frac{-\cos n\pi + 1}{n^2\pi} \qquad \leftarrow \cos n\pi = (-1)^n$$

$$= \frac{1 - (-1)^n}{n^2\pi}.$$

In like manner we find from (11) that

$$b_n = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx = \frac{1}{n}.$$

Note that $1 - (-1)^n = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd.}\end{cases}$

Therefore
$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right\}. \tag{13}$$

Note that a_n defined by (10) reduces to a_0 given by (9) when we set $n = 0$. But as Example 1 shows, this may not be the case *after* the integral for a_n is evaluated.

Convergence of a Fourier Series The following theorem gives sufficient conditions for convergence of a Fourier series at a point.

Theorem 12.2.1 Conditions for Convergence

Let f and f' be piecewise continuous on the interval $[-p, p]$; that is, let f and f' be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then for all x in the interval $(-p, p)$ the Fourier series of f converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier series converges to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.*

For a proof of this theorem you are referred to the classic text by Churchill and Brown.†

EXAMPLE 2 Convergence of a Point of Discontinuity

The function (12) in Example 1 satisfies the conditions of Theorem 12.2.1. Thus for every x in the interval $(-\pi, \pi)$, except at $x = 0$, the series (13) will converge to $f(x)$. At $x = 0$ the function is discontinuous, and so the series (13) will converge to

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}.$$

*In other words, for x a point in the interval and $h > 0$,

$$f(x+) = \lim_{h\to 0} f(x + h), \quad f(x-) = \lim_{h\to 0} f(x - h).$$

†Ruel V. Churchill and James Ward Brown, *Fourier Series and Boundary Value Problems* (New York: McGraw-Hill, 2000).

Periodic Extension

Observe that each of the functions in the basic set (1) has a different fundamental period;* namely, $2p/n$, $n \geq 1$, but since a positive integer multiple of a period is also a period we see that all of the functions have in common the period $2p$ (verify). Hence the right-hand side of (2) is $2p$-periodic; indeed, $2p$ is the fundamental period of the sum. We conclude that a Fourier series not only represents the function on the interval $(-p, p)$ but also gives the **periodic extension** of f outside this interval. We can now apply Theorem 12.2.1 to the periodic extension of f, or we may assume from the outset that the given function is periodic with period $T = 2p$; that is, $f(x + T) = f(x)$. When f is piecewise continuous and the right- and left-hand derivatives exist at $x = -p$ and $x = p$, respectively, then the series (8) converges to the average $[f(p-) + f(-p+)]/2$ at these endpoints and to this value extended periodically to $\pm 3p$, $\pm 5p$, $\pm 7p$, and so on. The Fourier series in (13) converges to the periodic extension of (12) on the entire x-axis. At 0, $\pm 2\pi$, $\pm 4\pi$, …, and at $\pm\pi$, $\pm 3\pi$, $\pm 5\pi$, …, the series converges to the values

We may assume that the given function f is periodic.

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi}{2} \quad \text{and} \quad \frac{f(\pi+) + f(\pi-)}{2} = 0,$$

respectively. The solid dots in **FIGURE 12.2.2** represent the value $\pi/2$.

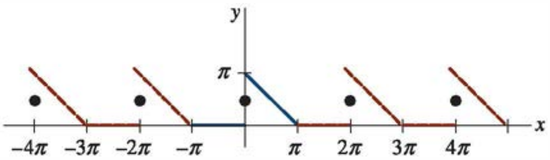


FIGURE 12.2.2 Periodic extension of the function f shown in Figure 12.2.1

Sequence of Partial Sums

It is interesting to see how the sequence of partial sums $\{S_N(x)\}$ of a Fourier series approximates a function. For example, the first three partial sums of (13) are

$$S_1(x) = \frac{\pi}{4}, \quad S_2(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x, \quad S_3(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x.$$

In **FIGURE 12.2.3** we have used a CAS to graph the partial sums $S_5(x)$, $S_8(x)$, and $S_{15}(x)$ of (13) on the interval $(-\pi, \pi)$. Figure 12.2.3(d) shows the periodic extension using $S_{15}(x)$ on $(-4\pi, 4\pi)$.

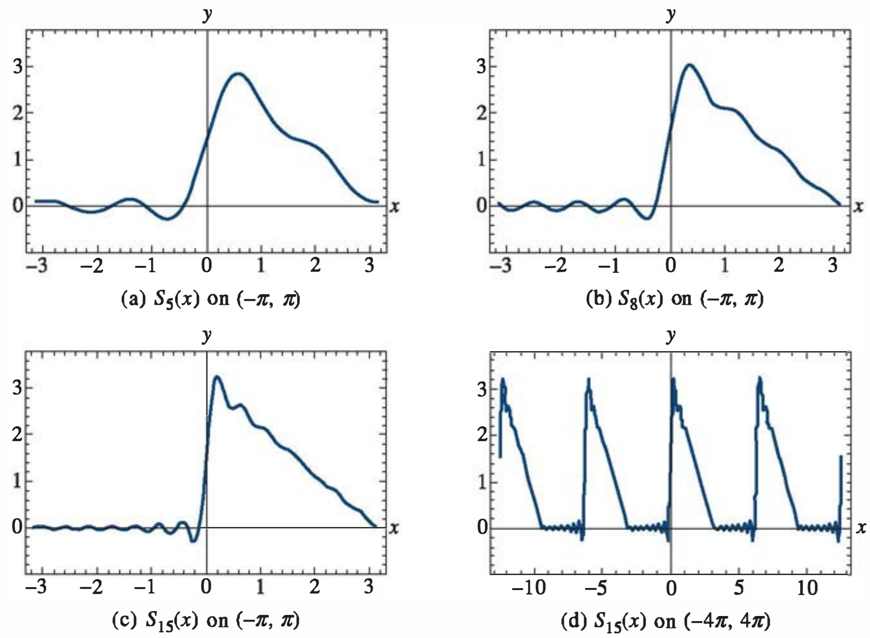


FIGURE 12.2.3 Partial sums of a Fourier series

*See Problem 21 in Exercises 12.1.

12.2 Exercises

Answers to selected odd-numbered problems begin on page ANS-29.

In Problems 1–16, find the Fourier series of f on the given interval.

1. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$
2. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 2, & 0 \le x < \pi \end{cases}$
3. $f(x) = \begin{cases} 1, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$
4. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$
5. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$
6. $f(x) = \begin{cases} \pi^2, & -\pi < x < 0 \\ \pi^2 - x^2, & 0 \le x < \pi \end{cases}$
7. $f(x) = x + \pi, \quad -\pi < x < \pi$
8. $f(x) = 3 - 2x, \quad -\pi < x < \pi$
9. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}$
10. $f(x) = \begin{cases} 0, & -\pi/2 < x < 0 \\ \cos x, & 0 \le x < \pi/2 \end{cases}$
11. $f(x) = \begin{cases} 0, & -2 < x < -1 \\ -2, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases}$
12. $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$
13. $f(x) = \begin{cases} 1, & -5 < x < 0 \\ 1 + x, & 0 \le x < 5 \end{cases}$
14. $f(x) = \begin{cases} 2 + x, & -2 < x < 0 \\ 2, & 0 \le x < 2 \end{cases}$
15. $f(x) = e^x, \quad -\pi < x < \pi$
16. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ e^x - 1, & 0 \le x < \pi \end{cases}$
17. Use the result of Problem 5 to show

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

and

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

18. Use Problem 17 to find a series that gives the numerical value of $\pi^2/8$.
19. Use the result of Problem 7 to show

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

20. Use the result of Problem 9 to show

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots.$$

21. The **root-mean-square value** of a function $f(x)$ defined over an interval (a, b) is given by

$$\text{RMS}(f) = \sqrt{\frac{\int_a^b f^2(x)\,dx}{b - a}}.$$

If the Fourier series expansion of f is given by (8), show that the RMS value of f over the interval $(-p, p)$ is given by

$$\text{RMS}(f) = \sqrt{\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)},$$

where a_0, a_n, and b_n are the Fourier coefficients in (9), (10), and (11), respectively.

12.3 Fourier Cosine and Sine Series

Introduction The effort expended in the evaluation of coefficients a_0, a_n, and b_n in expanding a function f in a Fourier series is reduced significantly when f is either an even or an odd function. A function f is said to be

$$\textbf{even if } f(-x) = f(x) \quad \text{and} \quad \textbf{odd if } f(-x) = -f(x).$$

On a symmetric interval such as $(-p, p)$, the graph of an even function possesses symmetry with respect to the y-axis, whereas the graph of an odd function possesses symmetry with respect to the origin.

Even and Odd Functions It is likely the origin of the words *even* and *odd* derives from the fact that the graphs of polynomial functions that consist of all even powers of x are symmetric with respect to the y-axis, whereas graphs of polynomials that consist of all odd powers of x are symmetric with respect to the origin. For example,

$$f(x) = x^{\overset{\text{even integer}}{\downarrow}2} \text{ is even since } f(-x) = (-x)^2 = x^2 = f(x)$$

$$f(x) = x^{\overset{\text{odd integer}}{\downarrow}3} \text{ is odd since } f(-x) = (-x)^3 = -x^3 = -f(x).$$

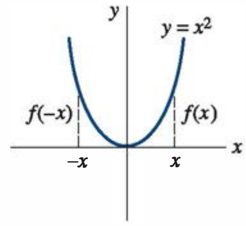


FIGURE 12.3.1 Even function

See **FIGURES 12.3.1** and **12.3.2**. The trigonometric cosine and sine functions are even and odd functions, respectively, since $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$. The exponential functions $f(x) = e^x$ and $f(x) = e^{-x}$ are neither even nor odd.

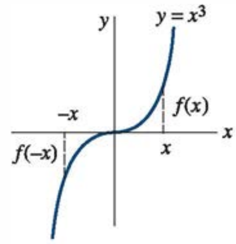


FIGURE 12.3.2 Odd function

Properties

The following theorem lists some properties of even and odd functions.

Theorem 12.3.1 Properties of Even/Odd Functions

(a) The product of two even functions is even.
(b) The product of two odd functions is even.
(c) The product of an even function and an odd function is odd.
(d) The sum (difference) of two even functions is even.
(e) The sum (difference) of two odd functions is odd.
(f) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
(g) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

PROOF OF (b): Let us suppose that f and g are odd functions. Then we have $f(-x) = -f(x)$ and $g(-x) = -g(x)$. If we define the product of f and g as $F(x) = f(x)g(x)$, then

$$F(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = F(x).$$

This shows that the product F of two odd functions is an even function. The proofs of the remaining properties are left as exercises. See Problem 56 in Exercises 12.3. ≡

Cosine and Sine Series

If f is an even function on the interval $(-p, p)$, then in view of the foregoing properties, the coefficients (9), (10), and (11) of Section 12.2 become

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = \frac{2}{p}\int_0^p f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\cos\frac{n\pi}{p}x}_{\text{even}}\,dx = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\sin\frac{n\pi}{p}x}_{\text{odd}}\,dx = 0.$$

Similarly, when f is odd on the interval $(-p, p)$,

$$a_n = 0, \qquad n = 0, 1, 2, \ldots, \qquad b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx.$$

We summarize the results in the following definition.

Definition 12.3.1 Fourier Cosine and Sine Series

(*i*) The Fourier series of an even function on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x, \tag{1}$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\, dx \tag{2}$$

$$a_n = \frac{2}{p}\int_0^p f(x) \cos \frac{n\pi}{p} x\, dx. \tag{3}$$

(*ii*) The Fourier series of an odd function on the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x, \tag{4}$$

where

$$b_n = \frac{2}{p}\int_0^p f(x) \sin \frac{n\pi}{p} x\, dx. \tag{5}$$

EXAMPLE 1 Expansion in a Sine Series

Expand $f(x) = x, \ -2 < x < 2$, in a Fourier series.

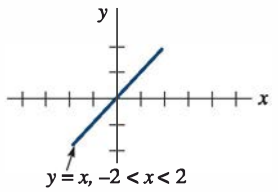


FIGURE 12.3.3 Odd function f in Example 1

SOLUTION Inspection of **FIGURE 12.3.3** shows that the given function is odd on the interval $(-2, 2)$, and so we expand f in a sine series. With the identification $2p = 4$, we have $p = 2$. Thus (5), after integration by parts, is

$$b_n = \int_0^2 x \sin \frac{n\pi}{2} x\, dx = \frac{4(-1)^{n+1}}{n\pi}.$$

Therefore

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} x. \tag{6}$$ ≡

The function in Example 1 satisfies the conditions of Theorem 12.2.1. Hence the series (6) converges to the function on $(-2, 2)$ and the periodic extension (of period 4) given in **FIGURE 12.3.4**.

