KENYATTA UNIVERSITY
INSTITUTE OF OPEN LEARNING

SMA 200
CALCULUS II

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DEPARTMENT OF MATHEMATICS

PREFACE

This **module** is designed primarily to provide the readers with the best preparation possible for the **Advanced** calculus examinations.

In its present form this **module** has developed from courses given by the author over the last thirty two years to the audience of mathematicians, physics and engineers in the University of Madras, Kenyatta University and the University of Nairobi.

It is hoped that it will be of great interest to students of pure and Applied Mathematics following **Advanced** calculus. Each lesson begins with a brief statement of definitions. Principles and important theorems followed by a set of solved and graded supplementary problems. Attention has been given to the lessons on applications of the theory.

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CONTENTS

CHAPTER 1

Integration of functions using Basic Formulae

1.1 Introduction

The process of integration is defined as the reverse or converse process of differentiation. In Differential Calculus, you have studied differentiation of functions and its applications. Now we shall learn the process of integration and its applications in the **Integral Calculus.** In this chapter we shall learn **TEN BASIC FORMULAE** for integration and integrate expressions using them.

1.2 Objectives of the chapter

By the end of this chapter you should be able to

- state the Ten Basic Formulae for integration.
- integrate simple functions using the Ten Basic Formulae only.

1.3 Meaning of Integration

Integration is defined as the reverse or the converse process of differentiation.

For example,

We know that $\frac{d}{dx}(\sin x) = \cos x$

Now we say that the integration of cos x is sin x.

We write this in symbols

$\int \cos x dx = \sin x$

(The symbol $\int$ is pronounced as "the integration of" . It was first used by the French Mathematician, Leibnitz. He used this symbol for the first letter 'S' of the word Summation)

1.4 Constant of Integration

Now $\frac{d}{dx}(\sin x) = \frac{d}{dx}(\sin x + c) = \cos x$. In fact any constant c, that is added disappears in the process of differentiation. Hence we always put a constant c after the process of integration. Hence we write $\int \cos x dx = \sin x + c$. Here c is called the **constant of integration.**

1.5 The Notation and the Integrand

If f (x) is any function of x, $\int f(x)dx$ stands for integral of f (x) with respect to x. the integral sign $\int$ cannot be divorced from dx if the integral is with respect to x.

The function f(x) in $\int f(x)dx$ is called the integrand of the integral. For example in $\int x^2 dx$, x^2 is called the integrand of the integral.

1.6 (A) Basic Formulae for Integration of Algebraic functions.

1. $$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \text{if n} \neq \text{-1}$$
2. $$\int \frac{1}{x} dx = \int x^{-1} dx = \ln \text{x} + \text{c}$$

For example

$$\int x^{10} dx = \frac{x^{11}}{11} + c$$

$$\int x^{-10} dx = \frac{x^{-9}}{-9} + c$$

1.7 (B) Basic Formulae for Integration of Exponential functions.

3. $\int a^x dx = \dfrac{a^x}{\ln a} + c$, a is any positive constant
4. $$\int e^x dx = e^x + c$$

For example

$$\int 2^x dx = \frac{2^x}{\ln 2} + c$$

$$\int 5^x dx = \frac{5^x}{\ln 5} + c$$

1.8 (C) Basic Formulae for Integration of Trigonometric functions

5. $\int \cos x dx = \sin x + c$, since $\dfrac{d}{dx}(\sin x + c) = \cos x$

6. $\int \sin x dx = -\cos x + c$, since $\frac{d}{dx}(\cos x + c) = -\sin x$

7. $\int \sec^2 x dx = \tan x + c$, since $\frac{d}{dx}(\tan x + c) = \sec^2 x$

8. $\int \sec x \tan x dx = \sec x + c$, since $\frac{d}{dx}(\sec x + c) = \sec x \tan x$

9. $\int \cos ec x \cot x dx = -\cos ec x + c$, since $\frac{d}{dx}(\cos ec x + c) = -\cos ec x \cot x$

10. $\int \cos ec^2 x dx = -\cot x + c$, since $\frac{d}{dx}\cot s = -\cos ec^2 x$

Note that we shall consider integration of tan x, cot x, sec x and cosec x afterwards.

1.9 Addition and subtraction Rule in Integration

Just like $\frac{d}{dx}(u + v - w) = \frac{d}{dx}u + \frac{d}{dx}v - \frac{d}{dx}w$,

we have $\int (u + v - w)dx = \int udx + \int vdx - \int wdx$

where u, v, w are functions of x.

For example

$$\int (x^2 + \sin x - e^x)dx = \int x^2 dx + \int \sin x dx - \int e^x dx$$
$$= \frac{x^3}{3} - \cos x - e^x + c$$

1.10 Integration of constant times a function

$$\int kf(x)dx = k\int f(x)dx$$

For example

$$\int 5x^3 dx = 5\int x^3 dx = \frac{5x^4}{4} + c$$
$$\int 8\cos x dx = 8\int \cos x dx = 8\sin x + c$$

Example 1

Find $\int (2x^5 + 3e^x + 4x + 5)dx$

Solution

$$\int (2x^5 + 3e^x + 4x + 5)dx = 2\int x^5 dx + 3\int e^x dx + 4\int x dx + 5\int dx$$
$$= \frac{2x^6}{6} + 3e^x + \frac{4x^2}{2} + 5x + c$$
$$= \frac{x^6}{3} + 3e^x + 2x^2 + 5x + c$$

We should put one constant c after the process of integration.

Note that

$$\boxed{\begin{aligned} &\int dx = x + c \\ &\int 5dx = 5\int dx = 5x + c \\ &\int x dx = \frac{x^2}{2} + c \end{aligned}}$$

Example 2

Find $\int (2x+3)(x-2)dx$

Solution

First we find the product and then we integrate

$\int (2x+3)(x-2)dx$

$= \int (2x^2 + x - 6)dx$

$= 2\frac{x^2}{3} + \frac{x^2}{2} - 6x + c$

Example 3

Find $\int \left(\frac{2x^3 + 5x^2 + 8x + 9}{x} \right) dx$

Solution

First we divide each term by x and then we integrate

$$\int \left(\frac{2x^3 + 5x^2 + 8x + 9}{x} \right) dx = \int \left(\frac{2x^3}{x} + \frac{5x^2}{x} + \frac{8x}{x} + \frac{9}{x} \right) dx$$

$$= \int \left(2x^3 + 5x + 8x + \frac{9}{x} \right) dx$$

$$= \frac{2x^3}{3} + \frac{5x^2}{2} + 8x + 9\ln x + c$$

Example 4

Find $\int \left(\frac{x^2 + 5x + 6}{(x+2)} \right) dx$

Solution

First we simplify the integrand $\frac{x^2 + 5x + 6}{x+2}$ and then we integrate.

$$\int \left(\frac{x^2 + 5x + 6}{(x+2)} \right) dx = \int \frac{(x+3)(x+2)}{(x+2)} dx$$

$$= \int (x+3) dx$$

$$= \frac{x^2}{2} + 3x + c$$

Example 5

Integrate $9x^3 + \frac{9}{x^3} + 2^x + 3e^x - x + 1$

Solution

$$\int (9x^3 + \frac{9}{x^3} + 2^x + 3e^x - x + 1) dx$$

$$= \frac{9x^4}{4} + \frac{9x^{-2}}{-2} + \frac{2^x}{\ln 2} + 3e^x - \frac{x^2}{2} + x + c$$

Example 6

Find $\int \left(2\cos x + 3\sin x + 4\sec^2 x - 5\cos ecx \cot x \right) dx$

Solution

$$\int \left(2\cos x + 3\sin x + 4\sec^2 x - 5\cos ecx \cot x \right) dx$$

$$= 2\sin x - 3\cos x + 4\tan x + 5\cos ecx + c$$

Example 7

Find $\int (\frac{2}{x} + 5\sec x \tan x - 6\cos ec^2 x + 4^x + 8)dx$

Solution

$\int (\frac{2}{x} + 5\sec x \tan x - 6\cos ec^2 x + 4^x + 8)dx$

$= 2\int \frac{1}{x} dx + 5\int \sec x \tan x dx - 6\int \cos ec^2 x dx + \int 4^x dx + 8\int dx$

$= 2\ln x + 5\sec x + 6\cot x + \frac{4^x}{\ln 4} + 8x + c$

Exercise 1

Using the ten basic formula for integration find

1. $\int \left(2x^5 + \frac{3}{x^5} - \frac{4}{x} + 2e^x + 4^x + 1 \right) dx$
2. $\int (3\sin x - 4\sec x \tan x + 5\sec^2 x) dx$
3. $\int (3\cos x - 4\cos ecx \cot x + 5\cos ec^2 x) dx$
4. $\int \left(x^{15} + \frac{1}{x^{15}} + 8x^4 - 3e^x + \frac{7}{x} + 7x + 8 \right) dx$
5. $\int \frac{(x^2 + 7x + 10)}{(x+5)} dx$
6. $\int \frac{x^2 + 6x + 7}{(x-1)} dx$
7. $\int \frac{3x^2 + 4x^3 + 8x^5 + 9x + 2}{x^2} dx$
8. $\int \frac{(x^2 + 3x + 1)(x^3 + 7x^2 + 2)}{x^2} dx$
9. If $\int (2x - 9)dx = 0$ find the values of x
10. If $\int (3x^2 + 2x - 12)dx = 0$ find the values of x
11. Find $\int \left(\frac{x^3 + 5x^2 + 6x}{(x^2 + 2x)} \right) dx$

12. Find $\int \frac{e^{2x} + e^{5x} - e^{x}}{e^{x}} dx$

You have learnt the following from this chapter

i). Integration is the reverse or converse process of differentiation.

ii). The meaning of "integrand" and constant of integration.

iii). The Ten Basic Formula for Integration, namely

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ if $n \neq -1$
2. $\int \frac{1}{x} dx = \ln x + c = \ln Ax$
3. $\int a^x dx = \frac{a^x}{\ln a} + c$,
4. $\int e^x dx = e^x + c$
5. $\int \cos x dx = \sin x + c$,
6. $\int \sin x dx = -\cos x + c$,
7. $\int \sec^2 x dx = \tan x + c$,
8. $\int \sec x \tan x dx = \sec x + c$,
9. $\int \cos ecx \cot x dx = -\cos ecx + c$,
10. $\int \cos ec^2 x dx = -\cot x + c$,

Further Reading

1. Pure mathematics (First Coure)
   By J. K Backhouse and others
   Longman Group Ltd
   Harlow, Essex, U.K

CHAPTER 2
Integration of Functions Containing First Degree Expressions – Extension of the Basic Formula.

2.1 Introduction

In chapter one you have learnt to integrate functions using the Ten Basic Formulae. Unlike differentiation of functions, integration is not so easy, however we have about six techniques of integration with which a given function can be integrated. One of the techniques is integration using the Extension formulae. Suppose that a first degree expression is present in the place of x in the Ten Basic formula, we can use conveniently the same Basic formulae and obtain the result. The new formulae are called "the extension formulae". In this chapter we shall integrate expressions containing the first degree, using the Ten Extension Formulae.

2.2 Objectives of the chapter

By the end of this chapter you should be able to

- write the Ten Extension Formulae corresponding to the Ten Basic formulae.
- apply the Ten Extension Formulae to integrate expressions containing first degree in x.

2.3 Definition of the Integrand

The expression to be integrated is written between the integral symbol and the corresponding dx symbol. This expression or the function is called the **Integrand** of the integral.

For example in $\int x^2 e^x dx$, the expression x^2 e^x is called the "integrand" of the integration.

Similarly in $\int \cos x dx$, cos x is called "the integrand" of the integral.

2.4 The extension of the Basic formulae

In each of the ten Basic formulae the integrand contains x. Suppose we replace x in both sides of the formula by a **first degree** expression in x of the form (2x + 3), (5x – 7), (3 - 4x) or (ax + b) in general. All the ten formulae hold good when we divide the final result by coefficient of x namely a. The Ten Extension Formulae are given in the following table:

Basic Formulae	**Corresponding Extension formulae with an example – Applicable for first degree in x only**
1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ (n $\neq$ -1)	$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$ e.g. $\int (2x+3)^{10} dx = \frac{(2x+3)^{11}}{11.2} + c$
2. $\int \frac{1}{x} dx = \ln x +$	$\int \frac{1}{ax+b} dx = \frac{\ln(ax+b)}{a} + c$ e.g. $\int \frac{1}{2x-3} dx = \frac{\ln(2x-3)}{2} + c$
3. $\int p^x dx = \frac{p^x}{\ln p}$ (p is positive)	$\int p^{ax+b} dx = \frac{p^{ax+b}}{a \ln p} + c$ e.g. $\int 3^{5x+1} dx = \frac{3^{5x+1}}{5 \ln 3} + c$
4. $\int e^x dx = e^x +$	$\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + c$ e.g. $\int e^{4-7x} dx = \frac{e^{4-7x}}{-7} + c$
5. $\int \cos x dx = \mathrm{si}$	$\int \cos(ax+b) dx = \frac{\sin(ax+b)}{a} + c$ e.g. $\int \cos 100x dx = \frac{\sin 100x}{100} + c$

6.	$\int \sin x dx = -c$	$\int \sin(ax+b)dx = \frac{-\cos(ax+b)}{a} + c$ e.g. $\int \sin(100x+5)dx = \frac{-\cos(100x+5)}{100} + c$
7.	$\int \sec^2 x dx = ta$	$\int \sec^2(ax+b)dx = \frac{\tan(ax+b)}{a} + c$ e.g. $\int \sec^2(3-4x)dx = \frac{\tan(3-4x)}{-4} + c$
8.	$\int \cos ec^2 x dx = -$	$\int \cos ec^2(ax+b)dx = \frac{-\cot(ax+b)}{a} + c$ e.g. $\int \cos ec^2(20x+1)dx = \frac{-\cot(20x+1)}{20} + c$
9.	$\int \sec x \tan xdx$	$\int \sec(ax+b)\tan(ax+b)dx = \frac{\sec(ax+b)}{a} + c$ e.g. $\int \sec 5x \tan 5xdx = \frac{\sec 5x}{5} + c$
10.	$\int \cos ecx \cot xdx$	$\int \cos ec(ax+b)\cot(ax+b)dx = \frac{-\cos ec(ax+b)}{a} + c$ e.g. $\int \cos ec\pi x \cot \pi x dx = \frac{-\cos ec\pi x}{\pi} + c$

2.5 Proof of the Ten Extension Formulae

We shall prove some of he Ten Extension formulae using the substitution

u = ax + b so that dx = $\frac{du}{a}$

Example 1

a). Prove that $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

b). Hence find $\int (4x+9)^{20} dx$

Solution

Let u = a x + b so that dx = $\dfrac{du}{a}$

a) Then $\int (ax+b)^n dx = \int \dfrac{u^n du}{a}$

$$= \frac{1}{a}\int u^n du \quad \text{(we use the Basic formula to get)}$$
$$= \frac{1}{a}\frac{u^{n+1}}{n+1} + c$$
$$= \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

b) Using $\int x^{20} dx = \dfrac{x^{21}}{21} + c$,

$$\int (4x+9)^{20} dx = \frac{(4x+9)^{21}}{21.4} + c,$$
$$= \frac{(4x+9)^{21}}{84} + c$$

Example 2

a). Prove that $\int \dfrac{1}{ax+b} dx = \dfrac{\ln(ax+b)}{a} + c$

b). Find $\int \dfrac{dx}{7-100x}$

Solution

a). We shall use the Basic formula $\int \dfrac{1}{x} dx = \ln x + c$ to prove this result

Let u = a x + b so that $\dfrac{du}{a}$ = dx

Then $\int \dfrac{dx}{ax+b} = \int \dfrac{du}{au} = \dfrac{1}{a}\ln u + c$

$$= \frac{\ln(ax+b)}{a} + c$$

b). Using he extension formula

$$\int \frac{dx}{7-100x} = \frac{\ln(7-100x)}{-100} + c$$

unless you are asked to prove $\int \frac{dx}{7-100x} = \frac{\ln(7-100x)}{-100} + c$ you can just write down the result without proof.

Example 3

Prove that $\int e^{ax+b}dx = \frac{e^{ax+b}}{a} + c$ where a and b are constants.

Proof

The integrand contains a linear function of x namely (a x + b)

Let u = a x + b (1) so that $\frac{du}{dx} = a$ (2)

and dx = $\frac{du}{a}$ (3)

Substitution of (1) and (3) in the given integral

$$\int e^{ax+b}dx = \int e^{u}\frac{du}{a}$$
$$= \frac{1}{a}\int e^{u}du$$
$$= \frac{1}{a}e^{u} + c$$
$$= \frac{e^{ax+b}}{a} + c$$

In the same way by taking the linear function of x as u or u = a x + b we can establish all the ten extension formulae.

Example 4

Find $\int (2x+3)^5 dx$

Solution

We compare this integration with $\int x^5 dx$

$$\int x^5 dx = \frac{x^6}{6} + c$$

By the extension technique

$$\int (2x+3)^5\,dx = \frac{(2x+3)^6}{6\times 2} + c = \frac{(2x+3)^6}{12} + c$$

Example 5

Find $\int \frac{1}{(5x-7)}dx$ (1)

Solution

We compare (1) with $\int \frac{1}{x}dx = \ln x + c$

By the extension technique to (a x + b),

$$\int \frac{1}{(5x-7)}dx \quad = \quad \frac{\ln(5x-7)}{5} + c$$

Example 6

Find $\int e^{3-4x}dx$ (1)

Solution

We compare (1) with the Basic formula

$$\int e^x dx = e^x + c$$

Then $\int e^{3-4x}dx \qquad = \frac{e^{3-4x}}{-4} + c$

Example 7

Find $\int \sin(2-3x)dx$ (1)

Solution

Comparing (1) with the Basic formula

$$\int \sin x dx = -\cos x + c$$

where $\int \sin(2-3x)dx \; = \frac{\cos(2-3x)}{3} + c$

Example 8

Find $\int \cos ec(5-4x)\cot(5-4x)dx$ (1)

Solution

Comparing (1) with the corresponding basic formula

$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$

We have $\int \operatorname{cosec}(5-4x)\cot(5-4x)dx = \dfrac{-\operatorname{cosec}(5-4x)}{-4} + c$

Note that the same (5 – 4x) should be present in both the places in (1), or else we cannot use the formula.

Example 9

Find $\int (3x+1)^{15} dx$ (1)

Solution

Using $\int (x)^{15} dx = \dfrac{x^{16}}{16} + c$ we have

$$\int (3x+1)^{15} dx = \frac{(3x+1)^{16}}{16 \times 3} + c$$
$$= \frac{(3x+1)^{16}}{48} + c$$

Note:
You should note that the extension formula is applicable only when the integrand is of first degree in x, such as 2x + 3, 5 – 2x or (a x + b) in general when a and b are constants.

Example 10

Find $\int \dfrac{1}{3x+8} dx$

Solution

Using the Basic formula $\int \dfrac{1}{x} dx = \ln x + c$

We have $\int \dfrac{1}{3x+8} dx = \dfrac{\ln(3x+8)}{3} + c$

Example 11

Find $\int \sec 5x \tan 5x dx$

Solution

Using the Basic formula $\int \sec x \tan x dx = \sec x + c$

We have $\int \sec 5x \tan 5x dx = \dfrac{\sec 5x}{5} + c$

Example 12

Find $\int\left[(1-x)^{10}+e^{2-3x}+\frac{1}{3-x}-\cos ec^2 7x\right]dx$

Solution

Using the corresponding Basic formula, we have

$$\int\left[(1-x)^{10}+e^{2-3x}+\frac{1}{3-x}-\cos ec^2 7x\right]dx=\frac{(1-x)^{11}}{-11}+\frac{e^{2-3x}}{-3}-\ln(3-x)+\cot 7x+c$$

Exercise 2

1. Cos (4x + 3)
2. Sin 3 x
3. Sin $\frac{1}{2}x$
4. e^{-2x}
5. e^{3x-2}
6. $\frac{1}{2x-5}$
7. $(4x-3)^3$
8. cos (5x + 4)
9. Sin (3 – 4x)
10. sec 3x tan 3x
11. cosec 2x cot 2x

Find the integrals of the following

12. $\int(\sin 13x+\cos 4x+e^{2x})dx$

13. $\int\left[2^{x+5}+e^{x-5}-\cos ec^2 5x+(2x+3)^5\right]dx$

14. $\int\left[\frac{1}{ax+b}+(ax+b)^{10}+\cos(ax+b)\right]dx$

15. $\int\left(\sec\frac{1}{2}x\tan\frac{1}{2}x+\cos ec^2\frac{1}{2}x\right)dx$

16. $\int\left[(3-2x)^2+\frac{1}{(3-2x)^2}+\frac{1}{3-2x}+\sin(3-2x)\right]dx$

Summary
You have learnt the following from this chapter

- to integrate expressions using the Ten Extension Formulae

Further Reading

1. Advanced Calculus
   By Watson Fulks
   John Wiley & sons
   New York   Brisbane.   Toronto

CHAPTER 3
Integration Using Substitutions.

