Integral Solution

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Question

The given integral is:

$$\int \frac{\tan^3(\ln(x))}{x}\,dx$$

Solution

Step 1: Substitution

Let's make the substitution $u = \ln(x)$, then $du = \frac{1}{x}\,dx$. So the integral becomes:

$$\int \tan^3(u)\,du$$

Step 2: Trigonometric Identity

Using the identity $\tan(u) = \frac{\sin(u)}{\cos(u)}$, we have:

$$\int \tan^3(u)\,du = \int \frac{\sin^3(u)}{\cos^3(u)}\,du$$

Step 3: Further Simplification

We can rewrite the integral as:

$$\int \frac{\sin(u)\cdot\sin^2(u)}{\cos^3(u)}\,du$$

$$= \int \frac{\sin(u)\cdot(1-\cos^2(u))}{\cos^3(u)}\,du$$

Step 4: Another Substitution

Let's make another substitution, $t = \cos(u)$, then $dt = -\sin(u)\,du$. So the integral becomes:

$$\int \frac{(t^2-1)}{t^3}\,dt$$

Step 5: Integration

Integrating, we get:

$$\ln|t| + \frac{1}{2t^2} + C$$

Step 6: Back Substitution

Substitute back $t = \cos(u)$ and $u = \ln(x)$:

$$\ln|\cos(\ln(x))| + \frac{1}{2\cos^2(\ln(x))} + C$$

Therefore, the solution to the given integral is:

$$\int \frac{\tan^3(\ln(x))}{x}\,dx = \ln|\cos(\ln(x))| + \frac{1}{2}\sec^2(\ln(x)) + C$$