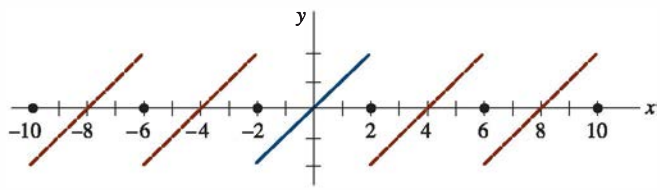


FIGURE 12.3.4 Periodic extension of the function f shown in Figure 12.3.3

EXAMPLE 2 Expansion in a Sine Series

The function $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$ shown in **FIGURE 12.3.5** is odd on the interval $(-\pi, \pi)$. With $p = \pi$ we have from (5)

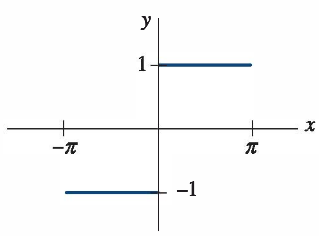


FIGURE 12.3.5 Odd function f in Example 2

$$b_n = \frac{2}{\pi}\int_0^{\pi} (1) \sin nx\, dx = \frac{2}{\pi}\frac{1 - (-1)^n}{n},$$

and so

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx. \tag{7}$$ ≡

Gibbs Phenomenon With the aid of a CAS we have plotted in **FIGURE 12.3.6** the graphs $S_1(x), S_2(x), S_3(x), S_{15}(x)$ of the partial sums of nonzero terms of (7). As seen in Figure 12.3.6(d) the graph of $S_{15}(x)$ has pronounced spikes near the discontinuities at $x = 0$, $x = \pi$, $x = -\pi$, and so on. This "overshooting" by the partial sums S_N from the function values near a point of discontinuity does not smooth out but remains fairly constant, even when the value N is taken to be large. This behavior of a Fourier series near a point at which f is discontinuous is known as the **Gibbs phenomenon**.

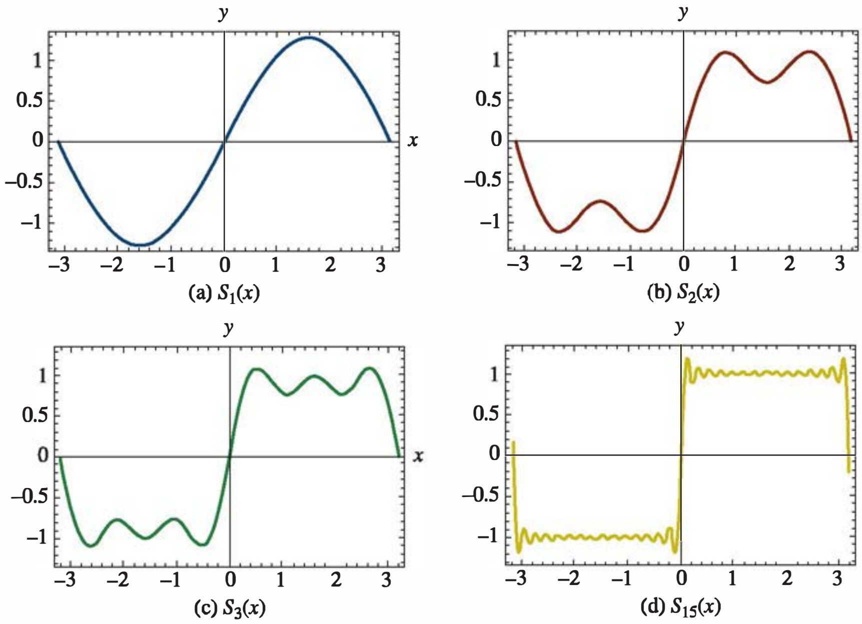


FIGURE 12.3.6 Partial sums of sine series (7) on the interval $(-\pi, \pi)$

The periodic extension of f in Example 2 onto the entire x-axis is a meander function (see page 241).

Half-Range Expansions Throughout the preceding discussion it was understood that a function f was defined on an interval with the origin as midpoint; that is, $(-p, p)$. However, in many instances we are interested in representing a function that is defined on an interval $(0, L)$ by a trigonometric series. This can be done in many different ways by supplying an arbitrary *definition* of the function on the interval $(-L, 0)$. For brevity we consider the three most important cases. If $y = f(x)$ is defined on the interval $(0, L)$, then:

(*i*) reflect the graph of the function about the y-axis onto $(-L, 0)$; the function is now even on the interval $(-L, L)$ (see **FIGURE 12.3.7**); or
(*ii*) reflect the graph of the function through the origin onto $(-L, 0)$; the function is now odd on the interval $(-L, L)$ (see **FIGURE 12.3.8**); or
(*iii*) define f on $(-L, 0)$ by $f(x) = f(x + L)$ (see **FIGURE 12.3.9**).

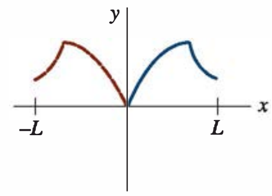


FIGURE 12.3.7 Even reflection

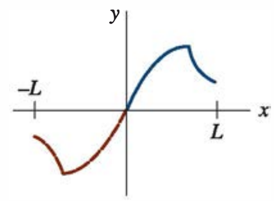
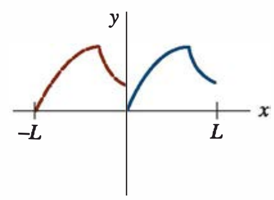


FIGURE 12.3.8 Odd reflection

Note that the coefficients of the series (1) and (4) utilize only the definition of the function on $(0, p)$, that is, for half of the interval $(-p, p)$. Hence in practice there is no actual need to make the reflections described in (*i*) and (*ii*). If f is defined on $(0, L)$, we simply identify the half-period as the length of the interval $p = L$. The coefficient formulas (2), (3), and (5) and the corresponding series yield either an even or an odd periodic extension of period $2L$ of the original function. The cosine and sine series obtained in this manner are known as **half-range expansions**. Lastly, in case (*iii*) we are defining the function values on the interval $(-L, 0)$ to be the same as the values on $(0, L)$. As in the previous two cases, there is no real need to do this. It can be shown that the set of functions in (1) of Section 12.2 is orthogonal on $[a, a + 2p]$ for any real number a. Choosing $a = -p$, we obtain the limits of integration in (9), (10), and (11) of that section. But for $a = 0$ the limits of integration are from $x = 0$ to $x = 2p$. Thus if f is defined over the interval $(0, L)$, we identify $2p = L$ or $p = L/2$. The resulting Fourier series will give the periodic extension of f with period L. In this manner the values to which the series converges will be the same on $(-L, 0)$ as on $(0, L)$.



$f(x) = f(x + L)$

FIGURE 12.3.9 Identity reflection

EXAMPLE 3 Expansion in Three Series

Expand $f(x) = x^2$, $0 < x < L$, **(a)** in a cosine series, **(b)** in a sine series, **(c)** in a Fourier series.

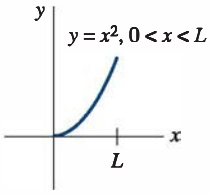


FIGURE 12.3.10 Function f in Example 3

SOLUTION The graph of the function is given in **FIGURE 12.3.10**.

(a) We have

$$a_0 = \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{3}L^2, \quad a_n = \frac{2}{L}\int_0^L x^2 \cos\frac{n\pi}{L}x\,dx = \frac{4L^2(-1)^n}{n^2\pi^2},$$

where integration by parts was used twice in the evaluation of a_n. Thus

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{L}x. \tag{8}$$

(b) In this case we must again integrate by parts twice:

$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{n\pi}{L}x\,dx = \frac{2L^2(-1)^{n+1}}{n\pi} + \frac{4L^2}{n^3\pi^3}[(-1)^n - 1].$$

Hence
$$f(x) = \frac{2L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n} + \frac{2}{n^3\pi^2}[(-1)^n - 1]\right\}\sin\frac{n\pi}{L}x. \tag{9}$$

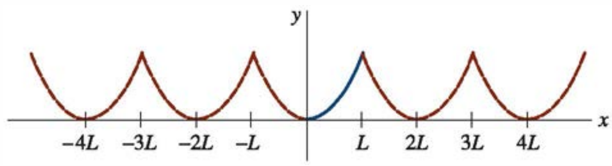
(c) With $p = L/2$, $1/p = 2/L$, and $n\pi/p = 2n\pi/L$, we have

$$a_0 = \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{3}L^2, \quad a_n = \frac{2}{L}\int_0^L x^2\cos\frac{2n\pi}{L}x\,dx = \frac{L^2}{n^2\pi^2}$$

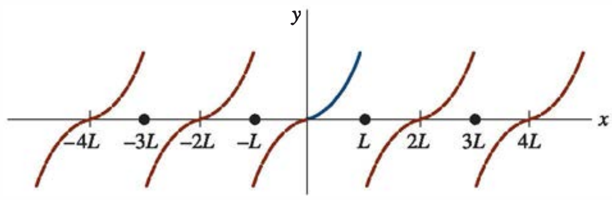
and
$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{2n\pi}{L}x\,dx = -\frac{L^2}{n\pi}.$$

Therefore
$$f(x) = \frac{L^2}{3} + \frac{L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}\cos\frac{2n\pi}{L}x - \frac{1}{n}\sin\frac{2n\pi}{L}x\right\}. \tag{10}$$ ≡

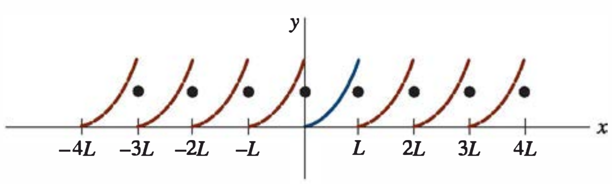
The series (8), (9), and (10) converge to the $2L$-periodic even extension of f, the $2L$-periodic odd extension of f, and the L-periodic extension of f, respectively. The graphs of these periodic extensions are shown in **FIGURE 12.3.11**.



(a) Cosine series



(b) Sine series



(c) Fourier series

FIGURE 12.3.11 Different periodic extensions of the function f in Example 3

Periodic Driving Force Fourier series are sometimes useful in determining a particular solution of a differential equation describing a physical system in which the input or driving force $f(t)$ is periodic. In the next example we find a particular solution of the differential equation

$$m\frac{d^2x}{dt^2} + kx = f(t) \tag{11}$$

by first representing f by a half-range sine expansion and then assuming a particular solution of the form

$$x_p(t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{p}t. \tag{12}$$

EXAMPLE 4 Particular Solution of a DE

An undamped spring/mass system, in which the mass $m = \frac{1}{16}$ slug and the spring constant $k = 4$ lb/ft, is driven by the 2-periodic external force $f(t)$ shown in **FIGURE 12.3.12**. Although the force $f(t)$ acts on the system for $t > 0$, note that if we extend the graph of the function in a 2-periodic manner to the negative t-axis, we obtain an odd function. In practical terms this means that we need only find the half-range sine expansion of $f(t) = \pi t,\ 0 < t < 1$. With $p = 1$ it follows from (5) and integration by parts that

$$b_n = 2\int_0^1 \pi t \sin n\pi t\, dt = \frac{2(-1)^{n+1}}{n}.$$

From (11) the differential equation of motion is seen to be

$$\frac{1}{16}\frac{d^2x}{dt^2} + 4x = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin n\pi t. \tag{13}$$

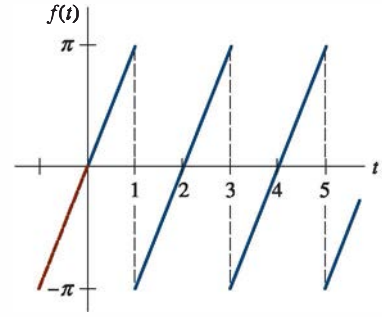


FIGURE 12.3.12 Periodic forcing function f in Example 4

To find a particular solution $x_p(t)$ of (13), we substitute (12) into the equation and equate coefficients of $\sin n\pi t$. This yields

$$\left(-\frac{1}{16}n^2\pi^2 + 4\right)B_n = \frac{2(-1)^{n+1}}{n} \quad \text{or} \quad B_n = \frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)}.$$

Thus

$$x_p(t) = \sum_{n=1}^{\infty}\frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)}\sin n\pi t. \tag{14}$$

≡

Observe in the solution (14) that there is no integer $n \geq 1$ for which the denominator $64 - n^2\pi^2$ of B_n is zero. In general, if there *is* a value of n, say N, for which $N\pi/p = \omega$, where $\omega = \sqrt{k/m}$, then the system described by (11) is in a state of pure resonance. In other words, we have pure resonance if the Fourier series expansion of the driving force $f(t)$ contains a term $\sin(N\pi/L)t$ (or $\cos(N\pi/L)t$) that has the same frequency as the free vibrations.