3.1 Introduction

In the previous chapters you have learnt to integrate expressions using the Ten Basic formulae and the extension of the formulae. In this chapter you will learn one of the most important techniques of integration of expressions consisting of two functions one of which is the derivative of the other.

3.2 Objectives of the Chapter:

By the end of this chapter you should be able to

- recognize a function u and its derivative u' in the integrand
- integrate the expression using the substitution u and u'.
- adjust the derivative so that the integrand consists of one function u and its derivative.
- derive the formulae for integration of tan x and cos x.

3.3 Integration using substitution.

When the integrand consists of two functions one of which is the derivative of the other function we use the substitution technique of integration. We call the main function as u and its derivative as u' or $\frac{du}{dx}$ so that u' dx will become du. The substitution transforms the integral into one of the basic formula of integration in the new variable u only. Hence we integrate easily using the basic formula. The technique is illustrated in the following examples.

Type 1

$$\int \frac{u'}{u} dx = \ln u + c$$

Example 1

Consider $\int \frac{3x^2 + \cos x}{x^3 + \sin x} dx$

$3x^2 + \cos x$ is the derivative of $x^3 + \sin x$.
Let us use the substitution

$u = x^3 + \sin x$ (1)

Then $\frac{du}{dx} = 3x^2 + \cos x$

Then $du = (3x^2 + \cos x)\, dx$. (2)

Using (1) and (2) the integral is transformed into

$\int \frac{du}{u}$ which comes under one of the basic formulae

Hence $\int \frac{3x^2 + \cos x}{x^3 + \sin x} dx = \frac{du}{u}$

$= \ln u + c$

$= \ln (x^3 + \sin x) + c$

Type 2

$$\int \frac{u'}{u^n} dx = \int \frac{du}{u^n} = \int u^{-n} du = \frac{u^{-n+1}}{-n+1} + c$$

Example 2

Consider $\int \frac{3x^2 + \cos x}{x^3 + \sin x} dx$

Again there are two functions

$x^3 + \sin x$ and its derivative $3x^2 + \cos x$ (leaving the index ten)

Let $u = x^3 + \sin x$ (1)

Then $\frac{du}{dx} = 3x^2 + \cos x$

or $du = (3x^2 + \cos x)\, dx$ (2)

Substitution of (1) and (2) in the given integral

$$\int \frac{\left(3x^2+\cos x\right)}{(x^3+\sin x)^{10}}dx = \int \frac{du}{u^{10}}$$

$$= \int u^{-10}du$$ (using the Basic formula we have)

$$= \frac{u^{-9}}{-9} + c$$

$$= \frac{1}{-9u^9} + c$$

$$= \frac{1}{-9(x^3+\sin x)^9} + c$$

Type 3

$$\int e^u\, u'\, dx = \int e^u du = e^u + c$$

Consider $\int e^{x^3 + \sin x}\,(3x^2 + \cos x)\,dx \int$

The integrand contains the function
$u = x^3 + \sin x$ and its derivative $(3x^2 + \cos x)$

Let $u = x^3 + \sin x$ (1)

$\frac{du}{dx} = 3x^2 + \cos x$

Then $du = (3x^2 + \cos x)\,dx$ (2)

Substitution of (1) and (2) in the given integral

$$\int e^{x^3 + \sin x}\,(3x^2 + \cos x)\,dx$$

$$= \int e^u du$$

using the basic formula

$$= e^u + c$$

$$= e^{x^3 + \sin x} + c$$

Type 4 $\int \mathbf{u^n\, u'\, dx} = \int u^n\, du = \dfrac{u^{n+1}}{n+1} + c$

Consider $\int (x^3 + 3x^2 + 6x + 2)^{15} (3x^2 + 6x + 6)\, dx$

The integrand contains the function

$u = x^3 + 3x^2 + 6x + 2$ and its derivative $(3x^2 + 6x + 6)$

Let $u = x^3 + 3x^2 + 6x + 2$

and $du = (3x^2 + 6x + 6)\, dx$ (2)

Substitution of (1) and (2) in the given integral

$$\int (x^3 + 3x^2 + 6x + 2)^{15} (3x^2 + 6x + 6)dx = \int u^{15}\, du$$

$$= \frac{u^{16}}{16} + c$$

$$= \frac{(x^3 + 3x^2 + 6x + 2)^{16}}{16} + c$$

Type 5

$\int$ **sin u u^1 dx form =** $\int$ **sin u du = - cos u + c**

Consider $\int \sin(x^3 + x^2 + 5)(3x^2 + 2x)\, dx.$.

The integrand contains a function

$u = x^3 + x^2 + 5$ (within sine) and its derivative

$u^1 = 3x^2 + 2x$

Let $u = x^3 + x^2 + 5$ (1)

Then $\dfrac{du}{dx} = 3x^2 + 2x$

or $du = (3x^2 + 2x)\, dx$ (2)

Substitution of (1) and (2) in the given integral

$$\int \sin(x^3 + x^2 + 5)(3x^2 + 2x)\,dx.$$

$= \int \sin u\, du$ which is a standard formula

$= -\cos u + c$

$= -\cos(x^3+x^2+5) + c$

Type 6

$\int \operatorname{cosec} u \cot u.\ u^1\,dx = \int \operatorname{cosec} u \cot u\,du = -\operatorname{cosec} u + c$

Consider $\int \operatorname{cosec} x^2 \cot x^2\,(2x)\,dx$

Hence $u = x^2$ and $\frac{du}{dx} = 2x$ or $du = 2x dx$

Substitution of u in the given integral gives

$\int \operatorname{cosec} u \cot u\,du = -\operatorname{cosec} u + c$ using formula

$= -\operatorname{cosec} x^2 + c$

Example 3

Find $\int \sec^2 x \tan x\,dx$

The integrand contains $u = \tan x$ and its derivative $\sec^2 x$

Let $u = \tan x$ (1)

Then $\frac{du}{dx} = \sec^2 x$

or $du = \sec^2 x\,dx$ (2)

Substitution of (1) and (2) in the given integral

$$\int \tan x \sec^2 x\,dx = \int u\,du$$

$$= \frac{u^2}{2} + c$$

$= \tan^2 x + c$

Example 4

Find $\int e^{\sin x} \cos x\,dx$

Let u = sin x, du = cos xdx

Hence $\int e^{\sin x} \cos xdx \quad = \int e^u\, du$

$= e^u + c$

$= e^{\sin x} + c$

Example 5

Find $\int (\tan x)^{10} \sec^2 xdx$

Let u = tan x, then du = $\sec^2$ xdx

Hence $\int (\tan x)^{10} \sec^2 xdx \quad = \int u^{10}\, du$

$$= \frac{u^{11}}{11} + c$$

$$= \frac{\tan^{11} x}{11} + c$$

4. Adjustment of the derivative

Sometimes the integrand may not have the exact derivative of the function u present. In such cases we shall adjust the derivative so that the function u and its exact derivative u^1 are present in the integrand. The following examples shall illustrate this principle.

Example 6

Find $\int \frac{x^2 + x + 1}{2x^3 + 3x^3 + 6x + 5} dx$

Solution

The integrand has one function

$u = 2x^3 + 3x^2 + 6x + 5$

Then $u^1 = 6x^2 + 6x + 6 = 6(x^2 + x + 10)$

The other function present in the integrand is $(x^2 + x + 1)$. We must multiply this second function by six and divide by six to get u^1.

Thus $\int \frac{x^2 + x + 1}{2x^3 + 3x^2 + 6x + 5} dx = \frac{1}{6} \int \frac{6x^2 + 6x + 1}{2x^3 + 3x^2 + 6x + 5} dx$

Now the integrand has u and its derivative u^1.

Let $u = 2x^3 + 3x^2 + 6x + 5$

$$\frac{du}{dx} = 6x^2 + 6x + 6 = 6(x^2 + x+1)$$

$$du = (6x^2+6x+6)\ dx$$

Then the given integral is transformed into

$$\frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\ln u + c$$

$$= \frac{1}{6}\ \ln\ (2x^3 + 3x^2 + 6x+5) + c$$

Example 7

Find $\int x^3 \sqrt{x^4 - 1}\ dx$

Solution

Let $u = x^4-1$

$$\frac{du}{dx} = 4x^3$$

$$du = 4x^3dx$$

Then $\int x^3 \sqrt{x^4 - 1}\ dx = \frac{1}{4}\int 4x^3 \sqrt{x^4 - 1}\ dx$

$$= \frac{1}{4}\int u^{\frac{1}{2}} du$$

$$= \frac{u^{\frac{3}{2}}}{6} + c$$

$$= \frac{1}{6}(x^4 - 1)^{3/2} + c$$

Example 8

Find $\int \sec^5 x \tan x\ dx$

Solution

If u = sec x, $\frac{du}{dx} = \sec \tan x$

Now $\int \sec^5 x \tan x \, dx = \int \sec^4 x \sec x \tan x dx$.

Let u = sec x du = sec x tan xdx

Then $\int \sec^5 \tan x dx = \int u^4 du$

$$= \frac{u^5}{5} + c$$

$$= \frac{\sec^5 x}{5} + c$$

Example 9

Find $\int \sin^3 8x \cos 8x dx$

Let u = sin 8x, $\frac{du}{dx}$ = 8 cos 8x

du = 8 cos 8xdx

Then $\int \sin^3 8x \cos 8x dx = \frac{1}{8}\int \sin^3 8x(8\cos 8x)dx$

$$= \frac{1}{8}\int u^3 du$$

$$= \frac{u^4}{32} + c$$

$$= \frac{\sin^4 8x}{32} + c$$

3.5 Integration of tan x and cot x (formulae)

a) $\int \tan x dx = \frac{\sin x}{\cos} dx$

Let u = cosx, $\frac{du}{dx}$ = -sinx,

du = -sinx dx,

Then $\int \tan xdx = -\int \frac{\text{-}\sin x}{\cos} dx$

$$= -\int \frac{-du}{u}$$

$= \text{-} \ln u + c$

$= \ln u^{-1} + c$

$$= \ln \frac{1}{u} + c$$

$$= \ln \frac{1}{\cos x} + c$$

$= \ln (\sec x) + c$

we write $\int \tan xdx = \ln |\sec x| + c$

showing that ln of negative values does not exist

b) Find $\int \cot xdx$

$$\int \cot xdx = \int \frac{\cos x}{\sin x} dx$$

Let u = sin x so that du = cos dx

Then $\int \cot xdx = \int \frac{du}{u} = \ln u + c$

$$\int \cot xdx = \ln (\sin x) + c$$

Exercise 3

Find the integrals of the following:

1. $\int \frac{3x^2 + 2}{x^3 + 2x + 1} dx$

2. $\int \frac{\cos x + e^x}{e^x + \sin x} dx$

3. $\int e^{x^2} x dx$

4. $\int x (x^2 - 3)^5 \, dx$

5. $\int \frac{x+1}{x^2 + 2x - 5} dx$

6. $\int 2x \sqrt{3x^2 - 5} \, dx$

7. $\int \frac{x-1}{(2x^2 - 4x + 1)5} dx$

8. $\int \frac{e^x}{e^x - 1} dx$

9. $\int \frac{(\ln x)^2}{x} dx$

10. $\int \tan^5 x \sec^2 x dx$

11. $\int \sec^2 x \tan^2 x dx$

12. $\int \sec^4 x \tan x dx$

Summary

You have learnt the following from this chapter:

1. When the integrand contains a function and its derivative you can integrate using the substitution:
2. The following are the formulae learnt:

a) $\int \frac{u'}{u} dx = \ln u + c$

b) $\int \frac{u'}{u^n} dx = \frac{u^{-n+1}}{1-n} + c$

c) $\int u^n u' dx = \frac{u^{n+1}}{1+n} + c$

d) $\int e^u \, u' \, dx = e^u + c$

e) $\int (\sin u) u' \, dx = -\cos u + c$

and similar trigonometric results.

f) $\tan x \, dx = \ln |\sec x| + c$

g) $\cot x \, dx = \ln |\sin x| + c$

Further Reading

1. Advanced Calculus
By Watson Fulks
John Wiley & sons
New York Brisbane. Toronto

CHAPTER 4
Integration by parts

In chapter three you have learnt to integrate an expression containing two functions one of which is the derivative of the other. If the two functions in the integrand are not related we use an important technique called integration by parts. We shall establish a formula using the product rule of differentiation and apply the same to integrate product of two functions.

4.2 Objectives of the Chapter:
By the end of this chapter you should be able to

- derive the formula for integration by parts
- apply the formula to integrate product of two functions.

4.3 Integration by parts – Formula
We shall establish the formula

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \qquad (1)$$

Formula (1) is also written as

$$\int udv = uv - \int vdu \qquad (2)$$

In formula (1) the integrand contains the product of two functions namely

u and $\dfrac{dv}{dx}$

The result (1) will be helpful only when

$\int v\dfrac{du}{dx}$ is easier than $\int u\dfrac{dv}{dx}$

Formula for integration by parts

We know that $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ (3)

Then $u\dfrac{dv}{dx} = \dfrac{d(uv)}{dx} - v\dfrac{du}{dx}$ (4)

Integrate both sides of (4) with respect to x

$$\boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx} \qquad (5)$$

or $\int udv = uv - \int vdu$

Note: The functions u and $\frac{dv}{dx}$ can be Logarithmic, Inverse trigonometric, Algebraic, Trigonometric or Exponential. We must arrange the two functions in proper order in the integrand before we proceed to apply the formula. You remember the order using the key word.: LIATE

where L, I, A, T and E represents the functions Logarithmic, Inverse trigonometric, Algebraic, Trigonometric or Exponential respectively.
The following examples illustrate the process of integration by parts.

Example 1
Find $\int e^x \, x \, dx$

Solution
The integrand consists of product of two functions e^x (Exponential) and x (Algebraic).
Remember the order LIATE
Algebraic function should come first and then only Exponential function must be written
If we do not follow this principle

$\int v \frac{du}{dx}$ will be more complicated than $\int u \frac{dv}{dx}$ (Remember LIATE !)

The required integration is

$\int x \, e^x \, dx$ (in proper order)

We let $u = x$

$\frac{du}{dx} = 1$ $\qquad$ $\frac{dv}{dx} = e^x$

$\frac{d^2u}{dx^2} = 0$ $\qquad$ $v = \int e^x dx$

$v = e^x$

(constant we can put at the end)

Substitution of the results in the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

We have $\int x\, e^x\, dx = xe^x - \int e^x\, 1\, dx$

$= xe^x - e^x + c$

Example 2

Find $\int x \cos dx$

Solution

(The order of LIATE is correct. Algebraic first and then comes Trigonometric; otherwise the formula will not be useful. Check yourself).

Let $u = x \quad \frac{dv}{dx} = \cos x, \quad dv = \cos x dx$

$\frac{du}{dx} = 1 \qquad v = \int \cos x dx$

$v = \sin x$

Using the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

We have $\int x \cos x dx \quad = x \sin x - \int \sin x + dx$

$= x \sin x + \cos x + c$

Example 3

Find $\int x^3 \ln x\, dx$

Solution

$\int x^3 \ln x dx = \int \ln x.\ x^3 dx$ writing in the order LIATE

Let $u = \ln x \qquad \underline{dv} = x^3$

$\frac{du}{dx} = \frac{1}{x} \qquad v = \int x^3 dx = \frac{x^4}{4}$

Substituting in the formula

$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ we have

$$\int \ln x(x^3)dx = \ln x.\frac{x^4}{4} - \int \frac{x^4}{4}\frac{1}{x}dx$$

$$\int x^3 \ln x dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4}dx$$

$$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + c$$

Example 4
Find $\int x^2 e^{2x} dx$

Solution
The order of $x^2 e^{2x}$ satisfies "LIATE"

Let $u = x^2$ $\qquad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 2x$ $\qquad v = \int e^{2x} dx$

$= \frac{e^{2x}}{2}$ (constant can be put at the end)

Using the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

we have $\int x^2 e^{2x} dx = \frac{x^2 e^{2x}}{2} - \int \frac{e^{2x}}{2} 2x dx$

$$= \frac{1}{2} x^2 e^{2x} - \int x e^{2x} dx \qquad (1)$$

Now we must integrate $\int x e^{2x} dx$

Again let $u = x \quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 1 \qquad v = \int e^{2x} dx = \frac{1}{2} e^{2x}$

Using the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

We have $\int xe^{2x} dx = \frac{1}{2} xe^{2x} - \int \frac{1}{2} e^{2x} \, 1 \, dx$

$$= \frac{1}{2} xe^{2x} - \frac{1}{4} e^{2x} \qquad (2)$$

Using the result (2) in (1)

$$\int x^2 e^{2x} \, dx = \frac{1}{2} x^2 e^{2x} - \left(\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x}\right)$$

$$= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + c$$

$$= \frac{1}{4} e^{2x} (2x^2 - 2x + 1) + c$$

Example 5

Find $\int x^2 \cos x \, dx$

Solution

(In this problem Algebraic Function x^2 comes before the Trigonometric function cos x. This satisfies the condition of LIATE).

Let $u = x^2$ $\qquad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 2x$ $\qquad v = \int \cos x dx = \sin x$

Using the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx \quad \text{we have}$$

$$\int x^2 \cos x dx = x^2 \sin x - \int \sin x \, 2x \, dx$$

$$= x^2 \sin x - 2 \int x \sin x dx \qquad (1)$$

Now we must integrate $\int x \sin x dx$

Using again integration by parts.

Let u = x $\quad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 1 \quad v = \int \sin x dx = -\cos x$

Applying the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx \text{ we have}$$

$$\int x \sin x dx = -x \cos x + \int \cos x \; 1 \; dx$$
$$= -x \cos x + \sin x \qquad (2)$$

Using the result (2) in (1) we have

$$\int x^2 \cos x \, dx = x^2 \sin x - 2(-x \cos x + \sin x) + c$$
$$= x^2 \sin x + 2x \cos x - 2 \sin x + c$$

Example 6

Find $\int x^3 \ln x \, dx$

Solution

To satisfy the condition for LIATE we write the integral as

$$\int \ln x \, (x^3) \, dx$$

Let u = ln x $\quad \frac{dv}{dv} = x^3$

$\frac{du}{dx} = \frac{1}{x} \quad v = \int x^3 \, dx = \frac{x^4}{4}$

Applying the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx \text{ we have}$$

$$\int (\ln x) \, x^3 \, dx = (\ln x)\frac{x^4}{4} - \int \frac{x^4}{4} \frac{1}{x} dx$$

$$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + c$$

4.4 Formula for integration of ln x, x>0

$$\int \ln x \, dx = x \ln x - x + c$$

Proof

Consider $\int \ln x \, dx = \int \ln x \; 1 \, dx$

Let $u = \ln x$ $\qquad$ $\frac{dv}{dx} = 1$

Then $\frac{du}{dx} = \frac{1}{x}$ $\qquad$ $v = \int 1 dx = x$

Using the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + c$$

Example 7

Find $\int (\ln x)^2 \, dx$

Solution

$$\int (\ln x)^2 \, dx = \int (\ln x)^2 .1 dx$$

Let $u = (\ln x)^2$ $\qquad$ $\frac{dv}{dx} = 1$

$\frac{du}{dx} = 2 \ln x \frac{1}{x}$ $\qquad$ $v = \int 1 \, dx = x$

Substitution of the results in the formula

$$\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$$

$$\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2x(\ln x)\frac{1}{x}\,dx$$

$$= x(\ln x)^2 - 2\int (\ln x)\,dx$$

$$= x(\ln x)^2 - 2(x \ln x - x) + c$$

$$= x\ln x\,(\ln x - 2) + 2x + c$$

Exercise 4

Find

1	$\int xe^{2x}\,dx$	7	$\int x^2 \ln x\,dx$
2	$\int x \sin x\,dx$	8	$\int \frac{\ln x}{x^2}\,dx$
3	$\int x \ln x\,dx$	9	$\int \ln 2x\,dx$
4	$\int \ln (x-1)\,dx$	10	$\int \tan^{-1} x\,dx$
5	$\int x \cos 3x\,dx$	11	$\int x^2 \sin x\,dx$
6	$\int e^x \sin x\,dx$	12	$\int x^{-3} \ln x\,dx$

Summary

You have learnt the following from this chapter:

1. Formula for integration by parts

$$\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$$

2. Apply the formula to integrate product of two functions.
3. The formula $\int \ln x\,dx = x \ln x - x + c$ where $x > 0$.

