Senan

$$\int \sqrt{\tan x}\, dx$$

let $\tan x = t^2$

$\sec^2 x\, dx = 2t\, dt$

$$dx = \frac{2t\, dt}{\sec^2 x}$$

$\sec^2 x = 1 + \tan^2 x$

$$dx = \frac{2t\, dt}{1+(t^2)^2}$$

$$\boxed{dx = \frac{2t\, dt}{1+t^4}}$$

$$\int \sqrt{\tan x}\, dx = \int \frac{t \times 2t\, dt}{1+t^4}$$

$$= \int \frac{2t^2}{1+t^4}\, dt$$

$\rightarrow$ divide num / den by t^2

why? $\longrightarrow$ by Experience

$$= \int \frac{2}{t^2 + 1/t^2}\, dt$$

Another Manipulation.
write 2 as 1+1

$$= \int \frac{1 + 1 + 0}{t^2 + 1/t^2}\, dt$$

add a zero in the numerator

$$= \int \frac{1 + 1/t^2 + 1 - 1/t^2}{t^2 + 1/t^2}\, dt$$

$$= \underbrace{\int \frac{1 + 1/t^2}{t^2 + 1/t^2}\, dt}_{I_1} + \underbrace{\int \frac{1 - 1/t^2}{t^2 + 1/t^2}\, dt}_{I_2}$$

$$I_1 = \int \frac{1 + 1/t^2}{\underbrace{t^2 + 1/t^2 - 2}_{(t - 1/t)^2} + 2}\, dt$$

$$= \int \frac{1 + 1/t^2}{(t - 1/t)^2 + 2}\, dt$$

let $u = t - 1/t$

$du = (1 + 1/t^2)\,dt$

Let $u = t - 1/t$

$du = \left(1 + 1/t^2\right)dt$

$$= \int \frac{du}{u^2 + 2}$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$= \int \frac{du}{u^2 + (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C_1$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t - 1/t}{\sqrt{2}}\right) + C_1$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}\, t}\right) + C_1$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2 \tan x}}\right) + C_1$$

$$I_2 = \int \frac{1 - 1/t^2}{\underbrace{t^2 + 1/t^2 + 2}_{(t + 1/t)^2} - 2}\, dt$$

$$= \int \frac{1 - 1/t^2}{(t + 1/t)^2 - 2}\, dt$$

let $s = t + 1/t$

$ds = (1 - 1/t^2)\, dt$

$$= \int \frac{ds}{s^2 - 2}$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$

$$= \int \frac{ds}{s^2 - (\sqrt{2})^2}$$

$$= \frac{1}{2\sqrt{2}} \ln \left| \frac{s - \sqrt{2}}{s + \sqrt{2}} \right| + C_2$$

$$= \frac{1}{2\sqrt{2}} \ln \left| \frac{t + 1/t - \sqrt{2}}{t + 1/t + \sqrt{2}} \right| + C_2$$

$$= \frac{1}{2\sqrt{2}} \ln \left| \frac{t^2 + 1 - \sqrt{2}\, t}{t^2 + 1 + \sqrt{2}\, t} \right| + C_2$$

$$= \frac{1}{2\sqrt{2}} \ln \left| \frac{\tan x + 1 - \sqrt{2 \tan x}}{\tan x + 1 + \sqrt{2 \tan x}} \right| + C_2$$

$$\int \sqrt{\tan x}\, dx = I_1 + I_2$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right)$$

$$+ \frac{1}{2\sqrt{2}} \ln\left|\frac{\tan x + 1 - \sqrt{2\tan x}}{\tan x + 1 + \sqrt{2\tan x}}\right|$$

$$+ C$$

where $C = C_1 + C_2$