Of course, if the $2p$-periodic extension of the driving force f onto the negative t-axis yields an even function, then we expand f in a cosine series.

12.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-29.

In Problems 1–10, determine whether the function is even, odd, or neither.

1. $f(x) = \sin 3x$
2. $f(x) = x\cos x$
3. $f(x) = x^2 + x$
4. $f(x) = x^3 - 4x$
5. $f(x) = e^{|x|}$
6. $f(x) = e^x - e^{-x}$
7. $f(x) = \begin{cases} x^2, & -1 < x < 0 \\ -x^2, & 0 \leq x < 1 \end{cases}$
8. $f(x) = \begin{cases} x + 5, & -2 < x < 0 \\ -x + 5, & 0 \leq x < 2 \end{cases}$
9. $f(x) = x^3, 0 \leq x \leq 2$
10. $f(x) = |x^5|$

In Problems 11–24, expand the given function in an appropriate cosine or sine series.

11. $f(x) = \begin{cases} \pi, & -1 < x < 0 \\ -\pi, & 0 \leq x < 1 \end{cases}$

12. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ 0, & -1 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$

13. $f(x) = |x|, -\pi < x < \pi$

14. $f(x) = x, -\pi < x < \pi$

15. $f(x) = x^2, -1 < x < 1$

16. $f(x) = x|x|, -1 < x < 1$

17. $f(x) = \pi^2 - x^2, -\pi < x < \pi$

18. $f(x) = x^3, -\pi < x < \pi$

19. $f(x) = \begin{cases} x - 1, & -\pi < x < 0 \\ x + 1, & 0 \le x < \pi \end{cases}$

20. $f(x) = \begin{cases} x + 1, & -1 < x < 0 \\ x - 1, & 0 \le x < 1 \end{cases}$

21. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ -x, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$

22. $f(x) = \begin{cases} -\pi, & -2\pi < x < -\pi \\ x, & -\pi \le x < \pi \\ \pi, & \pi \le x < 2\pi \end{cases}$

23. $f(x) = |\sin x|, -\pi < x < \pi$

24. $f(x) = \cos x, -\pi/2 < x < \pi/2$

In Problems 25–34, find the half-range cosine and sine expansions of the given function.

25. $f(x) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ 0, & \frac{1}{2} \le x < 1 \end{cases}$

26. $f(x) = \begin{cases} 0, & 0 < x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x < 1 \end{cases}$

27. $f(x) = \cos x, 0 < x < \pi/2$

28. $f(x) = \sin x, 0 < x < \pi$

29. $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \le x < \pi \end{cases}$

30. $f(x) = \begin{cases} 0, & 0 < x < \pi \\ x - \pi, & \pi \le x < 2\pi \end{cases}$

31. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \le x < 2 \end{cases}$

32. $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \end{cases}$

33. $f(x) = x^2 + x, \ 0 < x < 1$

34. $f(x) = x(2 - x), \ 0 < x < 2$

In Problems 35–38, expand the given function in a Fourier series.

35. $f(x) = x^2, 0 < x < 2\pi$

36. $f(x) = x, 0 < x < \pi$

37. $f(x) = x + 1, 0 < x < 1$

38. $f(x) = 2 - x, 0 < x < 2$

In Problems 39–42, suppose the function $y = f(x), 0 < x < L$, given in the figure is expanded in a cosine series, in a sine series, and in a Fourier series. Sketch the periodic extension to which each series converges.

39.

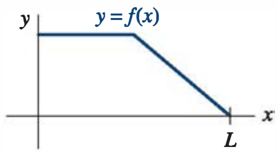


FIGURE 12.3.13 Graph for Problem 39

40.

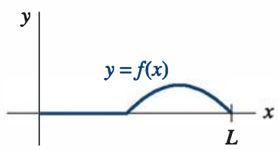


FIGURE 12.3.14 Graph for Problem 40

41.

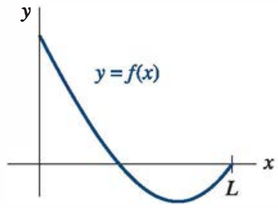


FIGURE 12.3.15 Graph for Problem 41

42.

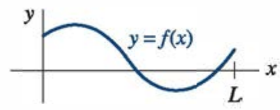


FIGURE 12.3.16 Graph for Problem 42

In Problems 43 and 44, proceed as in Example 4 to find a particular solution $x_p(t)$ of equation (11) when $m = 1, k = 10$, and the driving force $f(t)$ is as given. Assume that when $f(t)$ is extended to the negative t-axis in a periodic manner, the resulting function is odd.

43. $f(t) = \begin{cases} 5, & 0 < t < \pi \\ -5, & \pi < t < 2\pi \end{cases}; \quad f(t + 2\pi) = f(t)$

44. $f(t) = 1 - t, \ 0 < t < 2; \quad f(t + 2) = f(t)$

In Problems 45 and 46, proceed as in Example 4 to find a particular solution $x_p(t)$ of equation (11) when $m = \frac{1}{4}, k = 12$, and the driving force $f(t)$ is as given. Assume that when $f(t)$ is extended to the negative t-axis in a periodic manner, the resulting function is even.

45. $f(t) = 2\pi t - t^2, 0 < t < 2\pi; \quad f(t + 2\pi) = f(t)$

46. $f(t) = \begin{cases} t, & 0 < t < \frac{1}{2} \\ 1 - t, & \frac{1}{2} < t < 1 \end{cases}; \quad f(t + 1) = f(t)$

47. (a) Solve the differential equation in Problem 43, $x'' + 10x = f(t)$, subject to the initial conditions $x(0) = 0$, $x'(0) = 0$.
(b) Use a CAS to plot the graph of the solution $x(t)$ in part (a).

48. (a) Solve the differential equation in Problem 45, $\frac{1}{4}x'' + 12x = f(t)$, subject to the initial conditions $x(0) = 1$, $x'(0) = 0$.
(b) Use a CAS to plot the graph of the solution $x(t)$ in part (a).

49. Suppose a uniform beam of length L is simply supported at $x = 0$ and at $x = L$. If the load per unit length is given by $w(x) = w_0 x/L$, $0 < x < L$, then the differential equation for the deflection $y(x)$ is

$$EI\frac{d^4y}{dx^4} = \frac{w_0 x}{L},$$

where E, I, and w_0 are constants. See (4) in Section 3.9.
(a) Expand $w(x)$ in a half-range sine series.
(b) Use the method of Example 4 to find a particular solution $y(x)$ of the differential equation.

50. Proceed as in Problem 49 to find a particular solution $y(x)$ when the load per unit length is as given in FIGURE 12.3.17.

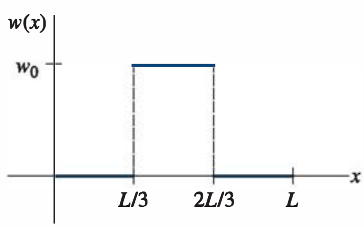


FIGURE 12.3.17 Graph for Problem 50

☰ Computer Lab Assignments

In Problems 51 and 52, use a CAS to graph the partial sums $\{S_N(x)\}$ of the given trigonometric series. Experiment with different values of N and graphs on different intervals of the x-axis. Use your graphs to conjecture a closed-form expression for a function f defined for $0 < x < L$ that is represented by the series.

51. $f(x) = -\dfrac{\pi}{4} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi}\cos nx + \frac{1 - 2(-1)^n}{n}\sin nx\right]$

52. $f(x) = -\dfrac{1}{4} + \dfrac{4}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi}{2}x$

☰ Discussion Problems

53. Is your answer in Problem 51 or in Problem 52 unique? Give a function f defined on a symmetric interval about the origin $(-a, a)$ that has the same trigonometric series as in Problem 51; as in Problem 52.

54. Discuss why the Fourier cosine series expansion of $f(x) = e^x$, $0 < x < \pi$ converges to e^{-x} on the interval $(-\pi, 0)$.

55. Suppose $f(x) = e^x$, $0 < x < \pi$ is expanded in a cosine series, and then $f(x) = e^x$, $0 < x < \pi$ is expanded in a sine series. If the two series are added and then divided by 2 (that is, the average of the two series) we get a series with cosines and sines that also represents $f(x) = e^x$ on the interval $(0, \pi)$. Is this a full Fourier series of f? [*Hint*: What does the averaging of the cosine and sine series represent on the interval $(-\pi, 0)$?]

56. Prove properties (a), (c), (d), (e), (f), and (g) in Theorem 12.3.1.

12.4 Complex Fourier Series

☰ Introduction As we have seen in the preceding two sections, a real function f can be represented by a series of sines and cosines. The functions $\cos nx$, $n = 0, 1, 2, \ldots$ and $\sin nx$, $n = 1, 2, \ldots$ are *real-valued* functions of a real variable x. The three different real forms of Fourier series given in Definitions 12.2.1 and 12.3.1 will be exceedingly important in Chapters 13 and 14 when we set about to solve linear partial differential equations. However, in certain applications; for example, the analysis of periodic signals in electrical engineering, it is actually more convenient to represent a function f in an infinite series of *complex-valued* functions of a real variable x such as the exponential functions e^{inx}, $n = 0, 1, 2, \ldots$, and where i is the imaginary unit defined by $i^2 = -1$. Recall for x a real number, Euler's formula

$$e^{ix} = \cos x + i\sin x \qquad \text{gives} \qquad e^{-ix} = \cos x - i\sin x. \tag{1}$$

In this section we are going to use the results in (1) to recast the Fourier series in Definition 12.2.1 into a **complex form** or **exponential form**. We will see that we can represent a real function by a complex series; a series in which the coefficients are complex numbers. To that end, recall that a complex number is a number $z = a + ib$, where a and b are real numbers, and $i^2 = -1$. The number $\bar{z} = a - ib$ is called the conjugate of z.

Complex Fourier Series If we first add the two expressions in (1) and solve for $\cos x$ and then subtract the two expressions and solve for $\sin x$, we arrive at

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \text{and} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{2}$$

Using (2) to replace $\cos(n\pi x/p)$ and $\sin(n\pi x/p)$ in (8) of Section 12.2, the Fourier series of a function f can be written

$$\begin{aligned}
&\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\frac{e^{in\pi x/p} + e^{-in\pi x/p}}{2} + b_n\frac{e^{in\pi x/p} - e^{-in\pi x/p}}{2i}\right] \\
&\quad = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)e^{in\pi x/p} + \frac{1}{2}(a_n + ib_n)e^{-in\pi x/p}\right] \\
&\quad = c_0 + \sum_{n=1}^{\infty}c_n e^{in\pi x/p} + \sum_{n=1}^{\infty}c_{-n}e^{-in\pi x/p},
\end{aligned} \tag{3}$$

where $c_0 = \frac{1}{2}a_0$, $c_n = \frac{1}{2}(a_n - ib_n)$, and $c_{-n} = \frac{1}{2}(a_n + ib_n)$. The symbols a_0, a_n, and b_n are the coefficients (9), (10), and (11) respectively, in Definition 12.2.1. When the function f is real,

c_n and c_{-n} are complex conjugates and can also be written in terms of complex exponential functions:

$$c_0 = \frac{1}{2} \cdot \frac{1}{p} \int_{-p}^{p} f(x)\, dx, \tag{4}$$

$$\begin{aligned} c_n &= \frac{1}{2}(a_n - ib_n) = \frac{1}{2}\left(\frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\, dx - i\frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\, dx\right) \\ &= \frac{1}{2p}\int_{-p}^{p} f(x)\left[\cos\frac{n\pi}{p}x - i\sin\frac{n\pi}{p}x\right] dx \\ &= \frac{1}{2p}\int_{-p}^{p} f(x)\, e^{-in\pi x/p}\, dx, \end{aligned} \tag{5}$$

$$\begin{aligned} c_{-n} &= \frac{1}{2}(a_n + ib_n) = \frac{1}{2}\left(\frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\, dx + i\frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\, dx\right) \\ &= \frac{1}{2p}\int_{-p}^{p} f(x)\left[\cos\frac{n\pi}{p}x + i\sin\frac{n\pi}{p}x\right] dx \\ &= \frac{1}{2p}\int_{-p}^{p} f(x)\, e^{in\pi x/p}\, dx. \end{aligned} \tag{6}$$

Since the subscripts of the coefficients and exponents range over the entire set of nonnegative integers $\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$, we can write the results in (3), (4), (5), and (6) in a more compact manner by summing over both the negative and nonnegative integers. In other words, we can use *one* summation and *one* integral that defines all three coefficients c_0, c_n, and c_{-n}.