Further Reading

1. Advanced Calculus
   Watson Fulks
   John Wiley & Sons
   New York. Brisbane. Toronto.

CHAPTER 5

Integration Using Trigonometric Identities.

5.1 Introduction

You have learnt some four important techniques of Integration so far. Now we shall learn in this chapter to integrate expressions using trigonometric identities and substitutions of trigonometric functions. Thus we transform the integrals to one of the basic formulae and then integrate easily.

5.2 Objectives of the chapter.

By the end of this chapter you should be able to: -

- use one of the important techniques of integration namely, integrating trigonometric expressions using trigonometric identities.

5.3 Review of some Important Trigonometric Identities Used in Integration.

i). $\text{Sin}^2 x + \cos^2 x = 1$

ii). $1 + \tan^2 x = \sec^2 x$

iii). $1 + \cot^2 x = \text{cosec}^2 x$

iv). $\sin 2x = 2 \sin x \cos x$

v). $\cos 2x = \cos^2 - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$

vi). $\sin (A + B) = \sin A \cos B + \cos A \sin B$ (1)

vii). $\sin (A - B) = \sin A \cos B - \cos A \sin B$ (2)

Adding (1) and (2) we get the formulae:

$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$ (3)

Subtracting (2) from (1),

$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$ (4)

viii). $\cos (A + B) = \cos A \cos B - \sin A \sin B$ (5)

ix). $\cos (A - B) = \cos A \cos B + \sin A \sin B$ (6)

adding (5) and (6)

$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$ (7)

subtracting (6) from (5) we have

$-2 \sin A \sin B = \cos (A + B) - \cos (A - B)$ (8)

Formulae (3), (4), (7) and (8) are very useful to resolve the product of two trigonometric functions into sum of two functions and integrate the sum of the functions easily. (Anyone of the formulae (3) or (4) is enough).

x). $\sin 3x = 3\sin x - 4\sin^3 x$

Hence $\mathbf{\sin^3 x} =$ $\frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

Similarly $\sin^3 30x = \frac{3}{4}\sin 10x - \frac{\sin 30x}{4}$

xi). $\text{Cos } 3x = -3\cos x + 4\cos^3 x$

$\mathbf{Cos^3 = \frac{3}{4}cosx + \frac{1}{4}cos3x}$

5.4 Integration using the fundamental identities

$\text{Sin}^2 x + \cos^2 x = 1$ (1)

$1 + \tan^2 x = \sec^2 x$ (2)

$1 + \cot^2 x = \text{cosec}^2$ (3)

Example 1

Find $\int \frac{\sin^2 x}{1+\cos x} dx$

Solution

Using the identity (1) $\sin^2 x + \cos^2 x = 1$

$$\int \frac{\sin^2 x}{1+\cos x} dx = \int \frac{(1-\cos^2 x)}{(1+\cos x)} dx$$

$$= \int \frac{(1-\cos x)(1+\cos x)}{(1+\cos x)} dx$$

$$= \int (1-\cos x) dx$$

$$= x - \sin x + c$$

Example 2

Find $\int \tan^2 x dx$

Solution

We use $1 + \tan^2 x = \sec^2 x$

Then $\int \tan^2 xdx = \int (\sec^2 x - 1)dx$

$= \int (\sec^2 xdx - \int 1dx$

$= \tan x - x + c$

Example 3

Find $\int \frac{\cos^2 x}{1 - \sin x} dx$

Solution

We have the identity $\cos^2 x = 1 - \sin^2 x$

Then $\int \frac{\cos^2 x}{1-\sin x}dx \quad = \int \frac{(1-\sin^2 x)}{(1-\sin x)}dx$

$= \int \frac{(1+\sin x)(1-\sin x)}{(1-\sin x)dx}dx$

$= \int (1+\sin x)dx$

$= \int 1dx + \int \sin xdx$

$= x - \cos x + c$

Example 4

Find $\int \cot^2 xdx$

Solution

Using the identity $1 + \cot^2 x = \operatorname{cosec}^2 x$, we have

$\int \cot^2 xdx = \int (\cos ec^2 x - 1)dx$

$= \int \cos ec^2 xdx - \int dx$

$= - \cot x - x + c$

5.5 Integration using the identity

$\cos 2x = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$= 1 - 2\sin^2 x.$

Example 5

Find = $\int \sin^2 x dx$

Solution

Using the identity cos 2x = 1 – 2 $\sin^2$ x, 2 $\sin^2$ x = 1 – cos 2x or $\sin^2$ x = $\frac{1}{2}-\frac{1}{2}\cos 2x$

$$\text{Hence} \int \sin^2 x dx = \int\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)dx$$
$$= \frac{1}{2}x-\frac{1}{2}\frac{\sin 2x}{2}+c$$
$$= \frac{1}{2}x-\frac{1}{4}\sin 2x+c$$

Example 6

Find = $\int \cos^2 10x dx$

Solution

Using the identity cos 2x = 2 $\cos^2$ x – 1

Cos 20 x = 2 $\cos^2$ 10x – 1, then $\cos^2$ 10x = $\frac{1}{2}+\frac{1}{2}\cos 20x$

$$\text{Then } \int \cos^2 10x dx = \int\left(\frac{1}{2}+\frac{1}{2}\cos 20x\right)dx$$

$$= \frac{1}{2}x+\frac{1}{40}\sin 20x+c$$

Example 7

Find = $\int \frac{\cos 6x}{(\cos 3x-\sin 3x)}dx$

Solution

Using the identity cos 2x = $\cos^2$ x – $\sin^2$ x

Cos 6 x = $\cos^2$ 3x – $\sin^2$ 3x

$$\text{Hence } \int \frac{\cos 6x}{(\cos 3x-\sin 3x)}dx = \int \frac{(\cos^2 3x-\sin^2 3x)}{(\cos 3x-\sin 3x)}dx$$

$$= \int \frac{(\cos 3x + \sin 3x)(\cos 3x - \sin 3x)dx}{(\cos 3x - \sin 3x)}$$

$$= \int (\cos 3x + \sin 3x)dx$$

$$= \frac{\sin 3x}{3} - \frac{\cos 3x}{3} + c$$

5.6 Integration using the identities,
2 sin A cos B = sin (A + B) + sin (A – B)
2 cos A cos B = cos (A + B) + cos (A – B)
-2 sin A sin B = cos (A+B) – cos (A – B)

These identities convert the product into sums. The following examples illustrate the use of these identities.

Example 8

Find $\int \sin 10x \cos 8x \, dx$

Solution
Using the identity 2 sin A cos B = sin (A + B) + sin (A – B)

We have $\int \sin 10x \cos 8x dx = \frac{1}{2}\int 2\sin 10x \cos 8x dx$

$$= \frac{1}{2}\int (\sin 18x + \sin 2x)dx$$

$$= \frac{1}{2}\left(-\frac{1}{18}\cos 18x - \frac{1}{2}\cos 2x\right) + c$$

$$= \frac{-1}{36}\cos 18x - \frac{1}{4}\cos 2x + c$$

Example 9

Find $= \int \cos 12x \sin 3x dx$

Solution

$$\int \cos 12x \sin 3x dx = \int \sin 3x \cos 12x dx$$

$$= \frac{1}{2}\int 2\sin 3x \cos 12x dx$$

$$= \frac{1}{2}\int [\sin 15x + \sin(-9x)]dx$$

$$= \frac{1}{2}\int (\sin 15x - \sin 9x)dx$$

$$= \frac{1}{2}\int (-\frac{1}{15}\cos 15x + \frac{1}{9}\cos 9x) + c$$

$$= -\frac{1}{30}\cos 15x + \frac{1}{18}\cos 9x + c$$

Example 10

Find = $\int \cos 5x \cos 7x dx$

Solution

$$\int \cos 5x \cos 7x dx = \frac{1}{2}\int 2\cos 5x \cos 7x dx$$

$$= \frac{1}{2}\int [\cos 12x + \cos(-2x)]dx$$

$$= \frac{1}{2}\int (\cos 12x + \cos 2x)dx$$

$$= \frac{1}{2}\left(\frac{1}{12}\sin 12x + \frac{1}{2}\sin 2x\right) + c$$

$$= \frac{1}{24}\sin 12x + \frac{1}{4}\sin 2x + c$$

Example 11

Find = $\int \sin 8x \sin 15x dx$

Solution

$$\int \sin 8x \sin 15x dx = -\frac{1}{2}\int -2\sin 8x \sin 15x dx$$

$$= -\frac{1}{2}\int [\cos 23x + \cos(-7x)]dx$$

$$= -\frac{1}{2}\int (\cos 23x - \cos 7x)dx$$

$$= -\frac{1}{46}\sin 23x + \frac{1}{14}\sin 7x + c$$

5.7 Integration using the identity

Sin 3x = 3 sin x – 4 $\sin^3$ x
Cos 3x = -3 cos x + 4 $\cos^3$ x

Example 12

Find = $\int \sin^3 6x dx$

Solution

Using the identity, sin 3x = 3 sin x – 4 $\sin^3$ x,
sin 18 x = 3 sin 6x – 4 $\sin^3$ 6 x

$$\sin^3 6x = \frac{3}{4}\sin 6x - \frac{1}{4}\sin 18x$$

Hence $\int \sin^3 6x dx = \frac{3}{4}\int \sin 6x dx - \frac{1}{4}\int \sin 18x dx$

$$= \frac{3}{4}\left(\frac{-\cos 6x}{6}\right) - \frac{1}{4}\left(\frac{-\cos 18x}{18}\right)$$

$$= \frac{\cos 18x}{72} - \frac{3\cos 6x}{24} + c$$

Example 13

Find = $\int \cos^3 9x dx$

Solution

Using the identity cos 3 x = - 3 cos x + 4 $\cos^3$ x
Cos 27 x = 3 cos 9 x +4 $\cos^3$ 9 x

Hence $\cos^3 9x = \frac{3}{4}\cos 9x + \frac{1}{4}\cos 27x$

$$\int \cos^3 9x dx = \frac{3}{4}\int \cos 9x dx + \frac{1}{4}\int \cos 27x dx$$

$$= \frac{3}{4}\left(\frac{\sin 9x}{9}\right) + \frac{1}{4}\left(\frac{\sin 27x}{27}\right) + c$$

$$= \frac{1}{12}\sin 9x + \frac{1}{108}\sin 27x + c$$

5.8 Formulae for integration when the integrand contains quadratic expressions in the denominator.

1. $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

2. $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

3. $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + c$$

4. $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} \quad \text{if } x > a$$

$$= \cosh^{-1}\left(\frac{-x}{a}\right) \quad \text{if } -x > a$$

5. $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \sin^{-1}\frac{x}{a}$$

Example 14

Prove that $\displaystyle\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

Proof

Let x = a tan θ so that θ = $\tan^{-1}\frac{x}{a}$

Then $x^2 + a^2 = a^2 \tan^2 \theta + a^2 = a^2 (1 + \tan^2 \theta) = a^2 \sec^2 \theta$

$dx = d(a \tan \theta) = a \sec^2 \theta \, d\theta$

Hence $\displaystyle\int \frac{dx}{x^2 + a^2} = \int \frac{a \sec^2 \theta d\theta}{a^2 \sec^2 \theta}$

$$= \frac{1}{a} \int d\theta$$

$$= \frac{1}{a} \theta$$

$$= \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

Example 13

Prove that $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \sin^{-1} \frac{x}{a} + c$

Proof

Let $x = a \sin \theta$ so that $dx = a \cos \theta \, d\theta$

and $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = \sqrt{a^2 \cos^2 \theta} = a \cos \theta$

$$= a \cos \theta$$

Hence $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \cos \theta d\theta}{a \cos \theta}$

$$= \int d\theta$$

$$= \theta + c$$

$$= \sin^{-1} \frac{x}{a} + c \quad \text{since } x = a \sin \theta$$

Example 14

Find $= \displaystyle\int \frac{dx}{25x^2 + 4}$

Solution

$$\int \frac{dx}{25x^2+4} = \int \frac{dx}{25\left(x^2+\frac{4}{25}\right)}$$

$$= \frac{1}{25}\int \frac{dx}{x^2+\left(\frac{2}{5}\right)^2}$$

$$= \frac{1}{25}\tan^{-1}\frac{2x}{5}+c$$

Example 15

Find $= \int \frac{dx}{x^2-3x+2}$

Solution

$x^2 - 3x + 2 \quad = x^2 - 3x + a^2 - a^2 - 2$

$\quad = (x - a)^2 - a^2 + 2$

$\quad = x^2 - 2ax + a^2 - a^2 + 2$

Equating coefficient of x on both sides 2a = 3 or a = $\frac{3}{2}$

Hence $x^2 - 3x + 2 = \left(x-\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2$

$$= \left(x-\frac{3}{2}\right)^2 - \frac{1}{4}$$

Then $\int \frac{dx}{x^2-3x+2} = \int \frac{dx}{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}$

$$= \frac{1}{2\left(\frac{1}{2}\right)} \ln\left|\frac{\left(x - \frac{3}{2}\right) - \frac{1}{2}}{\left(x - \frac{3}{2}\right) + \frac{1}{2}}\right| + c$$

$$\text{using}\left(\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c\right)$$

Hence $\displaystyle\int \frac{dx}{x^2 - 3x + 2} = \ln\left|\frac{x-2}{x-1}\right| + c$

Note: since $x^2 - 3x + 2 = (x - 2)(x - 1)$, we can resolve $\dfrac{dx}{x^2 - 3x + 2}$ into partial fraction and then integrate to get the same answer. Try this method yourself.

Example 16

Find = $\displaystyle\int \frac{dx}{\sqrt{2x^2 - 5x + 2}}$

Solution

Let $2x^2 + 5x + 1 = 2(x^2 + \frac{5}{2} + 1)$

$= 2[(x + a)^2 - a^2 + 1]$

$= 2[x^2 + 2ax + a^2 - a^2 + 1]$

Comparing the coefficient of x on both sides 4a = 5 or a = $\frac{5}{4}$.

Hence $2x^2 + 5x + 1 = 2\left[\left(x + \frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2 + 1\right]$

$$= 2\left[\left(x + \frac{5}{4}\right)^2 - \frac{9}{16}\right]$$

$$= 2\left[\left(x+\frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right]$$

$$\int \frac{dx}{\sqrt{2x^2-5x+2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(x+\frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{x+\frac{5}{4}}{\frac{3}{4}}\right) \text{ if } x > \frac{3}{4}$$

$$= \frac{1}{\sqrt{2}}\cosh^{-1}\frac{4x+5}{3} + c \text{ if } x > \frac{3}{4}$$

$$= \frac{1}{\sqrt{2}}\cosh^{-1}\frac{-4x-5}{3} + c \text{ if } -x > \frac{3}{4} \text{ or } x < -\frac{3}{4}.$$

Exercise 5
Find the integrals of the following

1. $\int \frac{\sin^2 x}{1-\cos x} dx$

2. $\int \cot^2 4x dx$

3. $\int \frac{\cos^2 x}{1+\sin x} dx$

4. $\int \sin^2 5x dx$

5. $\int \cos^2 5x dx$

6. $\int \frac{\cos 8x}{(\cos 4x - \sin 4x} dx$

7. $\int \frac{\cos 2ax}{(\cos ax - \sin ax} dx$

8. $\int$ cos 10x sin 4x dx

9. $\int$ cos 4x sin 9x dx

10. $\int \frac{dx}{5x^2+9x-8}$

11. $\int$ sin 5x cos 7x dx

12. $\int$ cos 8x cos 5x dx

13. $\int$ cos 3x cos 11x dx

14. $\int$ sin 4x sin 10x dx

15. $\int$ sin 12x sin 5x dx

16. $\int \sin^3$ 4x sin dx

17. $\int \sin^3$ 5x dx

18. $\int \cos^3$ 5x dx

19. $\int \cos^3$ 8x dx

20. $\int \frac{dx}{x^2+16}$

21. $\int \frac{dx}{x^2-16}$

22. $\int \frac{dx}{\sqrt{x^2}+25}$

23. $\int \frac{dx}{\sqrt{x^2}-25}$

24. $\int \frac{dx}{4-x^2}$

25. $\int \frac{dx}{\sqrt{16-x^2}}$

26. $\int \frac{dx}{3x^2+5x+1}$

27. $\int \frac{dx}{4x^2+12x+9}$

28. $\int \frac{dx}{\sqrt{9x^2-6x+1}}$

29. $\int \frac{dx}{\sqrt{9x^2+6x+1}}$

30. $\int \frac{dx}{\sqrt{25-x^2}}$

Summary

You have learnt the following from this chapter:

i). The technique of integration of trigonometric expressions using the following trigonometric identities.

1. $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

2. $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

3. $\tan^2 x = \sec^2 x - 1$

4. $\cot^2 x = \text{cosec}^2 x - 1$

5. 2 sin A cos B = sin (A + B) + sin (A – B)

6. 2 cos A cos B = cos (A + B) + cos (A – B)

7. -2 sin A sin B = cos (A+B) – cos (A – B)

8. $\sin 3x = 3 \sin x - 4 \sin^3 x$

9. $\cos 3x = -3 \cos x + 4 \cos^3 x$

ii). The technique of integration using the following formulae:

1. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

2. $\int \frac{1}{x^2 - a^2} + \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

3. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a}$

4. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \cosh^{-1}\frac{x}{a}$ if x > a

$= \cosh^{-1}\frac{x}{a}$ if x < -a

5. $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \sin^{-1} \frac{x}{a}$$

Further Reading
Additional pure mathematics
Harwood Clarke
Heinemann Educational Books Ltd.
London.

CHAPTER 6
Integration Using Partial Fractions

6.1 Introduction

Some algebraic expressions cannot be integrated directly. An important technique which we use for such expressions is to split it into partial fractions that can be integrated easily using the basic formulae.

6.2 Objectives of the chapter

By the end of this chapter you should be able to split an algebraic expression into partial fractions and integrate.

6.3 Expressions with only linear factors in the denominator

In this section we shall see $\int \frac{f(x)}{F(x)}dx$ where $\frac{f(x)}{F(x)}$ is a proper fraction (degree of x in f is less than that in F) and F(x) has only non-repeated linear factors. The technique of integration of such integral is illustrated in the following examples:

Example 1

Find $\int \frac{x-11}{(x+3)(x-4)}dx$ (1)

Solution

We write $\frac{x-11}{(x+3)(x-4)} = \frac{A}{x+3} + \frac{B}{x-4}$

$$= \frac{A(x-4)+B(x+3)}{(x+3)(x-4)}$$

Then x -11 = A (x-4) + B (x+3) (2)
We can put special values for x on both sides of (2)
Let x=4, then
4 - 11 = A (0) + B (7)
-7 = 7B, and B=-1

Let x = -3, then (2) becomes
- 3 – 11 = A (- 3 – 4) + B (-3 +3)
-14 = -7A and A=2

Hence $\dfrac{x-11}{(x+3)(x-4)} = \dfrac{2}{x+3} - \dfrac{1}{x-4}$

Then $\displaystyle\int \frac{(x-11)}{(x+3)(x-4)}dx = \int \frac{2}{x+3}dx - \int \frac{1}{x-4}dx$

We split the integrand into two partial fractions each can be integrated using the extension of the basic formulae for (ax + b)

Then $\displaystyle\int \frac{(x-11)}{(x+3)(x-4)}dx$ $= 2 \ln (x + 3) - \ln (x - 4) + A$

$= \ln (x+3)^2 - \ln (x - 4) + A$

$= \ln \dfrac{(x+3)^2}{x-4} + A$

6.4 Integration of expression with repeated linear factors in the denominator.

Consider $\displaystyle\int \frac{1}{(x+2)(x-1)^2}dx$ (1)

The integrand is a proper fraction and with repeated linear factors
It is written as

$$\frac{1}{(x+2)(x-1)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$= \frac{A(x-1)^2 + B(x+2)(x-1) + C(x+2)}{(x+2)(x-1)(x-1)}$$

Then $1 = A (x-1)^2 + B (x +2) (x - 1) + C (x + 2)$ (2)

Let us put x = 1 on both sides of (1) so that the terms containing A and B will become zero.

