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JEE Advanced 2019 - Paper 1 - Physics 16

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Problem *[Numerical Value]*

A parallel plate capacitor of capacitance C has spacing d between two plates having area A. The region between the plates is filled with N dielectric layers, parallel to its plates, each with thickness $\delta = \frac{d}{N}$. The dielectric constant of the m^{th} layer is $K_m = K\left(1 + \frac{m}{N}\right)$. For a very large N ($> 10^3$), the capacitance C is $\alpha\left(\frac{K\epsilon_0 A}{d \ln 2}\right)$. The value of α will be ______.
[ϵ_0 is the permittivity of free space]

What to Observe:

- A parallel plate capacitor has capacitance C.
- Plate spacing is d and plate area is A.
- The space between the plates is filled with N dielectric layers.
- Each dielectric layer is parallel to the plates and has thickness $\delta = \frac{d}{N}$.
- Dielectric constant of the m^{th} layer is $K_m = K\left(1 + \frac{m}{N}\right)$.
- N is very large ($N > 10^3$).
- Capacitance C is given in the form $C = \alpha\left(\frac{K\epsilon_0 A}{d \ln 2}\right)$.
- ϵ_0 is the permittivity of free space.

My Approach:

Capacitance

Thought

Based on what we know about equivalent capacitance calculation for n capacitors connected in series, we have:

$$\frac{1}{C_{\text{eq}}} = \sum_{i=1}^{n} \frac{1}{C_i}$$

We can use a similar logic in terms of integration for an elemental capacitor dC, and integrate it to get the result.
The reason for using integration here is that the given expression for C contains a $\ln 2$ term, which cannot be directly achieved using a normal summation. Such a value typically arises as a result of taking the limit of a sum — hence the need for integration.

The thickness of an elemental dielectric layer is $\delta = \frac{d}{N}$, so if the m-th plate is at a distance x, then:

$$x = m\delta = \frac{md}{N} \quad \Rightarrow \quad \frac{m}{N} = \frac{x}{d}$$

Thus, the dielectric constant of the m-th layer is:

$$K_m = K\left(1 + \frac{m}{N}\right) = K\left(1 + \frac{x}{d}\right)$$

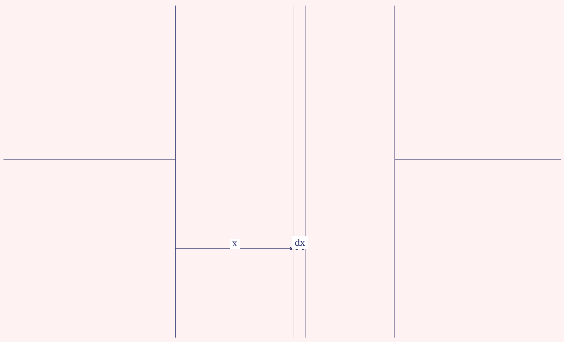

Figure 1. Logical Capacitor Diagram

Calculation

Since the capacitance of a dielectric layer is given by:

$$C = \frac{KA\epsilon_0}{d}$$

for an elemental layer of thickness dx, the capacitance is:

$$dC = \frac{K_m A\epsilon_0}{dx}$$

So the reciprocal of equivalent capacitance for series combination is:

$$\frac{1}{C_{\text{eq}}} = \int \frac{1}{dC} = \int \frac{dx}{K_m A\epsilon_0}$$

Substituting $K_m = K\left(1 + \frac{x}{d}\right)$, we get:

$$\frac{1}{C_{\text{eq}}} = \int_0^d \frac{dx}{K\left(1+\frac{x}{d}\right)A\epsilon_0}$$

Simplifying the integral:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{KA\epsilon_0}\int_0^d \frac{dx}{1+\frac{x}{d}}$$

Let $u = 1 + \frac{x}{d} \Rightarrow du = \frac{1}{d}dx \Rightarrow dx = d\,du$, when $x = 0$, $u = 1$; when $x = d$, $u = 2$. So:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{KA\epsilon_0}\int_1^2 \frac{d\,du}{u} = \frac{d}{KA\epsilon_0}\int_1^2 \frac{1}{u}\,du$$

$$\frac{1}{C_{\text{eq}}} = \frac{d}{KA\epsilon_0}[\ln u]_1^2 = \frac{d}{KA\epsilon_0}\ln 2.$$

Thus, the equivalent capacitance is:

$$C_{\text{eq}} = \frac{KA\epsilon_0}{d\ln 2}$$

We are given:

$$C_{\text{eq}} = \alpha\left(\frac{KA\epsilon_0}{d\ln 2}\right)$$

From the earlier calculation, we found:

$$C_{\text{eq}} = \frac{KA\epsilon_0}{d\ln 2}$$

Comparing both expressions, it is clear that:

$$\alpha = 1.$$

Conclusion

Based on detailed analysis and the computed expression for C_{eq}:

$$\boxed{\text{The value of } \alpha \text{ is } 1.00}$$

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