Definition 12.4.1 Complex Fourier Series

The **complex Fourier series** of functions f defined on an interval $(-p, p)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}, \tag{7}$$

where
$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)\, e^{-in\pi x/p}\, dx, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{8}$$

If f satisfies the hypotheses of Theorem 12.2.1, a complex Fourier series converges to $f(x)$ at a point of continuity and to the average

$$\frac{f(x+) + f(x-)}{2}$$

at a point of discontinuity.

EXAMPLE 1 Complex Fourier Series

Expand $f(x) = e^{-x}$, $-\pi < x < \pi$, in a complex Fourier series.

SOLUTION With $p = \pi$, (8) gives

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(in+1)x}\, dx \\ &= -\frac{1}{2\pi(in+1)}\left[e^{-(in+1)\pi} - e^{(in+1)\pi}\right]. \end{aligned}$$

We can simplify the coefficients c_n somewhat using Euler's formula:

$$e^{-(in+1)\pi} = e^{-\pi}(\cos n\pi - i\sin n\pi) = (-1)^n e^{-\pi}$$

and
$$e^{(in+1)\pi} = e^{\pi}(\cos n\pi + i\sin n\pi) = (-1)^n e^{\pi},$$

since $\cos n\pi = (-1)^n$ and $\sin n\pi = 0$. Hence

$$c_n = (-1)^n \frac{(e^{\pi} - e^{-\pi})}{2(in + 1)\pi} = (-1)^n \frac{\sinh \pi}{\pi} \frac{1 - in}{n^2 + 1}. \tag{9}$$

The complex Fourier series is then

$$f(x) = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{1 - in}{n^2 + 1} e^{inx}. \tag{10}$$ ≡

The series (10) converges to the 2π-periodic extension of f.

You may get the impression that we have just made life more complicated by introducing a complex version of a Fourier series. The reality of the situation is that in areas of engineering, the form (7) given in Definition 12.4.1 is sometimes more useful than that given in (8) of Definition 12.2.1.

Fundamental Frequency The Fourier series in Definitions 12.2.1 and 12.4.1 define a periodic function and the **fundamental period** of that function (that is, the periodic extension of f) is $T = 2p$. Since $p = T/2$, (8) of Section 12.2 and (7) become, respectively,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega x + b_n \sin n\omega x) \quad \text{and} \quad \sum_{n=-\infty}^{\infty} c_n e^{-in\omega x}, \tag{11}$$

where number $\omega = 2\pi/T$ is called the **fundamental angular frequency**. In Example 1 the periodic extension of the function has period $T = 2\pi$; the fundamental angular frequency is $\omega = 2\pi/2\pi = 1$.

Frequency Spectrum In the study of time-periodic signals, electrical engineers find it informative to examine various spectra of a wave form. If f is periodic and has fundamental period T, the plot of the points $(n\omega, |c_n|)$, where ω is the fundamental angular frequency and the c_n are the coefficients defined in (8), is called the **frequency spectrum** of f.

EXAMPLE 2 Frequency Spectrum

In Example 1, $\omega = 1$ so that $n\omega$ takes on the values $0, \pm 1, \pm 2, \ldots$. Using $|\alpha + i\beta| = \sqrt{\alpha^2 + \beta^2}$, we see from (9) that

$$|c_n| = \frac{\sinh \pi}{\pi} \frac{1}{\sqrt{n^2 + 1}}.$$

The following table shows some values of n and corresponding values of c_n.

n	-3	-2	-1	0	1	2	3
$\lvert c_n \rvert$	1.162	1.644	2.599	3.676	2.599	1.644	1.162

The graph in **FIGURE 12.4.1**, lines with arrowheads terminating at the points, is a portion of the frequency spectrum of f. ≡

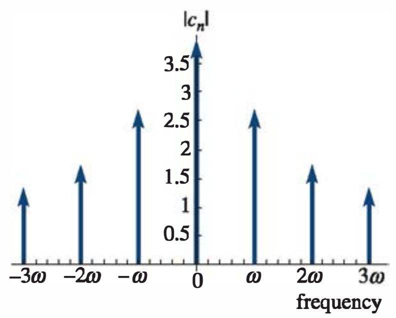


FIGURE 12.4.1 Frequency spectrum of f in Example 2

EXAMPLE 3 Frequency Spectrum

Find the frequency spectrum of the periodic square wave or periodic pulse shown in **FIGURE 12.4.2**. The wave is the periodic extension of the function f:

$$f(x) = \begin{cases} 0, & -\frac{1}{2} < x < -\frac{1}{4} \\ 1, & -\frac{1}{4} < x < \frac{1}{4} \\ 0, & \frac{1}{4} < x < \frac{1}{2}. \end{cases}$$

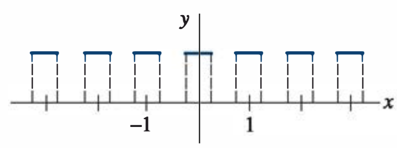


FIGURE 12.4.2 Periodic pulse in Example 3

SOLUTION Here $T = 1 = 2p$ so $p = \frac{1}{2}$. Since f is 0 on the intervals $(-\frac{1}{2}, -\frac{1}{4})$ and $(\frac{1}{4}, \frac{1}{2})$, (8) becomes

$$\begin{aligned} c_n &= \int_{-1/2}^{1/2} f(x)\, e^{2in\pi x}\, dx = \int_{-1/4}^{1/4} 1 \cdot e^{2in\pi x}\, dx \\ &= \frac{e^{2in\pi x}}{2in\pi}\Bigg]_{-1/4}^{1/4} \\ &= \frac{1}{n\pi} \frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}. \end{aligned}$$

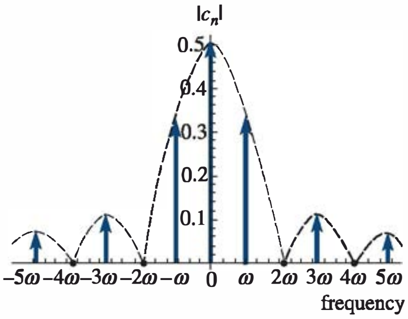


FIGURE 12.4.3 Frequency spectrum of f in Example 3

That is,
$$c_n = \frac{1}{n\pi}\sin\frac{n\pi}{2}. \quad \leftarrow \text{by (2)}$$

Since the last result is not valid at $n = 0$, we compute that term separately:
$$c_0 = \int_{-1/4}^{1/4} dx = \frac{1}{2}.$$

The following table shows some of the values of $|c_n|$, and **FIGURE 12.4.3** shows the

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\lvert c_n\rvert$	$\frac{1}{5\pi}$	0	$\frac{1}{3\pi}$	0	$\frac{1}{\pi}$	$\frac{1}{2}$	$\frac{1}{\pi}$	0	$\frac{1}{3\pi}$	0	$\frac{1}{5\pi}$

frequency spectrum of f. Since the fundamental frequency is $\omega = 2\pi/T = 2\pi$, the units $n\omega$ on the horizontal scale are $\pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$. The curved dashed lines were added in Figure 12.4.3 to emphasize the presence of the zero values of $|c_n|$ when n is an even nonzero integer. ≡

12.4 Exercises

Answers to selected odd-numbered problems begin on page ANS-30.

In Problems 1–6, find the complex Fourier series of f on the given interval.

1. $f(x) = \begin{cases} -1, & -2 < x < 0 \\ 1, & 0 < x < 2 \end{cases}$

2. $f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$

3. $f(x) = \begin{cases} 0, & -\frac{1}{2} < x < 0 \\ 1, & 0 < x < \frac{1}{4} \\ 0, & \frac{1}{4} < x < \frac{1}{2} \end{cases}$

4. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

5. $f(x) = x,\ 0 < x < 2\pi$

6. $f(x) = e^{-|x|},\ -1 < x < 1$

7. Find the frequency spectrum of the periodic wave that is the periodic extension of the function f in Problem 1.

8. Find the frequency spectrum of the periodic wave that is the periodic extension of the function f in Problem 3.

In Problems 9 and 10, sketch the given periodic wave. Find the frequency spectrum of f.

9. $f(x) = 4\sin x,\ 0 < x < \pi;\ f(x + \pi) = f(x)$ [*Hint*: Use (2).]

10. $f(x) = \begin{cases} \cos x, & 0 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases};\quad f(x + \pi) = f(x)$

11. **(a)** Show that $a_n = c_n + c_{-n}$ and $b_n = i(c_n - c_{-n})$.
(b) Use the results in part (a) and the complex Fourier series in Example 1 to obtain the Fourier series expansion of f.

12. The function f in Problem 1 is odd. Use the complex Fourier series to obtain the Fourier sine series expansion of f.

12.5 Sturm–Liouville Problem

≡ Introduction For convenience we present here a brief review of some of the ordinary differential equations that will be of importance in the sections and chapters that follow.

Linear equations	**General solutions**
$y' + \alpha y = 0,$	$y = c_1 e^{-\alpha x}$
$y'' + \alpha^2 y = 0, \quad \alpha > 0$	$y = c_1\cos\alpha x + c_2\sin\alpha x$
$y'' - \alpha^2 y = 0, \quad \alpha > 0$	$\begin{cases} y = c_1 e^{-\alpha x} + c_2 e^{\alpha x}, & \text{or} \\ y = c_1\cosh\alpha x + c_2\sinh\alpha x \end{cases}$

Cauchy–Euler equation	**General solutions, $x > 0$**
$x^2y'' + xy' - \alpha^2 y = 0, \quad \alpha \ge 0$	$\begin{cases} y = c_1x^{-\alpha} + c_2x^{\alpha}, & \alpha \ne 0 \\ y = c_1 + c_2\ln x, & \alpha = 0 \end{cases}$

Parametric Bessel equation ($\nu = 0$)	General solution, $x > 0$
$xy'' + y' + \alpha^2 xy = 0$	$y = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$

Legendre's equation ($n = 0, 1, 2, \ldots$)	Particular solutions are polynomials
$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$	$y = P_0(x) = 1,$
	$y = P_1(x) = x,$
	$y = P_2(x) = \frac{1}{2}(3x^2 - 1), \ldots$

Regarding the two forms of the general solution of $y'' - \alpha^2 y = 0$, we will, in the future, employ the following informal rule:

Use the exponential form $y = c_1 e^{-\alpha x} + c_2 e^{\alpha x}$ when the domain of x is an infinite or semi-infinite interval; use the hyperbolic form $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$ when the domain of x is a finite interval.

This rule will be useful in Chapters 13 and 14.

Eigenvalues and Eigenfunctions Orthogonal functions arise in the solution of differential equations. More to the point, an orthogonal set of functions can be generated by solving a two-point boundary-value problem involving a linear second-order differential equation containing a parameter λ. In Example 2 of Section 3.9 we saw that the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0, \tag{1}$$

possessed nontrivial solutions only when the parameter λ took on the values $\lambda_n = n^2\pi^2/L^2$, $n = 1, 2, 3, \ldots$ called **eigenvalues**. The corresponding nontrivial solutions $y = c_2 \sin(n\pi x/L)$ or simply $y = \sin(n\pi x/L)$ are called the **eigenfunctions** of the problem. For example, for (1) we have

$$\textit{BVP}: \quad y'' + \overset{\substack{\text{not an eigenvalue}\\ \downarrow}}{5}y = 0, \quad y(0) = 0, \quad y(L) = 0$$

$$\textit{Solution is trivial}: \quad y = 0.$$

$$\textit{BVP}: \quad y'' + \overset{\substack{\text{is an eigenvalue } (n = 2)\\ \downarrow}}{\frac{4\pi^2}{L^2}}y = 0, \quad y(0) = 0, \quad y(L) = 0$$

$$\textit{Solution is nontrivial}: \quad y = \sin(2\pi x/L).$$

For our purposes in this chapter it is important to recognize the set of functions generated by this BVP; that is, $\{\sin(n\pi x/L)\}$, $n = 1, 2, 3, \ldots$, is the orthogonal set of functions on the interval $[0, L]$ used as the basis for the Fourier sine series.