Then 1 = A (0) + B (0) + C (1+2), Hence C = $\dfrac{1}{3}$

Similarly put x = -2 on both sides of (1) so that the terms containing B and C become zero.

Then $1 = A (-2 - 1)^2 + B (0) + C (0)$, Hence A = $\dfrac{1}{9}$.

To get the value of B either we can give one more value other than 1 and –2 for x or else we can compare the coefficients of x^2 on both sides of (2)

0 = A + B

0 = $\dfrac{1}{9}$ + B then B = - $\dfrac{1}{9}$.

Hence the integral (1) becomes

$$\int \frac{1}{(x+2)(x-1)^2}dx = \int \frac{1}{9(x+2)}dx - \int \frac{1}{9(x-1)}dx + \int \frac{1}{3(x-1)^2}dx$$

$$= \frac{1}{9}\ln(x+2) - \frac{1}{9}\ln(x-1) - \frac{1}{3}(x-1)^{-1} + C$$

$$= \frac{1}{9}\ln\frac{x+2}{x-1} - \frac{1}{3(x-1)} + C$$

6.5 Integration of expressions with quadratic factor in the denominator – proper fraction.

Example 2

Consider $\int \frac{3x+7}{(x-1)(x^2+1)}dx$ (1)

The integrand is a proper fraction and has a quadratic factor. It is written as

$$\frac{3x+7}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\frac{3x+7}{(x-1)(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$$

Then $(3x + 7) = A(x^2 + 1) + (Bx + C)(x - 1)$ (2)

We can put x = 1 on both sides of (2) to get

$3(1) + 7 = A(1 + 1) + (B + C)(0)$

$10 = 2A$ or $A = 5$

Putting $x = 0$, $7 = A - C$, $C = -2$

Equating coefficient of x^2, $0 = A + B$ or $B = -A = -5$

Then $\frac{3x+7}{(x-1)(x^2+1)} = \frac{5}{x-1} + \frac{(-5x-2)}{x^2+1}$

Hence $\int \frac{3x+7}{(x-1)(x^2+1)}dx = 5\int \frac{dx}{x-1} - \int \frac{5x}{x^2+1}dx - 2\int \frac{dx}{x^2+1}$

$$= 5\ln(x-1) - \frac{5}{2}\ln(x^2+1) - 2\tan^{-1}x + C$$

6.6 Integrand is an improper fraction

If the degree of x in the Numerator is the same or greater that that of x in the denominator the Algebraic Fraction is called an improper fraction.

If the integrand $\frac{f(x)}{F(x)}$ is an improper fraction we shall divide f (x) by F (x) and the remainder is resolved into partial fractions. This principle is illustrated from the following example:

Example 3

Find $\int \frac{3x^2 - 2x - 7}{x^2 - x - 2} dx$

Solution

The integrand is an improper algebraic fraction. Hence using long division we find that the quotient is 3 and the remainder is (x – 1)

Then $\frac{3x^2 - 2x - 7}{x^2 - x - 2} = 3 + \frac{(x-1)}{(x-2)(x+1)}$

Note: sometimes instead of long division it is easier to proceed as follows:

$$\frac{3x^2 - 2x - 7}{x^2 - x - 2} = \frac{3(x^2 - x - 2) + x - 1}{(x^2 - x - 2)}$$

$$= 3 + \frac{x-1}{(x^2 - x - 2)}$$

$$= 3 + \frac{(x-1)}{(x-2)(x+1)}$$

Now we can resolve $\frac{x-1}{(x-2)(x+1)}$ into partial fractions.

Let $\frac{x-1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$

$$= \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}$$

(x – 1) = A (x + 1) + B (x – 2) (2)

Putting x = 2 on both sides of (2) A = $\frac{1}{3}$.

Putting x = -1 on both sides of (2), we have – 2 = B (-3) or $\frac{2}{3}$ = B

Then $\dfrac{x-1}{(x-2)(x+1)} = \dfrac{1}{3(x-2)} + \dfrac{2}{3(x+1)}$

Hence the integral (1), is written as

$$\int \frac{3x^2-2x-7}{x^2-x-2}dx = \int \left[3 + \frac{(x-1)}{(x-2)(x+1)}\right]dx$$

$$= \int \left[3 + \frac{1}{3(x-2)} + \frac{2}{3(x+1)}\right]dx$$

$$= 3x + \frac{1}{3}\ln(x-2) + \frac{2}{3}\ln(x+1) + C$$

Exercise 6

Find

1. $\displaystyle\int \frac{x-11}{(x+3)(x-4)}dx$

2. $\displaystyle\int \frac{x}{(25-x^2)}dx$

3. $\displaystyle\int \frac{3x^2-21x+24}{(x-1)(x-2)(x-3)}dx$

4. $\displaystyle\int \frac{5x^2-10x+11}{(x-3)(x^2+4)}$ dx

5. $\displaystyle\int \frac{x-5}{(x-2)^2}dx$

6. $\displaystyle\int \frac{5x+4}{(x-1)(x+2)^2}dx$

7. $\displaystyle\int \frac{3x+7}{x(x+2)(x-1)}dx$

8. $\displaystyle\int \frac{2x^3-x-1}{(x-3)(x^2+1)}$ dx

9. $\int \frac{x^3 + 2x^2 - 10x - 9}{(x^2 - 9)} dx$

10. $\int \frac{(x-2)}{(x^3+1)}$ dx

Hint: $a^3 + b^3 = (a + b)(a^2 - a b + b^2)$

11. $\int \frac{(x+3)}{x^3-8}$ dx

$a^3 - b^3 = (a - b)(a^2 + a b + b^2)$

12. $\int \frac{2x+1}{x^3-1} dx$

Summary

You have learnt the following from this chapter:

i). Splitting a fraction into partial fractions.

Type I: $\frac{2x+1}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$

Type II: $\frac{ax+b}{(x-a)(x-b)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-b)^2}$

Type III: $\frac{ax+b}{(x-a)(px^2+qx+r)} = \frac{A}{x-a} + \frac{Bx+C}{(px^2+qx+r)}$

Type IV: For improper algebraic fraction, we must divide and consider only the remainder for partial fraction.

ii). After partial fractions, integrand is transformed into the form of basic formula which can be integrated.

Further Reading

1. Additional pure mathematics
By L. Harwood Clarke.
Heinemann Educational Books Ltd London.

CHAPTER 7

Definite Integral and the Fundamental Theorem of Calculus

7.1 Introduction

You have learnt the integral of functions using seven techniques. In this chapter we shall learn the value of the integral between two limits for the variable. Such an integral is called a **Definite Integral.** The definite integral is evaluated using the fundamental theorem of the integral calculus namely $\int_a^b f(x)dx = G(b) - G(a)$ where

$\int f(x)dx = G(x)$

7.2 Objectives of the Chapter

By the end of this chapter you should be able to

- define the definite integral
- state the fundamental Theorem of the Integral calculus
- apply this theorem to evaluate the definite Integrals.

7.3. Definite Integral

Suppose that f(x) is integrable in an interval $[a \le x \le b]$ Draw the curve y = f(x)

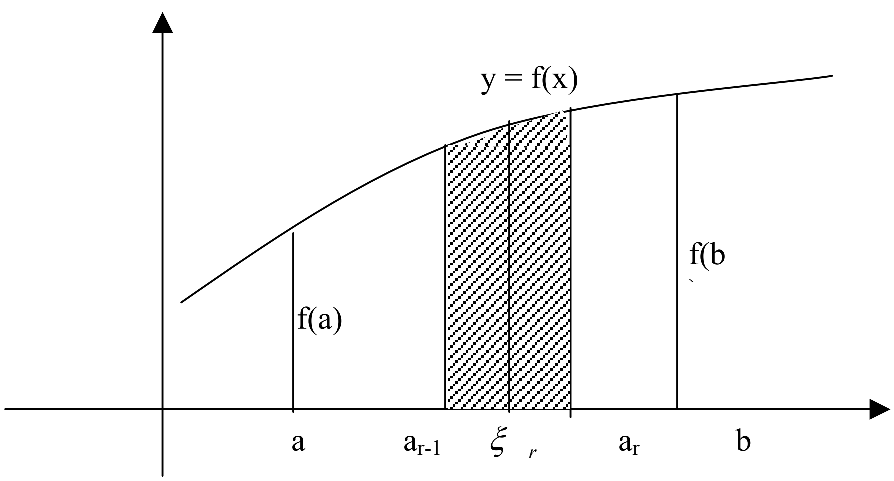


Figure 1

Divide the interval ab into n equal parts at a(=a_0), a_1, a_2 …….., a_n = l

The width of each interval $h = \frac{b-a}{n}$ consider a point ξ_r in the interval $a_{r-1} \le x \le a_r$

The integral $\int_a^b f(x)dx = \lim_{\substack{h\to 0 \\ n\to\infty}} \sum_{r=1}^{n} f(\xi_r)$ h is defined as the definite integral of f(x) in the interval $[a \le x \le b]$

Second definition of the definite Integral

If f(x) is continuous in the interval $[a \le x \le b]$ we definite the area enclosed by the curve y = f(x), the ordinates x = a, x = b, and the x axis as the definite integral of $\int_a^b f(x)dx$

7.4 Some important properties of definite Integral:

If f(x) is continuous in $[a \le x \le b]$

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. If f is integrable in $[a \le x \le b]$ and c is a point between a and b $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
4. If f(x) is integrable in $[a \le x \le b]$ and c is any constant then

$$\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x-c)dx$$

$$= \int_{a-c}^{b-c} f(x+c)dx$$

5. If f(x) is integrable in $[a \le x \le b]$ and m is any constant then

$$\int_a^b f(x)dx = \int_{ma}^{mb} f\left(\frac{x}{m}\right)dx$$

$$\int_a^b f(x)dx = \int_{\frac{a}{m}}^{\frac{b}{m}} f(mx)dx$$

7.5 Definition of an Indefinite Integral [or primitive of f(x)]

Suppose that f(x) is integrable in the interval $[a \le x \le b]$ then $\frac{d}{dx}\int_a^x f(t)dt = f(x)$ at each point x in the interval.

A function F(x) with the property that $F^1(x) = f(x)$ at every point x of the interval $[a \le x \le b]$ is called an indefinite integral or primitive of f(x)

Thus for a continuous function f(x) we have $F(x) = \int_a^x f(t)dt$

Consider $\int_a^b f(x)dx$ if $\int f(x)dx = G(x)$, we say in short, G(x) is the indefinite integral of $\int_a^b f(x)dx$

7.6 The fundamental Theorem of the integral calculus

If f(x) is continuous in $[a \le x \le b]$ and G is a primitive (indefinite integral) of f(x) then $\int_a^b f(x)dx = G(b) - G(a)$

This is called the fundamental theorem of the integral calculus

Proof

Let F be given by

$$F(x) = \int_a^x f(t)dt$$

Then $F(x) = G(x) + c$

Since $F(a) = \int_a^a (t)dt = 0$ we have

$$F(a) = 0 = G(a) + c$$

$$c = -G(a)$$

Hence $F(x) = G(x) + G(a)$

or $F(b) = \int_a^b f(x)dx = G(b) - G(a)$ where G is the primitive of f(x).

This is the Fundamental theorem of the Integral calculus.

$G(b) - G(a)$ is usually written as $[G(x)]_a^b$

7.7 Evaluation of definite integrals

Using the fundamental theorem of integral calculus we can evaluate definite integrals. The following examples will illustrate the evaluation of definite integrals.

Example 1

Evaluate $\int_2^3 (x^2 + 2x)dx$

Solution

$$\int_2^3 (x^2 + 2x)dx = \left[\frac{x^3}{3} + \frac{x^2}{1}\right]_2^3$$

$$= [9 + 9] - \left[\frac{8}{3} + \frac{4}{1}\right]$$

$$= 18 - \frac{20}{3}$$

$$= \frac{34}{3}$$

Example 2

Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x dx$

Solution

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx$$

$$= \left[\frac{1}{2} x + \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{4} + 0 \right) - (0 + 0) = \frac{\pi}{4}$$

Example 3

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin x} dx$

Solution

Cos x is the derivative of 1 + sin x

Put u = I + sin x, $\frac{du}{dx} = \cos x$

du = cos x dx

Hence $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin x} dx = \int_0^{\frac{\pi}{2}} \frac{du}{u}$

= ln u

(For definite integral we need not put the constant of integration)

$$= [\ln(1 + \sin x)]_0^{\frac{\pi}{2}}$$

$$= \{\ln(1 + 1)\} - \{\ln 1\}$$

$$= (\ln 2) - (0)$$

$$= \ln 2$$

Example 4

Evaluate $\int_0^3 e^{2x} dx$

Solution

$$\int_0^3 e^{2x} dx = \left[\frac{e^{2x}}{2}\right]_0^3$$

$$= \frac{e^6}{2} - \frac{1}{2}$$

$$= \frac{1}{2}\left(e^6 - 1\right)$$

Example 5

Evaluate $\int_0^{\frac{\pi}{2}} x \sin xdx$

Solution

Using integration by parts

$$\int_0^{\frac{\pi}{2}} x \sin xdx = [-x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos xdx$$

$$= [-x \cos x + \sin x]_0^{\frac{\pi}{2}}$$

$$= (0 + 1) - (0 + 0)$$

$$= 1$$

Example 6

Evaluate $\int_0^{\pi} \sin 8x \sin 3xdx$

Solution

$$\int_0^{\pi} \sin 8x \sin 3x dx = -\frac{1}{2}\int_0^{\pi} -2 \sin 8x \sin 3x dx$$

$$= -\frac{1}{2}\int_0^{\pi} \left(\cos 11x - \cos 5x\right) dx$$

$$= -\frac{1}{2}\left(\frac{\sin 11x}{11}\right)_0^{\pi} - \frac{1}{2}\left(\frac{\sin 5x}{5}\right)_0^{\pi}$$

$$= -\frac{1}{2}\left(0 - 0\right) - \frac{1}{2}\left(0 - 0\right)$$

$$= 0$$

Exercise 7

Evaluate the following

1. $\int_0^2 \frac{x^2+3x+8}{x}$

2. $\int_0^1 (x+3)(x-2)dx$

3. $\int_2^3 \left(x^5+2x^2\right)dx$

4. $\int_0^2 (2x+3)^7 dx$

5. $\int_0^{\pi} \cos 7x dx$

6. $\int_0^{\frac{\pi}{2}} \tan(2x+3)dx$

7. $\int_0^{\pi} \frac{\cos x+4x+3x^2}{x^3+2x^2+\sin x}dx$

8. $\int_0^{\frac{\pi}{2}} x\cos x dx$

9. $\int_0^1 e^{5x+3}dx$

10. $\int_1^2 \frac{dx}{7x+3}dx$

11. $\int_0^{\frac{\pi}{2}} \cos^2 3x dx$

12. $\int_0^{\pi} \sin^2 5x dx$

13. $\int_{-\pi}^{\pi} \sin^3 x dx$

14. $\int_0^{\frac{\pi}{2}} \frac{1+\sin x}{\cos^2 x}dx$

15. $\int_1^2 x^2 e^x dx$

16. $\int_0^2 \frac{dx}{(x-1)(x-2)}dx$

17. $\int_0^2 \frac{dx}{(x+1)(x^2+1)}$

18. $\int_1^3 \frac{dx}{(x-1)^2(x-2)}$

19. $\int_1^2 \frac{x^2+8x+15}{(x+3)}dx$

20. $\int_0^{\pi} x^2 \sin 2x dx$

Summary

You have learnt the following from this chapter:

a). The fundamental Theorem of the Integral calculus

If $\int f(x)dx = G(x)$ then

$$\int_a^b f(x)dx = G(b) - G(a)$$

b). $\int_a^b f(x)dx$ can be defined as the area enclosed by the four boundaries y = f(x), the x- axis and the ordinates x = a and x = b.

Further Reading

i).Additional Pure mathematics
Harwood Clarke
Heinemann Educational Books Ltd
London

CHAPTER 8
Area Under a Curve

8.1 Introduction

In the previous chapter we have learnt the meaning of the definite integrals. In this chapter we shall learn one of the important applications of the definite integrals namely calculating the area under a curve, x – axis and the ordinates x = a and x = b.

8.2 Objectives of the chapter

By the end of this chapter you should be able to: -

- establish the formula for finding the area under a curve the x – axis and the ordinates x = a and x = b.
- apply the formula to calculate exact area under different types of curves and also the areas between two curves.

8.3 Formula for the area under a curve and enclosed by the lines

the x – axis, and the ordinates x = a and x = b.

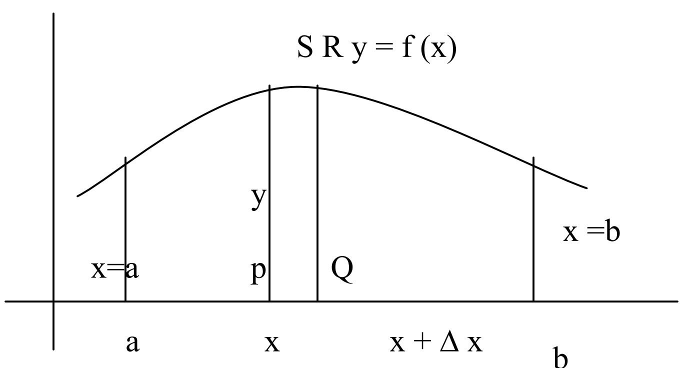


Figure 1

Consider the area under a curve y = f (x), and enclosed by the lines the x- axis and the ordinates x = a and x = b.
Divide the area into n equal strips of width Δx and height y.
The area of the strip PQRS = $y_r\Delta x$. Consider the total area of all the strips.
Total area required = $\sum y_r \Delta x$.

If Δx → 0 the area required is $A = \int_{x=a}^{x=b} y dx$ or $A = \int_a^b f(x)dx$

8.4 Area below x – axis

If the required area is below x – axis as in the figure 2

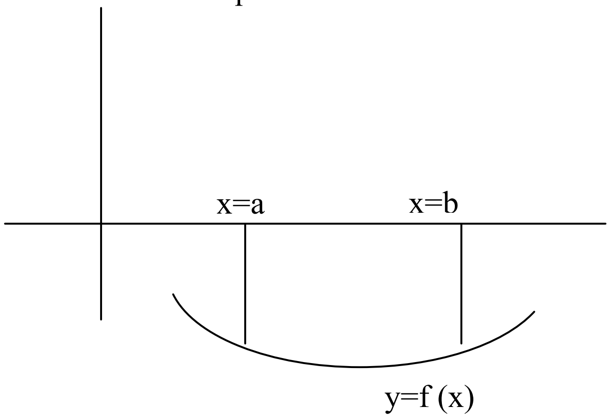


Figure 3

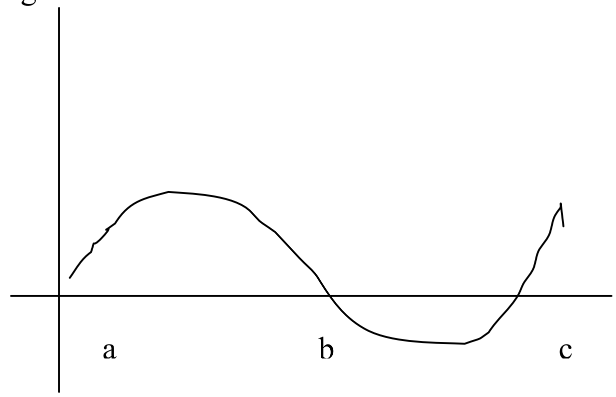


Figure 2

Then $A = \int(-y)dx$ since the y coordinate is negative. Hence $\int f(x)dx$ will be negative. Since we are interested only in the magnitude of the area we can take only the magnitude if the area is negative.