EXAMPLE 1 Eigenvalues and Eigenfunctions

It is left as an exercise to show, by considering the three possible cases for the parameter λ (zero, negative, or positive; that is, $\lambda = 0$, $\lambda = -\alpha^2 < 0$, $\alpha > 0$, and $\lambda = \alpha^2 > 0$, $\alpha > 0$), that the eigenvalues and eigenfunctions for the boundary-value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0 \tag{2}$$

are, respectively, $\lambda_n = \alpha_n^2 = n^2\pi^2/L^2$, $n = 0, 1, 2, \ldots$, and $y = c_1 \cos(n\pi x/L)$, $c_1 \neq 0$. In contrast to (1), $\lambda_0 = 0$ is an eigenvalue for this BVP and $y = 1$ is the corresponding eigenfunction. The latter comes from solving $y'' = 0$ subject to the same boundary conditions $y'(0) = 0$, $y'(L) = 0$. Note also that $y = 1$ can be incorporated into the family $y = \cos(n\pi x/L)$ by permitting $n = 0$. The set $\{\cos(n\pi x/L)\}$, $n = 0, 1, 2, 3, \ldots$, is orthogonal on the interval $[0, L]$. See Problem 3 in Exercises 12.5. ≡

Regular Sturm–Liouville Problem

The problems in (1) and (2) are special cases of an important general two-point boundary-value problem. Let p, q, r, and r' be real-valued functions continuous on an interval $[a, b]$, and let $r(x) > 0$ and $p(x) > 0$ for every x in the interval. Then

Solve:
$$\frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0 \tag{3}$$

Subject to:
$$A_1y(a) + B_1y'(a) = 0 \tag{4}$$

$$A_2y(b) + B_2y'(b) = 0 \tag{5}$$

is said to be a **regular Sturm–Liouville problem**. The coefficients in the boundary conditions (4) and (5) are assumed to be real and independent of λ. In addition, A_1 and B_1 are not both zero, and A_2 and B_2 are not both zero. The boundary-value problems in (1) and (2) are regular Sturm–Liouville problems. From (1) we can identify $r(x) = 1$, $q(x) = 0$, and $p(x) = 1$ in the differential equation (3); in boundary condition (4) we identify $a = 0$, $A_1 = 1$, $B_1 = 0$, and in (5), $b = L$, $A_2 = 1$, $B_2 = 0$. From (2) the identifications would be $a = 0$, $A_1 = 0$, $B_1 = 1$ in (4), and $b = L$, $A_2 = 0$, $B_2 = 1$ in (5).

The differential equation (3) is linear and homogeneous. The boundary conditions in (4) and (5), both a linear combination of y and y' *equal to zero at a point*, are also called **homogeneous**. A boundary condition such as $A_2y(b) + B_2y'(b) = C_2$, where C_2 is a nonzero constant, is **nonhomogeneous**. Naturally, a boundary-value problem that consists of a homogeneous linear differential equation and homogeneous boundary conditions is said to be homogeneous; otherwise it is nonhomogeneous. The boundary conditions (4) and (5) are said to be **separated** because each condition involves only a single boundary point. Boundary conditions are referred to as **mixed** if each condition involves both boundary points $x = a$ and $x = b$. For example, the periodic boundary conditions $y(a) = y(b)$, $y'(a) = y'(b)$ are mixed boundary conditions.

Because a regular Sturm–Liouville problem is a homogeneous BVP, it always possesses the trivial solution $y = 0$. However, this solution is of no interest to us. As in Example 1, in solving such a problem we seek numbers λ (eigenvalues) and nontrivial solutions y that depend on λ (eigenfunctions).

Properties

Theorem 12.5.1 is a list of some of the more important of the many properties of the regular Sturm–Liouville problem. We shall prove only the last property.

Theorem 12.5.1 Properties of the Regular Sturm–Liouville Problem

(a) There exist an infinite number of real eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$ such that $\lambda_n \to \infty$ as $n \to \infty$.

(b) For each eigenvalue there is only one eigenfunction (except for nonzero constant multiples).

(c) Eigenfunctions corresponding to different eigenvalues are linearly independent.

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on the interval $[a, b]$.

PROOF OF (d): Let y_m and y_n be eigenfunctions corresponding to eigenvalues λ_m and λ_n, respectively. Then

$$\frac{d}{dx}[r(x)y_m'] + (q(x) + \lambda_m p(x))y_m = 0 \tag{6}$$

$$\frac{d}{dx}[r(x)y_n'] + (q(x) + \lambda_n p(x))y_n = 0. \tag{7}$$

Multiplying (6) by y_n and (7) by y_m and subtracting the two equations gives

$$(\lambda_m - \lambda_n)p(x)y_my_n = y_m\frac{d}{dx}[r(x)y_n'] - y_n\frac{d}{dx}[r(x)y_m'].$$

Integrating this last result by parts from $x = a$ to $x = b$ then yields

$$(\lambda_m - \lambda_n)\int_a^b p(x)y_m y_n\,dx = r(b)[y_m(b)y_n'(b) - y_n(b)y_m'(b)] \\ - r(a)[y_m(a)y_n'(a) - y_n(a)y_m'(a)]. \quad (8)$$

Now the eigenfunctions y_m and y_n must both satisfy the boundary conditions (4) and (5). In particular, from (4) we have

$$A_1y_m(a) + B_1y_m'(a) = 0$$

$$A_1y_n(a) + B_1y_n'(a) = 0.$$

For this system to be satisfied by A_1 and B_1, not both zero, the determinant of the coefficients must be zero:

$$y_m(a)y_n'(a) - y_n(a)y_m'(a) = 0.$$

A similar argument applied to (5) also gives

$$y_m(b)y_n'(b) - y_n(b)y_m'(b) = 0.$$

Using these last two results in (8) shows that both members of the right-hand side are zero. Hence we have established the orthogonality relation

$$\int_a^b p(x)y_m(x)y_n(x)\,dx = 0, \quad \lambda_m \neq \lambda_n. \quad (9)$$ ≡

It can also be proved that the orthogonal set of eigenfunctions $\{y_1(x), y_2(x), y_3(x), \ldots\}$ of a regular Sturm–Liouville problem is complete on $[a, b]$. See page 658.

EXAMPLE 2 A Regular Sturm–Liouville Problem

Solve the boundary-value problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y(1) + y'(1) = 0. \quad (10)$$

SOLUTION You should verify that for $\lambda = 0$ and for $\lambda = -\alpha^2 < 0$, where $\alpha > 0$, the BVP in (10) possesses only the trivial solution $y = 0$. For $\lambda = \alpha^2 > 0$, $\alpha > 0$, the general solution of the differential equation $y'' + \alpha^2 y = 0$ is $y = c_1\cos\alpha x + c_2\sin\alpha x$. Now the condition $y(0) = 0$ implies $c_1 = 0$ in this solution and so we are left with $y = c_2\sin\alpha x$. The second boundary condition $y(1) + y'(1) = 0$ is satisfied if

$$c_2\sin\alpha + c_2\alpha\cos\alpha = 0.$$

Choosing $c_2 \neq 0$, we see that the last equation is equivalent to

$$\tan\alpha = -\alpha. \quad (11)$$

If we let $x = \alpha$ in (11), then **FIGURE 12.5.1** shows the plausibility that there exists an infinite number of roots of the equation $\tan x = -x$; namely, the x-coordinates of the points where the graph of $y = -x$ intersects the branches of the graph of $y = \tan x$. The eigenvalues of problem (10) are then $\lambda_n = \alpha_n^2$, where α_n, $n = 1, 2, 3, \ldots$, are the consecutive *positive* roots $\alpha_1, \alpha_2, \alpha_3, \ldots$ of (11). With the aid of a CAS it is easily shown that, to four rounded decimal places, $\alpha_1 = 2.0288$, $\alpha_2 = 4.9132$, $\alpha_3 = 7.9787$, and $\alpha_4 = 11.0855$, and the corresponding solutions are $y_1 = \sin 2.0288x$, $y_2 = \sin 4.9132x$, $y_3 = \sin 7.9787x$, and $y_4 = \sin 11.0855x$. In general, the eigenfunctions of the problem are $\{\sin\alpha_n x\}$, $n = 1, 2, 3, \ldots$.

With identifications $r(x) = 1$, $q(x) = 0$, $p(x) = 1$, $A_1 = 1$, $B_1 = 0$, $A_2 = 1$, and $B_2 = 1$ we see that (10) is a regular Sturm–Liouville problem. Thus $\{\sin\alpha_n x\}$, $n = 1, 2, 3, \ldots$ is an orthogonal set with respect to the weight function $p(x) = 1$ on the interval $[0, 1]$. ≡

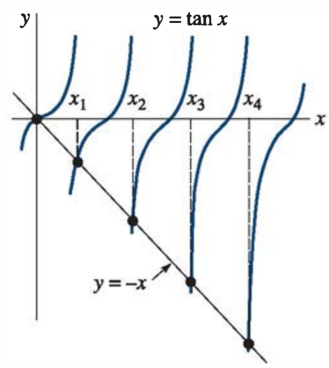


FIGURE 12.5.1 Positive roots of $\tan x = -x$ in Example 2

In some circumstances we can prove the orthogonality of the solutions of (3) without the necessity of specifying a boundary condition at $x = a$ and at $x = b$.

Singular Sturm–Liouville Problem There are several other important conditions under which we seek nontrivial solutions of the differential equation (3):

- $r(a) = 0$ and a boundary condition of the type given in (5) is specified at $x = b$; **(12)**
- $r(b) = 0$ and a boundary condition of the type given in (4) is specified at $x = a$; **(13)**
- $r(a) = r(b) = 0$ and no boundary condition is specified at either $x = a$ or at $x = b$; **(14)**
- $r(a) = r(b)$ and boundary conditions $y(a) = y(b)$, $y'(a) = y'(b)$. **(15)**

The differential equation (3) along with one of conditions (12) or (13) is said to be a **singular boundary-value problem**. Equation (3) with the conditions specified in (15) is said to be a **periodic boundary-value problem** because the boundary conditions are periodic. Observe that if, say, $r(a) = 0$, then $x = a$ *may* be a singular point of the differential equation, and consequently a solution of (3) may become unbounded as $x \to a$. However, we see from (8) that if $r(a) = 0$, then no boundary condition is required at $x = a$ to prove orthogonality of the eigenfunctions *provided* these solutions are bounded at that point. This latter requirement guarantees the existence of the integrals involved. By assuming the solutions of (3) are bounded on the closed interval $[a, b]$ we can see from inspection of (8) that

- If $r(a) = 0$, then the orthogonality relation (9) holds with no boundary condition at $x = a$; **(16)**
- If $r(b) = 0$, then the orthogonality relation (9) holds with no boundary condition at $x = b$;* **(17)**
- If $r(a) = r(b) = 0$, then the orthogonality relation (9) holds with no boundary conditions specified at either $x = a$ or $x = b$; **(18)**
- If $r(a) = r(b)$, then the orthogonality relation (9) holds with the periodic boundary conditions $y(a) = y(b)$, $y'(a) = y'(b)$. **(19)**

Self-Adjoint Form If we carry out the differentiation $\dfrac{d}{dx}[r(x)y']$, the differential equation in (3) is the same as

$$r(x)y'' + r'(x)y' + (q(x) + \lambda p(x))y = 0. \tag{20}$$

For example, Legendre's differential equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ is exactly of the form given in (20) with $r(x) = 1 - x^2$ and $r'(x) = -2x$. In other words, another way of writing Legendre's DE is

$$\frac{d}{dx}[(1 - x^2)y'] + n(n + 1)y = 0. \tag{21}$$

But if you compare other second-order DEs (say, Bessel's equation, Cauchy–Euler equations, and DEs with constant coefficients) you might believe, given the coefficient of y' is the derivative of the coefficient of y'', that few other second-order DEs have the form given in (3). On the contrary, if the coefficients are continuous and $a(x) \neq 0$ for all x in some interval, then *any* second-order differential equation

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0 \tag{22}$$

can be recast into the so-called **self-adjoint form** (3). To see this, we proceed as in Section 2.3 where we rewrote a linear first-order equation $a_1(x)y' + a_0(x)y = 0$ in the form $\dfrac{d}{dx}[\mu y] = 0$ by dividing the equation by $a_1(x)$ and then multiplying by the integrating factor $\mu = e^{\int P(x)dx}$ where, assuming no common factors, $P(x) = a_0(x)/a_1(x)$. So first, we divide (22) by $a(x)$. The first two terms are then $Y' + \dfrac{b(x)}{a(x)}Y + \cdots$ where, for emphasis, we have written $Y = y'$. Second, we multiply this equation by the integrating factor $e^{\int (b(x)/a(x))dx}$, where $a(x)$ and $b(x)$ are assumed to have no common factors

$$\underbrace{e^{\int (b(x)/a(x))dx}Y' + \frac{b(x)}{a(x)}e^{\int (b(x)/a(x))dx}Y}_{\text{derivative of a product}} + \cdots = \frac{d}{dx}\left[e^{\int (b(x)/a(x))dx}Y\right] + \cdots = \frac{d}{dx}\left[e^{\int (b(x)/a(x))dx}y'\right] + \cdots.$$

*Conditions (16) and (17) are equivalent to choosing $A_1 = 0$, $B_1 = 0$ in (4), and $A_2 = 0$, $B_2 = 0$ in (5), respectively.

In summary, by dividing (22) by $a(x)$ and then multiplying by $e^{\int(b(x)/a(x))dx}$ we get

$$e^{\int(b/a)dx}y'' + \frac{b(x)}{a(x)}e^{\int(b/a)dx}y' + \left(\frac{c(x)}{a(x)}e^{\int(b/a)dx} + \lambda\frac{d(x)}{a(x)}e^{\int(b/a)dx}\right)y = 0. \tag{23}$$

Equation (23) is the desired form given in (20) and is the same as (3):

$$\frac{d}{dx}\Big[\underbrace{e^{\int(b/a)dx}}_{r(x)}y'\Big] + \Big(\underbrace{\frac{c(x)}{a(x)}e^{\int(b/a)dx}}_{q(x)} + \lambda\underbrace{\frac{d(x)}{a(x)}e^{\int(b/a)dx}}_{p(x)}\Big)y = 0.$$

For example, to express $3y'' + 6y' + \lambda y = 0$ in self-adjoint form, we write $y'' + 2y' + \lambda\frac{1}{3}y = 0$ and then multiply by $e^{\int 2dx} = e^{2x}$. The resulting equation is

$$\underbrace{e^{2x}}_{r(x)}y'' + \underbrace{2e^{2x}}_{r'(x)}y' + \lambda\underbrace{\frac{1}{3}e^{2x}}_{p(x)}y = 0 \qquad \text{or} \qquad \frac{d}{dx}[e^{2x}y'] + \lambda\frac{1}{3}e^{2x}y = 0.$$

It is certainly not necessary to put a second-order differential equation (22) into the self-adjoint form (3) in order to *solve* the DE. For our purposes we use the form given in (3) to determine the weight function $p(x)$ needed in the orthogonality relation (9). The next two examples illustrate orthogonality relations for Bessel functions and for Legendre polynomials.

Note.

EXAMPLE 3 Parametric Bessel Equation

In Section 5.3 we saw that the general solution of the parametric Bessel differential equation $x^2y'' + xy' + (\alpha^2x^2 - n^2)y = 0$, $n = 0, 1, 2, \ldots$ is $y = c_1J_n(\alpha x) + c_2Y_n(\alpha x)$. After dividing the parametric Bessel equation by the lead coefficient x^2 and multiplying the resulting equation by the integrating factor $e^{\int(1/x)dx} = e^{\ln x} = x$, $x > 0$, we obtain the self-adjoint form

$$xy'' + y' + \left(\alpha^2x - \frac{n^2}{x}\right)y = 0 \qquad \text{or} \qquad \frac{d}{dx}[xy'] + \left(\alpha^2x - \frac{n^2}{x}\right)y = 0,$$

where we identify $r(x) = x$, $q(x) = -n^2/x$, $p(x) = x$, and $\lambda = \alpha^2$. Now $r(0) = 0$, and of the two solutions $J_n(\alpha x)$ and $Y_n(\alpha x)$ only $J_n(\alpha x)$ is bounded at $x = 0$. Thus in view of (16) above, the set $\{J_n(\alpha_i x)\}$, $i = 1, 2, 3, \ldots$, is orthogonal with respect to the weight function $p(x) = x$ on an interval $[0, b]$. The orthogonality relation is

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)\,dx = 0, \qquad \lambda_i \neq \lambda_j, \tag{24}$$

provided the α_i, and hence the eigenvalues $\lambda_i = \alpha_i^2$, $i = 1, 2, 3, \ldots$, are defined by means of a boundary condition at $x = b$ of the type given in (5):

$$A_2J_n(\lambda b) + B_2\alpha J_n'(\alpha b) = 0. \tag{25}$$

The extra factor of α in (25) comes from the Chain Rule:

$$\frac{d}{dx}J_n(\alpha x) = J_n'(\alpha x)\frac{d}{dx}\alpha x = \alpha J_n'(\alpha x).$$

≡

For any choice of A_2 and B_2, not both zero, it is known that (25) has an infinite number of roots $x_i = \alpha_i b$. The eigenvalues are then $\lambda_i = \alpha_i^2 = (x_i/b)^2$. More will be said about eigenvalues in the next chapter.

EXAMPLE 4 Legendre's Equation

From the result given in (21) we can identify $q(x) = 0$, $p(x) = 1$, and $\lambda = n(n + 1)$. Recall from Section 5.3 when $n = 0, 1, 2, \ldots$ Legendre's DE possesses polynomial solutions $P_n(x)$. Now we can put the observation that $r(-1) = r(1) = 0$ together with the fact that the

Legendre polynomials $P_n(x)$ are the only solutions of (21) that are bounded on the closed interval $[-1, 1]$, to conclude from (18) that the set $\{P_n(x)\}$, $n = 0, 1, 2, \ldots$, is orthogonal with respect to the weight function $p(x) = 1$ on $[-1, 1]$. The orthogonality relation is

$$\int_{-1}^{1} P_m(x)P_n(x)\, dx = 0, \qquad m \neq n. \qquad \equiv$$

Remarks

(*i*) A Sturm–Liouville problem is also considered to be singular when the interval under consideration is infinite. See Problems 9 and 10 in Exercises 12.5.

(*ii*) Even when the conditions on the coefficients p, q, r, and r' are as assumed in the regular Sturm–Liouville problem, if the boundary conditions are periodic, then property (b) of Theorem 12.5.1 does not hold. You are asked to show in Problem 4 of Exercises 12.5 that corresponding to each eigenvalue of the BVP

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L)$$

there exist two linearly independent eigenfunctions.

12.5 Exercises

Answers to selected odd-numbered problems begin on page ANS-30.

In Problems 1 and 2, find the eigenfunctions and the equation that defines the eigenvalues for the given boundary-value problem. Use a CAS to approximate the first four eigenvalues λ_1, λ_2, λ_3, and λ_4. Give the eigenfunctions corresponding to these approximations.

1. $y'' + \lambda y = 0,\ y'(0) = 0,\ y(1) + y'(1) = 0$
2. $y'' + \lambda y = 0,\ y(0) + y'(0) = 0,\ y(1) = 0$
3. Consider $y'' + \lambda y = 0$ subject to $y'(0) = 0$, $y'(L) = 0$. Show that the eigenfunctions are

$$\left\{1, \cos\frac{\pi}{L}x, \cos\frac{2\pi}{L}x, \ldots\right\}.$$

This set, which is orthogonal on $[0, L]$, is the basis for the Fourier cosine series.

4. Consider $y'' + \lambda y = 0$ subject to the periodic boundary conditions $y(-L) = y(L)$, $y'(-L) = y'(L)$. Show that the eigenfunctions are

$$\left\{1, \cos\frac{\pi}{L}x, \cos\frac{2\pi}{L}x, \ldots, \sin\frac{\pi}{L}x, \sin\frac{2\pi}{L}x, \sin\frac{3\pi}{L}x, \ldots\right\}.$$

This set, which is orthogonal on $[-L, L]$, is the basis for the Fourier series.

5. Find the square norm of each eigenfunction in Problem 1.
6. Show that for the eigenfunctions in Example 2,

$$\|\sin \alpha_n x\|^2 = \frac{1}{2}[1 + \cos^2\alpha_n].$$

7. **(a)** Find the eigenvalues and eigenfunctions of the boundary-value problem

$$x^2y'' + xy' + \lambda y = 0,\ y(1) = 0,\ y(5) = 0.$$

(b) Put the differential equation in self-adjoint form.
(c) Give an orthogonality relation.

8. **(a)** Find the eigenvalues and eigenfunctions of the boundary-value problem

$$y'' + y' + \lambda y = 0,\ y(0) = 0,\ y(2) = 0.$$

(b) Put the differential equation in self-adjoint form.
(c) Give an orthogonality relation.

9. **Laguerre's differential equation**

$$xy'' + (1 - x)y' + ny = 0, \quad n = 0, 1, 2, \ldots,$$

has polynomial solutions $L_n(x)$. Put the equation in self-adjoint form and give an orthogonality relation.

10. **Hermite's differential equation**

$$y'' - 2xy' + 2ny = 0, \quad n = 0, 1, 2, \ldots,$$

has polynomial solutions $H_n(x)$. Put the equation in self-adjoint form and give an orthogonality relation.

11. Consider the regular Sturm–Liouville problem:

$$\frac{d}{dx}[(1 + x^2)y'] + \frac{\lambda}{1 + x^2}y = 0, \quad y(0) = 0, \quad y(1) = 0.$$

(a) Find the eigenvalues and eigenfunctions of the boundary-value problem. [*Hint*: Let $x = \tan\theta$ and then use the Chain Rule.]
(b) Give an orthogonality relation.

12. **(a)** Find the eigenfunctions and the equation that defines the eigenvalues for the boundary-value problem

$$x^2y'' + xy' + (\lambda x^2 - 1)y = 0,$$
$$y \text{ is bounded at } x = 0,\ y(3) = 0.$$

(b) Use Table 5.3.1 of Section 5.3 to find the approximate values of the first four eigenvalues λ_1, λ_2, λ_3, and λ_4.

Discussion Problem

13. Consider the special case of the regular Sturm–Liouville problem on the interval $[a, b]$:

$$\frac{d}{dx}[r(x)y'] + \lambda p(x)y = 0, \quad y'(a) = 0, \quad y'(b) = 0.$$

Is $\lambda = 0$ an eigenvalue of the problem? Defend your answer.

Computer Lab Assignments

14. (a) Give an orthogonality relation for the Sturm–Liouville problem in Problem 1.
(b) Use a CAS as an aid in verifying the orthogonality relation for the eigenfunctions y_1 and y_2 that correspond to the first two eigenvalues λ_1 and λ_2, respectively.

15. (a) Give an orthogonality relation for the Sturm–Liouville problem in Problem 2.
(b) Use a CAS as an aid in verifying the orthogonality relation for the eigenfunctions y_1 and y_2 that correspond to the first two eigenvalues λ_1 and λ_2, respectively.

12.6 Bessel and Legendre Series

Introduction Fourier series, Fourier cosine series, and Fourier sine series are three ways of expanding a function in terms of an orthogonal set of functions. But such expansions are by no means limited to orthogonal sets of trigonometric functions. We saw in Section 12.1 that a function f defined on an interval (a, b) could be expanded, at least in a formal manner, in terms of any set of functions $\{\phi_n(x)\}$ that is orthogonal with respect to a weight function on $[a, b]$. Many of these orthogonal series expansions or generalized Fourier series derive from Sturm–Liouville problems that, in turn, arise from attempts to solve linear partial differential equations serving as models for physical systems. Fourier series and orthogonal series expansions (the latter includes the two series considered in this section) will appear in the subsequent consideration of these applications in Chapters 13 and 14.

12.6.1 Fourier–Bessel Series

We saw in Example 3 of Section 12.5 that for a fixed value of n the set of Bessel functions $\{J_n(\alpha_i x)\}$, $i = 1, 2, 3, \ldots$, is orthogonal with respect to the weight function $p(x) = x$ on an interval $[0, b]$ when the α_i are defined by means of a boundary condition of the form

$$A_2 J_n(\alpha b) + B_2 \alpha J_n'(\alpha b) = 0. \tag{1}$$

The eigenvalues of the corresponding Sturm–Liouville problem are $\lambda_i = \alpha_i^2$. From (7) and (8) of Section 12.1 the orthogonal series expansion or generalized Fourier series of a function f defined on the interval $(0, b)$ in terms of this orthogonal set is

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x), \tag{2}$$

where

$$c_i = \frac{\int_0^b x\, J_n(\alpha_i x) f(x)\, dx}{\|J_n(\alpha_i x)\|^2}. \tag{3}$$

The square norm of the function $J_n(\alpha_i x)$ is defined by (11) of Section 12.1:

$$\|J_n(\alpha_i x)\|^2 = \int_0^b x J_n^2(\alpha_i x)\, dx. \tag{4}$$

The series (2) with coefficients (3) is called a **Fourier–Bessel series**.