8.5 Combined area under a curve

If we wish to find the area under a curve between x = a and x = c (figure 3). We must find the area above x-axis and the area below x – axis separately and add the magnitudes of the two areas.

For example, let the area under the curve between x = a and x = b be 15 units and in between x = b to x = c be –5 units. Then the total area under the curve between x = a and x = c is:

$A = |15| + |-5|$

=15+5

=20 sq units

8.6 Area between two intersecting curves

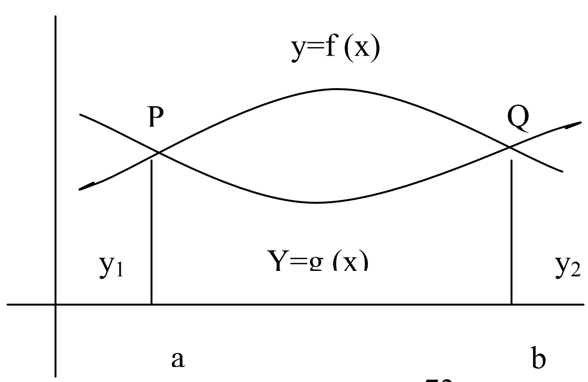


Figure 4

Let the equations of two curves be y = f (x) and y = g (x).

Suppose that they intersect at P and Q. Let the points of the intersection of the two curves be (a , y_1) and (b, y_2).

The magnitude of the area between the two curves is:

$$A = \int_{x=a}^{b} [f(x) - g(x)]dx$$

Proof

The area bounded by the four boundaries y = f (x), the x- axis, x = a and x = b is

$$\int_a^b f(x)dx$$

Similarly the area enclosed by the four boundaries y = g (x), the x – axis x = a and x = b is $\int_a^b g(x)dx$. Hence the shaded area between the two curves is:

$$A = \int_a^b f(x)dx - \int_a^b g[x]dx$$

$$= \int_a^b [f(x) - g(x)]dx$$

If this is negative we can take only the magnitude or the modulus of A.

The following examples illustrate all the types of areas under the curves.

Example 1

Find the area of the rectangle bounded by the line y = 2, the x – axis, the ordinate x = 1 and x = 5 using the formula. $A = \int ydx$

Solution

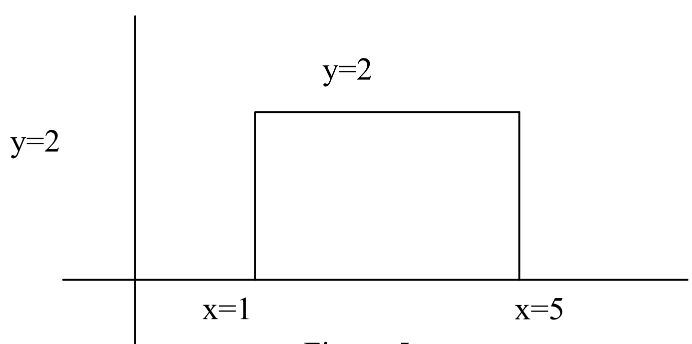


Figure 5

The curve in this case is y = 2.

Using the formula $A = \int_a^b y\,dx$ we have

$$A = \int_1^5 2dx$$

$$= [2x]_{x=1}^{x=5}$$

$$=(10) - (2)$$

$$= 8 \text{ sq units}$$

The area in this case is very simple since the area of the rectangle of length 4 units and breadth 2 units is 8 sq units.

Example 2

Sketch the curve $y = 2x^2$ from x = 0 to x = 5.
Evaluate the area enclosed by the curve, the x – axis and the ordinates x = 1 to x = 4

Solution

Let us prepare a table showing some points on the curve.

X	0	1	3	5
Y	0	2	18	50

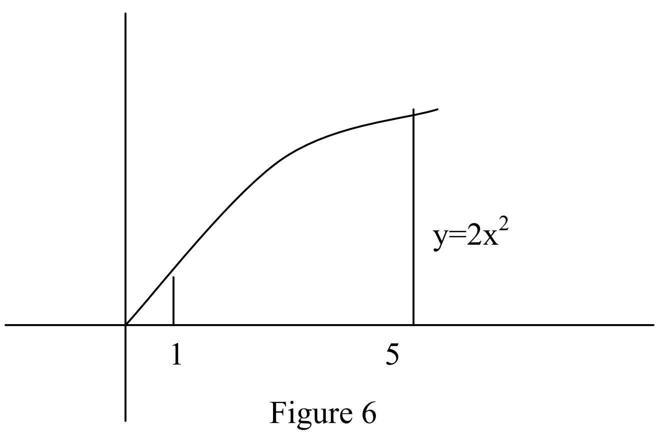


Figure 6

The above figure shows the curve $y = 2x^2$.
By formula,

$$A = \int_{x=a}^{b} y\,dx$$

$$= \int_1^4 2x^2 dx$$

$$= \left[\frac{2x^3}{3} \right]_{x=1}^{x=4}$$

$$= \left(\frac{128}{3} \right) - \left(\frac{2}{3} \right)$$

=42 sq units

Example 3

1. Sketch the curve $y = x^3 - 3x^2 + 2x$ from x = 0 to x = 3.
2. Find the area included between the curve $y = x^3 - 3x^2 + 2x$ and the axis of x from x = 0 to x = 2.
3. Evaluate $\int_0^2 \left(x^3 - 3x^2 + 2x\right)dx$. Interpret your result.

Solution

a) The following table shows some points on the curve.

X	0	1	2	0.5	1.5	2
Y	0	0	0	0.375	-0.375	0

The graph is shown in the figure

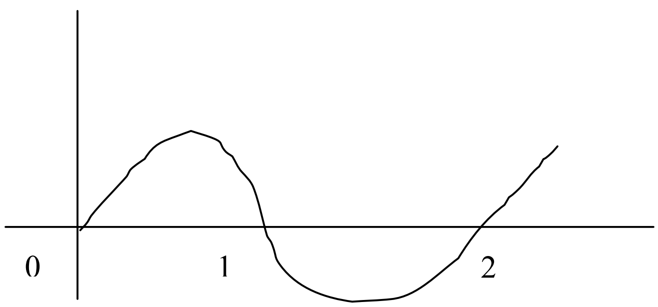


Figure 7

b) Part of the area is positive and the other part is negative. Hence we should find each area separately.

Area from x =0 to x = 1 is given by

$$A = \int_{x=0}^{1} \left(x^3 - 3x^2 + 2x\right)dx$$

$$= \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1 = \left(\frac{1}{4} - 1 + 1 \right) - 0$$

=1/4 sq units

Area from x = 1 to x = 2 is given by

$$A_2 = \int_1^2 \left(x^3 - 3x^2 + 2x \right) dx$$

$$= \left[\frac{x^4}{4} - x^3 + x^2 \right]_{x=1}^2$$

$$= (4 - 8 + 4) - \left(\frac{1}{4} - 1 + 1 \right)$$

$$= -\frac{1}{4} \text{ sq units}$$

Required area $= |A_1| + |A_2|$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2} \text{ sq units}$$

c) $$A = \int_0^2 \left(x^3 - 3x^2 + 2x \right) dx$$

$$= \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^2$$

$$= (4 - 8 + 4) - (0)$$

= 0 sq unit

If we evaluate the area directly from x = 0 to x =2 we get the area = 0.
This is because the area above x – axis is positive and that below the x- axis is negative. Hence we should not find the area directly from x = 0 to x = 2.

Example 4

Evaluate the area enclosed between the curve $y = x^2 - 5x + 4$ and the x- axis.

Solution
Here the ordinates are not given. Let us find the points of intersection of the curve with the x – axis.
Solving $y = x^2 - 5x + 4$ (the curve)
And $y = 0$ (the x – axis)
we have $x^2 - 5x + 4 = 0$
$(x - 1)(x - 4) = 0$
Hence $x = 1$ and $x = 4$.
If we sketch the curve $y = (x - 1)(x - 4)$ we get the following graph.

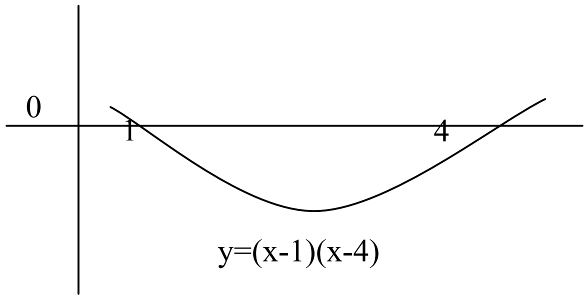


Figure 8

The entire area is below x – axis hence required area is:

$$A = \int_{x=1}^{4} \left(x^2 - 5x + 4\right) dx$$

$$= \left(\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right)_1^4$$

$$= \left(\frac{64}{3} - \frac{80}{2} + 16\right) - \left(\frac{1}{3} - \frac{5}{2} + 4\right)$$

$$= \frac{27}{6} \text{ sq units}$$

=4.5 sq units

Example 5
Find the area between the curve $x^2 = y$ and the straight line $y = 2$.

Solution
It is always better to sketch the curve to determine whether the required area is completely above the x – axis or below the x – axis or a portion of the area is above the x – axis and the other portion is below the x – axis.
The following figure shows the sketch of the curves:

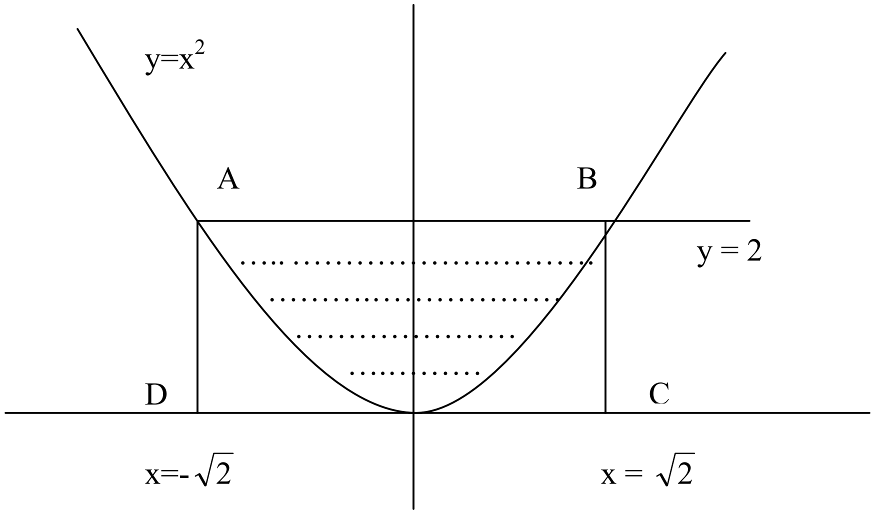


Figure 9

Solving $y = x^2$ and $y = 2$ we get $x^2 = 2$ and $x = \sqrt{2}$ and $x = -\sqrt{2}$ and $y = 2$. Since the complete area required is above the x – axis we get the required area.

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} y\,dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} x^2\,dx$$

$$= \left(\frac{x^3}{3}\right)_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{-2\sqrt{2}}{3}\right)$$

$\frac{4\sqrt{2}}{3}$ sq units.

This is the area below the curve and the x – axis.

The shaded area = Area of the rectangle ABCD - $\frac{4\sqrt{2}}{3}$

$$= 2\left(2\sqrt{2}\right) - \frac{4\sqrt{2}}{3}$$

$$= 4\sqrt{2} - \frac{-4\sqrt{2}}{3}$$

$$= \frac{8\sqrt{2}}{3} \text{ sq units}$$

Example 6
Find the area enclosed by the line y = 4x and the curve y = x^2.

Solution
The graph is shown in the figure below:

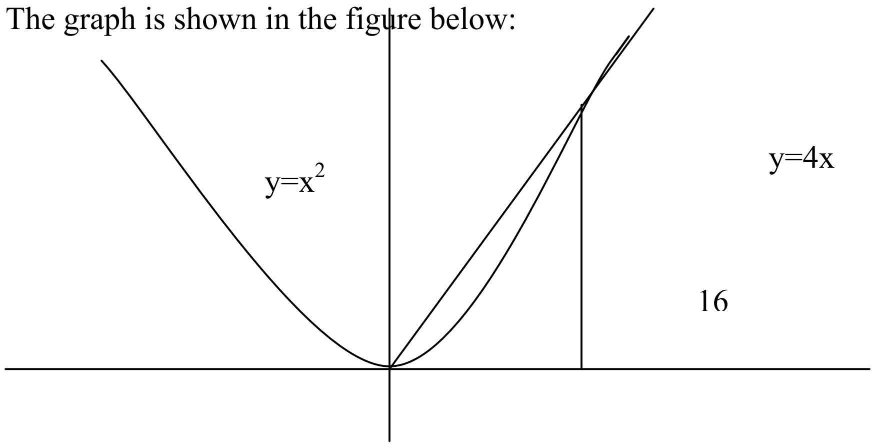


Figure 10

The points of intersection of the curve y = x^2 and the line y = 4x are given by solving the two equations.
Using $x^2 = 4x$ we have
$X^2 - 4x = 0$ or $x(x - 4) = 0$.
Hence x = 0 or x = 4 correspondingly y = 0 or y = 16.
The required area (shaded) = Area under the line (y = 4x) – area under the curve (y = x^2).

$$= \int_0^4 \left(4x - x^2\right) dx$$

$$= \left(\frac{4x^2}{2} - \frac{x^3}{3}\right)_0^4$$

= 32 – 64/3

= 32/3 sq units

Exercise 8

1. Find the area under the curve $y = x + 3x^2$ between $x = 1$ and $x = 2$.
2. Find the area enclosed by the curve $x^2y = 36$, the x – axis and the lines $x = 6$ and $x = 9$.
3. Find the area enclosed by the curve $y = \sqrt{x}$, the x – axis and the ordinates $x = 1$ and $x = 4$.
4. Find the area enclosed by the curve $y = x(x - 4)$ and the axis of x.
5. Find the area in the first quadrant enclosed by the line $y = x$ and the curve $y = x^3$.
6. Find the area between the curve $x^2 = y$ and the line $y = 2$.
7. Find the area in the positive quadrant enclosed by the curves $y = x^2$ and $y = x^3$.
8. Find the area common to the two curves $y^2 = 12x$ and $x^2 + y^2 = 24x$.
9. Find the area common to the circle $x^2 + y^2 = 25$ and the parabola $3x^2 = 16y$.
10. Find the area common to the circle $x^2 + y^2 = 16$ and the ellipse $x^2 + 3y^2 = 24$.

Summary

You have learnt the following from this chapter:

i. The area enclosed by a curve $y = f(x)$, the x – axis, the ordinate $x = a$ and the ordinate $x = b$ is given by

$$A = \int_a^b y\,dx$$

$$= \int_a^b f(x)\,dx$$

ii. Area above the x – axis is positive and the area below the x – axis is negative.

iii. If the required area consists of one portion above the x - axis and the other portion below the x – axis. We must find each area separately and add their magnitudes.

iv. If y = f (x) and y = g (x) are two curves and if they intersect, the numerical value of the area between the two curves. $\int_a^b [f(x) - g(x)dx]$ provided the common points of intersection have their x coordinates a and b.

Further reading

Additional pure mathematics
By L. Harwood Clarke
Heinemann Educational Books Ltd
London

CHAPTER 9

Improper Integrals

9.1 Introduction

In the chapter seven, we have learnt to evaluate the definite integrals whose limits are finite (a ≤ x ≤ b). In this chapter we shall learn to evaluate some integrals whose limits are infinite or the integrands become infinite even though the interval is finite. Such integrals are called improper integrals. We shall define improper integrals in detail in the following sections.

9.2 Objectives of the chapter.

By the end of this chapter you should be able to

- define the improper integrals of two kinds
- test the convergence and divergence of improper integrals
- test the absolute convergence and conditional convergence of improper integrals
- evaluate the Cauchy's principal value of the improper integral of the second kind.

9.3 Definition of improper Integrals

There are two kinds of improper Integrals

a). Improper Integrals of the first kind

When one or both the limits of a definite integral is infinite the integral is called an improper integrals of the first kind.

b). Improper integrals of the second kind

When the integrand has a singular point (the integrand becomes infinite) in the interval of the limits, the definite integral is called an improper integral of the second kind.

Consider $\int_0^3 \frac{x^3+2}{(x-1)}dx$

The interval of the limit is $0 \le x \le 3$

At the point x = 1, the integrand $\frac{x^3+2}{x-1}$ becomes infinite and hence

$\int_0^3 \frac{x^3+2}{(x-1)}dx$ is an improper integral of the second kind

$\int_{x=0}^{x=3} \frac{x^3+2}{x}dx$ and $\int_{x=0}^{x=3} \frac{x^3+2}{(x-3)}dx$ are also improper integrals of the second kind having singularities at the lower limit x = 0 and at the upper limit x = 3 respectively.

9.4 Convergence or divergence of the improper integrals of the type $\int_a^{\infty} f(x)dx$

The improper integral $\int_a^{\infty} f(x)dx$ is said to be convergent if $\lim_{b\to\infty} \int_a^b f(x)dx$

exists (or converges) to a finite limit and the limit is called **the value of the improper integral.**

If the limit does not exist we say that the improper integral **does not exist (or diverges)**

Example 1

a). When will you say that the improper integral $\int_1^{\infty} \frac{1}{x}dx$ is convergent or divergent.

b). Show that $\int_1^{\infty} \frac{1}{x}dx$ is divergent.

Solution

a). If $\lim_{b\to\infty} \int_1^b \frac{1}{x}dx$ converges (or exists) we say that $\int_1^{\infty} \frac{1}{x}dx$ converges otherwise we say that it diverges.

b). $\lim_{b\to\infty} \int_1^l \frac{1}{x}dx = \lim_{b\to\infty} [\ln x]_1^b$

$$= \lim_{b \to \infty} \ln b - \ln 1$$

$$= \lim_{b \to \infty} \ln b \quad \text{since } \ln 1 = 0$$

$$= \ln \infty$$

$$= \infty$$

Since $b \to \infty$ $\int_1^{\infty} \frac{1}{x} dx$ dose not exist, it diverges.

Example 2

Show that the improper integral $\int_1^{\infty} \frac{1}{x^2} dx$ is convergent

Solution

$\int_1^{\infty} \frac{1}{x^2} dx$ converges or diverges if $\lim_{b \to \infty} \int_1^{b} \frac{1}{x^2} dx$ exists or does not.

$$\lim_{b \to \infty} \int_1^{b} \frac{1}{x^2} dx = \left[\frac{x^{-1}}{-1} \right]_1^b$$

$$= \lim_{b \to \infty} \left[-\frac{1}{x} \right]_1^b$$

$$= \lim_{b \to \infty} \left[-\frac{1}{b} \right] - \left[-\frac{1}{1} \right]$$

$$= \lim_{b \to \infty} 1 - \frac{1}{b}$$

$$= 1 - \frac{1}{\infty}$$

$= 1$ (Limit exists and finite)

Hence $\int_1^{\infty} \frac{1}{x^2} dx$ is convergent.

9.5 Definition of absolute convergence and conditional convergence of an improper integral

i).The improper integral, $\int_a^{\infty} f(x)\,dx$ is said to be absolutely convergent if

$\int_a^{\infty} |f(x)|\,dx$ is convergent.

ii).The improper integral $\int_a^{\infty} f(x)\,dx$ is said to be conditionally convergent if

$\int_a^{b} f(x)\,dx$ is convergent but $\int_a^{\infty} |f(x)|\,dx$ is not convergent.

9.6 Cauchy's inequality for definite integrals

Cauchy's inequality states that $\left|\int_a^{b} f(x)\,dx\right| \le \int_a^{b} |f(x)|\,dx$

This inequality is very useful to determine the absolutely convergent and conditionally convergent improper integrals.

Example 3

Show that $\int_1^{\infty} \frac{\sin x}{x^2}\,dx$ is absolutely convergent.