Differential Recurrence Relations The differential recurrence relations that were given in (20) and (21) of Section 5.3 are often useful in the evaluation of the coefficients (3). For convenience we reproduce those relations here:

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x) \tag{5}$$

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x). \tag{6}$$

Square Norm The value of the square norm (4) depends on how the eigenvalues $\lambda_i = \alpha_i^2$ are defined. If $y = J_n(\alpha x)$, then we know from Example 3 of Section 12.5 that

$$\frac{d}{dx}[xy'] + \left(\alpha^2 x - \frac{n^2}{x}\right)y = 0.$$

After we multiply by $2xy'$, this equation can be written as

$$\frac{d}{dx}[xy']^2 + (\alpha^2 x^2 - n^2)\frac{d}{dx}[y]^2 = 0.$$

Integrating the last result by parts on $[0, b]$ then gives

$$2\alpha^2 \int_0^b xy^2\,dx = \left([xy']^2 + (\alpha^2 x^2 - n^2)y^2\right)\Big]_0^b.$$

Since $y = J_n(\alpha x)$, the lower limit is zero for $n > 0$ because $J_n(0) = 0$. For $n = 0$, the quantity $[xy']^2 + \alpha^2 x^2 y^2$ is zero at $x = 0$. Thus

$$2\alpha^2 \int_0^b xJ_n^2(\alpha x)\,dx = \alpha^2 b^2 [J_n'(\alpha b)]^2 + (\alpha^2 b^2 - n^2)[J_n(\alpha b)]^2, \tag{7}$$

where we have used the Chain Rule to write $y' = \alpha J_n'(\alpha x)$.

We now consider three cases of the boundary condition (1).

Case I: If we choose $A_2 = 1$ and $B_2 = 0$, then (1) is

$$J_n(\alpha b) = 0. \tag{8}$$

There are an infinite number of positive roots $x_i = \alpha_i b$ of (8) (see Figure 5.3.1) that define the α_i as $\alpha_i = x_i/b$. The eigenvalues are positive and are then $\lambda_i = \alpha_i^2 = x_i^2/b^2$. No new eigenvalues result from the negative roots of (8) since $J_n(-x) = (-1)^n J_n(x)$. (See page 277.) The number 0 is not an eigenvalue for any n since $J_n(0) = 0$ for $n = 1, 2, 3, \ldots$ and $J_0(0) = 1$. In other words, if $\lambda = 0$, we get the trivial function (which is never an eigenfunction) for $n = 1, 2, 3, \ldots$, and for $n = 0$, $\lambda = 0$ (or equivalently, $\alpha = 0$) does not satisfy the equation in (8). When (6) is written in the form $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$, it follows from (7) and (8) that the square norm of $J_n(\alpha_i x)$ is

$$\|J_n(\alpha_i x)\|^2 = \frac{b^2}{2} J_{n+1}^2(\alpha_i b). \tag{9}$$

Case II: If we choose $A_2 = h \geq 0$, $B_2 = b$, then (1) is

$$hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0. \tag{10}$$

Equation (10) has an infinite number of positive roots $x_i = \alpha_i b$ for each positive integer $n = 1, 2, 3, \ldots$. As before, the eigenvalues are obtained from $\lambda_i = \alpha_i^2 = x_i^2/b^2$. $\lambda = 0$ is not an eigenvalue for $n = 1, 2, 3, \ldots$. Substituting $\alpha_i b J_n'(\alpha_i b) = -hJ_n(\alpha_i b)$ into (7), we find that the square norm of $J_n(\alpha_i x)$ is now

$$\|J_n(\alpha_i x)\|^2 = \frac{\alpha_i^2 b^2 - n^2 + h^2}{2\alpha_i^2} J_n^2(\alpha_i b). \tag{11}$$

Case III: If $h = 0$ and $n = 0$ in (10), the α_i are defined from the roots of

$$J_0'(\alpha b) = 0. \tag{12}$$

Even though (12) is just a special case of (10), it is the only situation for which $\lambda = 0$ is an eigenvalue. To see this, observe that for $n = 0$, the result in (6) implies that $J_0'(\alpha b) = 0$ is equivalent to $J_1(\alpha b) = 0$. Since $x_1 = \alpha_i b = 0$ is a root of the last equation, $\alpha_1 = 0$, and because $J_0(0) = 1$ is nontrivial, we conclude from $\lambda_1 = \alpha_1^2 = x_1^2/b^2$ that $\lambda_1 = 0$ is an eigenvalue. But obviously we cannot use (11) when $\alpha_1 = 0$, $h = 0$, and $n = 0$. However, from the square norm (4) we have

$$\|1\|^2 = \int_0^b x\,dx = \frac{b^2}{2}. \tag{13}$$

For $\alpha_i > 0$ we can use (11) with $h = 0$ and $n = 0$:

$$\|J_0(\alpha_i x)\|^2 = \frac{b^2}{2} J_0^2(\alpha_i b). \tag{14}$$

The following definition summarizes three forms of the series (2) corresponding to the square norms in the three cases.

Definition 12.6.1 Fourier–Bessel Series

The **Fourier–Bessel series** of a function f defined on the interval $(0, b)$ is given by

(i)
$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \tag{15}$$

$$c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x)\, dx, \tag{16}$$

where the α_i are defined by $J_n(\alpha b) = 0$.

(ii)
$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \tag{17}$$

$$c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x)\, dx, \tag{18}$$

where the α_i are defined by $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$.

(iii)
$$f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_0(\alpha_i x) \tag{19}$$

$$c_1 = \frac{2}{b^2} \int_0^b x f(x)\, dx, \quad c_i = \frac{2}{b^2 J_0^2(\alpha_i b)} \int_0^b x J_0(\alpha_i x) f(x)\, dx, \tag{20}$$

where the α_i are defined by $J_0'(\alpha b) = 0$.

Convergence of a Fourier–Bessel Series

Sufficient conditions for the convergence of a Fourier–Bessel series are not particularly restrictive.

Theorem 12.6.1 Conditions for Convergence

Let f and f' be piecewise continuous on the interval $[0, b]$. Then for all x in the interval $(0, b)$, the Fourier–Bessel series of f converges to $f(x)$ at a point where f is continuous and to the average

$$\frac{f(x+) + f(x-)}{2}$$

at a point where f is discontinous.

EXAMPLE 1 Expansion in a Fourier–Bessel Series

Expand $f(x) = x$, $0 < x < 3$, in a Fourier–Bessel series, using Bessel functions of order one that satisfy the boundary condition $J_1(3\alpha) = 0$.

SOLUTION We use (15) where the coefficients c_i are given by (16) with $b = 3$:

$$c_i = \frac{2}{3^2 J_2^2(3\alpha_i)} \int_0^3 x^2 J_1(\alpha_i x)\, dx.$$

To evaluate this integral we let $t = \alpha_i x$, $dx = dt/\alpha_i$, $x^2 = t^2/\alpha_i^2$, and use (5) in the form $\frac{d}{dt}[t^2 J_2(t)] = t^2 J_1(t)$:

$$c_i = \frac{2}{9\alpha_i^3 J_2^2(3\alpha_i)} \int_0^{3\alpha_i} \frac{d}{dt}[t^2 J_2(t)]\, dt = \frac{2}{\alpha_i J_2(3\alpha_i)}.$$

Therefore the desired expansion is

$$f(x) = 2\sum_{i=1}^{\infty} \frac{1}{\alpha_i J_2(3\alpha_i)} J_1(\alpha_i x). \qquad \equiv$$

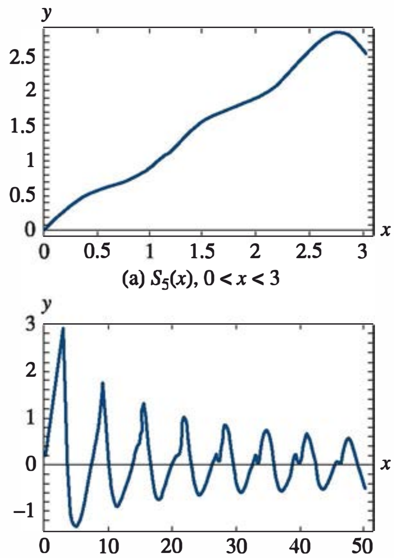
You are asked to find the first four values of the α_i for the foregoing Bessel series in Problem 1 in Exercises 12.6.

EXAMPLE 2 Expansion in a Fourier–Bessel Series

If the α_i in Example 1 are defined by $J_1(3\alpha) + \alpha J_1'(3\alpha) = 0$, then the only thing that changes in the expansion is the value of the square norm. Multiplying the boundary condition by 3 gives $3J_1(3\alpha) + 3\alpha J_1'(3\alpha) = 0$, which now matches (10) when $h = 3$, $b = 3$, and $n = 1$. Thus (18) and (17) yield, in turn,

$$c_i = \frac{18\alpha_i J_2(3\alpha_i)}{(9\alpha_i^2 + 8)J_1^2(3\alpha_i)}$$

and
$$f(x) = 18\sum_{i=1}^{\infty} \frac{\alpha_i J_2(3\alpha_i)}{(9\alpha_i^2 + 8)J_1^2(3\alpha_i)} J_1(\alpha_i x). \qquad \equiv$$



(a) $S_5(x)$, $0 < x < 3$

(b) $S_{10}(x)$, $0 < x < 50$

FIGURE 12.6.1 Partial sums of a Fourier–Bessel series

Use of Computers Since Bessel functions are "built-in functions" in a CAS, it is a straightforward task to find the approximate values of the α_i and the coefficients c_i in a Fourier–Bessel series. For example, in (9) we can think of $x_i = \alpha_i b$ as a positive root of the equation $hJ_n(x) + xJ_n'(x) = 0$. Thus in Example 2 we have used a CAS to find the first five positive roots x_i of $3J_1(x) + xJ_1'(x) = 0$ and from these roots we obtain the first five values of α_i: $\alpha_1 = x_1/3 = 0.98320$, $\alpha_2 = x_2/3 = 1.94704$, $\alpha_3 = x_3/3 = 2.95758$, $\alpha_4 = x_4/3 = 3.98538$, and $\alpha_5 = x_5/3 = 5.02078$. Knowing the roots $x_i = 3\alpha_i$ and the α_i, we again use a CAS to calculate the numerical values of $J_2(3\alpha_i)$, $J_1^2(3\alpha_i)$, and finally the coefficients c_i. In this manner we find that the fifth partial sum $S_5(x)$ for the Fourier–Bessel series representation of $f(x) = x$, $0 < x < 3$ in Example 2 is

$$\begin{aligned} S_5(x) = {} & 4.01844\, J_1(0.98320x) - 1.86937\, J_1(1.94704x) \\ & + 1.07106\, J_1(2.95758x) - 0.70306\, J_1(3.98538x) + 0.50343\, J_1(5.02078x). \end{aligned}$$

The graph of $S_5(x)$ on the interval (0, 3) is shown in **FIGURE 12.6.1(a)**. In Figure 12.6.1(b) we have graphed $S_{10}(x)$ on the interval (0, 50). Notice that outside the interval of definition (0, 3) the series does not converge to a periodic extension of f because Bessel functions are not periodic functions. See Problems 11 and 12 in Exercises 12.6.

12.6.2 Fourier–Legendre Series

From Example 4 of Section 12.5 we know that the set of Legendre polynomials $\{P_n(x)\}$, $n = 0, 1, 2, \ldots$, is orthogonal with respect to the weight function $p(x) = 1$ on the interval $[-1, 1]$. Furthermore, it can be proved that the square norm of a polynomial $P_n(x)$ depends on n in the following manner:

$$\|P_n(x)\|^2 = \int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n + 1}.$$

The orthogonal series expansion of a function in terms of the Legendre polynomials is summarized in the next definition.

Definition 12.6.2 Fourier–Legendre Series

The **Fourier–Legendre series** of a function f defined on the interval $(-1, 1)$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \tag{21}$$

where
$$c_n = \frac{2n + 1}{2}\int_{-1}^{1} f(x) P_n(x)\, dx. \tag{22}$$

Convergence of a Fourier–Legendre Series

Sufficient conditions for convergence of a Fourier–Legendre series are given in the next theorem.

Theorem 12.6.2 Conditions for Convergence

Let f and f' be piecewise continuous on the interval $[-1, 1]$. Then for all x in the interval $(-1, 1)$, the Fourier–Legendre series of f converges to $f(x)$ at a point where f is continuous and to the average

$$\frac{f(x+) + f(x-)}{2}$$

at a point where f is discontinuous.