Solution

By cauchy's inequality,

$$\left|\int_1^{\infty} \frac{\sin\ x}{x}\,dx\right| \le \int_1^{\infty} \left|\frac{\sin\ x}{x^2}\right| dx$$

$$\le \int_1^{\infty} \frac{1}{x^2}\,dx \qquad \text{since x is positive and } |\sin\ x| \le 1$$

$$\le \int_1^{\infty} x^{-2}\,dx$$

(1)

Now $\lim_{b \to \infty} \int_1^l x^{-2} dx = \lim_{b \to \infty} \left[\frac{-x}{-1} \right]_1^b$

$$= \lim_{b \to \infty} \left[-\frac{1}{x} \right]_1^b$$

$$= \lim_{b \to \infty} \left[-\frac{1}{b} \right] - \left[-\frac{1}{1} \right]$$

$$= \lim_{b \to \infty} 1 - \frac{1}{b}$$

$$= 1$$

Hence from (1) $\int_1^\infty \left| \frac{\sin x}{x^2} \right| dx \leq 1$

Then $\int_1^\infty \frac{\sin x}{x^2} dx$ is absolutely convergent.

Example 4

Show that $\int_0^\infty xe^{-x} dx$ is convergent.

Solution

$\int_0^\infty xe^{-x} dx$ is convergent if $\lim_{b \to \infty} \int_0^b xe^{-x} dx$ is convergent.

Using integration by parts

$$\lim_{b \to \infty} \int_0^l xe^{-x} dx = l \to \infty \left[-xe^{-x} \right]_0^b + \int_0^b e^{-x} dx$$

$$= \lim_{b \to \infty} \left[-be^{-b} + 1 - e^{-b} \right]$$

$$= \lim_{b \to \infty} \left[\frac{-b}{e^b} + 1 - \frac{1}{e^b} \right]$$

$= 1$ Since $\lim_{b \to \infty} \frac{-b}{e^b} \to 0$

Hence $\int_0^{\infty} xe^{-x}\,dx$ is convergent.

9.7 Theorem: if $\int_a^b f(x)dx$ is an absolutely convergent improper integral it is also convergent, where b can be finite or infinite.

Proof
By Cauchy's inequality,

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|dx$$

If $\int_a^b |f(x)|dx$ is convergent (or absolutely convergent) then it has an upper bound M

Hence $\left|\int_a^b f(x)dx\right| \le$ M

Then $\int_a^b f(x)dx \le$ M and hence it is convergent.

In other words $\int_a^b f(x)dx$ is convergent if $\int_a^b |f(x)|dx$ is convergent.

Example 5

Show that $\int_0^{\infty} e^{-x}\cos 2x\ dx$ is absolutely convergent.

Solution

We must show that $\int_0^{\infty} |e^{-x}\cos 2x|\ dx$ is convergent.

$$\int_0^\infty \left|e^{-x}\cos 2x\right| dx = \lim_{b\to\infty}\int_0^b \left|e^{-x}\cos 2x\right| dx$$

$$\leq \lim_{b\to\infty}\int_0^b \left|e^{-x}\right| dx$$

$$\leq \lim_{b\to\infty}\left[e^{-x}\right]_0^1$$

$$\leq \lim_{b\to\infty}\frac{1}{e^b} - 1$$

$$\leq -1$$

Since $\lim_{b\to\infty}\int_0^b \left|e^{-x}\cos 2x\right| dx$ is convergent since the limit exists

$\int_0^\infty \left|e^{-x}\cos 2x\right| dx$ is convergent.

Hence $\int_0^\infty e^{-x}\cos 2x$ is absolutely convergent.

9.8 Convergence and divergence of the improper integrals of the second kind.

Consider the integral $\int_a^b f(x)\ dx$ suppose that f(x) has a singularity at an interior point c of $[a \leq x \leq 1]$

Now $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$

(1)

The improper integral $\int_a^b f(x)\ dx$ is said to be convergent if both the improper integrals on the right side of (1) exist.

The sum of these is defined as **the value of the improper integral from a to b in (1)**

9.9 The Cauchy's principal value of $\int_a^b f(x)\ dx$

Let f(x) be defined on $[a \le x \le 1]$ except possibly at any interior point c, which is a singular point of f(x).

If $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ does not exist but

$\lim_{\varepsilon \to 0} \left[\int_a^{c-\varepsilon} f(x)\ dx + \int_{c+\varepsilon}^b f(x)\ dx \right]$ does exist. We call this limit

the Cauchy's principal value of $\int_a^b f(x)\ dx$

Thus the Cauchy's principal value of

$$\int_a^b f(x)\ dx = \lim_{\varepsilon \to 0} \left[\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx \right]$$

Example 6

a) Show that $\int_{-1}^2 \frac{1}{x}\ dx$ is improper.

b) Find the Cauchy's principle value of $\int_{-1}^2 \frac{1}{x}\ dx$

Solution

a) The integrand $\frac{1}{x}$ has a singularity at x = 0 in the interval $[-1 \le x \le 2]$]

Hence $\int_{-1}^2 \frac{1}{x}\ dx$ is an improper integral of the second kind.

b) $\int_{-1}^2 \frac{1}{x}\ dx = \int_{-1}^0 \frac{1}{x}\ dx + \int_0^2 \frac{1}{x} dx$ Clearly $\int_{-1}^0 \frac{1}{x}\ dx$ and $\int_0^2 \frac{1}{x}\ dx$ both diverge;

$$\int_{-1}^{0} \frac{1}{x}\, dx = [\ln x]_{-1}^{0} = \ln 0 - \ln |(-1)| = \ln 0 \quad \text{does not exist}$$

$$\int_{0}^{2} \frac{1}{x}\, dx = [\ln x]_{0}^{2} = \ln 2 - \ln(0) \quad \ln(0) \text{ does not exist.}$$

Hence $\int_{-1}^{2} \frac{1}{x}\, dx$ does not exist.

However $\int_{-1}^{-\varepsilon} \frac{1}{x}\, dx + \int_{\varepsilon}^{2} \frac{1}{x} dx$ exist when $\varepsilon \to 0$

Since $\int_{-1}^{\varepsilon} \frac{1}{x}\, dx = [\ln x]_{-1}^{-\varepsilon} = \ln |(-\varepsilon)| - \ln |(-1)| = \ln \varepsilon$

and $\int_{\varepsilon}^{2} \frac{1}{x}\, dx = [\ln x]_{\varepsilon}^{2} = \ln 2 - \ln \varepsilon$

Hence $\int_{-1}^{-\varepsilon} \frac{1}{x}\, dx + \int_{\varepsilon}^{2} \frac{1}{x} dx = \ln \varepsilon + \ln 2 - \ln \varepsilon$

$$= \ln 2$$

Thus the Cauchy's principal value of $\int_{-1}^{2} \frac{1}{x}\, dx = \ln 2$

Example 7

Show that $\int_{-2}^{3} \frac{dx}{x^4}$

i). does not converge
ii). does not have a principal value

Solution

The integral $\int_{-2}^{3} \frac{dx}{x^4}$ is improper since x = 0 is a singularity on the interval [-2 ≤ x ≤ 3]

Now $\int_{-2}^{3} \frac{dx}{x^4} = \int_{-2}^{0} \frac{dx}{x^4} + \int_{0}^{3} \frac{dx}{x^4}$

$$= \left[\frac{1}{-3x^3}\right]_{-2}^{0} + \left[\frac{1}{-3x^3}\right]_{0}^{3}$$

$$= -\infty - \frac{1}{24} - \frac{1}{81} - \infty$$

does not converge

The Cauchy's principal value of

$$\int_{-2}^{3} \frac{dx}{x^4} = \int_{-2}^{-\varepsilon} \frac{\text{dx}}{\text{x}^4} + \int_{\varepsilon}^{3} \frac{dx}{x^4} \quad \text{where } \varepsilon \to 0$$

$$= \left[\frac{1}{-3x^3}\right]_{-2}^{-\varepsilon} + \left[\frac{1}{-3x^3}\right]_{\varepsilon}^{3}$$

$$= \left(\frac{1}{3\varepsilon^3}\right) - \frac{1}{24} + \left(\frac{1}{-81}\right) + \frac{1}{3\varepsilon^3} \quad \text{where}$$

$\varepsilon \to 0$

$= \infty$

Thus even the principal value does not exist.

Exercise 9

1. Classify the following into
   Improper integrals of the first kind and
   Improper integrals of the second kind.

   a). $\displaystyle\int_{0}^{\infty} \frac{x^2+1}{(x-1)}\,dx$ b). $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-x}\sin x}{x}\,dx$

   c). $\displaystyle\int_{-2}^{3} \frac{dx}{x^4}$ d). $\displaystyle\int_{2}^{5} \frac{x^3}{(x-3)}\text{dx}$

2. When will you say that the improper integral $\displaystyle\int_{a}^{\infty} f(x)\,dx$ converges or diverges.

3. When will you say that the improper integral $\displaystyle\int_{-2}^{2} \frac{dx}{x^3}$ converges or diverges.

4. Define the Cauchy's principal value of the improper integral $\int_a^b f(x)\,dx$ if f (x) has a singularity at c where a < c < b

5. Define absolute convergence and conditional convergence of an improper integral $\int_a^b f(x)\,dx$

6. Show that $\int_0^1 \frac{dx}{x^5}$ is divergent .

7. Show that $\int_1^\infty \frac{dx}{x^5}$ is divergent.

8. Show that $\int_0^\infty e^{-x}\cos 3x\,dx$ is absolutely convergent.

9. Test whether $\int_1^\infty \frac{\sin x}{x^2}\,dx$ is absolutely convergent or not.

10. Find Cauchy's principal value of $\int_{-1}^{4} \frac{3}{x}dx$

11. Evaluate $\int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx$

12. Prove that $\int_0^\infty \cos^2 x\,dx$ converges

    (Hint: Make the substitution $u = x^2$)

13. Define $\int_{-\infty}^{a} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx$

14. Test the following integrals for convergence

    a). $\int_0^\infty \sin x\,dx$ b). $\int_1^3 \frac{\sqrt{x}}{\ln x}dx$

    c). $\int_0^\infty \frac{x^2}{(1+x^2)^2}dx$ d). $\int_0^\pi \frac{x}{\sin x}dx$

15. Test whether the principal value for the integral $\int_{-2}^{5} \frac{dx}{x^4}$ exists.

16. Show that $\int_{0}^{\infty} \frac{\sin x \cos 4x}{x}$ dx converges.

Summary

You have learnt the following from the chapter:

i. Definition of improper integrals of first and second kind.

ii. Convergence and divergence of improper integrals.

iii. Absolute convergence and conditional convergence of improper integrals.

iv. Cauchy's inequality for absolute convergence.

v. The cauchy's principal value of the improper integrals $\int_{a}^{b} f(x)\,dx$ when f(x) has a singularity at c where a $\leq$ c $\leq$ b

Further reading

1. Advanced Calculus
By Watson Fulks
John Wiley and sons
New York. Brisbane Toronto.

CHAPTER 10

Length of an Arc of a Curve

10.1 Introduction

If you ask a secondary school student to measure the length of an arc of a curve, he will use a string or a flexible material like a string placed upon the arc of the curve and he measures the length of the string placing against a ruler. Also an approximation of length could be found by marking a number of points on the arc and finding the sum of the lengths of all the chords that are small. Any mathematician could do the same method unless he knows the equation of the curve either in Cartesian form or Parametric or any other form. In this chapter we shall learn the method of finding the length of an arc of a curve.

10.2 Objectives of the chapter

By the end of this chapter you should be able to.

- derive an expression or formula for S, the arc length of a curve if the Cartesian or parametric equation of the curve is known.
- calculate accurately the length of the curve using the formula.

10.3 Length of an arc S of a curve in Cartesian equation is

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Proof

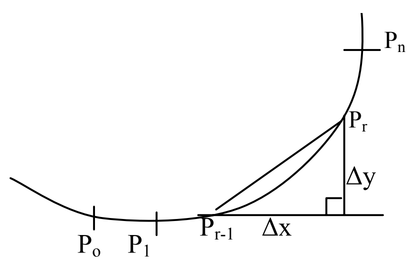


Figure 1

Suppose that an arc of a curve is divided into n parts by points p_0, p_1, p_2…….p_n. We shall assume that the sum of the lengths of chords, $p_0\, p_1 + p_1\, p_2 + \ldots\ldots p_{r-1}\, p_r + \ldots. p_{n-1}\, p_n$ is equal to the length of the arc in the limit when the length of the chords $\to 0$.

If Δ x and Δ y are the increments in x and y from p_{r-1} to p_r.

$$(p_{r-1}\, p_r)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$= \quad (\Delta x)^2\left[1+\left(\frac{\Delta y}{\Delta x}\right)^2\right]$$

Then $p_{r-1}\, p_r \quad = \sqrt{\left[1+\left(\frac{\Delta y}{\Delta x}\right)^2\right]}\Delta x$

Summing for all the chords and proceeding to the limit

Length of arc S = $\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$

where a and b are the values of x corresponding to the end points of the arc.

10.4 The length of an arc S of a curve in the Parametric form is

$$S = \int_{t_1}^{t_2} \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}dt$$

where t_1 and t_2 are the values of t corresponding to the end points of the arc.

Proof

We have seen in the previous section

$$(p_{r-1}p_r)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$= \left[\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2\right](\Delta t)^2$$

Then $p_{r-1}p = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}\, dt$

Summing for all the chords and proceeding to the limit.

Length of the arc S = $\int_{t_1}^{t_2} \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}dt$

Example 1

Find the length of the arc of parabola $y^2 = 4\,a\,x$ cut off by its latus rectum.

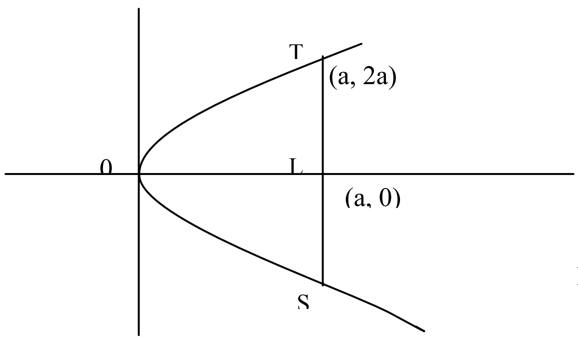


Figure 2

Solution

The equation of a parabola in parametric equation is $x = a\,t^2$, $y = 2\,a\,t$.

Latus Rectum is the line perpendicular to the axis of the parabola (in the Figure 2) at a distance a from the vertex. If TS is the Latus Rectum, the coordinates of L is (a, 0) and of T is $(a\,t^2, 2\,a\,t)$. Hence $a\,t^2 = a$ and $t = 1$. At the vertex 0, $a\,t^2 = 0$

Since arc SOT = Twice arc OT

we have S $= 2\int_{t=0}^{t=1} \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}\,dt$ (SOT = 2OT)

(arc = 2OT) $= 2\int \sqrt{(2at)^2 + (2a)^2}\,dt$

$$= 2\int \sqrt{4a^2t^2 + 4a^2}\,dt$$

$$= 4a\int \sqrt{1+t^2}\,dt$$

$$= 2a\left[t\sqrt{1+t^2} + \ln t + \sqrt{1+t^2}\right]_0^1$$ using integration by parts.

$$= 2a\left[\sqrt{2} + \ln(\sqrt{2+1}\right]$$

Example 2

a). Draw the curve $6\,y^2 = x\,(x-2)^2$

b). Find the point of intersection of the curve with x-axis.

c). Calculate the length of the loop of the curve $y^2 = \frac{1}{6}\,x\,(x-2)^2$

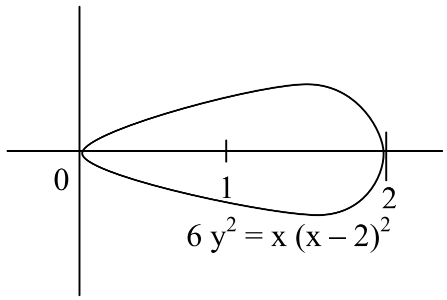


Figure 3

Solution

b) Let $6\,y^2 = x\,(x-2)^2$

Differentiating with respect to x

$$12\,y\,\frac{dy}{dx} = (x-2)^2 + 2\,x\,(x-2)$$

$$= (x-2)\;(3x-2)$$

$$\frac{dy}{dx} = \frac{(x-2)(3x-2)}{12y}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(x-2)^2(3x-2)^2}{24x(x-2)^2}$$

$$= \frac{(x-2)^2(3x-2)^2}{24x(x-2)^2}$$

$$= 1+\frac{(3x-2)^2}{24x}$$

Length of the loop $= 2\int_{x=0}^{a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= 2\int_0^2 \frac{(3x+2)}{\sqrt{24x}}dx$$

$$= 2\int_0^2 \left(\frac{3\sqrt{x}}{2\sqrt{6}} + \frac{2}{2\sqrt{6}\sqrt{x}} \right) dx$$

$$= \frac{2}{2\sqrt{6}} \left[\frac{2 \times 3x^{\frac{3}{2}}}{3} + 4\sqrt{x} \right]_0^2$$

$$= \frac{1}{\sqrt{6}} \left(2.2^{\frac{3}{2}} + 2.2^{\frac{3}{2}} \right)$$

$$= \frac{8}{\sqrt{3}}$$

Example 3
Show that the length of the circumference of a circle of radius a is 2 π a.

Solution

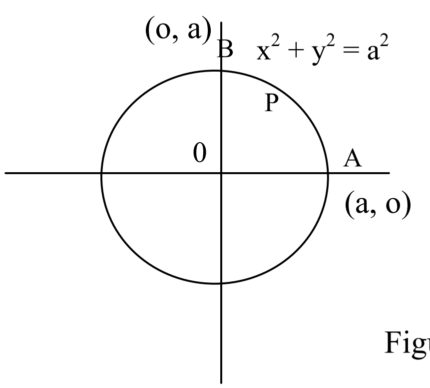


Figure 4

The equation of the circle of radius a is $x^2 + y^2 = a^2$. Differentiating with respect to x, we have $2x + 2y\frac{dy}{dx} = 0 \quad \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

length of arc APB = $S = \int_{x=0}^{x=a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_{x=0}^{a} \sqrt{1+\left(\frac{x^2}{y^2}\right)}dx$$

$$= \int_{x=0}^{a} \sqrt{\frac{x^2+y^2}{y^2}}dx$$

$$= \int_{0}^{a} \frac{a}{y}dx$$

$$= a\int_{0}^{a} \frac{1}{\sqrt{a^2-x^2}}dx$$

$$= a\left[\sin^{-1}\frac{x}{a}\right]_0^a$$

$$S = \frac{\pi a}{2}$$

the length of the circumference of the circle

= 4 times S

$$= 4\times\frac{\pi a}{2}$$

$$= 2\pi a$$

Example 4

Find the length of the arc in the first quadrant of $y = 2x^{\frac{3}{2}}$ from x = 0 to x = $\frac{1}{3}$.

Solution

The equation of the curve is given in Cartesian form. The length of the arc of a curve in Cartesian form is given by

$$S = \int_{a}^{b} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\frac{1}{3}} \sqrt{1 + \left(\frac{d}{dx} 2x^{\frac{3}{2}}\right)^2}\, dx$$

$$= \int_0^{\frac{1}{3}} \sqrt{1 + 9x}\, dx$$

$$= \int_0^{\frac{1}{3}} (9x + 1)^{\frac{1}{2}} dx$$

$$= \frac{(9x + 1)^{\frac{3}{2}}}{9 \cdot \frac{3}{2}}$$

using (ax + b) formula

$$= \frac{2}{27}[9x + 1]_0^{\frac{1}{3}}$$

$$= \frac{2}{27}[(3 + 1) - (1)]$$

$$= \frac{2}{9} \text{ units}$$

Example 5

Find the length of the arc from $\theta = 0$ to $\theta = \frac{\pi}{4}$ of the curve given by x = 3 cos θ,

y = 3 sin θ.