EXAMPLE 3 Expansion in a Fourier–Legendre Series

Write out the first four nonzero terms in the Fourier–Legendre expansion of

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 \le x < 1. \end{cases}$$

SOLUTION The first several Legendre polynomials are listed on page 282. From these and (22) we find

$$c_0 = \frac{1}{2}\int_{-1}^{1} f(x)P_0(x)\,dx = \frac{1}{2}\int_{0}^{1} 1\cdot 1\,dx = \frac{1}{2}$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} f(x)P_1(x)\,dx = \frac{3}{2}\int_{0}^{1} 1\cdot x\,dx = \frac{3}{4}$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} f(x)P_2(x)\,dx = \frac{5}{2}\int_{0}^{1} 1\cdot \frac{1}{2}(3x^2 - 1)\,dx = 0$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} f(x)P_3(x)\,dx = \frac{7}{2}\int_{0}^{1} 1\cdot \frac{1}{2}(5x^3 - 3x)\,dx = -\frac{7}{16}$$

$$c_4 = \frac{9}{2}\int_{-1}^{1} f(x)P_4(x)\,dx = \frac{9}{2}\int_{0}^{1} 1\cdot \frac{1}{8}(35x^4 - 30x^2 + 3)\,dx = 0$$

$$c_5 = \frac{11}{2}\int_{-1}^{1} f(x)P_5(x)\,dx = \frac{11}{2}\int_{0}^{1} 1\cdot \frac{1}{8}(63x^5 - 70x^3 + 15x)\,dx = \frac{11}{32}.$$

Hence $$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) + \cdots.$$ ≡

Like the Bessel functions, Legendre polynomials are built-in functions in computer algebra systems such as *Maple* and *Mathematica*, and so each of the coefficients just listed can be found using the integration application of such a program. Indeed, using a CAS, we further find that $c_6 = 0$ and $c_7 = -\frac{65}{256}$. The fifth partial sum of the Fourier–Legendre series representation of the function f defined in Example 3 is then

$$S_5(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) - \frac{65}{256}P_7(x).$$

The graph of $S_5(x)$ on the interval $(-1, 1)$ is given in **FIGURE 12.6.2**.

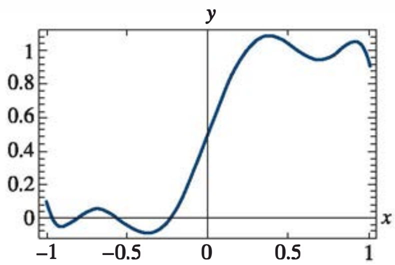


FIGURE 12.6.2 Partial sum $S_5(x)$ of Fourier–Legendre series in Example 3

Alternative Form of Series

In applications, the Fourier–Legendre series appears in an alternative form. If we let $x = \cos\theta$, then $x = 1$ implies $\theta = 0$, whereas $x = -1$ implies $\theta = \pi$. Since $dx = -\sin\theta\,d\theta$, (21) and (22) become, respectively,

$$F(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta) \tag{23}$$

$$c_n = \frac{2n+1}{2}\int_{0}^{\pi} F(\theta)P_n(\cos\theta)\sin\theta\,d\theta, \tag{24}$$

where $f(\cos\theta)$ has been replaced by $F(\theta)$.

12.6 Exercises

Answers to selected odd-numbered problems begin on page ANS-30.

12.6.1 Fourier–Bessel Series

In Problems 1 and 2, use Table 5.3.1 in Section 5.3.

1. Find the first four $\alpha_i > 0$ defined by $J_1(3\alpha) = 0$.
2. Find the first four $\alpha_i \geq 0$ defined by $J_0'(2\alpha) = 0$.

In Problems 3–6, expand $f(x) = 1$, $0 < x < 2$, in a Fourier–Bessel series using Bessel functions of order zero that satisfy the given boundary condition.

3. $J_0(2\alpha) = 0$
4. $J_0'(2\alpha) = 0$
5. $J_0(2\alpha) + 2\alpha J_0'(2\alpha) = 0$
6. $J_0(2\alpha) + \alpha J_0'(2\alpha) = 0$

In Problems 7–10, expand the given function in a Fourier–Bessel series using Bessel functions of the same order as in the indicated boundary condition.

7. $f(x) = 5x$, $0 < x < 4$
   $3J_1(4\alpha) + 4\alpha J_1'(4\alpha) = 0$
8. $f(x) = x^2$, $0 < x < 1$
   $J_2(\alpha) = 0$
9. $f(x) = x^2$, $0 < x < 3$
   $J_0'(3\alpha) = 0$
   [*Hint*: $t^3 = t^2 \cdot t$.]
10. $f(x) = 1 - x^2$, $0 < x < 1$
    $J_0(\alpha) = 0$

Computer Lab Assignments

11. (a) Use a CAS to graph $y = 3J_1(x) + xJ_1'(x)$ on an interval so that the first five positive x-intercepts of the graph are shown.
    (b) Use the root-finding capability of your CAS to approximate the first five roots x_i of the equation
    $$3J_1(x) + xJ_1'(x) = 0.$$
    (c) Use the data obtained in part (b) to find the first five positive values of α_i that satisfy
    $$3J_1(4\alpha) + 4\alpha J_1'(4\alpha) = 0.$$
    See Problem 7.
    (d) If instructed, find the first 10 positive values of α_i.
12. (a) Use the values of α_i in part (c) of Problem 11 and a CAS to approximate the values of the first five coefficients c_i of the Fourier–Bessel series obtained in Problem 7.
    (b) Use a CAS to graph the partial sums $S_N(x)$, $N = 1, 2, 3, 4, 5$, of the Fourier–Bessel series in Problem 7.
    (c) If instructed, graph the partial sum $S_{10}(x)$ for $0 < x < 4$ and for $0 < x < 50$.

Discussion Problems

13. If the partial sums in Problem 12 are plotted on a symmetric interval such as $(-30, 30)$, would the graphs possess any symmetry? Explain.
14. (a) Sketch, by hand, a graph of what you think the Fourier–Bessel series in Problem 3 converges to on the interval $(-2, 2)$.
    (b) Sketch, by hand, a graph of what you think the Fourier–Bessel series would converge to on the interval $(-4, 4)$ if the values α_i in Problem 7 were defined by $3J_2(4\alpha) + 4\alpha J_2'(4\alpha) = 0$.

12.6.2 Fourier–Legendre Series

In Problems 15 and 16, write out the first five nonzero terms in the Fourier–Legendre expansion of the given function. If instructed, use a CAS as an aid in evaluating the coefficients. Use a CAS to graph the partial sum $S_5(x)$.

15. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$
16. $f(x) = e^x$, $-1 < x < 1$
17. The first three Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$. If $x = \cos\theta$, then $P_0(\cos\theta) = 1$ and $P_1(\cos\theta) = \cos\theta$. Show that $P_2(\cos\theta) = \frac{1}{4}(3\cos 2\theta + 1)$.
18. Use the results of Problem 17 to find a Fourier–Legendre expansion (23) of $F(\theta) = 1 - \cos 2\theta$.
19. A Legendre polynomial $P_n(x)$ is an even or odd function, depending on whether n is even or odd. Show that if f is an even function on the interval $(-1, 1)$, then (21) and (22) become, respectively,
$$f(x) = \sum_{n=0}^{\infty} c_{2n} P_{2n}(x) \tag{25}$$
$$c_{2n} = (4n + 1)\int_0^1 f(x)P_{2n}(x)\,dx. \tag{26}$$
20. Show that if f is an odd function on the interval $(-1, 1)$, then (21) and (22) become, respectively,
$$f(x) = \sum_{n=0}^{\infty} c_{2n+1} P_{2n+1}(x) \tag{27}$$
$$c_{2n+1} = (4n + 3)\int_0^1 f(x)P_{2n+1}(x)\,dx. \tag{28}$$

The series (25) and (27) can also be used when f is defined on only the interval $(0, 1)$. Both series represent f on $(0, 1)$; but on the interval $(-1, 0)$, (25) represents an even extension, whereas (27) represents an odd extension. In Problems 21 and 22, write out the first four nonzero terms in the indicated expansion of the given function. What function does the series represent on the interval $(-1, 1)$? Use a CAS to graph the partial sum $S_4(x)$.

21. $f(x) = x$, $0 < x < 1$; (25)
22. $f(x) = 1$, $0 < x < 1$; (27)

Discussion Problems

23. Why is a Fourier–Legendre expansion of a polynomial function that is defined on the interval $(-1, 1)$ necessarily a finite series?
24. Use your conclusion from Problem 23 to find the finite Fourier–Legendre series of $f(x) = x^2$. The series of $f(x) = x^3$. Do *not* use (21) and (22).

12 Chapter in Review

Answers to selected odd-numbered problems begin on page ANS-30.

In Problems 1–10, fill in the blank or answer true/false without referring back to the text.

1. The functions $f(x) = x^2 - 1$ and $g(x) = x^5$ are orthogonal on the interval $[-\pi, \pi]$. _____

2. The product of an odd function f with an odd function g is an _____ function.

3. To expand $f(x) = |x| + 1$, $-\pi < x < \pi$, in an appropriate trigonometric series we would use a _____ series.

4. $y = 0$ is never an eigenfunction of a Sturm–Liouville problem. _____

5. $\lambda = 0$ is never an eigenvalue of a Sturm–Liouville problem. _____

6. If the function

$$f(x) = \begin{cases} x + 1, & -1 < x < 0 \\ -x, & 0 < x < 1 \end{cases}$$

is expanded in a Fourier series, the series will converge to _____ at $x = -1$, to _____ at $x = 0$, and to _____ at $x = 1$.

7. Suppose the function $f(x) = x^2 + 1$, $0 < x < 3$, is expanded in a Fourier series, a cosine series, and a sine series. At $x = 0$, the Fourier series will converge to _____, the cosine series will converge to _____, and the sine series will converge to _____.

8. The corresponding eigenfunction for the boundary-value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi/2) = 0$$

for $\lambda = 25$ is _____.

9. The set $\{P_{2n}(x)\}$, $n = 0, 1, 2, \ldots$ of Legendre polynomials of even degree is orthogonal with respect to the weight function $p(x) = 1$ on the interval $[0, 1]$. _____

10. The set $\{P_n(x)\}$, $n = 0, 1, 2, \ldots$ of Legendre polynomials is orthogonal with respect to the weight function $p(x) = 1$ on the interval $[-1, 1]$. Hence, for $n > 0$, $\int_{-1}^{1} P_n(x)\,dx =$ _____.

11. Without doing any work, explain why the cosine series of $f(x) = \cos^2 x$, $0 < x < \pi$, is the finite series

$$f(x) = \frac{1}{2} + \frac{1}{2}\cos 2x.$$

12. (a) Show that the set

$$\left\{ \sin\frac{\pi}{2L}x, \sin\frac{3\pi}{2L}x, \sin\frac{5\pi}{2L}x, \ldots \right\}$$

is orthogonal on the interval $[0, L]$.

(b) Find the norm of each function in part (a). Construct an orthonormal set.

13. Expand $f(x) = |x| - x$, $-1 < x < 1$, in a Fourier series.

14. Expand $f(x) = 2x^2 - 1$, $-1 < x < 1$, in a Fourier series.

15. Expand $f(x) = e^x$, $0 < x < 1$, in a cosine series. In a sine series.

16. In Problems 13, 14, and 15, sketch the periodic extension of f to which each series converges.

17. Find the eigenvalues and eigenfunctions of the boundary-value problem

$$x^2y'' + xy' + 9\lambda y = 0, \quad y'(1) = 0, \quad y(e) = 0.$$

18. Give an orthogonality relation for the eigenfunctions in Problem 17.

19. **Chebyshev's differential equation**

$$(1 - x^2)y'' - xy' + n^2y = 0$$

has a polynomial solution $y = T_n(x)$ for $n = 0, 1, 2, \ldots$. Specify the weight function $p(x)$ and the interval over which the set of Chebyshev polynomials $\{T_n(x)\}$ is orthogonal. Give an orthogonality relation.

20. Expand the periodic function shown in **FIGURE 12.R.1** in an appropriate Fourier series.

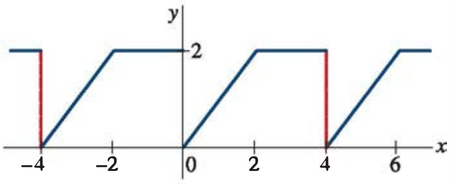


FIGURE 12.R.1 Graph for Problem 20

21. Expand $f(x) = \begin{cases} 1, & 0 < x < 2 \\ 0, & 2 < x < 4 \end{cases}$ in a Fourier–Bessel series, using Bessel functions of order zero that satisfy the boundary condition $J_0(4\alpha) = 0$.

22. Expand $f(x) = x^4$, $-1 < x < 1$, in a Fourier–Legendre series.

23. Suppose the function $y = f(x)$ is defined on the interval $(-\infty, \infty)$.

(a) Verify the identity $f(x) = f_e(x) + f_o(x)$, where

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2}.$$

(b) Show that f_e is an even function and f_o an odd function.

24. The function $f(x) = e^x$ is neither even nor odd. Use Problem 23 to write f as the sum of an even function and an odd function. Identify f_e and f_o.

25. Suppose f is an integrable $2p$-periodic function. Prove that for any real number a,

$$\int_0^{2p} f(x)\,dx = \int_a^{a+2p} f(x)\,dx.$$