Solution

The equation of the curve is given in parametric equation. In parametric equation, the length of arc is given by

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Here

$$S=\int_{\theta=0}^{\theta=\frac{\pi}{4}}\sqrt{\left(\frac{d3\cos\theta}{d\theta}\right)^2+\left(\frac{d3\sin\theta}{d\theta}\right)^2}\,d\theta$$

$$=\int_{0}^{\frac{\pi}{4}}\sqrt{(-3\sin\theta)^2+(3\cos\theta)^2}\,d\theta$$

$$=3\int_{0}^{\frac{\pi}{4}}\sqrt{(\sin^2\theta+\cos^2\theta)}\,d\theta$$

$$=3\int_{0}^{\frac{\pi}{4}}1\,d\theta$$

$$=3(\theta)_0^{\frac{\pi}{4}}$$

$$=\frac{3\pi}{4}\text{ units}$$

Length of an arc S of a curve in polar coordinates (r, θ) is

$$S=\int_{\alpha}^{\beta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

where values α and β are the values of θ corresponding to the end points of the curve.

Proof

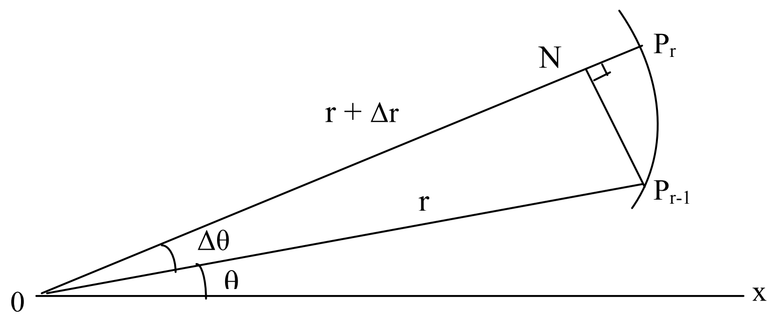


Figure 5

Let $o\,P_{r-1} = r$ and $o\,P_r = r + \Delta r$
Angle $x\,o\,P_{r-1} = \theta$ and angle $P_{r-1}\,o\,p_r = \Delta\theta$
Let $N\,P_{r-1}$ be perpendicular to $o\,P_r$.

Now $N\,P_{r-1} = r\,\Delta\theta$ and $N\,P_r = \Delta r$

Hence $(P_{r-1}\,P_r)^2 = (N\,P_r)^2 + (N\,P_{r-1})^2$
$= (\Delta r)^2 + (r\,\Delta\theta)^2$
$= (\Delta\theta)^2 \left[r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2 \right]$ when the limit $\Delta\theta \to 0$

$$S^2 = \left[r^2 + \left(\frac{\Delta r}{\Delta\theta}\right)^2 \right] (\Delta\theta)^2$$

$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

where α and β are the values of θ corresponding to at the end points of the curve.

Example 6
Find the length of the equiangular spiral $r = a\,e^{k\theta}$ from $\theta = 0$ to $\theta = 2\pi$.

Solution
The equation of the curve is given in polar form. The length of arc of a curve in polar form is given by

$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{\left(ae^{k\theta}\right)^2 + \left(\frac{dae^{k\theta}}{d\theta}\right)^2}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{a^2 e^{2k\theta} + \left(ae^{k\theta} k\right)} d\theta$$

$$= a\int_{0}^{2\pi} \sqrt{e^{2k\theta}(1+k^2)d\theta}$$

$$= a\sqrt{1+k^2}\int_{0}^{2\pi} e^{k\theta} d\theta$$

$$= a\sqrt{1+k^2}\left[\frac{e^{k\theta}}{k}\right]_0^{2\pi}$$

$$= \frac{a}{k}\sqrt{1+k^2}\left(e^{2k\pi} - 1\right)$$

Exercise 10

1. Find the length of the arc of y = ln sec x from x = $-\frac{1}{6}\pi$ to x = $\frac{1}{6}\pi$
2. Find the length of the arc of the curve y = ln sec x from x = 0 to x = $\frac{\pi}{3}$
3. Show that the length of the arc of the parabola y^2 = 4 ax cut off by the line 3y = 8x is a (ln 2 + $\frac{15}{16}$).
4. Sketch the astroid given by x = a $\cos^3$ t, y = a $\sin^3$ t and find the length of its circumference.
5. Find the length of the arc of the curve

   x = a (cos θ + θ sin θ)

   y = a (sin θ – θ cos θ)
6. Sketch the arc of the cycloid

   x = a (θ - sin θ)

   y = a (1 – cos θ)

from $\theta = 0$ to $\theta = 2\pi$. Find its length.

7. Find the length of the spiral of Archimedes $r = a\,\theta$ from $\theta = 0$ to $\theta = \pi$.
8. What is the length of the circumference of the cardioid $r = a\,(1 + \cos\theta)$.

Summary

You have learnt the following from this chapter.

a). The length of an arc S of a curve is

i). In Cartesian form $S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

ii). In parametric form $S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

iii). In polar form $S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

The limits being the end points of the curve

b). To apply the formula for various problems

Further Reading

Additional pure mathematics
Harwood Clarke
Heinemann Educational Books Ltd.
London.

CHAPTER 11

Volume and Surface Area of Solid of Revolution

11.1 Introduction

Consider an area under a curve between two ordinates. When this area is rotated about the axis of x the resulting solid is called a **Solid of Revolution.** A section of this solid by a plane perpendicular to the axis of x is a circle. When the area under a line of length *l* parallel to the axis of x is rotated you will obtain a right circular cylinder of height *l*.

The area under a line through the origin when rotated about the axis of x a right circular cone is obtained. We shall now find the volume and surface area of a solid of revolution, given the equation of the curve and the bounding ordinates.

11.2 Objectives of the chapter

By the end of this chapter you shall be able to:

- derive the formulae for the volume and the surface area of the solid of revolution.
- apply these formulae for finding the volumes and surface areas of common solids.

11.3 The formula for the volume of a solid of revolution

The curve y = f (x) bounded by the ordinate at x = a and x = b is rotated 360^0 or more about the x-axis a solid of revolution is generated.
The volume of this solid is given by

$$V = \pi \int_{x=a}^{x=b} y^2 dx$$

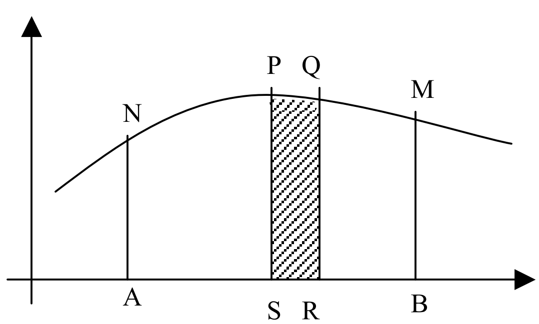


Figure 1

Consider the curve y = f (x) between the ordinates AN and BM. Let P be (x, y) and Q be (x + Δx, y + Δy) on the curve. PS and QR are perpendicular to x-axis and the distance RS =Δx where Δx → 0.

When the area ANMBA is rotated about x axis, the elementary area PSRQ rotates about OX, forming a slice of radius y and thickness Δx thus an elementary cylinder of volume $\pi y^2 \Delta x$ is generated.
The volume generated by the area ANMB is

V = Limits of the sum of the volume generated by such elementary areas as PSRQ.

$$= \lim_{\Delta \to 0} \sum \pi y^2 \Delta x \quad \text{where x varies from x = a to x = b.}$$

$$V = \int_{x=a}^{x=b} \pi y^2 dx$$

$$V = \pi \int_{x=a}^{b} y^2 dx$$

Similarly if a curve is rotated about the y axis between the lines y = c and y = d, the volume generated is given by

$$V = \pi \int_{y=c}^{y=d} x^2 dy$$

Example 1
Find the volume of a cone of height h and base radius r.
Solution

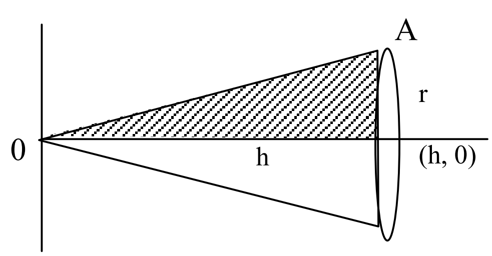


Figure 2

A cone is formed by the revolution of the area under the line OA about the x-axis.

The gradient of the line OA is $\frac{r}{h}$ and the equation of the line OA is y = $\frac{r}{h}$ x.

If V is the volume of the cone

$$V = \pi \int_{x=0}^{x=h} y^2 dx$$

$$= \pi \int \frac{r^2}{h^2} x^2 dx$$

$$= \frac{\pi r^2}{h^2} \left[\frac{x^3}{3} \right]_0^h$$

$$= \frac{\pi r^2}{h^2} \frac{h^3}{3}$$

$= \frac{1}{3} \pi r^2 h$ units

Example 2

Find the volume of a hemisphere of radius r units.

Hence show that the volume of a sphere of radius r is given by $V = \frac{4}{3} \pi r^3$

Solution

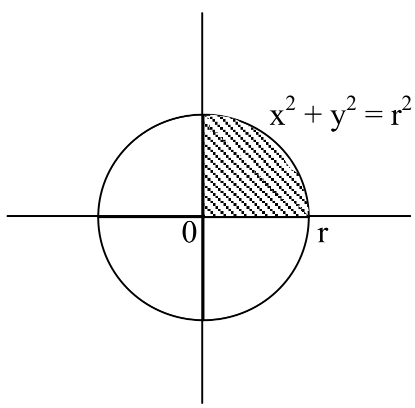


Figure 3

A hemisphere is formed by the revolution of the area of a quadrant of a circle $x^2 + y_2 = r^2$ about the x-axis.

The volume of the hemisphere

$$= \pi \int_{x=0}^{r} y^2 dx$$

$$= \pi \int_{0}^{r} \left(r^2 - x^2\right) dx$$

$$= \pi \left(r^2 x - \frac{x^3}{3}\right)_{x=0}^{r}$$

$$= \pi \left(r^3 - \frac{r^3}{3} \right)$$

$$= \frac{2\pi r^3}{3}$$

Hence the volume of sphere = twice $\frac{2\pi r^3}{3} = \frac{4\pi r^3}{3}$

Example 3
Find the volume obtained by rotating the area under the curve y = 1+ x between x= 1 and x = 2 about the axis of x.

Solution

Volume $= \pi \int_{x=1}^{2} y^2 dx$

$$= \pi \int_{x=1}^{2} \left(1 + x\right)^2 dx$$

$$= \pi \int_{1}^{2} \left(x^2 + 2x + 1\right) dx$$

$$= \pi\left(\frac{x^3}{3} + x^2 + x\right)_{x=1}^{x=2}$$

$$= \pi\left[\left(\frac{8}{3} + 4 + 2\right) - \left(\frac{1}{3} + 1 + 1\right)\right]$$

$$= \frac{19}{3}\pi$$

11.4 The Formula for the Surface Area of a Solid of Revolution

When an arc AB rotates about the axis of x a surface of revolution is generated. If arc AP = s and arc AQ = s + Δ s, arc PQ = Δ s.

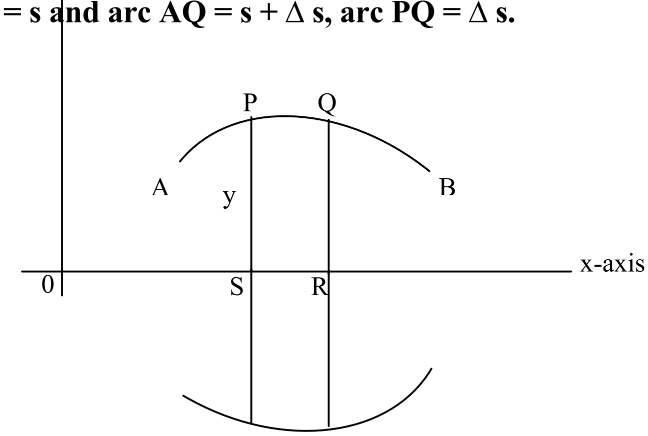


Figure 4

When the elementary area PSRQ rotates about x axis the curved surface area of the cylinder of radius y and height Δ s = 2πyΔ s, where Δ s – 0.
Hence the total surface area generated by the arc AB about x-axis.

=Limit ∑ 2 πΔ s from x = a to x = b.
Limit Δx → 0
Δy → 0

$\sum 2\pi y\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$ **. From x = a to x = b since $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$ (see 10.2)**

$$= \int_{x=a}^{x=b} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

This formula can be remembered as surface area is S $= 2\pi\int yds$

In parametric form

Surface areas $= 2\pi\int_{t=t1}^{t=t2} y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Example 4

Find the total surface area of a sphere of radius r.

Solution

A sphere is obtained by the revolution of a semi circle about its diameter.

Total surface area of a hemisphere $= 2\int_0^r 2\pi yds$

The equation of the circle is $x^2 + y^2 = r^2$

Then $y^2 = r^2 - x^2$ **differentiating with respect to x**

$$2y\frac{dy}{dx} = -2x$$

$\frac{dy}{dx} = -\frac{x}{y}$ **then** $1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{x^2}{y^2} = \frac{x^2+y^2}{y^2}$

But $x^2 + y^2 = r^2$, **Hence** $1+\left(\frac{dy}{dx}\right)^2 = \frac{r^2}{y^2}$

Then surface area of the sphere

$$= 2\int_{x=0}^{r} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\int_0^r 2\pi y.\frac{r}{y}dx$$

$$= 2\times 2\pi a\int_o^r dx$$

$$= 4\pi r[x]_o^r$$

$= 4\pi r^2$

Example 5

Find the surface area generated by the loop of the curve x=t², $y = t - \frac{t^3}{3}$ about the axis.

Solution

The curve cuts the x- axis at y=0 or $t - \frac{t^3}{3} = 0$, Hence t=0 or $t = \sqrt{3}$ or x=0 or x=3.

Eliminating t from x=t² and $y = t - \frac{t^3}{3}$ we have the Cartesian equation of the curve,

$9y^2 = x(3-x)^2$

$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ **since (d s)² = (d x)² + (d y)²**

$= 4t^2 + \left(1 - t^2\right)^2$

$= \left(1 + t^2\right)^2$

$\frac{ds}{dt} = 1 + t^2$

The required surface area = $2\pi \int yds$

$= 2\pi \int_{t=0}^{\sqrt{3}} \left(t - \frac{t^3}{3}\right)\left(1 + t^2\right)dt$

$= 2\pi \int_{0}^{\sqrt{3}} \left(t + \frac{2t^3}{3} - \frac{t^5}{3}\right)dt$

$= 2\pi \left[\frac{t^2}{2} + \frac{t^4}{6} - \frac{6^6}{18}\right]_0^{\sqrt{3}}$

$= \pi(3 + 3 - 3)$

$= 3\pi$ **Sqr units**

Example 6

Show that the curved surface area of a cone radius r, height h and slant height *l* is $\pi r l$.

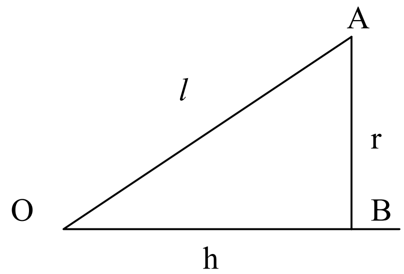


Figure 6

Solution

A cone is formed by the revolution of the area under the line OA. OB = height h, AB is the radius r and OA is the slant height *l*.

Gradient of OA is $\frac{r}{h}$ **and hence the equation of OA is** $y = \frac{r}{h}x$

Surface area S = $2\pi \int_{x=0}^{h} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, ds$

$$= \int_{x=0}^{h} \frac{r}{h}x\sqrt{1+\frac{r^2}{h^2}}dx$$

$$= 2\pi \frac{r}{h}\int_{o}^{h} x\frac{l}{h}dx \quad \textbf{since } h^2 + r^2 = l^2$$

$$= 2\pi \frac{r}{h}\frac{l}{h}\left[\frac{x^2}{2}\right]_0^h$$

= π r l

Exercise 11

1. **Find the volume generated by revolving about the x – axis that part of the curve** $y = \frac{1}{3}x^2$ **which its between the origin and x = 3.**

2. **The equation of the circle** $x^2 + y^2 = 9$ **is rotated about x- axis between the limits –3 and 3. Find the volume of the sphere by integration.**

3. **Sketch the curve $y = x^2 - x - 2$. The part of the curve lying between the points where it cuts the x- axis is rotated about that axis. Find the volume so generated.**

4. **Find the volume of the spheroid generated by rotating the eclipse $\frac{x^2}{4}+\frac{y^2}{9}$ about the mirror axis.**

5. **Find the volume when the loop of the curve $y^2 = x\,(2x - 1)^2$ revolves about the x – axis.**

6. **Find the surface area of the solid generated by the revolution of the loop of the curve.**

   $X = t^2$, $y=\frac{t^3}{3}$ about the x – axis.

7. **Find the surface area of the solid generated by the revolution of the loop of the curve. $3ay^2 = x\,(x - a)^2$.**

8. **Prove that the area of the curved surface of a sphere of radius *a* intercepted between two parallel planes at distances b and c from the center is $2\pi a(b-c)$.**

9. **Find the surface area generated when the eclipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves about major axis.**

10. **The area between the line y = x and the curve $y = x^2$ is rotated about the axis of x. Find the volume formed.**

Summary

You have learnt the following from this chapter.

1. **The formulae for the volume of a solid of revolution obtained by rotating a curve about x - axis between the ordinate at x = a and x = b is:**

$$V=\pi\int_a^b y^2dx$$

About y – axis between the lines x = c and x = d is:

$$V = \pi \int_{c}^{d} x^2 dy$$

2. **The formula for the surface area of a solid of revolution about x – axis is:**

   S = $2\pi \int yds$ **if *s* is known.**

   S = $2\pi \int_{t=t_1}^{t=t_2} y \sqrt{\left(\frac{dx}{dt}\right) + \left(\frac{dy}{dt}\right)^2}$ ***dt*** **if the equation of the curve is in parametric form.**

3. **Application of the formulae for any solid of revolution.**

Further Reading

2. Additional pure mathematics By L. Harwood Clarke.
   Heinemann Educational Books Ltd London.

3. Advanced calculus By Watson Fulks.
   John Wiley and sons New York. Brisbane. Toronto.

CHAPTER 12
Numerical integration:
Trapezoidal and Simpson Rule

12.1 Introduction

In chapter eight we have learnt how to find the area under a curve using definite integral. Suppose that the equation of the curve is not known or the known function of x cannot be integrated. In such cases an approximation for the area under a curve may be found by several numerical methods. In this chapter we shall consider two important methods namely Trapezoidal rule and Simpson rule to find approximate areas under a curve.

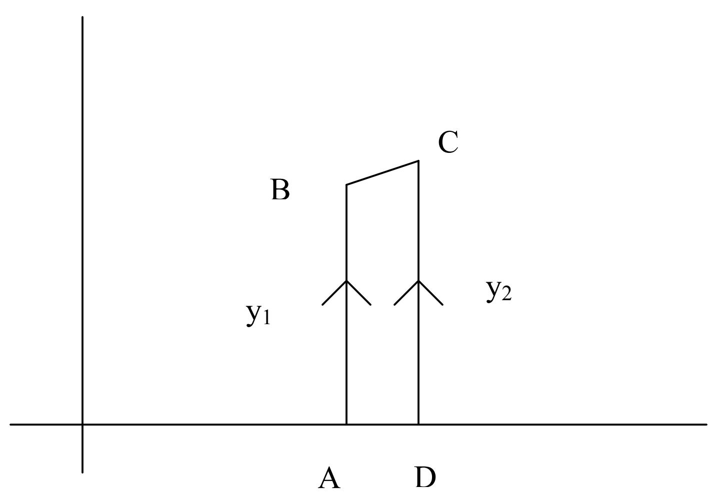
12.2 Objectives of the chapter:

By the end of this chapter you should be able to

- establish the Trapezoidal rule.
- derive the formula for the area under a parabola.
- establish Simpson's rule.
- apply the Trapezoidal rule and Simpson's rule.

12.3: Area of a trapezium

A quadrilateral in which one pair of opposite sides are parallel is called a trapezium. The area of a trapezium is the mean of parallel sides multiplied by the perpendicular distance between them.

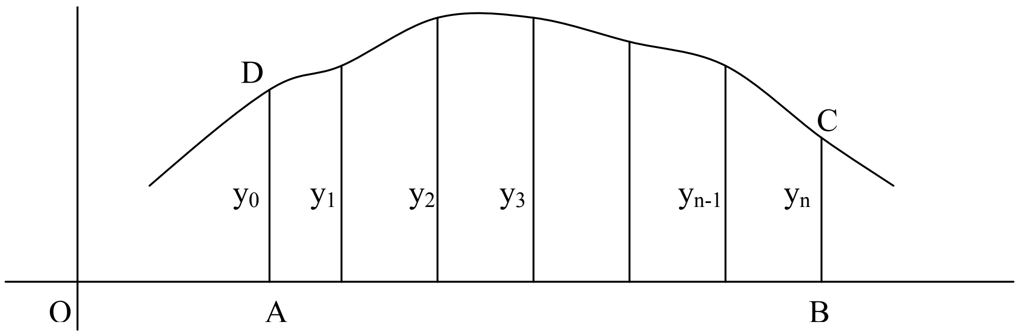


In the above figure A B C D is a trapezium in which A B is parallel to DC.
If AB=y_1, x = y_2 and the distance AD = h the area of the trapezium is

$= \dfrac{h(y_1 + y_2)}{2}$ or $\dfrac{h}{2}(y_1 + y_2)$

this

12.4: Trapezoidal rule



Consider a curve DC with end ordinates y_0 and y_n cutting x axis at A and B respectively. Divide the interval between A and B into n equal parts and erect ordinates at each point of division including the end points A and B. We wish to find the area ABCD under the curve DC. Let $\dfrac{AB}{n} = h$.

The area under curve is found by joining the ends of consecutive ordinates and by treating each trapezium so formed as an approximation for the area under the corresponding portion of the graph.

The area of the first trapezium is $\dfrac{h}{2}(y_0 + y_1)$

of the second is $\dfrac{h}{2}(y_1 + y_2)$

of the third is $\dfrac{h}{2}(y_2 + y_3)$

…………………………………

of the nth trapezium is $\dfrac{h}{2}(y_{n-1} + y_n)$

where h is the distance between consecutive ordinates.
Then an approximation for total area under the curve DC is

$$A = \frac{1}{2}h[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + ...(y_{n-1} + y_n)]$$

$$\mathbf{A} = \frac{1}{2}h(y_0 + 2y_1 + 2y_2 + 2y_3 + ... + 2y_{n-1} + y_n)$$

This is the trapezoidal rule

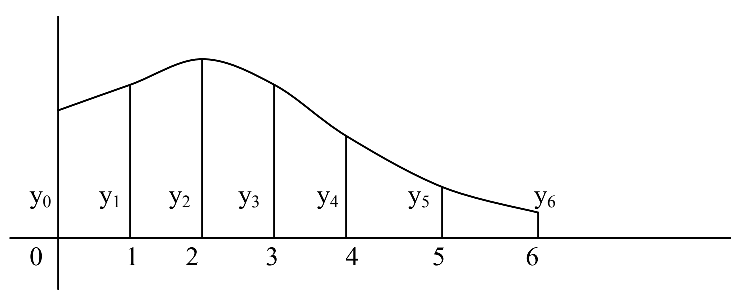
You can observe the following:

In finding the total sum,

i) y_0 and y_n remain the same

ii) $y_1, y_2, ... y_{n-1}$ are multiplied by two

iii) The total sum is multiplied by $\frac{h}{2}$, where $h = \frac{b-a}{n}$, to get the area under the curve



Given that OA is divided into 6 equal parts at and the corresponding ordinals are given below:

$x_0 = 0$ $y_0 = 8$

$x_1 = 1$ $y_1 = 12$

$x_2 = 2$ $y_2 = 14$

$x_3 = 3$ $y_3 = 11$

$x_4 = 4$	$y_4 = 9$
$x_5 = 5$	$y_5 = 3$
$x_6 = 6$	$y_6 = 1$

Solution

x=0 to 6 is divided into 6 equal parts; hence h $= \dfrac{6-0}{6}$

$= 1$

Total area = $\dfrac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6]$

$y_0 = 8$	$y_0 = 8$
$y_1 = 12$	$2y_1 = 24$
$y_2 = 14$	$2y_2 = 28$
$y_3 = 11$	$2y_3 = 22$
$y_4 = 9$	$2y_4 = 18$
$y_5 = 3$	$2y_5 = 6$
$y_6 = 1$	$y_6 = 1$
Total Sum	= 107

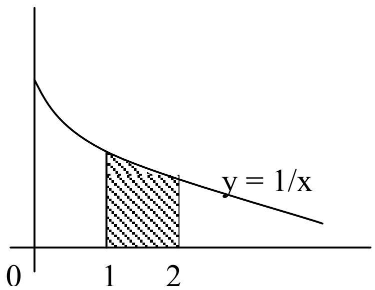
Total area $= \dfrac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6]$

$= \dfrac{1}{2}$ (107)

= 53.5 sq units

Example 2

Consider the curve y = $\dfrac{1}{x}$ in the figure given below between x = 1 and x = 2. The interval is divided into 5 equal parts. The ordinates at these points are y_0, y_1, y_2, y_3, y_4, y_5 which are given below:



y_0	y_1	y_2	y_3	y_4	y_5
1.0000	0.8333	0.7143	0.6250	0.5556	0.5000

a) Calculate $\int_1^2 \frac{1}{x}\,dx$ using trapezoidal rule with 5 intervals (Assume $\int \frac{1}{x}dx$ is not known)

b) Suppose that you know $\int \frac{1}{x}dx = \ln x + c$

Calculate the area under the curve.

c) Calculate the percentage error.

Solution

y_0 =1.0000 $\quad$ y_0 = 1.0000

y_1 = 0.8333 $\quad$ $2y_1$ = 1.6666

y_2 = 0.7143 $\quad$ $2y_2$ = 1.4286

y_3 = 0.6250 $\quad$ $2y_3$ = 1.2500

y_4 = 0.5556 $\quad$ $2y_4$ = 1.1112

y_5 =0.5000 $\quad$ y_5 = 0.5000

Total Sum = 6.9564

$$h = \frac{2-1}{5} = 0.2$$

Then $\int_1^2 \frac{1}{x}dx = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5]$

$= \frac{0.2}{2}[6.9564]$

$= \frac{1}{10}x6.9564$

$= 0.69564$

= 0.6956 to four decimals.

b) Now, $\int_1^2 \frac{1}{x}dx = [\ln x]_{x=1}^{x=2}$

$= \ln 2 - \ln 1$

$= 0.6931 - 0$

$= 0.6931$

c) To calculate the percentage error.

The percentage error = $\frac{0.6931 \sim 0.6956}{0.6931} x100$

$= 0.0025 \text{ x } 100$

$= 0.25\%$

[Percentage error = $\frac{\text{True value} \sim \text{Approximat value}}{\text{True value}} x100\ \%$]

Example3

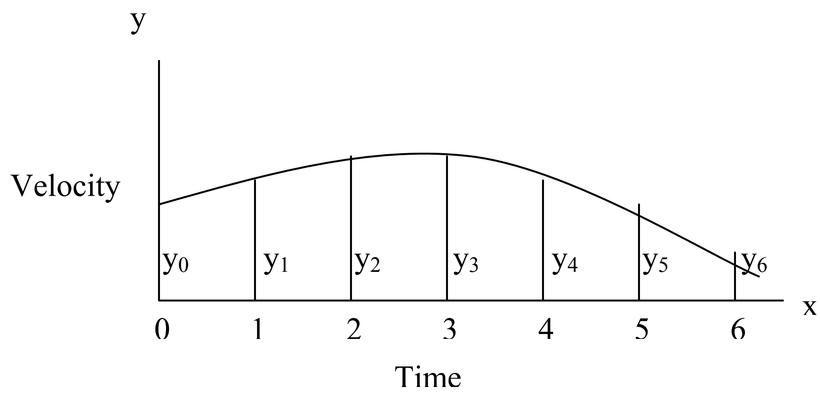
a) Find the distance traveled in a straight line by a particle in 6 seconds given that the velocity m per sec is connected with the time t sec as follows

t	0	1	2	3	4	5	6
v	10	12	15	16	11	5	3

b) Sketch the time velocity graph

(In velocity – time graph, area under graph represents the distance traveled)

Solution



Figure

We are required to find $\int_0^6 ydx$ or area under the curve between t = 0 and 6

y_0 =10	y_0 = 10
y_1 = 12	$2y_1$ = 24
y_2 = 15	$2y_2$ = 30
y_3 = 16	$2y_3$ = 32
y_4 = 11	$2y_4$ = 22
y_5 = 5	$2y_5$ = 10
y_6 = 3	y_6 = 3
Total Sum	131

$$h = \frac{6-0}{6} = 1 \text{ unit}$$

Total area under curve (distance travele)

$$= \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6\right]$$

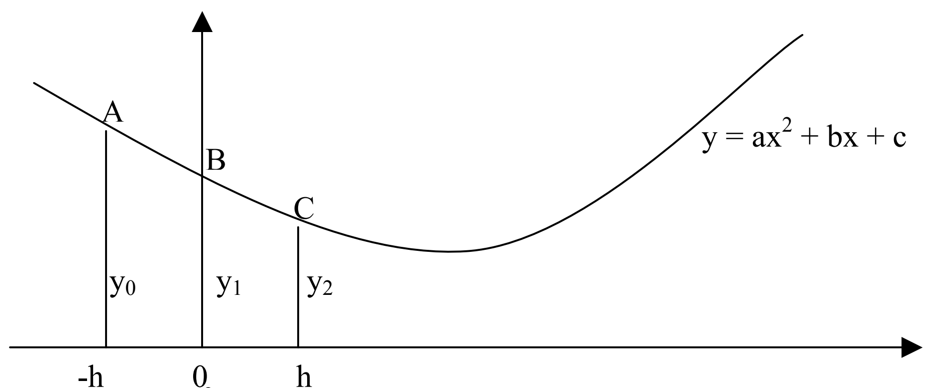
$$= \frac{1}{2} \quad (131)$$

= 65.5 metre

12.5 Principle of Simpsons Rule

The Trapezoidal Rule we have learnt is obtained by treating each **part of the curve** as a straight line, y = b x + c. Suppose that we now find a better approximation for each part of the curve in the form $y = ax^2 + b x + c$ which is a parabola.

Formula for the area under the curve $y = ax^2 + b x + c$ between x = -h and x = h



If the curve is $y = ax^2 + bx + c$ **the exact area** under the curve ABC is

$$A = \int_{-h}^{h} ax^2 + bx + c$$

$$= \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h}$$

$$= \frac{2ah^3}{3} + 2ch \qquad (1)$$

Now we shall find the area under the curve ABC in terms of the ordinates y_0, y_1, y_2

The curve is	$y = ax^2 + bx + c$	
When x = - h	$y_0 = ah^2 - bh + c$	
When x = 0	$y_1 = c$	(2)
When x = h	$y_2 = ah^2 + bh + c$	

Now $y_0 + y_2 = (ah^2 - bh + c) + (ah^2 + bh + c)$

$= 2ah^2 + 2c$

$= 2ah^2 + 2y_1$ using (2) (3)

From (3) $2ah^2 = y_0 + y_2 - 2y_1$ (4)

The actual area by integration, from (1) is

$$A = \frac{2ah^3}{3} + 2ch \quad (1)$$

$$= h\left(\frac{2ah^2}{3} + 2ch\right)$$

$$= \frac{h}{3}(2ah^2 + 6c) \quad (5)$$

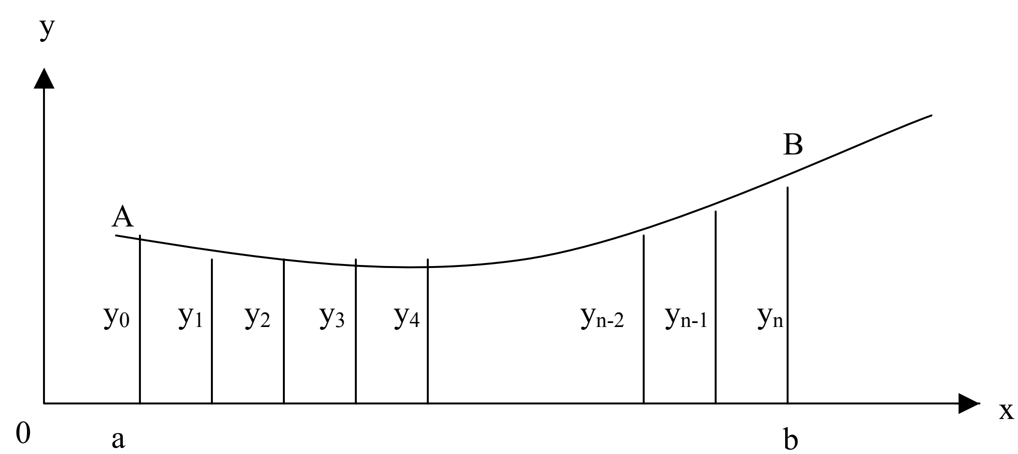
$$= \frac{h}{3}(y_0 - 2y_1 + y_2 + 6y_1) \text{ using (2) and (4) in (5)}$$

$$A = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Thus the area under the parabola ABC in terms of the ordinates is

$$A = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

12.6 Simpsons Rule for finding approximate area under a curve



As we have seen

The area between y_0 and y_2 $= \frac{1}{3} h(y_0 + 4y_1 + y_2)$

The area between y_2 and y_4 $= \frac{1}{3} h(y_2 + 4y_3 + y_4)$

The area between y_4 and y_6 $= \frac{1}{3} h(y_4 + 4y_5 + y_6)$

…………………………………………………………

The area between y_{n-2} and y_n $= \frac{1}{3} h(y_{n-2} + 4y_{n-1} + y_n)$

The total area under the curve AB is therefore

$$A = \frac{1}{3} h(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots\ldots. + 4y_{n-1} + y_n)$$

This is Simpsons Rule

You must understand the following to apply Simpsons rule.

i). Divide the interval ($a \leq x \geq b$) into **even** number of equal intervals, $h = \frac{b - a}{n}$

ii). In the addition, the first and the last ordinates y_0 and y_n remain the same

iii). Add four times the ordinates y_1, y_3, y_5, ,,,,,, and two times the ordinates y_2, y_4, y_6, (alternatively four times odd and two times even ordinates respectively) are added.

iv). The total sum is multiplied by one third of the width of equal interval (or multiply the sum by $\frac{h}{3}$)

To apply Simsons Rule the number of divisions of the interval must be even
In other words n should be an even positive integer to apply Simpsons Rule.

Example 4

Find $\int_0^6 y\,dx$, given the following values, using

i) Simpsons Rule for the interval ($0 \leq x \leq 6$)

ii) Trapezoidal Rule for the interval ($0 \leq x \leq 6$)

X	0	1	2	3	4	5	6
Y	8	12	14	11	9	3	1

Solution

i) Area under curve using Simpsons Rule

$Y_0 = 8$ $Y_0 = 8$

$Y_1 = 12$	$4Y_1 = 48$
$Y_2 = 14$	$2Y_2 = 28$
$Y_3 = 11$	$4Y_3 = 44$
$Y_4 = 9$	$2Y_4 = 18$
$Y_5 = 3$	$4Y_5 = 12$
$Y_6 = 1$	$Y_6 = 1$
Total Sum	159

$$h = \frac{6 - 0}{6} = 1$$

Approximate area in therefore.

$$A = \frac{1}{3} h(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$$

$$= \frac{1}{3}(1)(159)$$

$$= 53 \text{ sq units}$$

ii) Area under curve using Trapezoidal Rule (≤ 6)

$Y_0 = 8$	$Y_0 = 8$
$Y_1 = 12$	$2Y_1 = 24$
$Y_2 = 14$	$2Y_2 = 28$
$Y_3 = 11$	$2Y_3 = 22$
$Y_4 = 9$	$2Y_4 = 18$
$Y_5 = 3$	$2Y_5 = 6$
$Y_6 = 1$	$Y_6 = 1$

Total Sum 107

$$h = \frac{6 - 0}{6} = 1$$

Approximate area is therefore

$$A = \frac{1}{2} h (y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 4y_5 + y_6)$$

$$= \frac{1}{2} (1)(107)$$

$$= 53.5 \text{ sq units}$$

Exercise 12

1. State the formula for

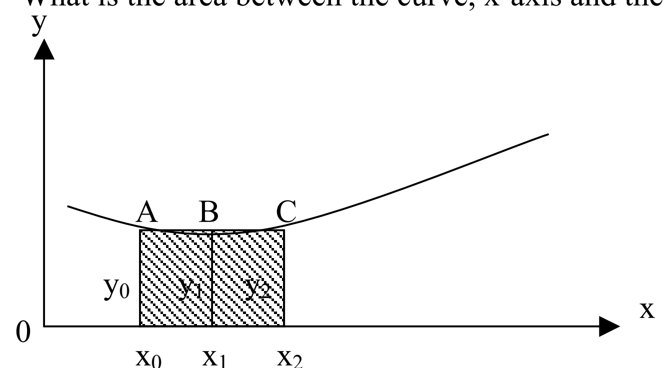
   i). Trapezoidal Rule

   ii). Simpsons Rule

2. What is the area of a trapezium whose ordinates are y_0 and y_1 and the distance between them is h.

3. Three consecutive ordinates are y_0 , y_1, and y_2, and each equal interval is h units.

   What is the area between the curve, x-axis and the ordinates y_0 and y_2

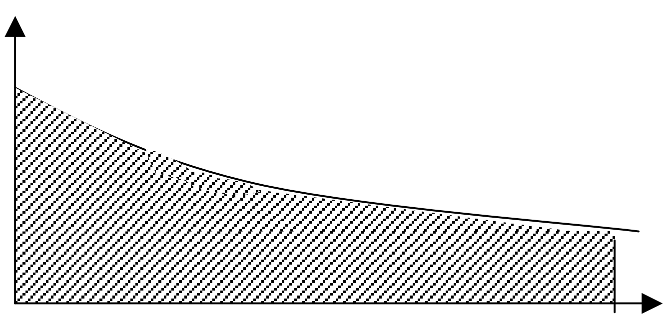


4. If y_0 = 2 units, y_1, = 3 units and y_2 = 5 units calculate the area included by the curve ABC, the ordinates y_0, y_2 and x-axis.

   i) If AB and BC are considered as straight lines

   ii) If the curve ABC is considered as a parabola $y = ax^2 + bx + c$.

5.



In the figure above, the equation of the curve is $y = \frac{1}{1 + x^2}$. Find the area under the curve between x = 0 and x = 1 using

i) Trapezoidal Rule with ten equal parts

ii) Simpsons Rule with 10 equal parts

6. A car starts from rest and its velocity in meter per second over the first ten seconds is given by the following table:

t	0	1	2	3	4	5	6	7	8	9	10
v	2	2.4	2.8	3.2	4.0	4.2	4.5	6.0	6.2	6.3	6.4

Find the distance traveled by the car at the end of 10 seconds.

7. Find an approximate value for $\int_0^3 (x^2 + x)dx$ using

i) Simpsons Rule by taking 10 equal intervals

ii) Trapezoidal Rule by taking 10 equal intervals

iii) Integration

8. Evaluate $\int_0^1 e^{-x^2} dx$ by Simpsons Rule taking ten intervals.

9. The distance moved by a body starting from rest is given by the following table.

Time t sec	0	1	2	3	4	5	6
Distance s met	0	5	17	34	57	84	115

Plot a velocity-time graph and estimate the speed after 6 seconds.

Summary

You have learnt the following from this chapter:

1. Area of trapezium = $\frac{h}{2}(y_0 + y_1)$ y_0 and y_1 are the ordinates and h is the distance between y_1 and y_2.
2. Area under a parabola $y = ax^2 + bx + c$ when there are two intervals bounded by y_0, y_1 and y_2 = $\frac{h}{3}(y_0 + 4y_1 + y_2)$.
3. Area under a curve using Trapezoidal Rule is

$$= \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + ... + 2y_{n-1} + y_n)$$

4. Area under a curve using Simpson's Rule is

$$A = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_2 + 2y_3 + ... + y_n).$$

where n should be an even integer.

5. Applying the Trapezoidal Rule and Simpsons Rule for finding the approximate area under a curve between x = a and x = b

Further Reading

1. Mathematics of Physics and Modern Engineering
   By Sokolnikoff
   Mc Graw – Hill Bok company, INC New York Toronto London

2. Additional Pure Mathematics
   By L. Harwood Clarke
   Heinmann Educational Books Ltd
   London